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Positivity

Cloning by positive maps in von Neumann algebras

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Abstract We investigate cloning in the general operator algebra framework in arbitrary dimension assuming only positivity instead of strong positivity of the cloning operation, generalizing thus results obtained so far under that stronger assumption. The weaker positivity assumption turns out quite natural when considering cloning in the general C^*-algebra framework.

Keywords Cloning states · Positive maps · von Neumann algebras

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1 Introduction

Cloning and broadcasting of quantum states has recently become an important topic in Quantum Information Theory. Since its first appearance in [5,14] in the form of a no-cloning theorem it has been investigated in various settings. The most interesting ones are the Hilbert space setup considered in [3,8], and the setup of generic probabilistic models considered in [1,2]. A common feature of these approaches consists in restricting attention to the finite-dimensional models; moreover, in the Hilbert space setup the map defining cloning or broadcasting is assumed to be completely positive. In [6] cloning and broadcasting are investigated in the general operator algebra framework,

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i.e. instead of the full algebra of all linear operators on a finite-dimensional Hilbert space an arbitrary von Neumann algebra on a Hilbert space of arbitrary dimension is considered; moreover, the cloning (broadcasting) operation is assumed to be a Schwarz (called also *strongly positive*) map instead of completely positive. The present paper can be viewed as a supplement to [6]; we follow the same approach weakening further the assumption on positivity of the cloning operation, namely, we assume only that it is positive. This weaker assumption is all we can hope for while considering the general problem of cloning in C^*-algebras (cf. [7]), thus the main theorem of the present paper (Theorem 8) is of importance for cloning in C^*-algebras. However, in our approach interesting results are obtained only for cloning, the problem of broadcasting in such a setup is still an open question.

It is probably worth mentioning that although cloning and broadcasting have their origins in quantum information theory, they are nevertheless purely mathematical objects concerning states on some C^*- or W^*-algebras, and thus their investigation is of independent mathematical interest. This is the approach taken in the present paper—we do not refer to any physical notions, however it is still possible (and hoped for) that the results obtained can find some physical applications.

The main results of this work are as follows. It is shown that all states cloneable by an operation are extreme points of the set of all states broadcastable by this operation, and a description of some algebra associated with the cloneable states is given. Moreover, for an arbitrary subset Γ of the normal states of a von Neumann algebra it is proved that the states in Γ are cloneable if and only if they have mutually orthogonal supports—the result obtained in [6] under the assumption that the cloning operation is a Schwarz map. Finally, the problem of uniqueness of the cloning operation is considered.

2 Preliminaries and notation

Let $\mathcal{M}$ be an arbitrary von Neumann algebra with identity $\mathbb{1}$ acting on a Hilbert space $\mathcal{H}$. The predual $\mathcal{M}_*$ of $\mathcal{M}$ is a Banach space of all *normal*, i.e. continuous in the σ-weak topology linear functionals on $\mathcal{M}$.

A *state* on $\mathcal{M}$ is a bounded positive linear functional $\rho\colon \mathcal{M} \to \mathbb{C}$ of norm one. For a normal state ρ its *support*, denoted by $\mathrm{s}(\rho)$, is defined as the smallest projection in $\mathcal{M}$ such that $\rho(\mathrm{s}(\rho)) = \rho(\mathbb{1})$. In particular, we have

$$\rho(\mathrm{s}(\rho)x) = \rho(x\,\mathrm{s}(\rho)) = \rho(x), \qquad x \in \mathcal{M},$$

and if $\rho(\mathrm{s}(\rho)x\,\mathrm{s}(\rho)) = 0$ for $\mathrm{s}(\rho)x\,\mathrm{s}(\rho) \geqslant 0$ then $\mathrm{s}(\rho)x\,\mathrm{s}(\rho) = 0$.

Let $\{\rho_\theta : \theta \in \Theta\}$ be a family of normal states on a von Neumann algebra $\mathcal{M}$. Define the support of this family by

$$e = \bigvee_{\theta\in\Theta} \mathrm{s}(\rho_\theta).$$

The family $\{\rho_\theta : \theta \in \Theta\}$ is said to be *faithful* if for each positive element $x \in \mathcal{M}$ from the equality $\rho_\theta(x) = 0$ for all $\theta \in \Theta$ it follows that $x = 0$. It is seen that the faithfulness of this family is equivalent to the relation $e = \mathbb{1}$; moreover, if $\rho_\theta(exe) = 0$ for all $\theta \in \Theta$ and $exe \geqslant 0$ then $exe = 0$.

By a W^*-algebra of operators acting on a Hilbert space we shall mean a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ with identity, closed in the weak-operator topology. A typical example (and in fact the only one utilized in the paper) is the algebra $p\mathcal{M}p$, where p is a projection in $\mathcal{M}$. For arbitrary $\mathcal{R} \subset \mathbb{B}(\mathcal{H})$ we denote by $W^*(\mathcal{R})$ the smallest W^*-algebra of operators on $\mathcal{H}$ containing $\mathcal{R}$.

A projection p in a W^*-algebra $\mathcal{M}$ is said to be *minimal* if it majorizes no other nonzero projection in $\mathcal{M}$. A W^*-algebra $\mathcal{M}$ is said to be *atomic* if the supremum of all minimal projections in $\mathcal{M}$ equals the identity of $\mathcal{M}$.

For $x, y \in \mathbb{B}(\mathcal{H})$ we define the *Jordan product* $x \circ y$ as follows

$$x \circ y = \frac{xy + yx}{2}.$$

(The same symbol "$\circ$" will also be used for a linear functional φ and a map T on $\mathcal{M}$, namely, $\varphi \circ T$ will stand for the functional defined as $(\varphi \circ T)(x) = \varphi(T(x)),\ x \in \mathcal{M}$, however, it should not cause any confusion.) Let $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ be a linear space. $\mathcal{A}$ is said to be a *JW^*-algebra* if it is weak-operator closed, contains an identity p, i.e. $pa = ap = a$ for each $a \in \mathcal{A}$, and is closed with respect to the Jordan product, i.e. for any $a, b \in \mathcal{A}$ we have $a \circ b \in \mathcal{A}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be W^*-algebras. A linear map $T : \mathcal{M} \to \mathcal{N}$ is said to be *normal* if it is continuous in the σ-weak topologies on $\mathcal{M}$ and $\mathcal{N}$, respectively. It is called *unital* if it maps the identity of $\mathcal{M}$ to the identity of $\mathcal{N}$.

Let T be a positive map on $\mathcal{M}$. [4, Proposition 3.2.4] yields that for any $x = x^* \in \mathcal{M}$ we have

$$\|T\| T(x^2) \geqslant T(x)^2.$$

For arbitrary $x \in \mathcal{M}$ we obtain, applying the inequality above to the selfadjoint elements $x + x^*$ and $i(x - x^*)$, the following Schwarz inequality

$$\|T\| T(x^* \circ x) \geqslant T(x)^* \circ T(x),$$

(cf. [11, Lemma 7.3]).

Let T be a normal positive unital map on a W^*-algebra $\mathcal{M}$. By analogy with the case where T is completely positive, we define the *multiplicative domain* of T as

$$\mathcal{A} = \{x \in \mathcal{M} : T(x^* \circ x) = T(x)^* \circ T(x)\}.$$

From [10, Theorem 1] it follows that $\mathcal{A}$ is a JW^*-subalgebra of $\mathcal{M}$.

Let $\mathcal{M}$ be a von Neumann algebra, and consider the tensor product $\mathcal{M} \overline{\otimes} \mathcal{M}$. We have obvious counterparts $\Pi_{1,2} : (\mathcal{M} \overline{\otimes} \mathcal{M})_* \to \mathcal{M}_*$ of the partial trace on $(\mathcal{M} \overline{\otimes} \mathcal{M})_*$ defined as

$$(\Pi_1 \widetilde{\rho})(x) = \widetilde{\rho}(x \otimes \mathbb{1}),\ (\Pi_2 \widetilde{\rho})(x) = \widetilde{\rho}(\mathbb{1} \otimes x),\ \widetilde{\rho} \in (\mathcal{M} \overline{\otimes} \mathcal{M})_*,\ x \in \mathcal{M}.$$

The main objects of our interest are the following two operations of *broadcasting* and *cloning* of states.

A linear map $K_*\colon \mathcal{M}_* \to (\mathcal{M}\overline{\otimes}\mathcal{M})_*$ sending states to states, and such that its dual $K\colon \mathcal{M}\overline{\otimes}\mathcal{M} \to \mathcal{M}$ is a unital positive map will be called a *channel*. (This terminology is almost standard, because by a "channel" is usually meant the map K as above, however, with some additional assumption of complete- or at least two-positivity.) A state $\rho \in \mathcal{M}_*$ is *broadcast* by channel K_* if $(\Pi_i K_*)(\rho) = \rho,\ i = 1, 2$; in other words, ρ is broadcast by K_* if for each $x \in \mathcal{M}$

$$\rho(K(x \otimes \mathbb{1})) = \rho(K(\mathbb{1} \otimes x)) = \rho(x).$$

A family of states is said to be *broadcastable* if there is a channel K_* that broadcasts each member of this family.

A state $\rho \in \mathcal{M}_*$ is *cloned* by channel K_* if $K_*\rho = \rho \otimes \rho$. A family of states is said to be *cloneable* if there is a channel K_* that clones each member of this family.

3 Broadcasting

The discussion in this section has an auxiliary character and is in main part a repetition for *positive* maps of the reasoning from [6] performed there for *Schwarz* maps. Its main purpose is to analyze some properties of broadcasting channels employed in Sect. 4.

Let $\mathcal{M}$ be a von Neumann algebra, and let $\Gamma \subset \mathcal{M}_*$ be a broadcastable family of states. Then there is a channel K_* which broadcasts the states in Γ. Denote by $\mathcal{B}(K_*)$ the set of *all* normal states broadcastable by K_*. We have $\Gamma \subset \mathcal{B}(K_*)$, thus our main object of interest will be the set $\mathcal{B}(K_*)$. In the rest of this section we assume that we are given a fixed channel K_*. Define maps $L, R\colon \mathcal{M} \to \mathcal{M}$ as

$$L(x) = K(x \otimes \mathbb{1}), \qquad R(x) = K(\mathbb{1} \otimes x), \qquad x \in \mathcal{M}.$$

Then L and R are unital normal positive maps on $\mathcal{M}$. Observe that for a state ρ broadcast by K_* we have, for each $x \in \mathcal{M}$,

$$(\rho \circ L)(x) = \rho(K(x \otimes \mathbb{1})) = \rho(x), \qquad (\rho \circ R)(x) = \rho(K(\mathbb{1} \otimes x)) = \rho(x),$$

i.e. $\rho \circ L = \rho \circ R = \rho$. Consequently,

$$\mathcal{B}(K_*) = \{\rho \text{ — normal state} : \rho \circ L = \rho \circ R = \rho\}.$$

Set

$$p = \bigvee_{\rho \in \mathcal{B}(K_*)} \mathrm{s}(\rho). \tag{1}$$

Then $\mathcal{B}(K_*)$ is a faithful family of states on the algebra $p\mathcal{M}p$.

Define maps $K^{(p)}: \mathcal{M}\overline{\otimes}\mathcal{M} \to p\mathcal{M}p$ and $L^{(p)}, R^{(p)}: \mathcal{M} \to p\mathcal{M}p$ by

$$K^{(p)}(\widetilde{x}) = pK(\widetilde{x})p, \qquad \widetilde{x} \in \mathcal{M}\overline{\otimes}\mathcal{M},$$
$$L^{(p)}(x) = pL(x)p, \qquad R^{(p)}(x) = pR(x)p, \qquad x \in \mathcal{M}.$$

Then $K^{(p)},\ L^{(p)},\ R^{(p)}$ are normal positive maps of norm one such that for each $x \in \mathcal{M}$

$$K^{(p)}(x \otimes \mathbb{1}) = L^{(p)}(x), \qquad K^{(p)}(\mathbb{1} \otimes x) = R^{(p)}(x),$$

and

$$K^{(p)}(\mathbb{1} \otimes \mathbb{1}) = L^{(p)}(\mathbb{1}) = R^{(p)}(\mathbb{1}) = p.$$

Moreover, we have

$$p - pL(p)p = p(\mathbb{1} - L(p))p \geqslant 0,$$

and for each $\rho \in \mathcal{B}(K_*)$ the L-invariance of ρ yields

$$\rho(p - pL(p)p) = 0,$$

which means that

$$pL(p)p = p, \tag{2}$$

since $\mathcal{B}(K_*)$ is faithful on $p\mathcal{M}p$. The same relation holds also for R, thus

$$L^{(p)}(p) = R^{(p)}(p) = p.$$

Consequently,

$$K^{(p)}(\mathbb{1} \otimes \mathbb{1}) = L^{(p)}(\mathbb{1}) = L^{(p)}(p) = R^{(p)}(\mathbb{1}) = R^{(p)}(p) = p.$$

Another description of $\mathcal{B}(K_*)$ is given by the following lemma.

Lemma 1 *The following formula holds*

$$\mathcal{B}(K_*) = \{\rho \text{ — normal state} : \rho \circ L^{(p)} = \rho \circ R^{(p)} = \rho\}.$$

Proof Assume that $\rho \in \mathcal{B}(K_*)$. Then, since $\mathrm{s}(\rho) \leqslant p$, we have for each $x \in \mathcal{M}$

$$\rho(x) = \rho(L(x)) = \rho(pL(x)p) = \rho(L^{(p)}(x)),$$

and the same holds for $R^{(p)}$.

Conversely, if $\rho \circ L^{(p)} = \rho \circ R^{(p)} = \rho$, then

$$\rho(\mathbb{1}) = \rho(L^{(p)}(\mathbb{1})) = \rho(p),$$

showing that $s(\rho) \leqslant p$, so for each $x \in \mathcal{M}$

$$\rho(L(x)) = \rho(pL(x)p) = \rho(L^{(p)}(x)) = \rho(x),$$

and by the same token $\rho \circ R = \rho$, which means that $\rho \in \mathcal{B}(K_*)$. □

For a map T on $\mathcal{M}$ denote by $\mathcal{F}(T)$ its fixed-point space, i.e.

$$\mathcal{F}(T) = \{x \in \mathcal{M} : T(x) = x\}.$$

Let $\mathcal{A}$ be the multiplicative domain of $K^{(p)}$.

Lemma 2 *The following relations hold*

(i) For each $x \in \mathcal{F}(L^{(p)})$ we have $x \otimes \mathbb{1} \in \mathcal{A}$,
(ii) For each $x \in \mathcal{F}(R^{(p)})$ we have $\mathbb{1} \otimes x \in \mathcal{A}$.

Proof It is enough to prove (i) since a proof of (ii) is analogous. Let $x \in \mathcal{F}(L^{(p)})$. Then $L^{(p)}(x) = x$ and $x = px = xp$. We have $\|L^{(p)}\| = 1$, and the Schwarz inequality for the map $L^{(p)}$ yields

$$x^* \circ x = L^{(p)}(x)^* \circ L^{(p)}(x) \leqslant L^{(p)}(x^* \circ x),$$

hence

$$p(x^* \circ x)p = x^* \circ x \leqslant L^{(p)}(x^* \circ x) = pL^{(p)}(x^* \circ x)p,$$

or in other words

$$p(L^{(p)}(x^* \circ x) - x^* \circ x)p \geqslant 0.$$

For an arbitrary $\rho \in \mathcal{B}(K_*)$ we have on account of the $L^{(p)}$-invariance of ρ

$$\begin{aligned} \rho(p(L^{(p)}(x^* \circ x) - x^* x)p) &= \rho(L^{(p)}(x^* \circ x) - x^* \circ x) \\ &= \rho(L^{(p)}(x^* \circ x)) - \rho(x^* \circ x) = 0, \end{aligned}$$

and since the family $\mathcal{B}(K_*)$ is faithful on the algebra $p\mathcal{M}p$ we obtain

$$p(L^{(p)}(x^* \circ x) - x^* \circ x)p = 0,$$

which amounts to the equality

$$L^{(p)}(x^* \circ x) = x^* \circ x.$$

Taking into account the definition of $K^{(p)}$ we get

$$\begin{aligned} K^{(p)}(x^* \otimes \mathbb{1}) \circ K^{(p)}(x \otimes \mathbb{1}) &= L^{(p)}(x^*) \circ L^{(p)}(x) = x^* \circ x = L^{(p)}(x^* \circ x) \\ &= pK(x^* \circ x \otimes \mathbb{1})p = K^{(p)}(x^* \circ x \otimes \mathbb{1}) = K^{(p)}((x^* \otimes \mathbb{1}) \circ (x \otimes \mathbb{1})), \end{aligned}$$

showing that $x \otimes \mathbb{1}$ belongs to $\mathcal{A}$. □

To simplify the notation let us agree on the following convention. For a positive map $T : \mathcal{M} \to p\mathcal{M}p$ such that

$$T(\mathbb{1}) = T(p) = p$$

denote by T_p the restriction $T|p\mathcal{M}p$, so that $T_p : p\mathcal{M}p \to p\mathcal{M}p$. Now the positive unital maps from $p\mathcal{M}p$ to $p\mathcal{M}p$ will also be denoted with the use of index p, thus T_p will stand for a positive map on the algebra $p\mathcal{M}p$ such that $T_p(p) = p$. To justify this abuse of notation let us define for such a map T_p the map T as follows

$$T(x) = T_p(pxp), \qquad x \in \mathcal{M}. \tag{3}$$

It is clear that we have $T|p\mathcal{M}p = T_p$, so for the consistency of our notation we only need the relation

$$T(x) = T(pxp), \qquad x \in \mathcal{M},$$

which is a consequence of the following well-known fact whose proof can be found e.g. in [9, Lemma 2]).

Lemma 3 *Let $T : \mathcal{M} \to \mathcal{M}$ be positive, and let e be a projection in $\mathcal{M}$ such that*

$$T(\mathbb{1}) = T(e) = e.$$

Then for each $x \in \mathcal{M}$

$$T(x) = T(ex) = T(xe) = eT(x) = T(x)e.$$

In the sequel while dealing with maps denoted by the same capital letter with or without index p we shall always assume that they are connected by relation (3). If T, T_p, V and V_p are maps as above then it is easily seen that

$$(TV)_p = T_p V_p,$$

in particular, for each positive integer m we have

$$(T^m)_p = (T_p)^m.$$

The same convention will be adopted also for states with supports contained in p, i.e. if φ is a state on $\mathcal{M}$ such that $\mathrm{s}(\varphi) \leqslant p$, then φ_p will denote its restriction to $p\mathcal{M}p$, and for an arbitrary state φ_p on $p\mathcal{M}p$ the state φ will be defined as

$$\varphi(x) = \varphi_p(pxp), \qquad x \in \mathcal{M}.$$

Now we fix attention on the algebra $p\mathcal{M}p$. In accordance with our convention, we define maps $L_p^{(p)}, R_p^{(p)} : p\mathcal{M}p \to p\mathcal{M}p$ as

$$L_p^{(p)} = L^{(p)}|p\mathcal{M}p, \qquad R_p^{(p)} = R^{(p)}|p\mathcal{M}p.$$

Clearly, $L_p^{(p)}$ and $R_p^{(p)}$ are normal positive unital maps such that for $\rho \in \mathcal{B}(K_*)$ the ρ_p are $L_p^{(p)}$- and $R_p^{(p)}$-invariant. It is obvious that $\mathcal{F}(L_p^{(p)}) = \mathcal{F}(L^{(p)})$ and $\mathcal{F}(R_p^{(p)}) = \mathcal{F}(R^{(p)})$. Let $\mathfrak{S}_p$ be the semigroup of normal positive maps on $p\mathcal{M}p$ generated by $L_p^{(p)}$ and $R_p^{(p)}$. Then $\mathcal{B}(K_*)_p$ defined as $\mathcal{B}(K_*)_p = \{\rho_p : \rho \in \mathcal{B}(K_*)\}$ is a faithful family of $\mathfrak{S}_p$-invariant normal states on $p\mathcal{M}p$. Denote by $\mathcal{F}(\mathfrak{S}_p)$ the fixed-point space of $\mathfrak{S}_p$, i.e.

$$\mathcal{F}(\mathfrak{S}_p) = \{x \in p\mathcal{M}p : S_p(x) = x \text{ for each } S_p \in \mathfrak{S}_p\}.$$

From the ergodic theorem for W^*-algebras (cf. [13]) we infer that $\mathcal{F}(\mathfrak{S}_p)$ is a JW^*-algebra, and there exists a normal faithful projection $\mathbb{E}_p$ from $p\mathcal{M}p$ onto $\mathcal{F}(\mathfrak{S}_p)$ such that

$$\mathbb{E}_p S_p = S_p \mathbb{E}_p = \mathbb{E}_p, \qquad \text{for each} \quad S_p \in \mathfrak{S}_p, \tag{4}$$

and

$$\rho_p \circ \mathbb{E}_p = \rho_p, \qquad \text{for each} \quad \rho \in \mathcal{B}(K_*).$$

Furthermore, $\mathbb{E}_p$ is positive and has the following property reminiscent of an analogous property of conditional expectation: for any $x \in p\mathcal{M}p$, $y \in \mathcal{F}(\mathfrak{S}_p)$ we have

$$\mathbb{E}_p(x \circ y) = (\mathbb{E}_p x) \circ y.$$

Moreover, if φ_p is an arbitrary $\mathbb{E}_p$-invariant normal state on $p\mathcal{M}p$ then from relation (4) we see that φ_p is $\mathfrak{S}_p$-invariant. Conversely, if φ_p is an arbitrary $\mathfrak{S}_p$-invariant normal state on $p\mathcal{M}p$ then another consequence of the ergodic theorem is that φ_p is also $\mathbb{E}_p$-invariant (this follows from the fact that for each $x \in p\mathcal{M}p$, $\mathbb{E}_p x$ lies in the σ-weak closure of the convex hull of $\{S_p x : S_p \in \mathfrak{S}_p\}$). Consequently, we have the following equivalence for a normal state φ_p on $p\mathcal{M}p$: φ_p is $\mathfrak{S}_p$-invariant if and only if it is $\mathbb{E}_p$-invariant.

Now we want to transfer these results from the algebra $p\mathcal{M}p$ to the algebra $\mathcal{M}$. Each element S_p of $\mathfrak{S}_p$ has the form

$$S_p = (L_p^{(p)})^{r_1}(R_p^{(p)})^{r_2} \dots (L_p^{(p)})^{r_{m-1}}(R_p^{(p)})^{r_m},$$

where the integers $r_1, \dots, r_m$ satisfy $r_1, r_m \geqslant 0$ and $r_2, \dots r_{m-1} > 0$, $m = 1, 2, \dots$. Consequently,

$$\begin{aligned} S_p &= ((L^{(p)})^{r_1})_p((R^{(p)})^{r_2})_p \dots ((L^{(p)})^{r_{m-1}})_p((R^{(p)})^{r_m})_p \\ &= ((L^{(p)})^{r_1}(R^{(p)})^{r_2} \dots (L^{(p)})^{r_{m-1}}(R^{(p)})^{r_m})_p, \end{aligned}$$

showing that S defined in accordance with our convention as

$$S(x) = S_p(pxp), \qquad x \in \mathcal{M},$$

is an element of the semigroup $\mathfrak{S}$ generated by the maps $L^{(p)}$ and $R^{(p)}$. Thus we have

$$\mathfrak{S}_p = \{S_p : S \in \mathfrak{S}\}.$$

It is easily seen that $\mathcal{F}(\mathfrak{S}) = \mathcal{F}(\mathfrak{S}_p)$, where $\mathcal{F}(\mathfrak{S})$ denotes the fixed-points of $\mathfrak{S}$. Again in accordance with our convention, we define a map $\mathbb{E}\colon \mathcal{M} \to \mathcal{F}(\mathfrak{S})$ by the formula

$$\mathbb{E}x = \mathbb{E}_p(pxp), \qquad x \in \mathcal{M}. \tag{5}$$

Then $\mathbb{E}$ is a normal positive projection onto the JW^*-algebra $\mathcal{F}(\mathfrak{S})$ such that $\mathbb{E}(\mathbb{1}) = p$. In the following proposition we obtain a characterization of $\mathcal{B}(K_*)$ in terms of the projection $\mathbb{E}$.

Proposition 4 *Let φ be a normal state on $\mathcal{M}$. The following conditions are equivalent*

(i) φ belongs to $\mathcal{B}(K_*)$,
(ii) φ is $\mathfrak{S}$-invariant,
(iii) $\varphi = \varphi \circ \mathbb{E}$.

Proof (i)$\Longrightarrow$(ii). It follows from Lemma 1.
(ii)$\Longrightarrow$(iii). We have

$$\varphi(\mathbb{1}) = \varphi(L^{(p)}(\mathbb{1})) = \varphi(p),$$

which means that $\mathrm{s}(\varphi) \leqslant p$. Consider the state φ_p. We have for each $x \in \mathcal{M}$

$$\varphi_p(L_p^{(p)}(pxp)) = \varphi(L^{(p)}(pxp)) = \varphi(pxp) = \varphi_p(pxp),$$

showing that φ_p is $L_p^{(p)}$-invariant. In the same way it is shown that φ_p is $R_p^{(p)}$-invariant, thus φ_p is $\mathfrak{S}_p$-invariant. Since the $\mathfrak{S}_p$-invariance of φ_p is equivalent to its $\mathbb{E}_p$-invariance, we have for each $x \in \mathcal{M}$

$$\varphi(x) = \varphi(pxp) = \varphi_p(pxp) = \varphi_p(\mathbb{E}_p(pxp)) = \varphi(\mathbb{E}_p(pxp)) = \varphi(\mathbb{E}x).$$

(iii)$\Longrightarrow$(i). Observe first that we have

$$\mathbb{E}L^{(p)} = \mathbb{E} = \mathbb{E}R^{(p)}.$$

Indeed, taking into account (4), the fact that $L^{(p)}$ has its range contained in $p\mathcal{M}p$, and Lemma 3 we obtain for each $x \in \mathcal{M}$

$$\begin{aligned}\mathbb{E}(L^{(p)}(x)) &= \mathbb{E}_p(p(L^{(p)}(x))p) = \mathbb{E}_p((L^{(p)}(x)) \\ &= \mathbb{E}_p(L_p^{(p)}(pxp)) = \mathbb{E}_p(pxp) = \mathbb{E}x,\end{aligned}$$

and similarly for the second equality. Now we have

$$\varphi(L^{(p)}(x)) = \varphi(\mathbb{E}(L^{(p)}(x))) = \varphi(\mathbb{E}x) = \varphi(x),$$

and the same for $R^{(p)}$, which shows that $\varphi \in \mathcal{B}(K_*)$. $\square$

It turns out that the map $K^{(p)}$ has a special form on the tensor product algebra $\mathcal{F}(\mathfrak{S})\overline{\otimes}\mathcal{F}(\mathfrak{S})$ (this is the weak closure of the algebra of *operators* $\{\sum_{i=1}^m x_i \otimes y_i : x_i, y_i \in \mathcal{F}(\mathfrak{S})\}$ acting on $\mathcal{H} \otimes \mathcal{H}$).

Proposition 5 *For each* $x, y \in \mathcal{F}(\mathfrak{S})$ *we have*

$$K^{(p)}(x \otimes y) = x \circ y. \tag{6}$$

Proof Considering the semigroups $\{(R_p^{(p)})^n : n = 0, 1, \dots\}$ and $\{(T_p^{(p)})^n : n = 0, 1, \dots\}$ generated by $R_p^{(p)}$ and $T_p^{(p)}$ we immediately notice that the fixed-point spaces of these semigroups are equal to $\mathcal{F}(R_p^{(p)})$ and $\mathcal{F}(T_p^{(p)})$, respectively, and the above-mentioned ergodic theorem shows that $\mathcal{F}(R_p^{(p)})$ and $\mathcal{F}(T_p^{(p)})$ are JW^*-algebras. Moreover,

$$\mathcal{F}(\mathfrak{S}) = \mathcal{F}(\mathfrak{S}_p) = \mathcal{F}(R_p^{(p)}) \cap \mathcal{F}(T_p^{(p)}).$$

Let $x, y \in \mathcal{F}(\mathfrak{S})$. Then by virtue of Lemma 2 we have $x \otimes \mathbb{1}, \mathbb{1} \otimes y \in \mathcal{A}$, and thus

$$\begin{aligned} K^{(p)}(x \otimes y) &= K^{(p)}((x \otimes \mathbb{1}) \circ (\mathbb{1} \otimes y)) = K^{(p)}(x \otimes \mathbb{1}) \circ K^{(p)}(\mathbb{1} \otimes y) \\ &= R^{(p)}(x) \circ T^{(p)}(y) = x \circ y, \end{aligned}$$

showing the claim. □

The next result is well-known in the case $p = \mathbb{1}$ (cf. [13, Lemma1]). Its proof for arbitrary p is similar, so we omit it.

Lemma 6 *For each* $\rho \in \mathcal{B}(K_*)$ *we have* $s(\rho) \in \mathcal{F}(\mathfrak{S})$.

4 Cloning

Let $\mathcal{M}$ be a von Neumann algebra, and let $K_* : \mathcal{M}_* \to (\mathcal{M} \overline{\otimes} \mathcal{M})_*$ be a channel. Denote by $\mathcal{C}(K_*)$ the set of all states cloneable by K_*, and put

$$e = \bigvee_{\rho \in \mathcal{C}(K_*)} \mathrm{s}(\rho).$$

The cloneable states and an associated algebra are described by

Theorem 7 *The following conditions hold true:*

1. *The states in* $\mathcal{C}(K_*)$ *have mutually orthogonal supports, and are extreme points of* $\mathcal{B}(K_*)$.
2. *The algebra* $e\mathcal{F}(\mathfrak{S})e$ *is an atomic abelian* W^**-subalgebra of* $\mathcal{F}(\mathfrak{S})$*, generated by* $\{\mathrm{s}(\rho) : \rho \in \mathcal{C}(K_*)\}$*, and such that* $e\mathcal{F}(\mathfrak{S})e \subset \mathcal{F}(\mathfrak{S})'$ *— the commutant of* $\mathcal{F}(\mathfrak{S})$.

Proof 1. Since $\mathcal{C}(K_*) \subset \mathcal{B}(K_*)$ we may use the analysis of the preceding section. In particular, we adopt the setup and notation introduced there. For each $\rho \in \mathcal{C}(K_*)$ we have $K_*\rho = \rho \otimes \rho$, so taking into account Proposition 5 we obtain the equality

$$\begin{aligned}\rho(x)\rho(y) &= \rho \otimes \rho(x \otimes y) = (K_*\rho)(x \otimes y) = \rho(K(x \otimes y)) \\ &= \rho(pK(x \otimes y)p) = \rho(K^{(p)}(x \otimes y)) = \rho(x \circ y) \qquad (7)\end{aligned}$$

for all $x, y \in \mathcal{F}(\mathfrak{S})$. The equality above yields that for each $z \in \mathcal{F}(\mathfrak{S})$ and any $\rho \in \mathcal{C}(K_*)$ we have

$$\rho(z^2) = \rho(z)^2. \qquad (8)$$

Let x be an arbitrary selfadjoint element of $\mathcal{F}(\mathfrak{S})$. For each $\rho \in \mathcal{C}(K_*)$ we have by (8)

$$\rho\big((x - \rho(x)\mathbb{1})^2\big) = \rho\big(x^2 - 2\rho(x)x - \rho(x)^2\mathbb{1}\big) = \rho(x^2) - \rho(x)^2 = 0,$$

which yields the equality

$$\mathrm{s}(\rho)\big(x - \rho(x)\mathbb{1}\big)^2\,\mathrm{s}(\rho) = 0,$$

i.e.

$$\mathrm{s}(\rho)x = \rho(x)\,\mathrm{s}(\rho). \qquad (9)$$

Since for an element x of a JW^*-algebra $x + x^*$ and $x - x^*$ are also elements of this algebra the equality above holds for all $x \in \mathcal{F}(\mathfrak{S})$ as well.
Let ρ and φ be two distinct states from $\mathcal{C}(K_*)$. Then by (9)

$$\mathrm{s}(\rho)\,\mathrm{s}(\varphi) = \rho(\mathrm{s}(\varphi))\,\mathrm{s}(\rho) \qquad \text{and} \qquad \mathrm{s}(\varphi)\,\mathrm{s}(\rho) = \varphi(\mathrm{s}(\rho))\,\mathrm{s}(\varphi),$$

which after taking adjoints yields the equality

$$\mathrm{s}(\rho)\,\mathrm{s}(\varphi) = \mathrm{s}(\varphi)\,\mathrm{s}(\rho).$$

Consequently,

$$\rho(\mathrm{s}(\varphi))\,\mathrm{s}(\rho) = \varphi(\mathrm{s}(\rho))\,\mathrm{s}(\varphi) = \mathrm{s}(\varphi)\,\mathrm{s}(\rho),$$

showing that either $\mathrm{s}(\rho) = \mathrm{s}(\varphi)$ or $\mathrm{s}(\rho)$ and $\mathrm{s}(\varphi)$ are orthogonal.
If $\mathrm{s}(\rho) = \mathrm{s}(\varphi)$ then on account of (9) we would have

$$\rho(x)\,\mathrm{s}(\rho) = \mathrm{s}(\rho)x = \mathrm{s}(\varphi)x = \varphi(x)\,\mathrm{s}(\varphi) = \varphi(x)\,\mathrm{s}(\rho)$$

for each $x \in \mathcal{F}(\mathfrak{S})$, i.e.

$$\rho|\mathcal{F}(\mathfrak{S}) = \varphi|\mathcal{F}(\mathfrak{S}). \qquad (10)$$

Let $\mathbb{E}$ be the projection onto $\mathcal{F}(\mathfrak{S})$ defined by formula (5). We have $\rho = \rho \circ \mathbb{E}$ and $\varphi = \varphi \circ \mathbb{E}$, thus equality (10) yields

$$\rho = \rho \circ \mathbb{E} = \varphi \circ \mathbb{E} = \varphi$$

contrary to the assumption that ρ and φ are distinct. Consequently, ρ and φ have orthogonal supports.
Now take an arbitrary $\rho \in \mathcal{C}(K_*)$, and assume that

$$\rho = \lambda\varphi_1 + (1-\lambda)\varphi_2,$$

for some $0 < \lambda < 1$ and $\varphi_1, \varphi_2 \in \mathcal{B}(K_*)$. Then

$$1 = \rho(\mathrm{s}(\rho)) = \lambda\varphi_1(\mathrm{s}(\rho)) + (1-\lambda)\varphi_2(\mathrm{s}(\rho)),$$

showing that $\varphi_1(\mathrm{s}(\rho)) = \varphi_2(\mathrm{s}(\rho)) = 1$, which means that $\mathrm{s}(\varphi_1), \mathrm{s}(\varphi_2) \leqslant \mathrm{s}(\rho)$. From equality (9) we obtain for $x \in \mathcal{F}(\mathfrak{S})$

$$\varphi_{1,2}(x) = \varphi_{1,2}(\mathrm{s}(\rho)x) = \rho(x)\varphi_{1,2}(\mathrm{s}(\rho)) = \rho(x),$$

giving the relation

$$\rho|\mathcal{F}(\mathfrak{S}) = \varphi_{1,2}|\mathcal{F}(\mathfrak{S}).$$

We have $\rho = \rho \circ \mathbb{E}$ and $\varphi_{1,2} = \varphi_{1,2} \circ \mathbb{E}$, and thus

$$\rho = \rho \circ \mathbb{E} = \varphi_{1,2} \circ \mathbb{E} = \varphi_{1,2},$$

showing that ρ is an extreme point of $\mathcal{B}(K_*)$.

2. From equality (9) we obtain that $\mathrm{s}(\rho) \in \mathcal{F}(\mathfrak{S})'$ for all $\rho \in \mathcal{C}(K_*)$, and that

$$ex = \sum_{\rho \in \mathcal{C}(K_*)} \rho(x)\,\mathrm{s}(\rho),$$

for all $x \in \mathcal{F}(\mathfrak{S})$, which means that $e\mathcal{F}(\mathfrak{S}) = e\mathcal{F}(\mathfrak{S})e$ is a W^*-algebra generated by $\{\mathrm{s}(\rho) : \rho \in \mathcal{C}(K_*)\}$. By virtue of Lemma 6 we have $s(\rho) \in \mathcal{F}(\mathfrak{S})$ for each $\rho \in \mathcal{C}(K_*)$ thus $e\mathcal{F}(\mathfrak{S})e$ is a subalgebra of $\mathcal{F}(\mathfrak{S})$, and $e\mathcal{F}(\mathfrak{S})e \subset \mathcal{F}(\mathfrak{S})'$. Finally, $\mathrm{s}(\rho)$ is a minimal projection in $e\mathcal{F}(\mathfrak{S})e$. Indeed, for any projection $f \in \mathcal{F}(\mathfrak{S})$ equality (8) yields

$$\rho(f) = 0 \text{ or } 1.$$

Now if q is a projection in $e\mathcal{F}(\mathfrak{S})e$ such that $q \leqslant \mathrm{s}(\rho)$ and $q \neq \mathrm{s}(\rho)$ we cannot have $\rho(q) = 1$, thus $\rho(q) = 0$, and the faithfulness of ρ on the algebra $\mathrm{s}(\rho)\mathcal{M}\,\mathrm{s}(\rho)$ yields $q = 0$. Consequently, algebra $e\mathcal{F}(\mathfrak{S})e$ being generated by minimal projections is atomic. □

Theorem 8 *Let Γ be an arbitrary subset of normal states on a von Neumann algebra $\mathcal{M}$. The following conditions are equivalent*

(i) Γ *is cloneable,*
(ii) *The states in* Γ *have mutually orthogonal supports.*

Proof The implication (i)$\Longrightarrow$(ii) follows from Theorem 7.

To prove (ii)$\Longrightarrow$(i) assume that the states from Γ have orthogonal supports $\{e_i\}$, that is $\Gamma = \{\rho_i\}$ and $\mathrm{s}(\rho_i) = e_i$. Define a channel $K_*\colon \mathcal{M}_* \to (\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{M})_*$ as follows

$$K_*\varphi = \sum_i \varphi(e_i)\, \rho_i \otimes \rho_i + \varphi(e^\perp)\, \widetilde{\omega}, \qquad \varphi \in \mathcal{M}_*, \tag{11}$$

where

$$e = \sum_i e_i,$$

and $\widetilde{\omega}$ is a fixed normal state on $\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{M}$. Since $\rho_i(e^\perp) = 0$, we have

$$K_*\rho_i = \sum_j \rho_i(e_j)\, \rho_j \otimes \rho_j + \rho_i(e^\perp)\, \widetilde{\omega} = \rho_i \otimes \rho_i,$$

showing that K_* clones the ρ_i. □

The above theorem yields an interesting corollary. Namely, a usual assumption about a channel is its complete (or at least two-) positivity, the assumption which we have tried to avoid in this work. It turns out that a stronger form of positivity gives the same cloneable states.

Corollary *Let* Γ *be an arbitrary subset of normal states on a von Neumann algebra* $\mathcal{M}$*. The following conditions are equivalent*

(i) Γ *is cloneable by a completely positive channel,*
(ii) Γ *is cloneable by a positive channel.*

Proof We need only to show the implication (ii)$\Longrightarrow$(i). From Theorem 8 (and with its notation) we know that the supports e_i of states ρ_i in Γ are mutually orthogonal. Define a channel K_* by formula (11). Then K_* clones the ρ_i. Its dual map $K : \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{M} \to \mathcal{M}$ has the form

$$K(\widetilde{x}) = \sum_i \rho_i \otimes \rho_i(\widetilde{x})\, e_i + \widetilde{\omega}(\widetilde{x})\, e^\perp, \qquad \widetilde{x} \in \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{M},$$

thus its range lies in the abelian von Neumann algebra generated by the projections e_i and $e^\perp$. Consequently, on account of [12, Corollary IV.3.5] K is completely positive. □

Finally, let us say a few words about the uniqueness of the cloning operation.

Proposition 9 *Let $\Gamma = \{\rho_i\}$ be an arbitrary subset of normal states on a von Neumann algebra $\mathcal{M}$ such that the states in Γ have mutually orthogonal supports. Put $e_i = \mathrm{s}(\rho_i)$, and assume that*

$$\sum_i e_i = \mathbb{1}.$$

Let the cloning channel K'_* be defined as in Theorem 8, i.e.

$$K'_*\varphi = \sum_i \varphi(e_i)\,\rho_i \otimes \rho_i, \qquad \varphi \in \mathcal{M}_*.$$

Then for each channel K_* that clones Γ we have

$$K'_* = (\mathbb{E} \otimes \mathbb{E})_* K_*,$$

where $\mathbb{E}$ is the projection from $\mathcal{M}$ onto $\mathcal{F}(\mathfrak{S})$ defined by means of the dual K of K_* as in Sect. 3 (it turns out that in our case $\mathbb{E}$ is actually a conditional expectation).

Proof Let K_* be a channel cloning Γ. Since $\Gamma \subset \mathcal{C}(K_*)$ we get

$$e = \bigvee_{\omega \in \mathcal{C}(K_*)} \mathrm{s}(\omega) = \mathbb{1}.$$

Theorem 7 asserts that the $\mathrm{s}(\omega)$ for $\omega \in \mathcal{C}(K_*)$ are minimal projections in $e\mathcal{F}(\mathfrak{S})e = \mathcal{F}(\mathfrak{S})$, thus the equality

$$\bigvee_i e_i = \mathbb{1} = \bigvee_{\omega \in \mathcal{C}(K_*)} \mathrm{s}(\omega)$$

yields $\Gamma = \mathcal{C}(K_*)$.

Denoting by K' the dual of K'_* we have for all $x, y \in \mathcal{M}$

$$K'(x \otimes y) = \sum_i \rho_i(x)\rho_i(y)\, e_i. \tag{12}$$

From Theorem 7 it follows that $\mathcal{F}(\mathfrak{S})$ is an abelian atomic W^*-algebra generated by $\{e_i\}$, i.e.

$$\mathcal{F}(\mathfrak{S}) = \left\{ \sum_i \alpha_i e_i : \alpha_i \in \mathbb{C},\ \sup_i |\alpha_i| < \infty \right\}.$$

For the projection $\mathbb{E}$ defined as in Sect. 3, and arbitrary $x \in \mathcal{M}$ we have

$$\mathbb{E}x = \sum_i \alpha_i e_i,$$

for some coefficients $\alpha_i \in \mathbb{C}$ depending on x. Consequently,

$$\rho_j(x) = \rho_j(\mathbb{E}x) = \sum_i \alpha_i \rho_j(e_i) = \alpha_j,$$

thus

$$\mathbb{E}x = \sum_i \rho_i(x)e_i.$$

(From the formula above it immediately follows that $\mathbb{E}$ is a conditional expectation.) Consequently, we obtain

$$\mathbb{E}x\mathbb{E}y = \sum_i \rho_i(x)\rho_i(y)e_i, \tag{13}$$

for all $x, y \in \mathcal{M}$.

By (6) we get for any $x, y \in \mathcal{M}$

$$K(\mathbb{E} \otimes \mathbb{E}(x \otimes y)) = K(\mathbb{E}x \otimes \mathbb{E}y) = \mathbb{E}x \circ \mathbb{E}y = \mathbb{E}x\mathbb{E}y, \tag{14}$$

and formulas (12), (13), and (14) yield

$$K'(x \otimes y) = K(\mathbb{E} \otimes \mathbb{E}(x \otimes y)),$$

for any $x, y \in \mathcal{M}$, which means that

$$K'_* = (\mathbb{E} \otimes \mathbb{E})_* K_*,$$

finishing the proof. □

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