

On an asymptotic series of Ramanujan

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Abstract An asymptotic series in Ramanujan's second notebook (Entry 10, Chap. 3) is concerned with the behavior of the expected value of $\phi(X)$ for large λ where X is a Poisson random variable with mean λ and ϕ is a function satisfying certain growth conditions. We generalize this by studying the asymptotics of the expected value of $\phi(X)$ when the distribution of X belongs to a suitable family indexed by a convolution parameter. Examples include the binomial, negative binomial, and gamma families. Some formulas associated with the negative binomial appear new.

Keywords Asymptotic expansion · Binomial distribution · Central moments · Cumulants · Gamma distribution · Negative binomial distribution · Poisson distribution · Ramanujan's notebooks

Mathematics Subject Classification (2000) Primary 34E05 · Secondary 60E05

1 An asymptotic series of Ramanujan

A version (modified from [2]) of Entry 10 in Chap. 3 of Ramanujan's second notebook reads

Theorem 1 *Let $\phi(x)$, $x \in [0, \infty)$, denote a function of at most polynomial growth as x tends to ∞ . Suppose there exist constants $x_0 > 0$ and $A \geq 1$, and a function $G(x)$ of at most polynomial growth as $x \rightarrow \infty$ such that for each nonnegative integer m and all $x > x_0$, the derivatives $\phi^{(m)}(x)$ exist and satisfy*

$$\left| \frac{\phi^{(m)}(x)}{m!} \right| \leq G(x) \left(\frac{A}{x} \right)^m. \quad (1)$$

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Assume that there exists a positive constant c such that

$$G(x) \gg e^{-c\sqrt{x}} \tag{2}$$

as $x \rightarrow \infty$. Put

$$\phi_\infty(x) = e^{-x} \sum_{k=0}^\infty \frac{x^k \phi(k)}{k!}.$$

Then for any fixed positive integer M ,

$$\phi_\infty(x) = \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor (n+1)/2 \rfloor + 1}^n b_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O(G(x)x^{-M}), \tag{3}$$

as x tends to ∞ , where $\lfloor (n+1)/2 \rfloor$ denotes the integer part of $(n+1)/2$ and the numbers b_{kn} are defined recursively by

$$\begin{aligned} b_{kk} &= 1, & k \geq 2; \\ b_{kn} &= 0, & n < k \text{ or } n > 2k - 2; \\ b_{k+1,n+1} &= nb_{k,n-1} + (n - k + 1)b_{kn}, & k \leq n \leq 2k - 1. \end{aligned}$$

This result may seem hard to penetrate at first glance. Its relevance, however, is easily appreciated through interesting examples such as [2, 3]

$$e^{-x} \sum_{k=0}^\infty \frac{\sqrt{k}x^k}{k!} = \sqrt{x} \left(1 - \frac{1}{8x} - \frac{7}{128x^2} + O\left(\frac{1}{x^3}\right) \right) \tag{4}$$

and

$$e^{-x} \sum_{k=0}^\infty \frac{x^k \log(k+1)}{k!} = \log(x) + \frac{1}{2x} + \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right),$$

both valid as $x \rightarrow \infty$; by choosing $\phi(x) = \log \Gamma(x+1)$ in (3), an asymptotic formula for the Shannon entropy of the Poisson distribution can also be obtained (see [6]).

The first goal of this note is a formal, probabilistic derivation of Theorem 1. The starting point is the observation that $\phi_\infty(x) = E\phi(U)$, where U is a Poisson random variable with mean x and E denotes expectation. Based on this we present in Sect. 2 a more general version (Theorem 2) of Theorem 1 by considering U distributed as some distribution other than the Poisson, e.g., a gamma distribution or a binomial distribution. As illustrations, we derive asymptotic expansions for digamma functions and for inverse moments of certain positive random variables. We prove Theorem 2 in Sect 3.

Noting that $\phi_\infty(x) = E\phi(U)$ where U has the $Po(x)$ distribution, we expand $\phi(U)$ as a Taylor series

$$\phi(U) = \phi(x) + \sum_{n=1}^{2M-2} \frac{(U-x)^n \phi^{(n)}(x)}{n!} + \dots,$$

and formally take the expectation term by term:

$$E\phi(U) = \phi(x) + \sum_{n=1}^{2M-2} \frac{E(U-x)^n \phi^{(n)}(x)}{n!} + \dots \tag{5}$$

The quantity $\mu_n = E(U-x)^n$ is the n th central moment of the $Po(x)$ distribution. The first few μ_n 's are

$$(\mu_1, \mu_2, \mu_3, \mu_4) = (0, x, x, 3x^2 + x),$$

and they obey the well-known recursion (see [19], Lemma 3, for example)

$$\mu_{n+1} = x \left(\frac{d\mu_n}{dx} + n\mu_{n-1} \right), \quad n \geq 2,$$

from which we obtain, by comparing the coefficients of x^{n-k+1} and using the definition of b_{kn} ,

$$\mu_n = \sum_{k=0}^n b_{kn} x^{n-k+1}, \quad n \geq 2. \tag{6}$$

The double sum in (3) is the result of substituting (6) in (5) and noting that $b_{kn} = 0$ if $k \leq \lfloor (n+1)/2 \rfloor$. Based on this it is also clear that μ_n is a polynomial in x of degree $\lfloor n/2 \rfloor$, which implies that, given the condition (1), the ‘‘leading term of the remainder’’ in (5),

$$\frac{E(U-x)^{2M-1} \phi^{(2M-1)}(x)}{(2M-1)!} = \frac{\mu_{2M-1} \phi^{(2M-1)}(x)}{(2M-1)!},$$

is $O(G(x)x^{-M})$.

The above derivation is, of course, strictly formal. However, it can be made rigorous under the stated conditions; see Berndt [2] and Evans [5]. Berndt actually proved a modification of (3) where the order of summation over k and n on the right hand side is inverted and certain higher order terms of the resulting sum are absorbed in the $O(G(x)x^{-M})$ term.

2 A general version

The formal derivation in Sect. 1 suggests that it is possible to generalize Theorem 1 if we let U have a suitable distribution other than the Poisson. Noting the key role played by the central moments of U , we give a version of Theorem 1 by imposing conditions on the moment generating function (mgf) of U . An introduction to moment generating functions can be found in probability texts such as Gut [9]. A useful property is that, if an mgf exists in a neighborhood of zero, then for all $m \geq 1$, the m th moment exists and can be obtained by differentiating the mgf m times.

Theorem 2 Let $\phi(x), x \in [0, \infty)$, denote a Borel measurable function that can be bounded in absolute value by a polynomial in x . Let M be a fixed positive integer. Suppose there exist a constant $A \geq 1$ and a function $G(x)$ of at most polynomial growth such that for $1 \leq m \leq 2M$ and all sufficiently large x , the derivatives $\phi^{(m)}(x)$ exist and satisfy

$$\left| \frac{\phi^{(m)}(x)}{m!} \right| \leq G(x) \left(\frac{A}{x} \right)^m. \tag{7}$$

Assume there exist $\eta \in (0, 1)$ and a constant B such that for all sufficiently large x ,

$$G(y) \leq BG(x) \quad \text{whenever } |y - x| \leq \eta x. \tag{8}$$

Let Ω be an unbounded subset of $[0, \infty)$ and let $U_x, x \in \Omega$, be a family of nonnegative random variables. Assume there exist a constant $\delta > 0$ and a function $g(s)$ such that for all $x \in \Omega$, the mgf of U_x exists in the interval $(-\delta, \delta)$ and satisfies

$$Ee^{sU_x} = e^{xg(s)}, \quad s \in (-\delta, \delta). \tag{9}$$

Assume $g'(0) = 1$ in addition. Then

$$E\phi(U_x) = \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor (n+1)/2 \rfloor + 1}^n c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O(G(x)x^{-M}), \tag{10}$$

as x tends to ∞ , where c_{kn} are constants that depend only on the function $g(s)$, and are determined by

$$E(U_x - x)^n = \sum_{k=0}^n c_{kn} x^{n-k+1}, \quad n \geq 2.$$

Evidently, Theorem 1 is the special case $g(s) = e^s - 1$, except for the assumption (8) on $G(x)$ which replaces (2). This new assumption does not appear very restrictive as we shall see from the examples later in this section; however it does make the proof of Theorem 2 more straightforward. We also relax the assumption on $\phi(x)$ slightly by requiring only $2M$ derivatives.

It should be emphasized that the function $g(s)$ in (9) does not depend on x . Also note that $g(s)$ is analytic in $s \in (-\delta, \delta)$ given the existence of the mgf. Aside from the Poisson, examples of distribution families that satisfy (9) include the binomial, negative binomial, and gamma families. In general, suppose Y_1, Y_2, \dots is a sequence of independent and identically distributed (i.i.d.), nonnegative, nondegenerate random variables whose mgf exists in a neighborhood of zero. Then the family of random variables $\{\sum_{k=1}^n Y_k, n = 1, 2, \dots\}$ (n is known as a convolution parameter) has mgf

$$E \exp\left(s \sum_{k=1}^n Y_k\right) = \exp\left(n \log\left(Ee^{sY_1}\right)\right), \quad n = 1, 2, \dots,$$

which is of the form (9) with $g(s) = (EY_1)^{-1} \log(Ee^{sY_1})$, if we index the family by its mean $x = nEY_1$. This shows that Theorem 2 is potentially applicable to a wide range of problems.

Example 1 For a fixed $p \in (0, 1)$, consider the family $U_x, x = p, 2p, 3p, \dots$, where U_x has the binomial distribution $\text{Bi}(n, p), n = x/p$. The first few central moments of U_x are given by $(q = 1 - p)$

$$(\mu_2, \mu_3, \mu_4) = (qx, (q - p)qx, 3q^2x^2 + q(1 - 6pq)x).$$

Since $\text{Bi}(n, p)$ is a sum of n i.i.d. Bernoulli(p) random variables, (9) is satisfied.

- Given r (real) and $a > 0$, let $\phi(x) = G(x) = (x + a)^{-r}$. It is easy to verify (7) and (8); thus we have

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} (k + a)^{-r} &= (np + a)^{-r} + \frac{qr(r + 1)}{2} (np)(np + a)^{-r-2} \\ &\quad - \frac{(q - p)qr(r + 1)(r + 2)}{6} (np)(np + a)^{-r-3} \\ &\quad + \frac{q^2r(r + 1)(r + 2)(r + 3)}{8} (np)^2 (np + a)^{-r-4} \\ &\quad + O(n^{-r-3}), \end{aligned} \tag{11}$$

as $n \rightarrow \infty$.

- If we let $\phi(x) = x^{-r}, x \geq 1$ and $\phi(x) = 0, x < 1$, then we obtain an asymptotic expansion for

$$\sum_{k=1}^n \binom{n}{k} p^k q^{n-k} k^{-r} \tag{12}$$

by simply substituting $a = 0$ in the right hand side of (11). When r is a positive integer, (12) is sometimes known as the r th inverse moment of the binomial. The problem of inverse moments has a long history in statistics (see, for example, Stephan [16], Grab and Savage [8], and David and Johnson [4]). More recently, expansions for (12) have been considered by Marciniak and Wesolowski [14] (see also Rempala [15]) for $r = 1$, and by Žnidarič [19] for general r . Žnidarič [19] also gives a brief historical account with many references.

A special case corresponding to $r = -1/2$ is

$$\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} \sqrt{k} = \sqrt{np} \left[1 - \frac{q/8}{np} + \frac{(q - p)q/16 - 15q^2/128}{(np)^2} + O\left(\frac{1}{n^3}\right) \right],$$

which is the binomial analog of (4) considered by Ramanujan [2].

- Let $\phi(x) = \log(x + \beta)$ for a fixed $\beta > 0$ and let $G(x) \equiv 1$. We have (as $n \rightarrow \infty$)

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} \log(k + \beta) \\ &= \log(np + \beta) - \frac{npq}{2} (np + \beta)^{-2} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{(q-p)q}{3}(np)(np+\beta)^{-3} - \frac{3q^2}{4}(np)^2(np+\beta)^{-4} \\
 &+ O(n^{-3}).
 \end{aligned}
 \tag{13}$$

The problem of approximating the left hand side of (13) appears in Krichevskiy [13] in an information theoretic context; see also Jacquet and Szpankowski [10, 11], who give an alternative derivation of (13) using the method of analytic poissonization and depoissonization. Flajolet [7] also considers similar problems using singularity analysis.

Example 2 For a fixed $p \in (0, 1)$, consider the negative binomial family $NB(n, p)$ whose probability mass function is $f(k; n, p) = \binom{n+k-1}{k} p^n q^k, k = 0, 1, \dots$, where $q = 1 - p$. The mean is nq/p and the first few central moments are

$$(\mu_2, \mu_3, \mu_4) = \left(\frac{nq}{p^2}, \frac{nq(1+q)}{p^3}, \left[3 + \frac{6}{n} + \frac{p^2}{nq} \right] \frac{(nq)^2}{p^4} \right).$$

Similar to the binomial case, as $NB(n, p)$ is a sum of n i.i.d. $\text{geometric}(p)$ random variables, (9) is satisfied and Theorem 2 is applicable for an appropriate $\phi(x)$.

- Take $\phi(x) = (x + a)^{-r}, a > 0$. We have, as $n \rightarrow \infty$,

$$\begin{aligned}
 &\sum_{k=0}^{\infty} \binom{n+k-1}{k} p^n q^k (k+a)^{-r} \\
 &= \left(\frac{nq}{p} + a \right)^{-r} + \frac{r(r+1)}{2p} \left(\frac{nq}{p} \right) \left(\frac{nq}{p} + a \right)^{-r-2} \\
 &\quad - \frac{(1+q)r(r+1)(r+2)}{6p^2} \left(\frac{nq}{p} \right) \left(\frac{nq}{p} + a \right)^{-r-3} \\
 &\quad + \frac{r(r+1)(r+2)(r+3)}{8p^2} \left(\frac{nq}{p} \right)^2 \left(\frac{nq}{p} + a \right)^{-r-4} \\
 &\quad + O(n^{-r-3}).
 \end{aligned}
 \tag{14}$$

- As in the binomial case, we obtain an asymptotic expansion for

$$\sum_{k=1}^{\infty} \binom{n+k-1}{k} p^n q^k k^{-r}
 \tag{15}$$

for real r by substituting $a = 0$ in the right hand side of (14). Expansions for (15) have been considered by Marciniak and Wesolowski [14] and Rempala [15] for the special case $r = 1$, and by Wuyungaowa and Wang [18] for integer $r \geq 0$.

- Let $\phi(x) = \log(x + \beta)$ for a fixed $\beta > 0$. We have

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} p^n q^k \log(k + \beta) = \log \left(\frac{nq}{p} + \beta \right) - \frac{1}{2p} \left(\frac{nq}{p} \right) \left(\frac{nq}{p} + \beta \right)^{-2}$$

$$\begin{aligned}
 &+ \frac{(1+q)}{3p^2} \left(\frac{nq}{p}\right) \left(\frac{nq}{p} + \beta\right)^{-3} \\
 &- \frac{3}{4p^2} \left(\frac{nq}{p}\right)^2 \left(\frac{nq}{p} + \beta\right)^{-4} + O(n^{-3}).
 \end{aligned}$$

Example 3 Consider the gamma family $\text{Gam}(x, 1)$, $x > 0$, whose density function is $f(u; x) = u^{x-1} e^{-u} / \Gamma(x)$, $u > 0$. The mean is x and the first few central moments are

$$(\mu_2, \mu_3, \mu_4) = (x, 2x, 3x^2 + 6x).$$

The moment generating function is $(1 - s)^{-x}$, $s < 1$, which is of the form (9) with $g(s) = -\log(1 - s)$.

Take $\phi(x) = G(x) = x \log(x)$. We have

$$\frac{1}{\Gamma(x)} \int_0^\infty u \log(u) u^{x-1} e^{-u} du = x \log(x) + \frac{1}{2} - \frac{1}{12x} + O\left(\frac{\log(x)}{x^2}\right),$$

as $x \rightarrow \infty$. Noting $\Gamma(x + 1) = x\Gamma(x)$, we may write

$$\frac{\Gamma'(x + 1)}{\Gamma(x + 1)} = \log(x) + \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{\log(x)}{x^3}\right), \tag{16}$$

which is a familiar asymptotic formula for the digamma function ([1], p. 259). By expanding for one more term we can replace $O(x^{-3} \log(x))$ by $O(x^{-3})$ in (16). A full asymptotic expansion can be recovered by applying (10) and using the following recursion between the central moments of $\text{Gam}(x, 1)$ (see [17]):

$$\mu_k = (k - 1)(\mu_{k-1} + x\mu_{k-2}), \quad k \geq 2.$$

3 Proof of Theorem 2

Our proof follows Berndt [2]. In the setting of Theorem 2 we have

Lemma 1 *Let $h(u)$, $u \in [0, \infty)$, be a Borel measurable function that can be bounded in absolute value by a polynomial. Then for a fixed $t \in (0, 1)$, both $E I(U_x < tx)h(U_x)$ and $E I(U_x > x/t)h(U_x)$ tend to 0 exponentially fast as x tends to ∞ , where $I(\cdot)$ is the indicator function.*

Proof Observe that $xg(s)$, the cumulant generating function of U_x , is an analytic function of s (real) in a neighborhood of zero. Because $g(0) = 0$, $g'(0) = 1$ and $t \in (0, 1)$, we may choose $r, \epsilon > 0$ small enough such that both $g(r) < r/t$ and $g(r + \epsilon) < r/t$. Since $|h(u)|$ is bounded by a polynomial, there exists a constant D such that $|h(u)| < e^{\epsilon u} + D$ for all $u \in [0, \infty)$. We have

$$\begin{aligned}
 |E I(U_x > x/t)h(U_x)| &\leq E(e^{\epsilon U_x} + D)e^{r(U_x - x/t)} \\
 &= e^{x[g(r+\epsilon) - r/t]} + D e^{x[g(r) - r/t]},
 \end{aligned}$$

which tends to zero exponentially as $x \rightarrow \infty$. The proof for $EI(U_x < tx)h(U_x)$ is similar and hence omitted. \square

Proof of Theorem 2 Throughout we assume that x is sufficiently large. Define intervals $I_1 = [0, (1 - \eta)x)$, $I_2 = [(1 - \eta)x, (1 + \eta)x)$ and $I_3 = [(1 + \eta)x, \infty)$, where η is as specified in (8).

By Lemma 1, both

$$EI(U_x \in I_1)\phi(U_x) \tag{17}$$

and

$$EI(U_x \in I_3)\phi(U_x) \tag{18}$$

tend to zero exponentially as $x \rightarrow \infty$.

Consider the Taylor polynomial

$$\psi(y) = \sum_{k=0}^{2M-1} \frac{\phi^{(k)}(x)}{k!} (y - x)^k.$$

Since for any $y \in I_1$,

$$|\psi(y)| \leq \sum_{k=0}^{2M-1} \left| \frac{\phi^{(k)}(x)}{k!} \right| x^k \equiv q(x),$$

we have

$$|EI(U_x \in I_1)\psi(U_x)| \leq q(x)EI(U_x \in I_1).$$

From (7) it follows that $q(x)$ has at most polynomial growth as $x \rightarrow \infty$; by Lemma 1 we know that

$$EI(U_x \in I_1)\psi(U_x) \tag{19}$$

tends to zero exponentially as $x \rightarrow \infty$.

Similarly, for any $y \in I_3$, we have

$$|\psi(y)| \leq \sum_{k=0}^{2M-1} \left| \frac{\phi^{(k)}(x)}{k!} \right| y^k,$$

and hence

$$|EI(U_x \in I_3)\psi(U_x)| \leq \sum_{k=0}^{2M-1} \left| \frac{\phi^{(k)}(x)}{k!} \right| EI(U_x \in I_3)U_x^k.$$

By Lemma 1, each of $EI(U_x \in I_3)U_x^k$, $k \leq 2M - 1$, tends to zero exponentially as $x \rightarrow \infty$. By (7), each of $\phi^{(k)}(x)$ has at most polynomial growth as $x \rightarrow \infty$. Overall

$$EI(U_x \in I_3)\psi(U_x) \tag{20}$$

tends to zero exponentially as $x \rightarrow \infty$.

For any $y \in I_2$, there exists some point ζ between x and y such that

$$\begin{aligned} |\phi(y) - \psi(y)| &= \left| \frac{\phi^{(2M)}(\zeta)}{(2M)!} \right| |y - x|^{2M} \\ &\leq G(\zeta) \left(\frac{A}{(1 - \eta)x} \right)^{2M} |y - x|^{2M} \\ &\leq BG(x) \left(\frac{A}{(1 - \eta)x} \right)^{2M} |y - x|^{2M}, \end{aligned}$$

where (7) and (8) are used in the inequalities. Letting $C = B[A/(1 - \eta)]^{2M}$, we have

$$|EI(U_x \in I_2)[\phi(U_x) - \psi(U_x)]| \leq C \frac{G(x)}{x^{2M}} E|U_x - x|^{2M}.$$

We now consider the n th central moment of U_x , $\mu_n = E(U_x - x)^n$, as a function of x . (Note that the mean of U_x is x as $EU_x = xg'(0) = x$.) Expand $xg(s)$ around $s = 0$ to get

$$xg(s) = \sum_{j=1}^{\infty} \frac{xg^{(j)}(0)s^j}{j!}.$$

Note that the coefficient $xg^{(j)}(0)$ is the j th cumulant of U_x , and, according to the well-known relation between central moments and cumulants (see [12] or [17], for example)

$$\mu_n = \sum_{j=0}^{n-2} \binom{n-1}{j} \mu_j xg^{(n-j)}(0), \quad n \geq 2, \tag{21}$$

with $\mu_0 = 1$ and $\mu_1 = 0$. Based on (21), it is easy to show by induction that μ_n is a polynomial in x of degree at most $\lfloor n/2 \rfloor$, its coefficients depending only on the function $g(s)$. Hence, for large x we have $E(U_x - x)^{2M} = O(x^M)$ and

$$EI(U_x \in I_2)[\phi(U_x) - \psi(U_x)] = O(G(x)x^{-M}).$$

Combined with the exponentially small items (17), (18), (19) and (20), this gives

$$E[\phi(U_x) - \psi(U_x)] = O(G(x)x^{-M}).$$

It remains to calculate $E\psi(U_x)$. We have, by the definition of c_{kn} ,

$$\begin{aligned} E\psi(U_x) &= \sum_{n=0}^{2M-1} E(U_x - x)^n \frac{\phi^{(n)}(x)}{n!} \\ &= \phi(x) + \sum_{n=2}^{2M-1} \sum_{k=0}^n c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} \end{aligned}$$

$$\begin{aligned}
&= \phi(x) + \sum_{n=2}^{2M-1} \sum_{k=\lfloor(n+1)/2\rfloor+1}^n c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} \\
&= \phi(x) + \sum_{n=2}^{2M-2} \sum_{k=\lfloor(n+1)/2\rfloor+1}^n c_{kn} x^{n-k+1} \frac{\phi^{(n)}(x)}{n!} + O(G(x)x^{-M}).
\end{aligned}$$

Note that the inner sum over k is curtailed because the degree of μ_n is at most $\lfloor n/2 \rfloor$, i.e., $c_{kn} = 0$ if $k \leq \lfloor (n+1)/2 \rfloor$. As a consequence of (7), the term corresponding to $n = 2M - 1$ in the outer sum is written as $O(G(x)x^{-M})$ in the last equality. The proof of (10) is now complete. \square

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