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Positivity

The positive aspects of smoothness in Banach lattices

Marek Wójtowicz

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Abstract Let X be a Banach lattice, and let $x \in X\backslash\{0\}$. We study the structure of the set $\mathrm{Grad}(x)$, of all supporting functionals of x. If X is a Dedekind σ-complete Banach lattice, there is an isometry from $\mathrm{Grad}(x)$ onto $\mathrm{Grad}(|x|)$; hence the elements x and $|x|$ are smooth simultaneously. And if, additionally, X^* is strictly monotone then $\mathrm{Grad}(|x|)$ consists of positive functionals. As a by-product of our results we obtain that an arbitrary Banach lattice X is strictly monotone whenever its dual X^* is smooth.

Keywords Smooth point · Strictly monotone Banach lattice · Orlicz space · Marcinkiewicz space · M-ideal

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1 Introduction and notations

We use standard notations and for notions undefined here we refer the reader to the monographs [1,15].

Here and below X denotes a *fixed* real Banach lattice, X^* is its topological dual, and S_X denotes the unit sphere of X. The ideal of order continuous elements of X is denoted by X_a. For $x \in X\backslash\{0\}$ the symbol $\mathrm{Grad}(x)$ stands for the set of all supporting functionals of x, i.e., $\mathrm{Grad}(x) = \{f \in S_{X^*} : f(x) = \|x\|\}$. If Y is a sublattice of X, or X is a sublattice of Y, and $x \in Y$, then the symbol $\mathrm{Grad}_Y(x)$ denotes the set $\{\phi \in S_{Y^*} : \phi(x) = \|x\|\}$, where Y^* is the topological dual of Y. In this paper Y will be either an ideal of X or $Y = X^{**}$.

M. Wójtowicz (✉)
Instytut Matematyki, Uniwersytet Kazimierza Wielkiego, Pl. Weyssenhoffa 11,
85-072 Bydgoszcz, Poland
e-mail: mwojt@ukw.edu.pl

Let us recall that an element $x \in X\backslash\{0\}$ is said to be smooth if the set $\mathrm{Grad}(x)$ is a singleton. Over the last 30 years, there have been published many papers devoted to smooth points in concrete Banach lattices (see, e.g., [4,7–9] and the references therein). Most of the proofs are too technical, and there is no general point of view both on the structure of $\mathrm{Grad}(x)$ and smooth points of X.

The purpose of this paper is to show that in typical cases (e.g., for the class of real Köthe spaces) the structures of the sets of $\mathrm{Grad}(x)$ and $\mathrm{Grad}(|x|)$ are "similar" (Theorem 1b). It is rather clear that a positive element x is smooth if its (unique) supporting functional is positive (in Lemma 1 of Sect. 3 we prove a more general result). In this case $\mathrm{Grad}(x) \subset (X^*)^+$, but the latter inclusion also happens to be true if X^* is strictly monotone (Theorem 2). In Theorem 3 we prove that, under a few natural assumptions about the order continuous part X_a of X, an element $x \in X_a$ is smooth in X_a iff it is smooth in X. Moreover, in Corollary 2 we show that the smoothness of X^* implies the strict monotonicity of X.

Let us fix further notation. A functional $f \in X^*$ is said to be *order continuous*, if the condition $x_\alpha \downarrow 0$ in X implies $|f|(x_\alpha) \to 0$. The band of order continuous elements of X^* is denoted by X_n^*. If W is a nonempty subset of X, the symbol $W^\perp$ denotes the annihilator (in X^*) of W.

A function $N : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if it is convex, continuous, with $N(0) = 0$ and $N \not\equiv 0$. Let (Ω, Σ, μ) be a fixed σ-finite measure space, and let $L_0(\mu)$ denote the linear lattice of all (classes of) real μ-measurable functions on Ω. The function N determines a functional $\rho_N : L_0(\mu) \to [0, \infty)$ defined by the rule $\rho_N(x) = \int_\Omega N(|x(\omega)|)d\mu$. The subspace

$$L_N := \{x \in L_0(\mu) : \rho_N(\lambda x) < \infty \textit{ for some } \lambda > 0\}$$

is called an Orlicz space. It is a Dedekind complete Banach lattice with respect to the Luxemburg norm $\|x\| = \inf\{\lambda > 0 : \rho_N(x/\lambda) \le 1\}$, and its order continuous part $(L_N(\mu))_a$ equals $E_N := \{x \in L_0(\mu) : \rho(\lambda x) < \infty \textit{ for all } \lambda > 0\}$ (see, e.g., [3,14]).

2 The results

The following theorem allows us to reduce the studies of smoothness of arbitrary points of some Banach lattices to smoothness of their moduli. In particular, this is so in the class of Banach function lattices, e.g., Orlicz lattices.

Theorem 1 (a) *Let X be an arbitrary Banach lattice, and let $x \in X^+\backslash\{0\}$. If the element x is smooth, then* $\mathrm{Grad}(x) = \{f\}$*, where $f \ge 0$.*

(b) *Let X be a Dedekind σ-complete Banach lattice, and let an element $x = x^+ - x^- \in X$ be such that $x^+, x^- \ne 0$. Then there is a linear auto-isometry θ of X such that its adjoint θ^* maps* $\mathrm{Grad}(x)$ *onto* $\mathrm{Grad}(|x|)$*. Hence, the elements x and $|x|$ are smooth simultaneously.*

For x a positive element of X, we set

$$\mathrm{Grad}(x)^+ := \{0 \le f \in X^* : \ \|f\| = 1, \ f(x) = \|x\|\} = \mathrm{Grad}(x) \cap (X^*)^+.$$

Let us notice that, since x is positive, its norm is attained by a positive element f of X^*, and hence the set $\mathrm{Grad}(x)^+$ is not empty.

We shall say that X is *positively smooth* if every $x \in X^+$ is smooth (by Theorem 1(a) we then also have $\mathrm{Grad}(x) = \mathrm{Grad}(x)^+$). From Theorem 1(b) we immediately obtain the following corollary.

Corollary 1 *Let X be a Dedekind σ-complete Banach lattice. Then X is smooth if (and only if) it is positively smooth.*

In other words, Corollary 1 says that, to check if a Dedekind σ-complete Banach lattice X is smooth, *it is enough to check if for every $x \in X^+$ the set* $\mathrm{Grad}(x)^+$ *is a singleton.*

Kurc [10, p. 156] defines X to be *order smooth* if, for every positive $x \in X$, the set $\mathrm{Grad}(x)^+$ does not contain nontrivial order intervals. Hence, by Corollary 1, for X Dedekind σ-complete, we have

$$smooth \iff positively\ smooth \Rightarrow order\ smooth. \qquad (*)$$

Moreover, by [10, Theorem 1], if X^* is order smooth, then X is strictly monotone (i.e., the condition $0 \le x < y$ in X implies $\|x\| < \|y\|$). Since X^* is Dedekind complete, from (*) we immediately obtain the following connection between smoothness and strict monotonicity.

Corollary 2 *Let X be a Banach lattice such that its dual X^* is smooth. Then X is strictly monotone. In particular, if X is reflexive and smooth, then X^* is strictly monotone.*

Remark 1 The implications in the above corollary cannot be reversed, in general. In the non-reflexive case, the Banach lattice $X = \ell_1$ is strictly monotone, yet its dual $X^* = \ell_\infty$ is non-smooth.

For the reflexive-space case, let us consider the ℓ_2-sum $X = (\sum_{n=1}^{\infty} X_n)_2$ of the two-dimensional Banach lattices $X_n = \ell_\infty^{(2)}$, $n = 1, 2, \ldots$, endowed with the sup-norm. Then its dual $X^* = (\sum_{n=1}^{\infty} X_n^*)_2$ is strictly monotone, because every X_n^* is lattice isometric to $\ell_1^{(2)}$, while X is evidently non-smooth.

It should be clear that if X is a Banach lattice and $x \in X^+$, then a supporting functional f of x need not be positive (the simplest example is $x = (0, 1) \in X = \ell_1^{(2)}$ and $f = (1, -1) \in X^* = \ell_\infty^{(2)}$; cf. Lemma 1 in Sect. 3). But, by Theorem 1(a), if x is additionally smooth then $\mathrm{Grad}(x) \subset (X^*)^+$. In the next theorem we present yet another sufficient condition for the latter inclusion to hold.

Theorem 2 *Let X be an arbitrary Banach lattice such that its dual X^* is strictly monotone. Then, for every positive element x of X, we have the equality*

$$\mathrm{Grad}(x) = \mathrm{Grad}(x)^+. \qquad (**)$$

Remark 2 In the above theorem, X is not necessarily Dedekind σ-complete, and hence we do not claim there is a connection between $\mathrm{Grad}(x)$ and $\mathrm{Grad}(|x|)$. However, if π denotes the canonical (and order) embedding of X into X^{**} then, by Theorem 1(b), there is an isometry from $\mathrm{Grad}_{X^{**}}(\pi(x))$ onto $\mathrm{Grad}_{X^{**}}(\pi(|x|))$.

Since the dual X^* of every AM-space X is an AL-space, X^* is strictly monotone. Hence, by Theorem 2, we obtain the following corollary.

Corollary 3 *Let K be a completely regular topological space, and let $C_b(K)$ denote the real space of all continuous and bounded functions on K. Then every positive $x \in C_b(K)$ fulfils equality* $(**)$.

For the purpose of the next theorem, let us recall that a closed subspace W of a Banach space Y is an M-ideal [5], if there is a projection $P : Y^* \to Y^*$ with range $W^\perp$ such that $\|y^*\| = \|Py^*\| + \|(I - P)y^*\|$ for all $y^* \in Y^*$.

Theorem 3 *Let X be a real Banach lattice such that its order continuous part X_a is order dense in X. If X_a is an M-ideal then, for every element $x \in X_a \backslash \{0\}$, the set* $\mathrm{Grad}(x)$ *consists of order continuous functionals on X:*

$$\mathrm{Grad}(x) \subset X_n^*. \tag{***}$$

Moreover, if the topological dual X_a^* (of X_a) is strictly monotone, then

$$\mathrm{Grad}(|x|) = \mathrm{Grad}(|x|)^+, \text{ for all } x \in X_a,$$

and there are isometries T and S mapping $\mathrm{Grad}(x)$ *onto* $\mathrm{Grad}(|x|)$ *and* $\mathrm{Grad}(x)$ *onto* $\mathrm{Grad}_{X_a}(x)$*, respectively. In particular, the elements x and $|x|$ are smooth points both of X and X_a simultaneously.*

The above theorem has an interpretation in two classes of Banach lattices.

Let N be an Orlicz function. It is known that E_N is the order continuous part of L_N and is order dense in L_N (see [3, Theorem 1.25] and [14, p.145], respectively). Furthermore E_N is an M-ideal in L_N (see [3, Theorems 1.47 and 1.48]). Similarly, the order continuous part $(M(\Psi))_a$ of a Marcinkiewicz space $M(\Psi)$ has the same properties whenever $(M(\Psi))_a \neq \{0\}$ (see [6] for details, or [12, Lemma 1 and p. 266]).

Although the smooth points in Orlicz and Marcinkiewicz spaces have been already described [2,9], the corollary below adds a new information about their sets. Its proof follows immediately from Theorem 3 and the remarks following it.

Corollary 4 *Let N be a finite Orlicz function. An element $x \in E_N$ (as well as its modulus $|x|$) is smooth in E_N if and only if it is smooth in L_N.*

Similarly, if $(M(\Psi))_a \neq \{0\}$, then an element $x \in (M(\Psi))_a$ is smooth in $(M(\Psi))_a$ and in $M(\Psi)$ simultaneously.

3 Proofs

In the proofs we shall apply the following result. It shows that if an element x is positive, then the set $\mathrm{Grad}(x)$ contains a positive functional.

Lemma 1 *Let x be a strictly positive element of X, and let f be a supporting functional of x: $\|x\| = f(x)$ and $\|f\| = 1$. Then the positive part f^+ of f is a supporting functional of x, too. More exactly:*

(i) $1 = \|f^+ + f^-\| = \|f^+\|$ *and* $f^+(x) = \|x\|$, *whence*
(ii) $f^-(x) = 0$.

In particular, the condition $f^- = 0$ (i.e., $f \geq 0$) holds if, either the dual lattice X^* is strictly monotone or x is strictly positive on X^* (i.e., the condition $h > 0$ for $0 \leq h \in X^*$ implies $h(x) > 0$).

Remark 3 By [1, Theorem 4.85], an element $x \in X^+$ is strictly positive on X^* iff x is a quasi-interior point, i.e., the ideal A_x generated by x is norm dense in X. It is well known [1, pp. 267–268] that every separable Banach lattice X has strictly positive points on X^*.

Proof of Lemma 1. From $1 = \|f\| = \||f|\|$ we obtain

$$f^+(x) \leq f^+(x) + f^-(x) = |f|(x) \leq \|x\|.$$

Moreover,

$$\|x\| = f^+(x) - f^-(x) \leq f^+(x) \leq \|x\|.$$

Thus, $\|x\| = f^+(x)$, $f^-(x) = 0$, and $\|f^+\| \leq \|f\| = 1$, whence

$$\|f^+\| = 1 = \|f\| = \||f|\| = \|f^+ + f^-\|. \tag{1}$$

This proves parts (i) and (ii) of the lemma.

Now, if X^* is strictly monotone then the inequality $f^+ \leq f^+ + f^-$ and condition (1) imply that $f^+ = f^+ + f^-$. Hence $f^- = 0$. The second particular case is obvious.

The following result is known to specialists in Banach lattices. We include its proof for the convenience of the reader.

Lemma 2 *Let X be a Banach lattice such that its order continuous part X_a is order dense in X, and let X_a^* denote the topological dual of X_a. Then:*

(i) *The Banach lattices X_n^* and X_a^* are order isometric: the isometry from X_n^* onto X_a^* is of the form: $r(f) = f_{|X_a}$.*
(ii) *Every element $f \in X^*$ has the form $f = f_n + f_s$, where f_n is order continuous and f_s vanishes on X_a, i.e., $X^* = X_n^* + (X_a)^\perp$.*

Proof of Lemma 2. (i) By [13, Corollary 1.2], every $\varphi \in X_a^*$ has a unique and norm-preserved extension $e(\varphi)$ to an element of X_n^*. Let $f \in X_n^*$ be fixed, and set $\varphi := f_{|X_a}$. Then $\varphi \in X_a^*$. By the uniqueness of the extension e, we have $e(\varphi) = f$; moreover, $\|\varphi\| \leq \|f\|$. We must also have $\|\varphi\| = \|f\|$ since, otherwise, $\|f\| = \|e(\varphi)\| < \|f\|$, a contradiction. Hence, $\|r(f)\| = \|\varphi\| = \|f\|$, as claimed.

(ii) We have $X^* = X_n^* + X_s^*$, where X_s^* is the disjoint complement of the band X_n^*. Since X_a is order dense in X, [15, Theorem 88.11] implies that $X_s^* = (X_a)^{\perp}$.

Proof of Theorem 1. Part (a) follows immediately from Lemma 1.

For the proof of part (b), let P denote the order projection onto the principal band $(x^+)^{dd}$. Set $\theta := P - P^c$, where $P^c = I - P$ and I is the identity on X. Since $x^- \wedge x^+ = 0$, we have that $x^- \in (x^+)^d = P^c(X)$, and so $P(x^-) = 0$ and $P(x^+) = x^+$; similarly, $P^c(x^+) = 0$ and $P^c(x^-) = x^-$. Hence,

$$\theta(x) = (P - P^c)(x^+ - x^-) = x^+ + x^- = |x|; \quad \textit{similarly}, \ \theta(|x|) = x. \tag{2}$$

Since for every $y \in X$ the elements $P(y)$ and $P^c(y)$ are disjoint, we obtain (see [1, Ex. 3, p. 21])

$$|\theta(y)| = |P(y) - P^c(y)| = |P(y) + P^c(y)| = |y|,$$

whence $\|\theta(y)\| = \||\theta(y)|\| = \||y|\| = \|y\|$. Thus, θ is an isometry of X, and θ maps X onto X because it is an involution, i.e., $\theta^2 = I$. Thus, θ^* is an auto-isometry of X^*.

Now let $f \in \mathrm{Grad}(x) : f(x) = \|x\|$ and $\|f\| = 1$. Then, by (2), we obtain

$$\theta^*(f)(|x|) = f(\theta(|x|)) = f(x) = \|x\| = \||x|\|;$$

thus $\theta^*(f) \in \mathrm{Grad}(|x|)$ because θ^* is an isometry. Hence, $\theta^*(\mathrm{Grad}(x)) \subset \mathrm{Grad}(|x|)$. Similarly, by (2), we obtain the reversed inclusion $\mathrm{Grad}(|x|) \subset \theta^*(\mathrm{Grad}(x))$.

Proof of Theorem 2. We have to prove the nontrivial inclusion $\mathrm{Grad}(x) \subset \mathrm{Grad}(x)^+$. Since $x \geq 0$, from the second part of Lemma 1 we obtain that every element of $\mathrm{Grad}(x)$ is positive, hence it is an element of $\mathrm{Grad}(x)^+$. That is, $\mathrm{Grad}(x) \subset \mathrm{Grad}(x)^+$, as claimed.

Proof of Theorem 3. Let $x \in X_a \setminus \{0\}$, and let $f \in \mathrm{Grad}(x)$. By Lemma 2 (ii), $f = f_n + f_s$, where $f_{s|X_a} = 0$ and $f_n \in X_n^*$. Hence

$$\|x\| = f(x) = f_n(x). \tag{3}$$

Since X_a is an M-ideal,

$$1 = \|f\| = \|f_n\| + \|f_s\|. \tag{4}$$

Thus from (3) and (4) we obtain $\|f_n\| = 1$ and $f_s = 0$, whence $f = f_n \in X_n^*$. So that, $\mathrm{Grad}(x) \subset X_n^*$, as claimed.

By Lemma 2 (i), there is an order isometry r from X_n^* onto X_a^*. For $x \in X_a$, set

$$\mathrm{Grad}_{X_a}(x) := \{\varphi \in X_a^* : \varphi(x) = \|x\|, \|\varphi\| = 1\}.$$

From the form of r we easily obtain that r maps $\mathrm{Grad}(x)$ onto $\mathrm{Grad}_{X_a}(x)$. Moreover, by Theorem 1, there is an auto-isometry θ of X_a such that its adjoint, θ^*, maps $\mathrm{Grad}_{X_a}(x)$ onto $\mathrm{Grad}_{X_a}(|x|_{X_a})$, where $|x|_{X_a}$ denotes the modulus of x in the (sub)lattice X_a. But since X_a is an ideal of X, the moduli $|x|_{X_a}$ and $|x|$ coincide. Hence, the composition $S := \theta^* r$ is an isometry mapping $\mathrm{Grad}(x)$ onto $\mathrm{Grad}_{X_a}(|x|)$, and so $T := r^{-1}S$ maps isometrically $\mathrm{Grad}(x)$ onto $\mathrm{Grad}(|x|)$. In particular, the elements x and $|x|$ are smooth (both in X_a and in X) simultaneously.

Finally, Theorem 2 implies that $\mathrm{Grad}_{X_a}(|x|) \subset (X_a^*)^+$, whence, by $(***)$,

$$\mathrm{Grad}(|x|) = r^{-1}(\mathrm{Grad}_{X_a}(|x|)) \subset (X_n^*)^+ \subset (X^*)^+.$$

By definition, $\mathrm{Grad}(|x|) = \mathrm{Grad}(|x|)^+$.

The proof of Theorem 3 is complete.

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