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Absolutely continuous operators on function spaces and vector measures

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Abstract Let (Ω, Σ, μ) be a finite atomless measure space, and let E be an ideal of $L^0(\mu)$ such that $L^\infty(\mu) \subset E \subset L^1(\mu)$. We study absolutely continuous linear operators from E to a locally convex Hausdorff space (X, ξ). Moreover, we examine the relationships between μ-absolutely continuous vector measures $m : \Sigma \to X$ and the corresponding integration operators $T_m : L^\infty(\mu) \to X$. In particular, we characterize relatively compact sets $\mathcal{M}$ in $ca_\mu(\Sigma, X)$ (= the space of all μ-absolutely continuous measures $m : \Sigma \to X$) for the topology $\mathcal{T}_s$ of simple convergence in terms of the topological properties of the corresponding set $\{T_m : m \in \mathcal{M}\}$ of absolutely continuous operators. We derive a generalized Vitali–Hahn–Saks type theorem for absolutely continuous operators $T : L^\infty(\mu) \to X$.

Keywords Function spaces · Absolutely continuous operators · Integration operators · Countably additive vector measures · Absolutely continuous vector measures · Mackey topologies · Order-bounded topology

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1 Introduction and terminology

For terminology concerning vector lattices and function spaces we refer the reader to [1], [2], [10]. Throughout the paper we assume that (Ω, Σ, μ) is a complete finite atomless measure space and $L^0(\mu)$ denotes the corresponding space of μ-equivalence classes of all Σ-measurable real valued-functions defined on Ω. Let E be an

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ideal of $L^0(\mu)$ such that $L^\infty(\mu) \subset E \subset L^1(\mu)$, and let $E^\sim$ and $E_n^\sim$ stand for the order dual and order continuous dual of E respectively. Then $E_n^\sim$ separates the points of E and it can be identified with the Köthe dual E' of E through the mapping $E' \ni v \mapsto \varphi_v \in E_n^\sim$, where $\varphi_v(u) = \int_\Omega uv d\mu$ for all $u \in E$. It is known that the Mackey topology $\tau(E, E_n^\sim)(= \tau(E, E'))$ is a locally solid Lebesgue topology.

The so-called *order-bounded topology* τ_0 can be defined on E as the finest locally convex topology on E for which every order interval in E is a bounded set (see [11]). A local base $\mathcal{B}_0$ at zero for τ_0 is the class of all absolutely convex subsets of E that absorb all order bounded sets in E. Then τ_0 coincides with the Mackey topology $\tau(E, E^\sim)$. Note that if $u_n, u \in E$ and $u_n \to u$ uniformly on Ω, then $u_n \to u$ for τ_0.

From now on we assume that (X, ξ) is a locally convex Hausdorff space (for short, lcHs) and let $\mathcal{P}_\xi$ denote the set of all ξ-continuous seminorms on X. By X'_ξ we denote the topological dual of (X, ξ). We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K\rangle$.

Recall that a linear operator $T : E \to X$ is said to be *order-bounded* (resp. *order-weakly compact*), if for each $u \in E^+$, the set $T([-u, u])$ is ξ-bounded (resp. relatively $\sigma(X, X'_\xi)$-compact) in X (see [6]).

Proposition 1.1 *For a linear operator $T : E \to X$ the following statements are equivalent:*

(i) *T is order-bounded.*
(ii) *T is (τ_0, ξ)-continuous.*

Proof (i)$\Longrightarrow$(ii) Assume that T is order-bounded. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$. We shall show that there is $V \in \mathcal{B}_0$ such that $T(V) \subset B_p(\varepsilon)(= \{x \in X : p(x) \le \varepsilon\})$. Indeed, let $V = T^{-1}(B_p(\varepsilon))$. Since $T(V) \subset T(T^{-1}(B_p(\varepsilon)) \subset B_p(\varepsilon)$, it is enough to show that V absorbs every order interval in E. Given $u \in E^+$ there is $r_u > 0$ such that $T([-u, u]) \subset B_p(r_u)$. Then for $\lambda_u = \frac{\varepsilon}{r_u}$ and for all $v \in [-u, u]$ we get $p(T(\lambda_u v)) = \lambda_u p(T(v)) \le \varepsilon$, so $\lambda_u v \in V$. This means that $\lambda_u[-u, u] \subset V$, i.e., V absorbs $[-u, u]$, as desired.

(ii)$\Longrightarrow$(i) Assume that T is (τ_0, ξ)-continuous and $p \in \mathcal{P}_\xi$. Then there is $V_p \in \mathcal{B}_0$ such that $T(V_p) \subset B_p(1)$. Given $u \in E^+$ there exists $\lambda_u > 0$ such that $\lambda_u[-u, u] \subset V_p$. Hence $T(\lambda_u[-u, u]) \subset T(V_p) \subset B_p(1)$, so $T([-u, u]) \subset B_p(\frac{1}{\lambda_u})$. It follows that the set $T([-u, u])$ is ξ-bounded in X. □

Following [13] a linear operator $T : E \to X$ is said to be *absolutely continuous* if for each $u \in E, T(\mathbb{1}_{A_n} u) \to 0$ for ξ whenever $\mu(A_n) \to 0, (A_n) \subset \Sigma$. Absolutely continuous operators on Orlicz spaces and Frechét function spaces have been examined by Orlicz and Wnuk (see [12, 13]).

In Sect. 2 we study absolutely continuous operators $T : E \to X$. We show that a linear operator $T : E \to X$ is absolutely continuous if and only if T is $(\tau(E, E_n^\sim), \xi)$-continuous. We characterize relatively compact sets in the space $\mathcal{L}_{\tau,\xi}(E, X)$ of all $(\tau(E, E_n^\sim), \xi)$-continuous linear operators $T : E \to X$, provided with the topology of simple convergence. In Sect. 3 we examine the relationships between μ-absolutely continuous vector measures $m : \Sigma \to X$ and the corresponding integration operators $T_m : L^\infty(\mu) \to X$.

2 Absolutely continuous operators on function spaces

We start with the following result.

Proposition 2.1 *Assume that $T : E \to X$ is an absolutely continuous linear operator. Then T is (τ_0, ξ)-continuous.*

Proof In view of Proposition 1.1 it is sufficient to show that $T([-u, u])$ is ξ-bounded in X for every $u \in E^+$. For this purpose one can repeat the proof of Theorem 1 in [13].

Now we present a characterization of absolutely continuous operators on E. □

Proposition 2.2 *For a linear operator $T : E \to X$ the following statements are equivalent:*

(i) $x' \circ T \in E_n^\sim$ *for each* $x' \in X'_\xi$.
(ii) T *is* $(\sigma(E, E_n^\sim), \sigma(X, X'_\xi))$*-continuous.*
(iii) T *is* $(\tau(E, E_n^\sim), \xi)$*-continuous.*
(iv) T *is smooth, i.e.,* $T(u_\alpha) \to 0$ *for* ξ *whenever* $u_\alpha \xrightarrow{(o)} 0$ *in* E.
(v) T *is* σ*-smooth, i.e.,* $T(u_n) \to 0$ *for* ξ *whenever* $u_n \xrightarrow{(o)} 0$ *in* E.
(vi) T *is absolutely continuous.*

Proof (i)$\Longleftrightarrow$(ii) See [1, Theorem 9.26].

(ii)$\Longrightarrow$(iii) Assume that T is $(\sigma(E, E_n^\sim), \sigma(X, X'_\xi))$-continuous. It follows that T is $(\tau(E, E_n^\sim), \tau(X, X'_\xi))$-continuous (see [1, Exercise 11, p. 149]), and hence T is $(\tau(E, E_n^\sim), \xi)$-continuous because $\xi \subset \tau(X, X'_\xi)$.

(iii)$\Longrightarrow$(iv) Assume that T is $(\tau(E, E_n^\sim), \xi)$-continuous, and let (u_α) be a net in E such that $u_\alpha \xrightarrow{(o)} 0$ in E. Then $u_\alpha \to 0$ for $\tau(E, E_n^\sim)$ because $\tau(E, E_n^\sim)$ is a Lebesgue topology on E. Hence $T(u_\alpha) \to 0$ for ξ, as desired.

(iv)$\Longrightarrow$(v) It is obvious.

(v)$\Longleftrightarrow$(vi) It is enough to repeat the reasoning in the proof of Proposition 4 in [13] and use Proposition 2.1 and the fact that $u_n \to 0$ in E for τ_0 whenever $u_n \to 0$ uniformly on Ω.

(v)$\Longrightarrow$(i) It is obvious. □

Corollary 2.3 *Every absolutely continuous operator $T : E \to X$ is order-weakly compact.*

Proof Note that for each $u \in E^+$, the order interval $[-u, u]$ in E is relatively $\sigma(E, E_n^\sim)$-compact because $\tau(E, E_n^\sim)$ is a Lebesgue topology (see [2], Theorem 6.62]). Hence by Proposition 2.2 the set $T([-u, u])$ is relatively $\sigma(X, X'_\xi)$-compact in X, as desired. □

Let $\mathcal{L}_{\tau,\xi}(E, X)$ stand for the space of all $(\tau(E, E_n^\sim), \xi)$-continuous linear operators from E to X, equipped with the topology $\mathcal{T}_s$ of simple convergence. Let $\mathcal{P}_\xi$ be the family of all ξ-continuous seminorms on X. Then $\mathcal{T}_s$ is generated by the family $\{q_{p,u} : p \in \mathcal{P}_\xi, u \in E\}$ of seminorms, where $q_{p,u}(T) = p(T(u))$ for all $T \in \mathcal{L}_{\tau,\xi}(E, X)$.

The following result will be of importance (see [15, Theorem 2]).

Theorem 2.4 *Let $\mathcal{K}$ be a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}$. If C is a $\sigma(X'_\xi, X)$-closed and ξ-equicontinuous subset of X'_ξ, then $\{x' \circ T : T \in \mathcal{K}, x' \in C\}$ is a $\sigma(E_n^\sim, E)$-compact subset of $E_n^\sim$.*

Now we can state a characterization of relative $\mathcal{T}_s$-compactness in $\mathcal{L}_{\tau,\xi}(E, X)$.

Theorem 2.5 *Let $\mathcal{K}$ be a subset of $\mathcal{L}_{\tau,\xi}(E, X)$. Then the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\mathcal{T}_s$-compact.*
(ii) *$\mathcal{K}$ is $(\tau(E_n^\sim, E), \xi)$-equicontinuous and for each $u \in E$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in X.*

Proof (i)$\Longrightarrow$(ii) Assume that $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact. Let W be an absolutely convex and ξ-closed neighbourhood of 0 for ξ in X. Then the polar W^0 of W (with respect to the dual pair $\langle E, E'_\xi\rangle$), is a $\sigma(X'_\xi, X)$-closed and ξ-equicontinuous subset of X'_ξ (see [1, Theorem 9.21]). Then by Theorem 2.4 the set $H = \{x' \circ T : T \in \mathcal{K}, x' \in W^0\}$ in $E_n^\sim$ is $\sigma(E_n^\sim, E)$-compact. Hence in view of the Nakamo theorem (see [2, Corollary 6.31]) the $\sigma(E_n^\sim, E)$-closed absolutely convex hull (abs conv H)$^-$ of H is $\sigma(E_n^\sim, E)$-compact in $E_n^\sim$. The the polar $V = ((\text{absconv } H)^-)^0$ (with respect to the dual pair $\langle E, E_n^\sim\rangle$) is a $\tau(E, E_n^\sim)$-neighbourhood of 0 in E and $H \subset V^0$. Then for each $T \in \mathcal{K}$ we have that $\{x' \circ T : x' \in W^0\} \subset V^0$, i.e., if $x' \in W^0$, then $|x'(T(u))| \leq 1$ for all $u \in V$. This means that for each $T \in \mathcal{K}$ we have $W^0 \subset T(V)^0$. Hence $T(V) \subset T(V)^{00} \subset W^{00} = W$ for each $T \in \mathcal{K}$, i.e., $\mathcal{K}$ is $(\tau(E, E_n^\sim), \xi)$-equicontinuous.

Clearly, for each $u \in E$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in X.

(ii)$\Longrightarrow$(i) It follows from [3, Chap. 3, § 3.4, Corollary 1], [4, Chap. 3.2.2, Corollary, p. 89]. □

Corollary 2.6 *Assume that $\mathcal{K}$ is a relatively $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(E, X)$. Then $\mathcal{K}$ is uniformly μ-absolutely continuous, i.e., for each $u \in E$ and $p \in \mathcal{P}_\xi$ we have*

$$\sup_{T\in\mathcal{K}} p(T(\mathbb{1}_{A_n} u)) \longrightarrow 0 \quad whenever \quad \mu(A_n) \longrightarrow 0, \ (A_n) \subset \Sigma.$$

Proof In view of Theorem 2.4, $\mathcal{K}$ is $(\tau(E, E_n^\sim), \xi)$-equicontinuous. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ be given. Then there exists a $\tau(E, E_n^\sim)$-neigbourhood V of 0 in E such that for each $T \in \mathcal{K}$ we have $p(T(u)) \leq \varepsilon$ for all $u \in V$. Let $u \in E$ and $\mu(A_n) \to 0$ and let $u_n = \mathbb{1}_{A_n} u$ for $n \in \mathbb{N}$. Note that $u_n \to 0(\mu)$ and $|u_n(\omega)| \leq |u(\omega)|\mu$-a.e. for all $n \in \mathbb{N}$. Hence by the Riesz theorem for every subsequence (u_{k_n}) of (u_n) there exists a subsequence $(u_{l_{k_n}})$ of (u_{k_n}) such that $u_{l_{k_n}}(\omega) \to 0\mu$-a.e. This means that $u_{l_{k_n}} \xrightarrow{(o)} 0$ in E (see [10, Chap. 10, §1]). Hence $u_{l_{k_n}} \to 0$ for $\tau(E, E_n^\sim)$ because $\tau(E, E_n^\sim)$ is a Lebesgue topology. It follows that $u_n \to 0$ for $\tau(E, E_n^\sim)$. Then there exists $n_\varepsilon \in \mathbb{N}$ such $u_n \in V$ for $n \geq n_\varepsilon$, and hence $\sup_{T\in\mathcal{K}} p(T(\mathbb{1}_{A_n} u)) \leq \varepsilon$ for $n \geq n_\varepsilon$. □

3 Absolutely continuous vector measures

Let (X, ξ) be a quasicomplete lcHs and $m : \Sigma \to X$ be a ξ-bounded vector measure (i.e., the range of m is ξ-bounded in X) and $m(A) = 0$ if $\mu(A) = 0$, $A \in \Sigma$ (in symbols, $m \ll \mu$).

For $u \in L^\infty(\mu)$ let $\|u\|_\infty = \text{ess sup}_{\omega\in\Omega} |u(\omega)|$. Given $u \in L^\infty(\mu)$, let (s_n) be a sequence in $\mathcal{S}(\mu)$ (= the space of all μ-simple functions on Ω) such that $\|u - s_n\|_\infty \to 0$ (see [10, Chap. 1, §6, Theorem 3]). Define

$$\int_\Omega u dm := \xi - \lim \int_\Omega s_n \, dm.$$

Then the integral $\int_\Omega u dm$ is well defined and the corresponding integration operator $T_m : L^\infty(\mu) \to X$ given by $T_m(u) = \int_\Omega u dm$ is $(\|\cdot\|_\infty, \xi)$-continuous and linear, and for each $x' \in X'_\xi$,

$$x'\left(\int_\Omega u dm\right) = \int_\Omega u \, d(x' \circ m) \quad \text{for} \quad u \in L^\infty(\mu),$$

(see [9], [14, Lemma 6]). Conversely, let $T : L^\infty(\mu) \to X$ be a $(\|\cdot\|_\infty, \xi)$-continuous linear operator, and let $m(A) = T(\mathbb{1}_A)$ for $A \in \Sigma$. Then $m : \Sigma \to X$ is a ξ-bounded vector measure such that $m \ll \mu$ (called the representing measure of T) and $T_m(u) = T(u)$ for all $u \in L^\infty(\mu)$.

An important example of a quasicomplete lcHs is the space $\mathcal{L}(Y, Z)$ of all bounded linear operators between Banach spaces Y and Z, provided with the strong operator topology.

Recall that a vector measure $m : \Sigma \to X$ is said to be *μ-absolutely continuous* $m(A_n) \to 0$ for ξ whenever $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$ (see [5, Definition 3, p. 11]).

Now we characterize μ-absolutely continuous measures in terms of the properties of the corresponding integration operators.

Proposition 3.1 *Assume that (X, ξ) is a quasicomplete lcHs. Let $m : \Sigma \to X$ be a ξ-bounded vector measure such that $m \ll \mu$. Then the following statements are equivalent:*

- (i) *$x' \circ m \in ca_\mu(\Sigma)$ for each $x' \in X'_\xi$.*
- (ii) *$x' \circ T_m \in L^\infty(\mu)^\sim_n$ for each $x' \in X'_\xi$.*
- (iii) *T_m is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-continuous.*
- (iv) *T_m is σ-smooth.*
- (v) *T_m is absolutely continuous.*
- (vi) *m is μ-absolutely continuous.*

Proof (i)$\Longrightarrow$(ii) Let $x' \in X'_\xi$ and $x' \circ m \in \text{ca}_\mu(\Sigma)$. Then by the Radon–Nikodym theorem there exists $v_{x'} \in L^1(\mu)$ such that $(x' \circ m)(A) = \int_A v_{x'} d\mu$ for all $A \in \Sigma$. It follows that

$$(x' \circ T_m)(u) = \int_\Omega u \, d(x' \circ m) = \int_\Omega u v_{x'} d\mu \quad \text{for all} \quad u \in L^\infty(\mu),$$

and this means that $x' \circ T_m \in L^\infty(\mu)^\sim_n$.

(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) $\Longleftrightarrow$(v) See Proposition 2.1.

(v)$\Longrightarrow$(vi) Assume that T_m is absolutely continuous, and let $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. Then $m(A_n) = T_m(\mathbb{1}_{A_n}) \to 0$ for ξ, as desired.

(vi)$\Longrightarrow$(i) It is obvious. □

As a consequence of Proposition 3.1 we get the following Pettis type theorem for countably additive measures (see [5, Theorem 1, p. 10]).

Corollary 3.2 *Assume that (X, ξ) is a quasicomplete lcHs. Let $m : \Sigma \to X$ be a ξ-countably additive measure. Then the following statements are equivalent:*

(i) $m \ll \mu$.
(ii) *m is μ-absolutely continuous.*

Let $ca(\Sigma, X)$ stand for the space of all ξ-countably additive measures $m : \Sigma \to X$. By $ca_\mu(\Sigma, X)$ we denote the subspace of $ca(\Sigma, X)$ consisting of all $m \in ca(\Sigma, X)$ that are μ-absolutely continuous. Denote by $\mathcal{T}_s$ the topology of simple convergence in $ca(\Sigma, X)$. Then $\mathcal{T}_s$ is generated by the family $\{q_{p,A} : p \in \mathcal{P}_\xi, A \in \Sigma\}$ of seminorms, where

$$q_{p,A}(m) := p(m(A)) \qquad \text{for all} \qquad m \in ca(\Sigma, X).$$

Proposition 3.3 *$ca_\mu(\Sigma, X)$ is a closed set in $(ca(\Sigma, X), \mathcal{T}_s)$.*

Proof Let $m \in ca(\Sigma, X)$ and $m \in cl_{\mathcal{T}_s}(ca_\mu(\Sigma, X))$. Then there is a net (m_α) in $ca_\mu(\Sigma, X)$ such that $m_\alpha \to m$ for $\mathcal{T}_s$, i.e., for each $p \in \mathcal{P}_\xi$ and $A \in \Sigma$ we have $q_{p,A}(m - m_\alpha) = p(m(A) - m_\alpha(A)) \underset{\alpha}{\longrightarrow} 0$. Assume that $\mu(A) = 0$. Then $m_\alpha(A) = 0$ for all α, and it follows that $p(m(A)) = 0$ for each $p \in \mathcal{P}_\xi$, i.e., $m(A) = 0$. In view of Corollary 3.2 $m \in ca_\mu(\Sigma, X)$. □

Now we establish some terminology (see [14, pp. 92–93]). For $p \in \mathcal{P}_\xi$ let $X_p = (X, p)$ be the associated seminormed space. Denote by $(\widetilde{X}_p, \|\cdot\|_{\widetilde{p}})$ the completion of the quotient normed space $X/p^{-1}(0)$. Let $\Pi_p : X_p \to X/p^{-1}(0) \subset \widetilde{X}_p$ be the canonical quotient map.

Given a vector measure $m : \Sigma \to X$ with $m \ll \mu$, let $m_p : \Sigma \to \widetilde{X}_p$ be given by

$$m_p(A) := (\Pi_p \circ m)(A) \ \text{ for } \ A \in \Sigma.$$

Then m_p is a Banach space-valued measure on Σ. We define the p-variation $\|m\|_p$ of m by

$$\|m\|_p(A) := \|m_p\|(A) \quad \text{for} \quad A \in \Sigma,$$

where $\|m_p\|$ denotes the semivariation of $m_p : \Sigma \to \widetilde{X}_p$. Note that m is ξ-bounded if and only if $\|m\|_p(\Omega) < \infty$ for each ξ-continuous seminorm p on X. Moreover, we have (see [14, Lemma 7]):

$$\|m\|_p(\Omega) = \|T_m\|_p := \sup\left\{ p\left(\int_\Omega u\,dm\right) : u \in L^\infty(\mu), \|u\|_\infty \le 1 \right\}. \tag{3.1}$$

For a subset $\mathcal{M}$ of $ca_\mu(\Sigma, X)$ let

$$\mathcal{K}_{\mathcal{M}} = \{T_m \in \mathcal{L}_{\tau,\xi}(L^\infty(\mu), X) : m \in \mathcal{M}\}.$$

Now we are ready to state a characterization of relative compactness in the space $(ca_\mu(\Sigma, X), \mathcal{T}_s)$ in terms of the topological properties of the set $\mathcal{K}_{\mathcal{M}}$ (see [8, Theorem 7], [15, Theorem 8], [16, Theorem 2.1]).

Theorem 3.4 *Let (X, ξ) be a quasicomplete lcHs. Then for a set $\mathcal{M}$ in $ca_\mu(\Sigma, X)$ the following statements are equivalent:*

(i) *$\mathcal{K}_{\mathcal{M}}$ is a relatively compact set in $(\mathcal{L}_{\tau,\xi}(L^\infty(\mu), X), \mathcal{T}_s)$.*
(ii) *$\mathcal{K}_{\mathcal{M}}$ is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-equicontinuous and for each $u \in L^\infty(\mu)$, the set $\{T_m(u) : m \in \mathcal{M}\}$ is relatively ξ-compact in X.*
(iii) *$\mathcal{M}$ is uniformly μ-absolutely continuous and for each $A \in \Sigma$, the set $\{m(A) : m \in \mathcal{M}\}$ is relatively ξ-compact in X.*
(iv) *$\mathcal{M}$ is a relatively compact set in $(ca_\mu(\Sigma, X), \mathcal{T}_s)$.*

Proof (i)$\Longleftrightarrow$(ii) See Theorem 2.5.

(ii)$\Longrightarrow$(iii) Assume that (ii) holds and let $\mu(A_n \to 0, (A_n) \subset \Sigma$. Then using Proposition 3.1 and Corollary 2.6 for each $p \in \mathcal{P}_\xi$ we have

$$\sup_{m\in\mathcal{M}} p(m(A_n)) = \sup_{m\in\mathcal{M}} p(T_m(\mathbb{1}_{A_n})) \xrightarrow[n]{} 0.$$

This means that the family $\mathcal{M}$ is uniformly μ-absolutely continuous.

(iii)$\Longrightarrow$(iv) Assume that (iii) holds. Then $\mathcal{M} \subset ca_\mu(\Sigma, X) \subset ca(\Sigma, X)$ and $\mathcal{M}$ is a uniformly ξ-countably additive set in $ca(\Sigma, X)$. Hence by [8, Theorem 7] $\mathcal{M}$ is a relatively compact set in $(ca(\Sigma, X), \mathcal{T}_s)$. Since $ca_\mu(\Sigma, X)$ is closed in $(ca(\Sigma, X), \mathcal{T}_s)$, we obtain that $\mathcal{M}$ is a relatively compact set in $(ca_\mu(\Sigma, X), \mathcal{T}_s)$.

(iv)$\Longrightarrow$(i) Assume that $\mathcal{M}$ is a relatively compact set in $(ca_\mu(\Sigma, X), \mathcal{T}_s)$, and let (T_{m_α}) be a net in $\mathcal{K}_{\mathcal{M}}$. Without loss of generality, we can assume that $m_\alpha \to m$ for $\mathcal{T}_s$, where $m \in ca_\mu(\Sigma, X)$. We shall show that $T_{m_\alpha} \to T_m$ in $(\mathcal{L}_{\tau,\xi}(L^\infty(\mu), X), \mathcal{T}_s)$. Indeed, let $p \in \mathcal{P}_\xi$ and fix $\varepsilon > 0$. Since $\mathcal{M}$ is a $\mathcal{T}_s$-bounded subset of $ca_\mu(\Sigma, X)$, for each $A \in \Sigma$ we have $\sup_\alpha p(m_\alpha(A)) = \sup_\alpha q_{p,A}(m_\alpha) < \infty$. Hence, since the mapping $\Pi_p : X \to \widetilde{X}_p$ is $(p, \|\cdot\|_p^\sim)$-continuous, we obtain that $\sup_\alpha \|(m_\alpha)_p(A)\|_p^\sim = \sup_\alpha \|(\Pi_p \circ m_\alpha)(A)\|_p^\sim < \infty$. In view of the Nikodym boundedness theorem (see [5, Theorem 1, p. 14]) and 3.1 we get

$$c = \sup_\alpha \|T_{m_\alpha}\|_p = \sup_\alpha \|m_\alpha\|_p(\Omega) < \infty.$$

Let $u \in L^\infty(\mu)$ be given and choose $s_0 \in \mathcal{S}(\mu)$ such that $\|u - s_0\|_\infty \le \frac{\varepsilon}{3a}$, where $a = \max(c, \|T_m\|_p)$. Then there exists α_0 such that $p(T_{m_\alpha}(s_0) - T_m(s_0)) \le \frac{\varepsilon}{3}$ for $\alpha \ge \alpha_0$. Hence for $\alpha \ge \alpha_0$ we get

$$\begin{aligned} &p(T_{m_\alpha}(u) - T_m(u)) \\ &\quad \le p(T_m(u - s_0)) + p(T_m(s_0) - T_{m_\alpha}(s_0)) + p(T_{m_\alpha}(s_0) - T_{m_\alpha}(u)) \end{aligned}$$

$$\leq \|T_m\|_p \cdot \|u - s_0\|_\infty + p(T_m(s_0) - T_{m_\alpha}(s_0)) + \|T_{m_\alpha}\|_p \cdot \|s_0 - u\|_\infty$$
$$\leq a \cdot \frac{\varepsilon}{3a} + \frac{\varepsilon}{3} + a \cdot \frac{\varepsilon}{3a} = \varepsilon.$$

This means that $T_{m_\alpha} \underset{\alpha}{\longrightarrow} T_m$ in $(\mathcal{L}_{\tau,\xi}(L^\infty(\mu), X), \mathcal{T}_s)$, as desired. □

Recall that the general Vitali–Hahn–Saks theorem (see [7, Theorem 2.14']) says that if (m_k) is a sequence of μ-absolutely continuous measures on a σ-algebra Σ taking values in a lcHs (X, ξ), and $m(A) := \xi - \lim m_k(A)$ for each $A \in \Sigma$, then $m : \Sigma \to X$ is a μ-absolutely continuous measure and the family $\{m_k : k \in \mathbb{N}\}$ is uniformly μ-absolutely continuous.

Now we shall state a generalized Vitali–Hahn–Saks theorem for operators from $L^\infty(\mu)$ to a quasicomplete lcHs (X, ξ).

Theorem 3.5 *Assume that (X, ξ) is a quasicomplete lcHs. Let $m_k : \Sigma \to X$ be μ-absolutely continuous measures for $k \in \mathbb{N}$ and assume that $m(A) := \xi - \lim m_k(A)$ exists for each $A \in \Sigma$. Then the following statements hold:*

(i) *$m : \Sigma \to X$ is a μ-absolutely continuous measure, and the integration operator $T_m : L^\infty(\mu) \to X$ is absolutely continuous.*
(ii) *$T_m(u) = \xi - \lim_k T_{m_k}(u)$ for all $u \in L^\infty(\mu)$.*
(iii) *The family $\{T_{m_k} : k \in \mathbb{N}\}$ is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-equicontinuous.*
(iv) *The family $\{T_{m_k} : k \in \mathbb{N}\}$ is uniformly absolutely continuous.*

Proof In view of the general Vitali–Hahn–Saks theorem (see [7, Theorem 2.14']) $m : \Sigma \to X$ is μ-absolutely continuous, and by Proposition 3.1 $T_m : L^\infty \to X$ is absolutely continuous.

Let $p \in \mathcal{P}_\xi$ and fix $\varepsilon > 0$. We show that $p(T_{m_k}(u) - T_m(u)) \to 0$ for each $u \in L^\infty(\mu)$. Indeed, since $p(m_k(A) - m(A)) \to 0$ for all $A \in \Sigma$, we have

$$\|\Pi_p(m_k(A) - m(A))\|_p^\sim \to 0, \text{ i.e., } \|(m_k)_p(A) - m_p(A)\|_p^\sim \to 0 \text{ for all } A \in \Sigma.$$

It follows that $\sup_k \|(m_k)_p(A)\|_p^\sim < \infty$ for all $A \in \Sigma$, and in view of the Nikodym boundedness theorem (see [5, Theorem 1, p. 14]) and 3.1 we get

$$a = \sup_k \|T_{m_k}\|_p = \sup_k \|m_k\|_p(\Omega) < \infty.$$

Let $u \in L^\infty(\mu)$ be given and choose $s_0 \in \mathcal{S}(\mu)$ such that $\|u - s_0\|_\infty \leq \frac{\varepsilon}{3a}$, where $a = \max(c, \|T_m\|_p)$. Then there is $k_0 \in \mathbb{N}$ such that $p(T_{m_k}(s_0) - T_m(s_0)) \leq \frac{\varepsilon}{3}$ for $k \geq k_0$. Hence for $k \geq k_0$ we have

$$p(T_{m_k}(u) - T_m(u - s_0))$$
$$\leq p(T_m(u - s_0)) + p(T_m(s_0) - T_{m_k}(s_0)) + p(T_{m_k}(s_0) - T_{m_k}(u))$$
$$\leq \|T_m\|_p \cdot \|u - s_0\|_\infty + p(T_m(s_0) - T_{m_k}(s_0)) + \|T_{m_k}\|_p \cdot \|s_0 - u\|_\infty$$
$$\leq a \cdot \frac{\varepsilon}{3a} + \frac{\varepsilon}{3} + a \cdot \frac{\varepsilon}{3a} = \varepsilon.$$

It follows that $T_{m_k} \to T$ for $\mathcal{T}_s$ in $\mathcal{L}_{\tau,\xi}(L^\infty(\mu), X)$. Since $\{T_{m_k} : k \in \mathbb{N}\} \cup \{T\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(L^\infty(\mu), X)$, by Theorem 2.5 the set $\{T_{m_k} : k \in \mathbb{N}\}$ is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-equicontinuous, and by Corollary 2.6 it is uniformly absolutely continuous. □

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