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Existence and uniqueness of positive solutions for the Neumann p-Laplacian

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Abstract We consider a nonlinear Neumann problem driven by the p-Laplacian and with a Carathéodory reaction which satisfies only a unilateral growth restriction. Using the principal eigenvalue of an eigenvalue problem involving the Neumann p-Laplacian plus an indefinite potential, we produce necessary and sufficient conditions for the existence and uniqueness of positive smooth solutions.

Keywords p-Laplacian · Nonlinear strong maximum principle · Positive solutions · Unilateral growth restriction

Mathematics Subject Classification (2000) 35J65 · 35J70 · 35J92

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann problem:

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$$\begin{cases} -\Delta_p u(z) = f\big(z, u(z)\big) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \quad u > 0. \end{cases} \tag{1.1}$$

Here Δ_p denotes the p-Laplace differential operator, defined by

$$\Delta_p u = \operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) \quad \forall u \in W^{1,p}(\Omega),$$

with $p \in (1, +\infty)$. Also, the reaction $f(z, \zeta)$ is a Carathéodory function, i.e., for all $\zeta \in \mathbb{R}$, $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, $\zeta \longmapsto f(z, \zeta)$ is continuous.

We are interested in the existence and uniqueness of positive solutions when the nonlinearity $f(z, \cdot)$ is only unilaterally restricted (only from above). Problems like this were studied primarily in the context of semilinear (i.e., $p = 2$) equations with Dirichlet boundary conditions. We mention the works of Amann [2], Brézis and Oswald [4], Dancer [6], de Figueiredo [7], Hess [16], Krasnoselskii [19], Laetsch [20], and Simpson and Cohen [24]. Extensions to the Dirichlet p-Laplacian can be found in the works of Guo [14], Guo and Webb [15] and Kamin and Veron [18], but for special classes of equations, such as logistic equations. To the best of our knowledge, there are no such results for the Neumann p-Laplacian. Some other existence results for Neumann p-Laplacian problems, but with no information on the sign of solutions can be found in Gasiński and Papageorgiou [9–11] and with some sign information on the solution (but without uniqueness) can be found in Gasiński and Papageorgiou [12,13].

As it is remarked in de Figueiredo [7], the problem of uniqueness for elliptic equations, is in general a difficult one and requires special structure on the reaction term. Our work here is closely related to that of Brézis and Oswald [4]. In fact our result is a twofold generalization of that in [4]. First, we pass from the Laplacian (semilinear equation; i.e., $p = 2$) to the p-Laplacian (nonlinear equation; i.e., $p \in (1, +\infty)$). Second, we pass from the Dirichlet to the Neumann boundary condition. We should mention that sufficient conditions for the uniqueness of the positive solutions of the Dirichlet p-Laplacian were obtained by Belloni and Kawohl [3], were the authors exploited in a direct way the convexity of the energy functional $u \longmapsto \varphi(u)$ in u^p.

2 An eigenvalue problem

In this section we discuss the first eigenvalue of the nonlinear eigenvalue problem involving the negative Neumann p-Laplacian plus an indefinite potential. This quantity plays a central role in our subsequential considerations, but it is also of independent interest.

The eigenvalue problem under consideration is the following:

$$\begin{cases} -\Delta_p u(z) + \beta(z)\big|u(z)|^{p-2}u(z) = \widehat{\lambda}\big|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

Proposition 2.1 *If $\beta \in L^{\infty}(\Omega)$, then problem (2.1) has a smallest eigenvalue $\widehat{\lambda}_1 = \widehat{\lambda}_1(\beta) \in \mathbb{R}$ which is simple, has a corresponding L^p-normalized eigenfunction $\widehat{u}_1 \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$ with $\widehat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$.*

Proof Let $\xi : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional, defined by

$$\xi(u) = \|\nabla u\|_p^p + \int_\Omega \beta |u|^p \, dz$$

and let $M \subseteq W^{1,p}(\Omega)$ be the C^1-Banach manifold, defined by

$$M = \left\{ u \in W^{1,p}(\Omega) : \|u\|_p = 1 \right\}.$$

We set

$$\widehat{\lambda}_1 = \widehat{\lambda}_1(\beta) = \inf \left\{ \xi(u) : \ u \in M \right\}. \tag{2.2}$$

Because for $u \in M$, we have

$$\left| \int_\Omega \beta |u|^p \, dz \right| \leqslant \int_\Omega |\beta||u|^p \, dz \leqslant \|\beta\|_\infty \|u\|_p^p = \|\beta\|_\infty,$$

so

$$\xi(u) = \|\nabla u\|_p^p + \int_\Omega \beta |u|^p \, dz \geqslant \|\nabla u\|_p^p - \|\beta\|_\infty \geqslant -\|\beta\|_\infty \quad \forall u \in M.$$

Thus $\widehat{\lambda}_1 \geqslant -\|\beta\|_\infty$. We will show that the infimum in (2.2) is realized at a $\widehat{u}_1 \in W^{1,p}(\Omega)$, with $\|\widehat{u}_1\|_p = 1$. To this end, let $\{u_n\}_{n \geqslant 1} \subseteq M$ be a minimizing sequence, i.e.,

$$\xi(u_n) \longrightarrow \widehat{\lambda}_1.$$

Clearly the sequence $\{u_n\}_{n \geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} \widehat{u}_1 \quad \text{in } W^{1,p}(\Omega), \tag{2.3}$$

$$u_n \longrightarrow \widehat{u}_1 \quad \text{in } L^p(\Omega). \tag{2.4}$$

From (2.3) and (2.3), we have

$$\|\nabla \widehat{u}_1\|_p^p \leqslant \liminf_{n \to +\infty} \|\nabla u_n\|_p^p \quad \text{and} \quad \lim_{n \to +\infty} \int_\Omega \beta |u_n|^p \, dz = \int_\Omega \beta |\widehat{u}_1|^p \, dz,$$

so

$$\xi(\widehat{u}_1) \leqslant \widehat{\lambda}_1.$$

It is clear from (2.3) that $\|\widehat{u}_1\|_p = 1$, i.e., $\widehat{u}_1 \in M$. Hence $\xi(\widehat{u}_1) = \widehat{\lambda}_1$.

The Lagrange multiplier rule (see, e.g., Papageorgiou and Kyritsi [23, p. 76]) implies that $\widehat{\lambda}_1$ is an eigenvalue of problem (2.1), with the corresponding eigenfunction $\widehat{u}_1 \in W^{1,p}(\Omega)$. Using the Moser iteration technique, we show that $\widehat{u}_1 \in L^\infty(\Omega)$ (see, e.g., Hu and Papageorgiou [17]) and the nonlinear regularity theorem of Lieberman [21], implies that $\widehat{u}_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Moreover, since

$$\xi(|u|) = \xi(u) \quad \forall u \in M,$$

we infer that $\widehat{u}_1$ does not change sign and we may assume that $\widehat{u}_1 \geqslant 0$. Invoking the nonlinear maximum principle of Vázquez [25], we conclude that

$$\widehat{u}_1(z) > 0 \quad \forall z \in \overline{\Omega}.$$

Next, we show the simplicity of $\widehat{\lambda}_1$. So, let $\widehat{v}_1 \in W^{1,p}(\Omega)$ be another eigenfunction corresponding to $\widehat{\lambda}_1$. As above, we show that $\widehat{v}_1 \in C^1(\overline{\Omega})$ and $\widehat{v}_1(z) > 0$ for all $z \in \overline{\Omega}$. We introduce

$$R(\widehat{u}_1, \widehat{v}_1)(z) = \|\nabla\widehat{u}_1(z)\|^p - \|\nabla\widehat{v}_1(z)\|^{p-2}\left(\nabla\widehat{v}_1(z),\ \nabla\left(\frac{\widehat{u}_1(z)^p}{\widehat{v}_1(z)^{p-1}}\right)\right)_{\mathbb{R}^N}. \tag{2.5}$$

From the generalized Picone identity of Allegretto and Huang [1] and the nonlinear Green's identity (see Casas and Fernández [5]), we have

$$\begin{aligned}
0 &\leqslant \int_\Omega R(\widehat{u}_1, \widehat{v}_1)\,dz \\
&= \int_\Omega \left[\|\nabla\widehat{u}_1\|^p - \|\nabla\widehat{v}_1\|^{p-2}\left(\nabla\widehat{v}_1,\ \nabla\left(\frac{\widehat{u}_1^p}{\widehat{v}_1^{p-1}}\right)\right)_{\mathbb{R}^N}\right]dz \\
&= \int_\Omega \left[\|\nabla\widehat{u}_1\|^p + \Delta_p\widehat{v}_1\left(\frac{\widehat{u}_1^p}{\widehat{v}_1^{p-1}}\right)\right]dz \\
&= \int_\Omega \left[\|\nabla\widehat{u}_1\|^p + \big(\beta(z) - \widehat{\lambda}_1\big)\widehat{v}_1^{p-1}\frac{\widehat{u}_1^p}{\widehat{v}_1^{p-1}}\right]dz \\
&= \int_\Omega \left[\|\nabla\widehat{u}_1\|^p + \beta\widehat{u}_1^p\right]dz - \widehat{\lambda}_1\|\widehat{u}_1\|_p^p \\
&= \xi(\widehat{u}_1) - \widehat{\lambda}_1\|\widehat{u}_1\|_p^p = 0,
\end{aligned}$$

so

$$\int_\Omega R(\widehat{u}_1, \widehat{v}_1)\, dz = 0$$

and thus

$$R(\widehat{u}_1, \widehat{v}_1) = 0 \quad \forall z \in \overline{\Omega},$$

so finally $\widehat{u}_1 = k\widehat{v}_1$ for some $k > 0$ (see Allegretto and Huang [1]). This proves that $\widehat{\lambda}_1$ is simple (i.e., it is a principal eigenvalue). □

From the above proof, we have

$$\begin{aligned}\widehat{\lambda}_1(\beta) &= \inf\left\{\int_\Omega \|\nabla u\|^p\, dz + \int_\Omega \beta|u|^p\, dz :\ u \in W^{1,p}(\Omega),\ \|u\|_p = 1\right\}\\ &= \inf\left\{\int_\Omega \|\nabla u\|^p\, dz + \int_{\{u\neq 0\}} \beta|u|^p\, dz :\ u \in W^{1,p}(\Omega),\ \|u\|_p = 1\right\}. \end{aligned} \tag{2.6}$$

Note that in the second infimum in (2.6), the integral $\int_{\{u\neq 0\}} \beta|u|^p\, dz$ makes sense even when β is only a measurable function and there exists $\widehat{c} > 0$, such that

$$\beta(z) \leqslant \widehat{c} \quad \text{for almost all } z \in \Omega$$

or

$$\beta(z) \geqslant -\widehat{c} \quad \text{for almost all } z \in \Omega.$$

In the first case $\widehat{\lambda}_1(\beta) \in [-\infty, +\infty)$ and in the second case $\widehat{\lambda}_1(\beta) \in (-\infty, +\infty]$.

In what follows by $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ we denote the nonlinear map, defined by

$$\langle A(u), y\rangle = \int_\Omega \|\nabla u\|^{p-2}(\nabla u,\ \nabla y)_{\mathbb{R}^N}\, dz \quad \forall u, y \in W^{1,p}(\Omega).$$

This map is continuous and maximal monotone (see [8] or [23]).

3 Existence of positive solutions

In this section we prove the existence of a positive smooth solution. The hypotheses on the reaction f are the following:

$\underline{H_f}$: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that

(i) for all $\zeta \geqslant 0$, $f(\cdot, \zeta) \in L^\infty(\Omega)$ and there exists $c > 0$, such that

$$f(z, \zeta) \leqslant c(1 + \zeta^{p-1}) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0;$$

(ii) for almost all $z \in \Omega$, the function $\zeta \longrightarrow \frac{f(z,\zeta)}{\zeta^{p-1}}$ is strictly decreasing on $(0, +\infty)$;

(iii) if $\eta(z) = \lim_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{p-1}}$, then $\widehat{\lambda}_1(-\eta) > 0$;

(iv) if $\eta_0(z) = \lim_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}}$, then $\widehat{\lambda}_1(-\eta_0) < 0$.

Remark 3.1 Since we are looking for positive solutions and hypotheses H_f concern only the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, by truncating if necessary, we may (and will) assume that

$$f(z, \zeta) = f(z, 0) \quad \text{for almost all } \zeta \leqslant 0.$$

Note that H_f(i) is a unilateral growth condition. Hypothesis H_f(ii) implies that both functions η and η_0 are measurable. Moreover, we have

$$\frac{f(z, \zeta)}{\zeta^{p-1}} \leqslant f(z, 1) \leqslant \|f(\cdot, 1)\|_\infty = \widehat{c} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 1,$$

so

$$\eta(z) \leqslant \widehat{c} \quad \text{for almost all } z \in \Omega$$

and thus

$$\widehat{\lambda}_1(-\eta) \in (-\infty, +\infty].$$

Similarly, we have

$$\frac{f(z, \zeta)}{\zeta^{p-1}} \geqslant f(z, 1) \geqslant -\|f(\cdot, 1)\|_\infty = -\widehat{c} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in (0, 1],$$

so

$$\eta_0(z) \geqslant -\widehat{c} \quad \text{for almost all } z \in \Omega$$

and thus

$$\widehat{\lambda}_1(-\eta_0) \in [-\infty, +\infty).$$

If $\eta, \eta_0 \in L^\infty(\Omega)$, then $\widehat{\lambda}_1(-\eta), \widehat{\lambda}(-\eta_0) \in \mathbb{R}$ and are the principal eigenvalues of (2.1) when $\beta = -\eta$ and $\beta = -\eta_0$ respectively. If $f(z, \zeta) = f(\zeta)$ (autonomous case), then hypotheses H_f(iii) and H_f(iv) are equivalent to saying that

$$\eta < \widehat{\lambda}_1 = 0 < \eta_0$$

(recall that the first eigenvalue of the negative Neumann p-Laplacian (i.e., problem (2.1) with $\beta \equiv 0$) is zero).

Example 3.2 Let

$$f(\zeta) = \lambda\big(\zeta^{p-1} - \zeta^{q-1}\big) \quad \forall \zeta \geqslant 0,$$

with $1 < p < q, \lambda > 0$. Then f satisfies hypotheses H_f. This function corresponds to the equidiffusive p-logistic equation and $\eta_0 = \lambda > 0$, $\eta = -\infty$. More generally, let

$$f(\zeta) = \begin{cases} \zeta^{p-1} - \zeta^{q-1} & \text{if} \quad \zeta \in [0, 1], \\ \zeta^{p-1} - e^{\zeta-1} & \text{if} \quad \zeta \geqslant 1, \end{cases}$$

with $1 < p \leqslant q$. Note that this f has no polynomial growth restriction from below.

We introduce the following truncation–perturbation of f:

$$g(z, \zeta) = \begin{cases} f(z, 0) & \text{if} \quad \zeta \leqslant 0, \\ f(z, \zeta) + \zeta^{p-1} & \text{if} \quad \zeta > 0, \end{cases} \tag{3.1}$$

This is a Carathéodory function. We set

$$F(z, \zeta) = \int_0^\zeta f(z, s)\, ds \quad \text{and} \quad G(z, \zeta) = \int_0^\zeta g(z, s)\, ds.$$

Note that hypothesis H_f(i) and (3.1) imply that

$$G(z, \zeta) \leqslant c_1(1 + \zeta^p) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R} \tag{3.2}$$

and some $c_1 > 0$. Because of (3.2), we see that we can introduce the functional $\widehat{\varphi}\colon W^{1,p}(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, defined by

$$\widehat{\varphi} = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p - \int_\Omega G(z, u)\, dz \quad \forall u \in W^{1,p}(\Omega).$$

Proposition 3.3 *If hypotheses H_f hold, then $\widehat{\varphi}$ is coercive, i.e., $\widehat{\varphi}(u) \longrightarrow +\infty$ as $\|u\| \rightarrow +\infty$.*

Proof We argue by contradiction. So, suppose that we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$, such that

$$\|u_n\| \longrightarrow +\infty \quad \text{and} \quad \widehat{\varphi}(u_n) \leqslant M_1 \quad \forall n \geqslant 1, \tag{3.3}$$

for some $M_1 > 0$. We have

$$\frac{1}{p}\left(\|\nabla u_n\|_p^p + \|u_n\|_p^p\right) \leqslant c_2\left(1 + \|u_n\|_p^p\right) \quad \forall n \geqslant 1, \tag{3.4}$$

for some $c_2 > 0$ (see (3.2) and (3.3)).

It is clear from (3.3) and (3.4) that $\|u_n\|_p \longrightarrow +\infty$. We set

$$y_n = \frac{u_n}{\|u_n\|_p} \quad \forall n \geqslant 1.$$

Then

$$\|y_n\|_p = 1 \quad \forall n \geqslant 1 \tag{3.5}$$

and from (3.4), we have

$$\frac{1}{p}\big(\|\nabla y_n\|_p^p + \|y_n\|_p^p\big) \leqslant c_2\bigg(\frac{1}{\|u_n\|_p^p} + 1\bigg),$$

so the sequence $\{y_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

So, passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega), \tag{3.6}$$

$$y_n \longrightarrow y \quad \text{in } L^p(\Omega), \tag{3.7}$$

hence $\|y\|_p = 1$. We have

$$\begin{aligned} \frac{1}{p}\big(\|\nabla y_n\|_p^p + \|y_n\|_p^p\big) &\leqslant \frac{M_1}{\|u_n\|_p^p} + \int\limits_{\Omega} \frac{G(z,u_n)}{\|u_n\|_p^p}\,dz \\ &\leqslant \frac{M_1}{\|u_n\|_p^p} + \int\limits_{\{u_n>0\}} \bigg(\frac{F(z,u_n)}{\|u_n\|_p^p} + \frac{1}{p}y_n^p\bigg)dz + \int\limits_{\{u_n\leqslant 0\}} \frac{f(z,0)u_n}{\|u_n\|_p^p}\,dz \end{aligned} \tag{3.8}$$

(see (3.1)). Note that

$$\frac{f(z,\zeta)}{\zeta^{p-1}} \geqslant f(z,1) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in (0,1]$$

(see hypothesis H_f(ii)). Hence

$$f(z,\zeta) \geqslant f(z,1)\zeta^{p-1} \geqslant -\big\|f(\cdot,1)\big\|_\infty \zeta^{p-1} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in (0,1].$$

So, it follows that

$$f(z,0) \geqslant 0 \quad \text{for almost all } z \in \Omega.$$

Then

$$\int_{\{u_n \leqslant 0\}} \frac{f(z,0)u_n}{\|u_n\|_p^p}\, dz \leqslant 0 \quad \forall n \geqslant 1. \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$\frac{1}{p}\left(\|\nabla y_n\|_p^p + \|y_n\|_p^p\right) \leqslant \frac{M_1}{\|u_n\|_p^p} + \frac{1}{p}\|y_n^+\|_p^p + \int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\, dz \quad \forall n \geqslant 1. \tag{3.10}$$

Suppose that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq L^p(\Omega)$ is bounded. Then $y \leqslant 0$. From hypothesis H_f(i), we have

$$F(z,\zeta) \leqslant c_3(1+\zeta^p) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0 \tag{3.11}$$

and some $c_3 > 0$. Then using (3.11), we have

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\, dz \leqslant \frac{c_3|\Omega|_N}{\|u_n\|_p^p} + c_3\|y_n^+\|_p^p$$

($|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$), so

$$\limsup_{n\to+\infty} \int_\Omega \frac{F(z,u_n^+)}{\|u_n\|_p^p}\, dz \leqslant 0$$

(see (3.7) and recall that $y \leqslant 0$). So, if in (3.10) we pass to the limit as $n \to +\infty$, we obtain

$$\frac{1}{p}\left(\|\nabla y\|_p^p + \|y\|_p^p\right) \leqslant 0,$$

so $y = 0$, which contradicts (3.5) and (3.7).

Therefore we may assume that $\|u_n^+\|_p \longrightarrow +\infty$. From the inequality in (3.3), we have

$$\frac{1}{p}\|\nabla y_n^+\|_p^p \leqslant \frac{M_1}{\|u_n^+\|_p^p} + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|_p^p}\, dz \quad \forall n \geqslant 1 \tag{3.12}$$

(see (3.9)). We have

$$\int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p}\, dz = \int_{\{y^+=0\}} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p}\, dz + \int_{\{y>0\}\cap\{y_n>0\}} \frac{F(z, u_n^+)}{(u_n^+)^p} (y_n^+)^p\, dz \quad \forall n \geqslant 1. \tag{3.13}$$

Since

$$y_n^+ \longrightarrow y^+ \quad \text{in } L^p(\Omega)$$

(see (3.7)), by passing to a further subsequence if necessary, we may also assume that

$$y_n^+(z) \longrightarrow y^+(z) \quad \text{for almost all } z \in \Omega. \tag{3.14}$$

From (3.11), we have

$$\left| \int_{\{y^+=0\}} \frac{F(z, u_n^+)}{\|u_n^+\|_p^p}\, dz \right| \leqslant c_3 \int_{\{y^+=0\}} \left(\frac{1}{\|u_n^+\|_p^p} + (y_n^+)^p \right) dz \longrightarrow 0. \tag{3.15}$$

Note that

$$u_n^+(z) \longrightarrow +\infty \quad \text{almost everywhere on } \{y^+ > 0\} \tag{3.16}$$

and

$$\chi_{\{y>0\}\cap\{y_n>0\}}(z) \longrightarrow \chi_{\{y>0\}}(z) \quad \text{almost everywhere in } \Omega. \tag{3.17}$$

Moreover, we claim that

$$\limsup_{\zeta \to +\infty} \frac{F(z, \zeta)}{\zeta^p} \leqslant \frac{1}{p} \eta(z) \quad \text{for almost all } z \in \Omega. \tag{3.18}$$

Indeed, first let $z \in \{\eta > -\infty\} \setminus D$, with $|D|_N = 0$ be such that

$$\frac{f(z, \zeta)}{\zeta^{p-1}} \longrightarrow \eta(z) \quad \text{as } \zeta \to +\infty.$$

(see hypotheses H_f(ii) and (iii)). For a given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon, z) > 0$, such that

$$f(z, \zeta) \leqslant \big(\eta(z) + \varepsilon\big)\zeta^{p-1} \quad \forall \zeta \geqslant M_2,$$

so

$$F(z,\zeta) \leqslant \frac{1}{p}\big(\eta(z)+\varepsilon\big)\zeta^p \quad \forall \zeta \geqslant M_2,$$

thus

$$\frac{F(z,\zeta)}{\zeta^p} \leqslant \frac{1}{p}\big(\eta(z)+\varepsilon\big) \quad \forall \zeta \geqslant M_2$$

and so

$$\limsup_{\zeta\to+\infty} \frac{F(z,\zeta)}{\zeta^p} \leqslant \frac{1}{p}\big(\eta(z)+\varepsilon\big).$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$ to conclude that

$$\limsup_{\zeta\to+\infty} \frac{F(z,\zeta)}{\zeta^p} \leqslant \frac{1}{p}\eta(z) \quad \text{for almost all } z \in \{\eta > -\infty\}.$$

If $z \in \{\eta = -\infty\} \setminus D$, with $|D|_N = 0$ is such that

$$\frac{f(z,\zeta)}{\zeta^{p-1}} \longrightarrow -\infty = \eta(z) \quad \text{as } \zeta \to +\infty,$$

then for every $\xi > 0$, we can find $M_3 = M_3(\xi, z) > 0$, such that

$$f(z,\zeta) \leqslant -\xi\zeta^{p-1} \quad \forall \zeta \geqslant M_3,$$

so

$$\frac{F(z,\zeta)}{\zeta^p} \leqslant -\frac{\xi}{p} \quad \forall \zeta \geqslant M_3$$

and thus

$$\limsup_{\zeta\to+\infty} \frac{F(z,\zeta)}{\zeta^p} \leqslant -\frac{\xi}{p}.$$

Since $\xi > 0$ was arbitrary, we let $\xi \to +\infty$ to conclude that

$$\lim_{\zeta\to+\infty} \frac{F(z,\zeta)}{\zeta^p} = -\infty \quad \text{for almost all } z \in \{\eta = -\infty\}.$$

Therefore, finally we have proved (3.18).

Using Fatou's lemma in (3.18) (which is legitimate because of (3.11)) as well as (3.16), (3.17) and (3.14), we have

$$\limsup_{n\to+\infty} \int\limits_{\{y>0\}\cap\{y_n>0\}} \frac{F(z,u_n^+)}{(u_n^+)^p}(y_n^+)^p\,dz \leqslant \frac{1}{p}\int\limits_{\{y>0\}} \eta y^p\,dz = \frac{1}{p}\int\limits_{\{y^+\neq 0\}} \eta (y^+)^p\,dz. \tag{3.19}$$

Hence, if in (3.13) we pass to the limit as $n \to +\infty$ and use (3.15) and (3.19), we obtain

$$\limsup_{n\to+\infty} \int\limits_{\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \leqslant \frac{1}{p}\int\limits_{\{y^+\neq 0\}} \eta (y^+)^p\,dz. \tag{3.20}$$

Returning to (3.12), taking limits as $n \to +\infty$ and using (3.6) and (3.20), we have

$$\|\nabla y^+\|_p^p \leqslant \frac{1}{p}\int\limits_{\{y^+\neq 0\}} \eta (y^+)^p\,dz. \tag{3.21}$$

If $y^+ = 0$, then from (3.10), we have

$$\frac{1}{p}\left(\|\nabla y^-\|_p^p + \|y^-\|_p^p\right) \leqslant 0,$$

so $y^- = 0$, i.e., $y = 0$ which contradicts (3.6).

So $y^+ \neq 0$ and then from (3.21), it follows that

$$\widehat{\lambda}_1(-\eta) \leqslant 0$$

(see (2.6)), which contradicts hypothesis H_f(iii). This proves that $\widehat{\varphi}$ is coercive. □

Proposition 3.4 *If hypotheses H_f hold, then $\widehat{\varphi}$ is sequentially weakly lower semicontinuous.*

Proof From the expression of $\widehat{\varphi}$ and since the norm in a Banach space is sequentially weakly lower semicontinuous, it suffices to show that the integral functional $\psi\colon W^{1,p}(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$, defined by

$$\psi(u) = -\int\limits_{\Omega} G(z,u)\,dz$$

is sequentially weakly lower semicontinuous. To this end, we need to show that for every $\lambda \in \mathbb{R}$, the sublevel set

$$L_\lambda = \left\{u \in W^{1,p}(\Omega):\ \psi(u) \leqslant \lambda\right\}$$

is sequentially weakly closed. To this end, let $\{u_n\}_{n\geqslant 1} \subseteq L_\lambda$ and assume that

$$u_n \overset{w}{\longrightarrow} u \quad \text{in } W^{1,p}(\Omega).$$

Then

$$u_n \longrightarrow u \quad \text{in } L^p(\Omega)$$

(by the Sobolev embedding theorem) and since $L^p(\Omega)$ is a Banach lattice, we also have that

$$u_n^{\pm} \longrightarrow u^{\pm} \quad \text{in } L^p(\Omega). \tag{3.22}$$

We may also assume that

$$u_n^{\pm}(z) \longrightarrow u^{\pm}(z) \quad \text{almost everywhere in } \Omega. \tag{3.23}$$

We have

$$\lambda \geqslant -\int_\Omega G(z,u_n)\,dz = -\int_\Omega F(z,u_n^+)\,dz - \frac{1}{p}\|u_n^+\|_p^p - \int_\Omega f(z,0)(-u_n^-)\,dz. \tag{3.24}$$

Note that

$$\frac{1}{p}\|u_n^+\|_p^p \longrightarrow \frac{1}{p}\|u^+\|_p^p \tag{3.25}$$

and

$$\int_\Omega f(z,0)(-u_n^-)\,dz \longrightarrow \int_\Omega f(z,0)(-u^-)\,dz \tag{3.26}$$

(see (3.22)). Also, from (3.23) and Fatou's lemma, we have

$$\liminf_{n\to+\infty}\left(-\int_\Omega F(z,u_n^+)\,dz\right) = -\limsup_{n\to+\infty}\int_\Omega F(z,u_n^+)\,dz \geqslant -\int_\Omega F(z,u^+)\,dz. \tag{3.27}$$

Then, from (3.24) and using (3.25) and (3.27), in the limit as $n \to +\infty$, we have

$$\lambda \geqslant -\int_\Omega F(z,u^+)\,dz - \frac{1}{p}\|u^+\|_p^p - \int_\Omega f(z,0)(-u^-)\,dz = -\int_\Omega G(z,u)\,dz,$$

so $u \in L_\lambda$ and so ψ is sequentially weakly lower semicontinuous. ◻

Now we are ready to establish the existence of positive solutions.

Proposition 3.5 *It hypotheses H_f hold, then problem* (1.1) *has a positive solution $u_0 \in C^1(\overline{\Omega})$ with $u_0(z) > 0$ for all $z \in \overline{\Omega}$.*

Proof Propositions 3.3, 3.4 and the Weierstrass theorem, imply that we can find $u_0 \in W^{1,p}(\Omega)$, such that

$$\widehat{\varphi}(u_0) = \inf\left\{\widehat{\varphi}(u) :\ u \in W^{1,p}(\Omega)\right\} = \widehat{m}. \tag{3.28}$$

Claim 1. $u_0 \geqslant 0$, $u_0 \neq 0$.

Note that, if $u_0^- \neq 0$, then

$$\begin{aligned}\widehat{\varphi}(u_0^+) &= \frac{1}{p}\|\nabla u_0^+\|_p^p - \int\limits_\Omega F(z, u_0^+)\,dz \\ &< \frac{1}{p}\|\nabla u_0\|_p^p + \frac{1}{p}\|u_0^-\|_p^p - \int\limits_\Omega F(z, u_0^+)\,dz - \int\limits_\Omega f(z, 0)(-u_0^-)\,dz \\ &= \widehat{\varphi}(u_0)\end{aligned}$$

(see (3.6) and recall that $f(z, 0) \geqslant 0$ for almost all $z \in \Omega$), which contradicts (3.28). Therefore $u_0 \geqslant 0$.

Next we show that $u_0 \neq 0$. By hypothesis H_f(iv) and (2.6), we see that we can find $u \in W^{1,p}(\Omega)$, such that

$$\|\nabla u\|_p^p - \int\limits_{\{u\neq 0\}} \eta_0 |u|^p\,dz < 0, \tag{3.29}$$

with $\|u\|_p$=1. Replacing u with $|u| \in W^{1,p}(\Omega)$ if necessary, we may assume that $u \geqslant 0$, $u \neq 0$. Let $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ be a sequence, such that

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega)$$

(see e.g., Gasiński and Papageorgiou [8, p. 189]). Since

$$u_n^+ \longrightarrow u^+ = u \quad \text{in } W^{1,p}(\Omega),$$

we may assume that $u_n \geqslant 0$ for all $n \geqslant 1$. Let us set

$$\widehat{u}_n = \min\{u, u_n\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \forall n \geqslant 1.$$

Then

$$\widehat{u}_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega)$$

(see e.g., Gasiński and Papageorgiou [8, p. 198]). We may also assume that

$$\widehat{u}_n(z) \longrightarrow u(z) \quad \text{for almost all } z \in \Omega.$$

By virtue of hypothesis H_f(iv), we have

$$\eta_0(z) \geqslant f(z,1) \geqslant -\|f(\cdot,1)\|_\infty \quad \text{for almost all } z \in \Omega,$$

so

$$\eta_0(z)\widehat{u}_n(z)^p \chi_{\{u_n \neq 0\}}(z) \geqslant -\|f(\cdot,1)\|_\infty u(z)^p \quad \text{for almost all } z \in \Omega. \tag{3.30}$$

Note that $\|f(\cdot,1)\|_\infty u^p \in L^1(\Omega)$. Also, we have

$$\eta_0(z)\widehat{u}_n(z)^p \chi_{\{u_n \neq 0\}}(z) \longrightarrow \eta_0(z)u(z)^p \chi_{\{u \neq 0\}}(z) \quad \text{for almost all } z \in \Omega. \tag{3.31}$$

From (3.30), (3.31) and Fatou's lemma, we have

$$\liminf_{n\to+\infty} \int\limits_{\{u_n \neq 0\}} \eta_0 \widehat{u}_n^p \, dz \geqslant \int\limits_{\{u \neq 0\}} \eta_0 u^p \, dz. \tag{3.32}$$

Since $\widehat{u}_n \longrightarrow u$ in $W^{1,p}(\Omega)$, we have

$$\|\nabla \widehat{u}_n\|_p^p \longrightarrow \|\nabla \widehat{u}\|_p^p.$$

From (3.29), (3.32) and (3.33), we see that

$$\|\nabla \widehat{u}_n\|_p^p - \int\limits_{\{\widehat{u}_n \neq 0\}} \eta_0 \widehat{u}_n^p \, dz < 0 \quad \text{for large } n \geqslant 1. \tag{3.33}$$

This means that we can find $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\|\nabla u\|_p^p - \int\limits_{\{u \neq 0\}} \eta_0 u^p \, dz < 0, \quad u \geqslant 0. \tag{3.34}$$

Moreover, dividing with $\|u\|_p^p$ if necessary, we may assume that $\|u\|_p = 1$. For $\zeta > 0$, we have

$$F(z,\zeta) = \int_0^1 \frac{d}{dt} F(z,t\zeta)\, dt = \int_0^1 f(z,t\zeta)\zeta \, dt,$$

so, using hypothesis H_f(ii), we have

$$\frac{F(z,\zeta)}{\zeta^p} = \int_0^1 \frac{f(z,t\zeta)}{\zeta^{p-1}}\,dt \geqslant \frac{f(z,\zeta)}{\zeta^{p-1}} \int_0^1 t^{p-1}\,dt, = \frac{1}{p}\frac{f(z,\zeta)}{\zeta^{p-1}}$$

and thus

$$\liminf_{\zeta\to 0^+} \frac{F(z,\zeta)}{\zeta^p} \geqslant \frac{1}{p}\eta_0(z) \quad \text{for almost all } z \in \Omega. \tag{3.35}$$

Consider $u \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\|u\|_p = 1$ satisfying (3.34). For $r \in (0,1)$ small, we will have $ru(z) \in [0,1]$ for almost all $z \in \Omega$. Then, using hypothesis H_f(iii), we have

$$\begin{aligned}\frac{F(z,ru(z))}{r^p} &= \frac{1}{r^p}\int_0^{ru(z)} f(z,s)\,ds \geqslant -\frac{1}{r^p}\|f(\cdot,1)\|_\infty \int_0^{ru(z)} s^{p-1}\,ds\\ &\geqslant -\frac{\|f(\cdot,1)\|_\infty}{p}u(z)^p \geqslant -\frac{\|f(\cdot,1)\|_\infty}{p}\|u\|_\infty^p.\end{aligned} \tag{3.36}$$

From (3.35), (3.36) and Fatou's lemma, we have

$$\liminf_{r\to 0^+} \int_{\{u\neq 0\}} \frac{F(z,ru)}{r^p}\,dz \geqslant \frac{1}{p}\int_{\{u\neq 0\}} \eta_0 u^p\,dz,$$

so, using also (3.34), we have

$$\frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \frac{F(z,ru)}{r^p}\,dz < 0 \quad \text{for small } r \in (0,1),$$

thus

$$\widehat{\varphi}(ru) < 0 \quad \text{for small } r \in (0,1)$$

(recall that $ru \geqslant 0$ and see (3.1)). Using also (3.28), we see that

$$\widehat{m} = \widehat{\varphi}(u_0) < 0 = \widehat{\varphi}(0)$$

and so $u_0 \neq 0$.

This completes the proof of Claim 1.

Claim 2. $u_0 \in L^\infty(\Omega)$

For $k \geqslant 1$, we introduce the truncation

$$f_k(z,\zeta) = \begin{cases} f(z,0) & \text{if } \zeta \leqslant 0, \\ \max\left\{f(z,\zeta),\ -k\zeta^{p-1}\right\} & \text{if } \zeta > 0. \end{cases} \tag{3.37}$$

Evidently this is a Carathéodory function, $f_k(\cdot,\zeta) \in L^\infty(\Omega)$ for all $\zeta \in \mathbb{R}$ and

$$\left|f_k(z,\zeta)\right| \leqslant c_4\left(1+|\zeta|^{p-1}\right) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}, \tag{3.38}$$

for some $c_4 > 0$. We set

$$\eta_0^k(z) = \liminf_{\zeta \to 0^+} \frac{f_k(z,\zeta)}{\zeta^{p-1}} \quad \text{and} \quad \eta^k(z) = \limsup_{\zeta \to +\infty} \frac{f_k(z,\zeta)}{\zeta^{p-1}}.$$

Since

$$f_k(z,\zeta) \geqslant f(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R},$$

we see that $\eta_0^k \geqslant \eta_0$ and so

$$\widehat{\lambda}_1(-\eta_0^k) \leqslant \widehat{\lambda}_1(-\eta_0) < 0 \quad \forall k \geqslant 1$$

(see (2.6) and hypothesis H_f(iv)).

Moreover, from (3.37), we see that $\eta^k \searrow \eta$ and so

$$\widehat{\lambda}_1(-\eta^k) \longrightarrow \widehat{\lambda}_1(-\eta) > 0$$

(see (2.6) and hypothesis H_f(iii)). Hence, we have

$$\widehat{\lambda}_1(-\eta^k) > 0 \quad \text{for large } k \geqslant 1.$$

Reasoning as before, we obtain $u_{0k} \in W^{1,p}(\Omega)$, $u_{0k} \geqslant 0$, $u_{0k} \neq 0$ which minimizes $\widehat{\varphi}_k$ (here $\widehat{\varphi}_k$ is defined as $\widehat{\varphi}$ with $f(z,\zeta)$ replaced by $f_k(z,\zeta)$). Note that because of (3.38), we have $\widehat{\varphi}_k \in C^1(W^{1,p}(\Omega))$ and so for $k \geqslant 1$ large, we have

$$\widehat{\varphi}_k'(u_{0k}) = 0,$$

so

$$A(u_{0k}) = N_{f_k}(u_{0k})$$

with $N_{f_k}(u)(\cdot) = f_k\left(\cdot, u(\cdot)\right)$ for all $u \in W^{1,p}(\Omega)$ (recall that $u_{0k} \geqslant 0$). Thus

$$\begin{cases} -\Delta_p u_{0k}(z) = f_k\left(z, u_{0k}(z)\right) \text{ in } \Omega, \\ \frac{\partial u_{0k}}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Nonlinear regularity theory implies that $u_{0k} \in C^1(\overline{\Omega})$ for all $k \geqslant 1$ (see Lieberman [21] and Gasiński and Papageorgiou [8, pp. 738–739]). We set

$$v_k = \min\{u_0, u_{0k}\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \forall k \geqslant 1.$$

Since u_{0k} is a minimizer of $\widehat{\varphi}_k$, we have

$$\widehat{\varphi}_k(u_{0k}) \leqslant \widehat{\varphi}_k(h) \quad \forall h \in W^{1,p}(\Omega).$$

So, if we choose $h = \max\{u_0, u_{0k}\} \in W^{1,p}(\Omega)$, then

$$\begin{aligned} &\frac{1}{p} \int\limits_{\{u_{0k}<u_0\}} \|\nabla u_{0k}\|^p \, dz - \int\limits_{\{u_{0k}<u_0\}} F_k(z, u_{0k}) \, dz \\ &\leqslant \frac{1}{p} \int\limits_{\{u_{0k}<u_0\}} \|\nabla u_0\|^p \, dz - \int\limits_{\{u_{0k}<u_0\}} F_k(z, u_0) \, dz, \end{aligned}$$

so

$$\begin{aligned} &\frac{1}{p} \int\limits_{\{u_{0k}<u_0\}} \big(\|\nabla u_{0k}\|^p - \|\nabla u_0\|^p\big) \, dz \\ &\quad \leqslant \int\limits_{\{u_{0k}<u_0\}} \big(F_k(z, u_{0k}) - F_k(z, u_0)\big) \, dz. \end{aligned} \tag{3.39}$$

We have

$$\begin{aligned} \widehat{\varphi}(v_k) - \widehat{\varphi}(u_0) &= \frac{1}{p} \int\limits_{\{u_{0k}<u_0\}} \big(\|\nabla u_{0k}\|^p - \|\nabla u_0\|^p\big) \, dz \\ &\quad - \int\limits_{\{u_{0k}<u_0\}} \big(F(z, u_{0k}) - F(z, u_0)\big) \, dz \\ &\leqslant \int\limits_{\{u_{0k}<u_0\}} \big(F_k(z, u_{0k}) - F_k(z, u_0) - F(z, u_{0k}) - F(z, u_0)\big) \, dz \end{aligned} \tag{3.40}$$

(see (3.39)).

But on $\{u_{0k} < u_0\}$, we have

$$F_k(z, u_{0k}) - F_k(z, u_0) - F(z, u_{0k}) + F(z, u_0) = \int\limits_{u_{0k}}^{u_0} \big(- f_k(z, s) + f(z, s)\big) \, dz \leqslant 0 \tag{3.41}$$

(recall that $f_k \geqslant f$). Using (3.41) in (3.40), we obtain

$$\widehat{\varphi}(v_k) \leqslant \widehat{\varphi}(u_0) = \widehat{m},$$

so

$$\widehat{\varphi}(v_k) = \widehat{m}.$$

Since $v_k \in L^\infty(\Omega)$, we conclude that Claim 2 holds.

Next, let $h \in C^1(\overline{\Omega})$ and $t \in (-1, 1)$. We set

$$w(h) = \int_\Omega \big(G(z, u_0 + th) - G(z, u_0) - g(z, u_0)h\big)\, dz,$$

so

$$\begin{aligned} \big|w(h)\big| &\leqslant \int_\Omega \int_0^1 \big|g(z, u_0 + th) - g(z, u_0)\big|\, dt\ |h|\, dz \\ &\leqslant \int_0^1 \big\|N_g(u_0 + th) - N_g(u_0)\big\|_{p'}\, dt\ \|h\|, \end{aligned}$$

where $N_g(u)(\cdot) = g(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$. Here we have used Fubini's theorem and Hölder's inequality. Hypotheses H_f(i), (ii) and the fact that $h \in C^1(\overline{\Omega})$, imply that

$$\big|g\big(z,\ u_0(z) + th(z)\big)\big| \leqslant \widehat{a}(z) \quad \text{for almost all } z \in \Omega,\ \text{all } t \in (-1, 1),$$

with $\widehat{a} \in L^\infty(\Omega)$. From this it follows that

$$\int_0^1 \big\|N_g(u_0 + th) - N_g(u_0)\big\|_p\, dt \longrightarrow 0 \quad \text{as } \|h\| \to 0.$$

Therefore

$$\frac{|w(h)|}{\|h\|} \longrightarrow 0 \quad \text{as } \|h\| \to 0$$

and so we see that the Gâteaux derivative exists at u_0 in every direction $h \in C^1(\overline{\Omega})$ and is equal to $A(u_0) - N_f(u_0)$ (recall that $u_0 \geqslant 0$ and see (3.1)). Moreover, by virtue of (3.28), we have

$$\big\langle \widehat{\varphi}'_G(u_0),\ h\big\rangle = \big\langle A(u_0) - N_g(u_0),\ h\big\rangle = 0 \quad \forall h \in C^1(\overline{\Omega}).$$

Since the embedding $C^1(\overline{\Omega}) \subseteq W^{1,p}(\Omega)$ is dense, it follows that

$$A(u_0) = N_f(u_0),$$

so

$$\begin{cases} -\Delta_p u_0(z) = f\big(z, u_0(z)\big) \text{ in } \Omega, \\ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega \end{cases} \tag{3.42}$$

(see Motreanu and Papageorgiou [22]).

Note that $f(\cdot, u_0(\cdot)) \in L^\infty(\Omega)$ (see hypothesis H_f(i) and recall that by Claim 2, $u_0 \in L^\infty(\Omega)$). So, from nonlinear regularity theory, we have that $u_0 \in C^1(\overline{\Omega})$. By virtue of hypothesis H_f(ii), we have

$$\frac{f(z, u_0(z))}{u_0(z)^{p-1}} \geqslant \frac{f(z, \|u_0\|_\infty)}{\|u_0\|_\infty^{p-1}} \quad \text{for almost all } z \in \{u_0 > 0\},$$

so

$$f\big(z, u_0(z)\big) \geqslant \frac{-\|f(\cdot, \|u_0\|_\infty)\|_\infty}{\|u_0\|_\infty^{p-1}} u_0(z)^{p-1} \quad \text{for almost all } z \in \{u_0 > 0\}.$$

Also, recall that $f(z, 0) \geqslant 0$ for almost all $z \in \Omega$. Therefore from (3.42), it follows that

$$\Delta_p u_0(z) \leqslant c_5 u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega,$$

with $c_5 > 0$ and so

$$u_0(z) \; > \; 0 \quad \forall z \in \overline{\Omega}$$

(see Vázquez [25]). □

4 Uniqueness of positive solutions

Next we show the uniqueness of positive solutions for problem (1.1). In fact, we show that hypotheses H_f(iii) and (iv) are both necessary and sufficient for the existence and uniqueness of positive solutions for problem (1.1).

Proposition 4.1 *If hypotheses H_f hold, then problem* (1.1) *has a unique positive solution $u_0 \in C^1(\overline{\Omega})$, such that $u_0(z) > 0$ for all $z \in \overline{\Omega}$.*

Proof Let u, v be two positive solutions for problem (1.1). Then we have $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see Motreanu and Papageorgiou [22]) and moreover, as before through nonlinear regularity theory and the nonlinear maximum principle of Vázquez [25], we have that $u(z) > 0$ and $v(z) > 0$ for all $z \in \overline{\Omega}$. Let

$$R(u, v)(z) = \left\|\nabla u(z)\right\|^{p} - \left\|\nabla v(z)\right\|^{p-2}\left(\nabla v(z),\ \nabla\left(\frac{u(z)^{p}}{v(z)^{p-1}}\right)\right)_{\mathbb{R}^N}. \tag{4.1}$$

From Allegretto and Huang [1], we know that

$$R(u, v)(z) \geqslant 0 \quad \forall z \in \overline{\Omega}.$$

Using the nonlinear Green's identity (see Casas and Fernández [5]), we have

$$\begin{aligned}\int_{\Omega} \frac{f(z, u)}{u^{p-1}}(u^{p} - v^{p})\, dz &= -\int_{\Omega} \Delta_{p} u\left(u - \frac{v^{p}}{u^{p-1}}\right) dz \\ &= \int_{\Omega} \|\nabla u\|^{p-2}\left(\nabla u,\ \nabla u - \nabla\left(\frac{v^{p}}{u^{p-1}}\right)\right)_{\mathbb{R}^N} dz \\ &= \|\nabla u\|_{p}^{p} - \int_{\Omega} \|\nabla u\|^{p-2}\left(\nabla u,\ \nabla\left(\frac{v^{p}}{u^{p-1}}\right)\right)_{\mathbb{R}^N} dz \\ &= \|\nabla u\|_{p}^{p} - \|\nabla v\|_{p}^{p} + \int_{\Omega} R(v, u)\, dz. \end{aligned} \tag{4.2}$$

Similarly, interchanging the roles of u and v, we also have

$$\int_{\Omega} \frac{f(z, v)}{v^{p-1}}(v^{p} - u^{p})\, dz = \|\nabla v\|_{p}^{p} - \|\nabla u\|_{p}^{p} + \int_{\Omega} R(u, v)\, dz. \tag{4.3}$$

Adding (4.2) and (4.3), using hypothesis H_f(ii) and recalling that $R \geqslant 0$, we obtain

$$0 \geqslant \int_{\Omega} \left(\frac{f(z, u)}{u^{p-1}} - \frac{f(z, v)}{v^{p-1}}\right)(u^{p} - v^{p})\, dz = \int_{\Omega} \big(R(v, u) + R(u, v)\big)\, dz \geqslant 0,$$

so

$$\int_{\Omega} \big(R(v, u) + R(u, v)\big)\, dz = 0$$

and thus

$$R(v, u) = R(u, v) = 0 \quad \text{for almost all } z \in \Omega,$$

thus

$$u = kv,$$

for some $k > 0$ (see Allegretto and Huang [1]). Hypothesis H_f(ii) implies that $k = 1$ and so $u = v$. □

As we already remarked, we are going to show that hypotheses H_f(iii) and (iv) are also necessary for the uniqueness of positive solutions for problem (1.1).

Proposition 4.2 *If $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying hypotheses H_f*(i) *and* (ii) *and problem* (1.1) *has a unique positive solution $u_0 \in W^{1,p}(\Omega)$, then $\widehat{\lambda}_1(-\eta_0) < 0 < \widehat{\lambda}_1(-\eta)$, where*

$$\eta_0(z) = \lim_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} \quad \textit{and} \quad \eta(z) = \lim_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{p-1}}$$

Proof Note that $u_0 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see e.g., Hu and Papageorgiou [17]) and as before via nonlinear regularity (see Lieberman [21]) and the nonlinear maximal principle (see Vázquez [25]), we have $u_0 \in C^1(\overline{\Omega})$ with $u_0(z) > 0$ for all $z \in \overline{\Omega}$.

Using (2.6) and hypothesis H_f(ii), we have

$$\begin{aligned}
\widehat{\lambda}_1(-\eta_0) &\leqslant \frac{\|\nabla u_0\|_p^p - \int_\Omega \eta_0 u_0^p \, dz}{\|u_0\|_p^p} \\
&= \frac{\int_\Omega f(z,u_0)u_0 \, dz - \int_\Omega \eta_0 u_0^p \, dz}{\|u_0\|_p^p} \\
&< \frac{\int_\Omega \eta_0 u_0^p \, dz - \int_\Omega \eta_0 u_0^p \, dz}{\|u_0\|_p^p} \\
&= 0.
\end{aligned}$$

This proves that $\widehat{\lambda}_1(-\eta_0) < 0$.

Next, let

$$\beta(z) = -\frac{f(z, \|u_0\|_\infty + 1)}{(\|u_0\|_\infty + 1)^{p-1}}.$$

Then $\beta \in L^\infty(\Omega)$. By virtue of Proposition 2.1, problem (2.1) with this particular weight β, has a principal eigenfunction $\widehat{u}_1 \in C^1(\overline{\Omega})$, such that $\widehat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$. Let $k > 0$ be large enough, such that $u_0 < k\widehat{u}_1 = \widetilde{u}_1$. As before (see the proof of Proposition 4.1), we have

$$\int_\Omega \frac{f(z,u_0)}{u_0^{p-1}}(u_0^p - \widetilde{u}_1^p) \, dz = \|\nabla u_0\|_p^p - \|\nabla \widetilde{u}_1\|_p^p + \int_\Omega R(\widetilde{u}_1, u_0) \, dz \tag{4.4}$$

and

$$\int_\Omega \big(\widehat{\lambda}_1(\beta) - \beta\big)(\widetilde{u}_1^p - u_0^p) \, dz = \|\nabla \widetilde{u}_1\|_p^p - \|\nabla u_0\|_p^p + \int_\Omega R(u_0, \widetilde{u}_1) \, dz. \tag{4.5}$$

Adding (4.4) and (4.5), we obtain

$$\int_{\Omega} \left(\frac{f(z, u_0)}{u_0^{p-1}} + \beta - \widehat{\lambda}_1(\beta) \right) (u_0^p - \widetilde{u}_1^p)\, dz = \int_{\Omega} \big(R(\widetilde{u}_1, u_0) + R(u_0, \widetilde{u}_1) \big)\, dz \geqslant 0. \tag{4.6}$$

Note that by virtue of hypothesis H_f(ii), we have

$$\frac{f(z, u_0)}{u_0^{p-1}} \;>\; \frac{f(z, \|u_0\|_\infty + 1)}{(\|u_0\|_\infty + 1)^{p-1}} = -\beta(z) \quad \text{for almost all } z \in \Omega,$$

so

$$\frac{f(z, u_0)}{u_0^{p-1}} + \beta(z) \;>\; 0 \quad \text{for almost all } z \in \Omega. \tag{4.7}$$

Also, recall that

$$\big(u_0^p - \widetilde{u}_1^p \big)(z) < 0 \quad \text{for almost all } z \in \Omega. \tag{4.8}$$

So, using (4.7) and (4.8) in (4.6), we infer that

$$\widehat{\lambda}_1(\beta) \;>\; 0.$$

But $\beta \leqslant -\eta$ (see hypothesis H_f(ii)) and so $\widehat{\lambda}_1(\beta) \leqslant \widehat{\lambda}_1(-\eta)$. Hence $\widehat{\lambda}_1(-\eta) > 0$. ☐

So, summarizing the situation for problem (1.1), we can state the following theorem.

Theorem 4.3 *If $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function which satisfies hypotheses H_f(i) and (ii), then problem (1.1) admits a unique positive solution if and only if*

$$\widehat{\lambda}_1(-\eta_0) < 0 < \widehat{\lambda}_1(\eta),$$

where

$$\eta_0(z) = \lim_{\zeta \to 0^+} \frac{f(z, \zeta)}{\zeta^{p-1}} \quad \textit{and} \quad \eta(z) = \lim_{\zeta \to +\infty} \frac{f(z, \zeta)}{\zeta^{p-1}}.$$

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