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Investigating demand models with more flexible elasticity functions: empirical insights from rail demand analysis

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Abstract
Models of rail travel demand take one of three generic functional forms: the generalised cost (GC) function where price and the various time variables are combined into a composite term through the use of appropriate values of time; the generalised journey time and fare (GJT-Fare) approach, where the time-related variables are combined into a single term and fare remains separate; and the specification of separate elasticities for all terms, termed the separate components (SC) approach. This research extends that reported by Wardman and Toner (Transportation 47:75–108, 10.1007/s11116-017-9850-7, 2020) in exploring more flexible functional forms where appropriate parameterisation of the standard GC, GJT and SC models allows them to have more general elasticity properties. Whilst the aforementioned study discounted the standard GC approach on the grounds of inferior fit and undesirable elasticity properties, the analysis reported here, based on large datasets, finds best-fitting more flexible models to have elasticity properties that resemble those of the GC approach. Indeed, the preferred functions can deliver elasticities that are somewhat different from those of the GJT-Fare approach that has long provided the basis of official rail demand forecasts in Great Britain. In addition, the study adds to the existing evidence base by providing credible and precise elasticities for GJT and fare, and importantly for the rarely estimated GC and SC elasticities, reaffirms the need of GC models to directly estimate demand consistent values of time, and indicates that the weights currently attached to headway and interchange in formulating GJT are in need of significant amendment. Although the context is rail in Great Britain, the results have relevance to demand analysis of other modes and in other countries as well as to other transport modelling approaches.

Keywords Demand elasticities · Railways · Generalised cost · Revealed preference · Functional form

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Introduction

Research background

The rail travel market in Great Britain provides a fertile context for the econometric analysis of travel behaviour stemming from two key attractions of the availability of very large amounts of time-series demand data and ample variation in the key demand drivers.[1] This is evidenced in the large amount of empirical evidence covering a wide range of variables (Wardman 2022a, 2022b; Wardman and Batley 2021; Rail Delivery Group 2018).

The price and timetable related features of journey time, service frequency and interchange that characterise train services can be represented within demand models in different ways. At one extreme is the generalised cost (GC) function, which collapses these variables into a single term, in contrast to what might be termed the separate components (SC) approach where each variable enters the demand model separately. Between the two is the specification of the timetable related variables within a composite Generalised Journey Time (GJT) term alongside a separate fare variable. These different representations possess somewhat different elasticity properties, as subsequently discussed, whereupon we might expect that a desirable feature of any demand analysis would be the identification, where possible, of the most appropriate of these generic specifications rather than default adoption of one of them.

In this context, Wardman and Toner (2020) reported innovative research that tested whether the longstanding practice in transport planning, analysis and appraisal of adopting the GC formulation could be justified or whether the GJT-Fare approach, widely used in the railway industry in Great Britain, or the SC representation, typical of discrete choice modelling, are preferable in terms of explanatory performance and the credibility of implied elasticities and forecasts.

They found strong support for the GJT-Fare specification over the GC formulation, and indeed demonstrated that the credibility of the latter was critically dependent upon being able to directly estimate the value of time used in constructing GC. The SC approach, rarely adopted in econometric rail demand analysis in Great Britain, emerged as promising.

Wardman and Toner (2020) concluded:

> *"We have restricted our investigations to the standard constant elasticity models that dominate the analysis and forecasting practice based around aggregate models of the direct demand form. More flexible functions which allow the GC elasticity to vary can be estimated which also permit closer approximation of the GC, GJT-Fare, and the SC models through appropriate parameterisation of each model. For example, we might reject the strong elasticity variation forced by the conventional GC approach and here tested, but a more limited degree of variation might be supported using a more flexible functional form. Equally, interaction terms within the GJT-Fare and SC approaches can bestow elasticity properties similar to the GC approach".*

[1] The car and bus markets exhibit insufficient variations in journey times over time to estimate robust time elasticities, which is apparent in the lack of evidence, whilst demand data at best lacks spatial detail and at worst does not exist. Whilst time and price vary cross-sectionally, there is a longstanding preference for demand data with a time-series dimension since it is much less likely to suffer from the endogeneity issues that are typically suspected and indeed observed to afflict pure cross-sectional models.

The research reported here was inspired by these recommendations and provides original insights into the performance of more flexible demand functions.

Elasticity properties of conventional demand functions

We here set out the three principal demand specifications, already mentioned, of the typical constant elasticity form, along with their implied elasticity properties.

The Passenger Demand Forecasting Handbook (PDFH) sets out a forecasting framework and recommended demand parameters that have been in use in Great Britain since the early 1980s and are regularly updated according to latest evidence (Rail Delivery Group 2018). The timetable related aspects of service quality are combined into a composite GJT,[2] specified at the level of station-to-station movements:

$$GJT_{ij} = T_{ij} + \mu H_{ij} + \tau I_{ij} \tag{1}$$

T_{ij}, H_{ij} and I_{ij} respectively denote the journey time, service headway and number of interchanges for travel between station i and j, and the μ and τ parameters convert headway and interchange into equivalent amounts of travel time. The other key variable of interest here is fare (F)[3] with the demand function almost invariably specified as:

$$V_{ij} = \varphi GJT_{ij}^{\alpha} F_{ij}^{\beta} \tag{2}$$

V denotes the volume of rail demand, α and β are the elasticities to GJT and fare respectively, and other relevant explanatory variables are included in the demand model but not elaborated here.

Whilst the GJT elasticity is constant, the implied elasticities to its constituent variables can vary appreciably as follows:

$$\eta_{Tij} = \alpha \frac{T_{ij}}{GJT_{ij}} \tag{3}$$

$$\eta_{Hij} = \alpha \frac{\mu H_{ij}}{GJT_{ij}} \tag{4}$$

$$\eta_{Iij} = \alpha \frac{\tau}{GJT_{ij}} \tag{5}$$

where η_T, η_H, and η_I are the implied elasticities to time, headway and interchange on a specific flow.[4] Note that I is often zero whereupon it makes more sense to deal in term

[2] This function dominates econometric rail demand analysis in Great Britain, and a recent meta-analysis of British time-related elasticities illustrates the very large amount of available empirical evidence (Wardman 2022a).

[3] Since rail demand data relates to inter-station movements, access and egress times and costs cannot be included. Whilst other terms relating to crowding, station facilities and rolling stock quality could be included, historic and detailed data on them is generally not readily available, whilst reliability is entered as a separate term with its own elasticity.

[4] The large variations in actual implied time, headway and interchange elasticities are illustrated in Table 9.

of the proportionate change in rail demand after a change in the number of interchanges which is how the demand responsiveness measure is calculated throughout.

Following Wilson (1969), the GC composite term can here be specified as:

$$GC_{ij} = v_{ij} GJT_{ij} + F_{ij} \tag{6}$$

GJT is converted into money terms using the value of time (v). The latter can be expected to vary by flow, not least due to journey purpose and distance characteristics, and over time due to income variation.

The use of GC is commonplace in transport planning although it is rarely used in the rail market in Great Britain. If the GC demand model is specified in the conventional constant elasticity form:

$$V_{ij} = \kappa GC_{ij}^{\lambda} \tag{7}$$

then the implied elasticities to GJT and F are:

$$\eta_{GJTij} = \lambda \frac{v_{ij} GJT_{ij}}{GC_{ij}} \tag{8}$$

$$\eta_{Fij} = \lambda \frac{F_{ij}}{GC_{ij}} \tag{9}$$

If instead the composite term is expressed in time units, termed generalised time (GT), its elasticity would be the same as that for GC since it is simply a change of units, and the same implied elasticity relationships would apply. Again, the elasticities to the constituent variables can vary considerably even though the elasticity to the composite variable is constant.[5]

In the extensive world of discrete choice modelling, composite terms such as GC or GJT are rarely estimated.[6] Instead, separate terms are specified for each variable and the analogous SC demand model would here be:

$$V_{ij} = \varpi T_{ij}^{\omega} H_{ij}^{\psi} e^{\phi I_{ij}} F_{ij}^{\beta} \tag{10}$$

Interchange is entered in exponential form since it can be zero. The elasticities to time, headway and interchange are here directly estimated rather than inferred, although in this typical functional form would be constant in stark contrast to the appreciable variation implied by the GC and GJT-Fare formulations.

Objectives of this research

The use of composite terms, as demonstrated, forces what can be appreciable variation in the elasticities to the constituent variables without any explicit empirical justification. In contrast, the typical constant elasticity position for the SC model is potentially restrictive.

5 Table 8 illustrates the large variations in actual implied GJT and fare elasticities.

6 An exception is forecasting applications of choice models, as in TAG Unit 2.1 (Department for Transport 2020), where composite terms such as GC are used rather than separate components.

The aim of this paper is to explore whether there is empirical justification for variant specifications of the GC, GJT-Fare and SC functions with more flexible properties which, through appropriate parameterisation, allow a function to move towards or be a special case of an 'adjacent' function.[7] Specifically, we test model formulations that:

- Allow the elasticity properties of the GC model to move towards those of the GJT-Fare model.
- Allow the elasticity properties of the GJT-Fare model either to move towards those of the GC model or separately towards those of the SC model.
- Allow the elasticity properties of the SC model to move towards those of the GJT-Fare model and of the GC model.

We are not aware of such previous research, and although focussed upon the rail market the research has more general relevance. Whilst there are more aspects of potential elasticity variation than are encompassed within the functions here explored, such as how price and time elasticities vary with the levels they take, with each other and with factors such as distance, the competitive environment, ticket type, geography and over time, the emphasis here is *specifically* upon generalising the Wardman and Toner (2020) analysis.[8]

Structure of paper

Section "Candidate functional forms" sets out various functional forms to be tested to address the aims of the research. Section "Data and foundation models" discusses the data used in this research and reports standard models that illustrate that we have a firm foundation for the intended econometric investigation. The more flexible GC models, GJT-Fare models and SC models that have been estimated are reported in Sections "Generalising the GC approach towards the GJT-Fare approach: results" to "Generalising the SC approach towards the GJT-Fare approach: results". Section "Synthesis" provides a synthesis of the findings and concluding remarks are presented in Section "Conclusions".

Candidate functional forms

Generalising the GC approach towards the GJT-Fare approach

A more general specification of the GC demand function, termed a damped negative exponential function, is:

$$V_{ij} = \kappa e^{\alpha GC_{ij}^{\beta}} \tag{11}$$

This has implied GJT and fare elasticities of:

[7] Bruzelius (1981) addressed the appropriate functional form of GC models in terms of consistency with conventional economic theory. In contrast, the research reported here is entirely empirical.

[8] We should though acknowledge that econometric demand modelling in the published literature rarely explores elasticity variation.

$$\eta_{GJTij} = \alpha\beta \frac{v_{ij} GJT_{ij}}{GC_{ij}^{1-\beta}} \tag{12}$$

$$\eta_{Fij} = \alpha\beta \frac{F_{ij}}{GC_{ij}^{1-\beta}} \tag{13}$$

As β tends to zero, the model tends to the standard GC function and the strong dependency of the GJT and fare elasticities upon the proportion these variables respectively form of GC as set out in Eqs. (8) and (9). As β tends to one, the effect of the GC proportion is diminished and at β = 1 the GJT and fare elasticities are independent of the proportion they form of GC albeit with the potentially undesirable feature that they are then directly proportional to the levels of GJT and fare respectively. The latter property can be addressed if the GC function is generalised to be:

$$GC_{ij} = v_{ij} GJT_{ij}^{\theta} + F_{ij}^{\gamma} \tag{14}$$

The elasticities to GJT and fare then become:

$$\eta_{GJTij} = \alpha\beta \frac{\theta v_{ij} GJT_{ij}^{\theta}}{GC_{ij}^{1-\beta}} \tag{15}$$

$$\eta_{Fij} = \alpha\beta \frac{\gamma F_{ij}^{\gamma}}{GC_{ij}^{1-\beta}} \tag{16}$$

This seems an attractive GC formulation, allowing flexible elasticity properties whilst retaining some dependence on the proportion that GJT and fare form of GC. If β is 1, the GJT and fare elasticities respectively tend to those of the GJT-Fare approach as θ and γ tend to zero.

Generalising the GJT-Fare approach towards the GC approach

The GJT-Fare approach can be generalised to allow the GJT and fare elasticities to depend upon GC by appropriately entering GC into Eq. (2). However, this would complicate the interpretation of results since the GJT and fare elasticities would enter the model directly and additionally through GC. A pragmatic solution is to specify GC to be some reference route-specific and time-invariant level whereupon it can be treated as an interaction effect in a straightforward manner. The GJT-Fare model would then be specified as:

$$V_{ij} = \kappa e^{\gamma \left(\frac{v_{ij} GJT_{ij}}{\overline{GC}_{ij}} \right)^{\delta} + \theta \left(\frac{F_{ij}}{\overline{GC}_{ij}} \right)^{\omega}} \tag{17}$$

where $\overline{GC}_{ij}$ is the reference level of GC on the flow in question.[9] The elasticities to GJT and Fare would then be:

[9] The data sets cover the years 2009 through to 2016 and the first year is taken as the reference.

$$\eta_{GJTij} = \gamma\delta\left(\frac{v_{ij}GJT_{ij}}{\overline{GC}_{ij}}\right)^{\delta} \tag{18}$$

$$\eta_{Fij} = \theta\omega\left(\frac{F_{ij}}{\overline{GC}_{ij}}\right)^{\omega} \tag{19}$$

An attraction compared to amending the GC approach is that the impacts of GC on the GJT and fare elasticities are readily different. If the standard GC approach is justified then both δ and ω would be one. At the other extreme, as δ and ω respectively tend to zero then the elasticity properties tend to the GJT-Fare approach.

An alternative approach would be to specify the demand function as:

$$V_{ij} = \kappa GJT_{ij}^{\gamma+\delta\Pi_{GJTij}} F_{ij}^{\theta+\omega\Pi_{Fij}} \tag{20}$$

where Π_{GJTij} and Π_{Fij} are reference levels of the proportion that GJT and fare respectively form of GC on a route, whereupon the GJT and fare elasticities are:

$$\eta_{GJTij} = \gamma + \delta\Pi_{GJTij} \tag{21}$$

$$\eta_{Fij} = \theta + \omega\Pi_{Fij} \tag{22}$$

This function allows the GJT and fare elasticities to depend upon the proportion that their variables form of GC to the extent that is empirically warranted. Wardman and Toner (2020) did allow the GJT and fare elasticities to vary respectively with categories of Π_{GJTij} and Π_{Fij} but did not estimate the continuous function of Eq. (20).

Generalising the GJT-Fare approach towards the SC approach

The same approach can here be used as in Section "Generalising the GC approach towards the GJT-Fare approach" where the GC approach was generalised towards the GJT-Fare approach. If GJT is specified as:

$$V_{ij} = \kappa e^{\alpha GJT_{ij}^{\beta}} \tag{23}$$

then the implied time, headway and interchange elasticities are:

$$\eta_{Tij} = \alpha\beta\frac{T_{ij}}{GJT_{ij}^{1-\beta}} \tag{24}$$

$$\eta_{Hij} = \alpha\beta\frac{\mu H_{ij}}{GJT_{ij}^{1-\beta}} \tag{25}$$

$$\eta_{Iij} = \alpha\beta\frac{\tau}{GJT_{ij}^{1-\beta}} \tag{26}$$

where the interchange elasticity is again specified as the proportionate change in demand after a change in interchange. As β tends to zero then the standard elasticities of the GJT-Fare method are approached and if β equals one then the GJT-Fare approach collapses to the SC approach albeit with elasticities dependent upon the level of the variable in question.

Also following the approach in Section "Generalising the GC approach towards the GJT-Fare approach", the GJT function could be generalised to amend the impacts of each variable on its elasticity:

$$GJT_{ij} = T_{ij}^{\theta} + \mu H_{ij}^{\varpi} + \tau I_{ij}^{\psi} \tag{27}$$

whereupon the implied elasticities are:

$$\eta_{Tij} = \alpha\beta \frac{\theta T_{ij}^{\theta}}{GJT_{ij}^{1-\beta}} \tag{28}$$

$$\eta_{Hij} = \alpha\beta \frac{\mu\varpi H_{ij}^{\varpi}}{GJT_{ij}^{1-\beta}} \tag{29}$$

$$\eta_{Iij} = \alpha\beta \frac{\tau\psi I_{ij}^{\psi-1}}{GJT_{ij}^{1-\beta}} \tag{30}$$

If β tends to one, θ and ϖ tend to zero, and ψ tends to one then the GJT-Fare approach approximates the standard SC approach.

Generalising the SC approach towards the GJT-Fare approach

We can use the same procedures in generalising the SC approach to have elasticity properties that approximate the GJT-Fare approach as used in Section "Generalising the GJT-Fare approach towards the GC approach" in generalising the GJT-Fare approach towards the GC approach. The demand function would be specified as:

$$V_{ij} = \kappa e^{\omega\left(\frac{T_{ij}}{\overline{GJT}_{ij}}\right)^{\gamma} + \psi\left(\frac{\mu H_{ij}}{\overline{GJT}_{ij}}\right)^{\delta} + \phi\left(\frac{\tau}{\overline{GJT}_{ij}}\right)^{\theta} I} \tag{31}$$

where $\overline{GJT}_{ij}$ is the reference level of GJT on the flow in question invariant across years. The elasticities to time, headway and interchange would then be:

$$\eta_{Tij} = \omega\gamma\left(\frac{T_{ij}}{\overline{GJT}_{ij}}\right)^{\gamma} \tag{32}$$

$$\eta_{Hij} = \psi\delta\left(\frac{\mu H_{ij}}{\overline{GJT}_{ij}}\right)^{\delta} \tag{33}$$

$$\eta_{Iij} = \phi\left(\frac{\tau}{GJT_{ij}}\right)^{\theta} \tag{34}$$

As γ, δ and θ tend to 1 (0) then the respective elasticities tend to those implied by the GJT-Fare (SC) approach.

Alternatively, incremental terms can be specified for each of time, headway and interchange, along the lines of the generalisation of the GJT-Fare approach in Section "Generalising the GJT-Fare approach towards the GC approach". This would take the form:

$$V_{ij} = \kappa T_{ij}^{\omega+\gamma\Pi_{Tij}} H_{ij}^{\psi+\delta\Pi_{Hij}} e^{\phi I_{ij}+\theta I_{ij}\Pi_{Iij}} \tag{35}$$

The Π_{Tij}, Π_{Hij} and Π_{Iij} denote the reference proportions that time, headway and interchange form of GJT on each route but which are invariant over years. The elasticities to time, headway and interchange would then be:

$$\eta_{Tij} = \omega + \gamma\Pi_{Tij} \tag{36}$$

$$\eta_{Hij} = \psi + \delta\Pi_{Hij} \tag{37}$$

$$\eta_{Iij} = \phi + \theta\Pi_{Iij} \tag{38}$$

These elasticities can then have the dependence upon the proportion that they respectively form of GJT to the extent that is empirically warranted. It would be a straightforward matter to re-define the Π terms to reflect the proportion that time, headway and interchange form of GC.

Data and foundation models

The data here analysed represents annual demand between stations covering 2009 through to 2016 for flows with sufficient variations in timetable-related service quality and fares to support the econometric analysis of the issues here under investigation.[10] These are:

- 3964 Non-London long distance flows over 20 miles and 1327 short distance flows up to 20 miles covering non-season tickets;
- 2113 Non-London season ticket flows up to 60 miles;
- 564 longer distance non-season ticket flows to and from airports.

Table 1 reports fixed-effects panel models estimated to these large datasets for the three conventional functional forms set out in Section "Elasticity properties of conventional demand functions". The previous paper (Wardman and Toner 2020) compared fixed and random-effects models and concluded that, "random-effects specifications were rejected convincingly based on the Hausman test". Not only do we here have far more

[10] Whilst London based flows are highly important from a revenue perspective, the variations in GJT over time are very limited and do not support the investigation here being undertaken.

Table 1 Base GJT-Fare, GC and SC Models

	Urban Non-London Non-Seasons			Urban Non-London Seasons			Inter-Urban Non-London Non-Seasons			Airports		
	GJT-Fare	GC	SC	GJT-Fare	GC	SC	GJT-Fare	GC	SC	GJT-Fare	GC	SC
GJT	−0.98 (21.9)			−0.91 (11.0)			−0.93 (32.5)			−0.75 (7.7)		
S_H	0.35 (9.2)			0.67 (4.4)			0.79 (13.1)			1.82 (3.6)		
S_I	n.s.			n.s.			2.12 (17.6)			2.69 (4.0)		
Time			−0.54 (12.1)			−0.51 (6.1)			−0.55 (22.7)			−0.33 (4.0)
Headway			−0.19 (9.2)			−0.25 (6.6)			−0.13 (13.7)			−0.17 (6.4)
Interchange			−0.45 (7.0)			−0.19 (3.0)			−0.41 (36.2)			−0.32 (13.2)
Fare	−1.19 (33.0)		−1.18 (32.7)	−0.57 (17.5)		−0.57 (17.5)	−1.02 (73.3)		−1.02 (73.1)	−0.87 (25.0)		−0.88 (25.2)
GC		−1.99 (34.9)			−1.05 (13.8)			−1.85 (84.9)			−1.61 (27.4)	
ν		5.33 (15.1)			4.95 (5.9)			6.95 (36.5)			6.09 (12.2)	
GVA	0.89 (8.0)	1.70 (14.2)	0.90 (8.1)	n.a.	n.a.	n.a.	0.74 (14.4)	1.62 (29.5)	0.74 (14.5)	1.01 (6.9)	1.76 (11.5)	1.02 (7.0)
Emp	0.17 (6.3)	0.18 (6.4)	0.18 (6.5)	1.00	1.00	1.00	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
Trend	0.028 (28.5)	0.028 (28.6)	0.028 (28.3)	0.023 (15.5)	0.025 (17.4)	0.023 (15.8)	0.036 (92.2)	0.036 (92.3)	0.036 (91.6)	0.056 (54.6)	0.056 (54.9)	0.057 (54.9)
Pop	1.00	1.00	1.00	n.a.	n.a.	n.a.	1.00	1.00	1.00	1.00	1.00	1.00

Table 1 (continued)

	Urban Non-London Non-Seasons			Urban Non-London Seasons			Inter-Urban Non-London Non-Seasons			Airports		
	GJT-Fare	GC	SC	GJT-Fare	GC	SC	GJT-Fare	GC	SC	GJT-Fare	GC	SC
Fuel	0.07 (4.4)	0.08 (4.6)	0.08 (4.7)	0.25 (8.2)	0.24 (7.6)	0.25 (8.2)	0.26	0.26	0.26	0.25	0.25	0.25
APM	−0.04 (9.4)	−0.04 (9.3)	−0.04 (9.6)	−0.02 (2.7)	−0.02 (2.4)	−0.02 (2.6)	−0.03 (20.2)	−0.03 (17.8)	−0.03 (19.6)	n.s	n.s	n.s
RSS	211.82	216.62	213.30	1984.77	1994.87	1986.80	455.40	460.05	457.83	60.78	61.10	61.10
Adj R^2	0.979	0.978	0.978	0.912	0.912	0.912	0.986	0.986	0.986	0.986	0.986	0.986
Observations	10,616			16,904			31,712			4512		

The figures in parentheses here and in subsequent tables indicate t ratios cross-sections, which further favours the fixed-effects approach, but the extensively used non-linear least squares estimation is readily applied to the fixed-effects specification.

The dependent variable is the logarithm of annual demand between stations. The independent variables enter in logarithmic form with the exception of the time trend, the performance measure and the number of interchanges.[11] Fare is defined as revenue per trip which will represent an average across a range of different ticket types, particularly on the longer distance flows, modified to the extent to which railcards that provide fare discounts are used. The CPI was used to allow for inflation.

We would be placing the GJT-Fare approach, and indeed the GC approach, at a disadvantage if we did not test whether PDFH's recommended μ and τ parameters used in constructing GJT were appropriate. Hence scales have been estimated to both these parameters, denoted S_H and S_I in Table 1, which indicate that better explanations of demand can be achieved with different μ for all flow types and different τ for long distance and airport flows. The variations from current recommendations, which are based on dated and specifically Stated Preference evidence, are substantial and lead to large improvements in the residual sum of squares (RSS) from the 214.31, 1985.17, 461.14 and 61.24 respectively in models without these scales. In all subsequent modelling, these estimated scales on headway and interchange are used to create revised GJT measures which in turn will impact upon GC measures. The GJT and fare elasticities are estimated very precisely and are plausible across the four flow categories and in line with previous evidence (Wardman 2022a, 2022b).

Turning to the GC models, the ν in 2009 pence per minute are directly estimated using non-linear least squares,[12] with the headway and interchange components of GJT scaled according to the findings of the GJT-Fare model.

The GC elasticities are estimated with a very high degree of precision and are broadly similar to the sum of the estimated GJT and fare elasticities. A number of points can be made about the ν estimates. Firstly, they are very precisely estimated. Secondly, whilst we might expect ν to increase with GVA over time and with distance across routes, it was not possible to additionally estimate credible income and distance elasticities for ν. As a result, the income elasticity of ν was constrained to be one, in line with widespread official appraisal guidance, and the distance elasticity of ν was fixed at 0.15, on the basis of meta-analysis evidence (Abrantes and Wardman 2011; Wardman et al. 2016).[13] Thirdly, the scales applied to the μ and τ parameters made little difference to the ν estimates. Finally, the ν are consistent with the findings of Wardman and Toner (2020) in not resembling standard behavioural values (Arup et al. 2015; Wardman et al. 2016). It would seem that the ν relevant at the margin to behavioural change are not the average values typically reported

[11] Correlations of estimated coefficients were mostly low, with the largest being between the estimates of the GVA elasticity and the time trend which tended to be around −0.5 and, in the GC model, the estimates of the GVA elasticity and υ which were also around that magnitude.

[12] Non-linear least squares is also required to estimate the non-linear in parameters demand functions of Eqs. (11), (17), (23) and (31). In order to assist convergence in the 'challenging' estimation of these functions and to avoid local optima, extensive grid searches covering a range of pre-specified levels of relevant parameters were conducted using ordinary least squares to identify the set of estimates that yielded a best fit. These then served as sensible starting values in the iterative non-linear least squares estimation.

[13] The income effect is an index varying from 1 in 2009 whilst the distance term is specified relative to the mean level across flows. Hence the reported ν is for 2009 and the mean distance. Without the distance and income effects, the estimates of the GC elasticity and ν were little different, at -1.96 and 5.18, -1.20 and 6.98, -1.83 and 6.84 and -1.64 and 6.26 respectively. The model fits were slightly better when the income and distance elasticity constraints were introduced.

in empirical studies that estimate them. Imposing the latter in constructing GC would place the GC model at a distinct disadvantage and would imply GJT (Fare) elasticities that are somewhat larger (smaller) and inconsistent with the findings of the GJT-Fare approach.

It is encouraging to find that all three timetable related variables have significant, and mainly highly significant, coefficient estimates in each of the four SC models, and the elasticities are credible. This SC model specification is essentially an extension of the constant elasticity PDFH approach. In contrast, the elasticities of discrete choice models tend to depend upon the level of the variable in question, and an analogous approach here would be to specify time, headway and fare in the exponential form used for interchange in Eq. (10). When this was done, the fit was inferior for all but airport flows, with respective RSS of 224.45, 1999.97, 491.68 and 60.43.

Turning to the other variables commonly included in rail demand models, there was strong support for the inclusion of a time trend over and above the GVA per capita term specified at the NUTS3 level relevant to the origin station. Rail demand grew strongly in the period, despite moderate GVA growth, and this can be attributed to increases in rail attractiveness due to the digital revolution and rail travel being in a position to exploit increasing possibilities to use travel time in a worthwhile manner (Wardman et al. 2020; Wardman and Lyons 2016) and also inter-temporal changes in socio-economic, demographic and land-use characteristics that would have benefitted rail travel (Williams and Jahanshahi 2018). Accounting for these impacts with relevant explanatory variables available over time at the level of station-to-station movements is an issue that continues to challenge those studies whose main purpose is to determine the impact of external factors. The unaccounted for annual growth is here substantial, varying between 2.3% for season tickets to 5.7% for airport access.

The GVA per capita elasticities for the GJT-Fare and SC models are credible and in line with other evidence. Those for the GC models are impacted by the presence of GVA within the v function; setting the v income elasticity to zero brings the GC models' GVA elasticities in line with the other models and makes very little difference to the other parameter estimates or model fit.

Population at the origin (Pop) defined at local authority level enters non-season models and employment at the destination (Emp) also defined at the local authority level is the main driver of commuting and replaces GVA in the season ticket model. We would expect these elasticities to be 1 and they are constrained to be such. Allowing free estimation leads to somewhat different, and not credible, values due to correlation with GVA and the time trend. It was though possible to obtain credible employment elasticities for the short distance non-season ticket models, reflecting the use of non-season tickets by some commuters.

The remaining two factors are fuel cost and a variable termed average performance minutes (APM) which represents the average amount of late arrival time (Wardman and Batley 2021). APM is unlogged, in line with current industry modelling practice. A significant and correct sign effect can be recovered for all but airport flows. Fuel cost cross-elasticities are expected to be relatively low which makes them difficult to estimate. Nonetheless, significant estimates are obtained for two sets of flow and for the other two the effect is isolated by constraining the parameter to PDFH recommendations.

The estimated models do not distinguish between short and long run effects. Specifying dynamic effects would add further complexity to some already challenging non-linear estimations in a context where we would not expect large differential impacts across the various model forms being compared. Moreover, annual data is analysed and it is generally accepted that, at least for non-season demand, a large proportion of the long run effect is achieved within a year (Rail Delivery Group 2018). An exception is for season tickets, where the behavioural responses of home and workplace relocation can take substantially longer, and in this market we have to assume that our comparative findings based on one-year effects would not be materially altered if allowance had been made for the longer term effects.

This section cannot conclude without making a comparison of the statistical performance of the three model forms in Table 1. Where the models have different numbers of parameters, they can be assessed using the F test to compare the restricted and unrestricted RSS:

$$F = \frac{\left(RSS_R - RSS_U\right)/r}{RSS_U/(n-k)} \tag{39}$$

with r denoting the number of restrictions imposed, n the number of observations and k the number of estimated parameters.

Comparing the GJT-Fare and GC models, the calculated F statistic far exceeds the tabulated F value at the usual 5% level for all four flow types, indicating that the GJT-Fare model is statistically superior.

As for the comparison of the GJT-Fare and SC models, they have the same number of estimated parameters for long distance and airport flows whereupon the former is superior due to its lower RSS. Note that when the scales within GJT were not estimated, F tests indicated that the SC model would be superior, thereby denoting the importance of estimating the scales. For the two short distance flow types, the GJT-Fare model achieves a better fit than the SC model despite having one fewer parameter. However, F tests would again indicate the latter model to be superior if the scales within GJT were not estimated.

These comparative fits are in stark contrast to Wardman and Toner (2020) where across all seven model types estimated the SC model was best followed by GJT-Fare model. These results might well have stemmed from the use of unscaled GJT measures.

In summary, the results across the standard models reported in Table 1 are highly plausible, generally very precisely estimated and largely consistent with each other. These findings, along with ample variation in the variables expressed as proportions of the composite GC and GJT terms, provide a very firm foundation for the investigation of the flexible functional forms set out in Section "Candidate functional forms" which is the primary purpose of the research reported here.

Generalising the GC approach towards the GJT-Fare approach: results

Table 2 reports GC model results generalised using demand Eq. (11) along with the standard GC function of Eq. (6) and also with the more general function of Eq. (14).

The estimation of the more flexible demand Eq. (11) with GC Eq. (6) achieves a better fit, as would be expected, than the standard GC model of Table 1 in all cases except for Model I which did not converge. The evidence indicates that for all but airport flows the estimate of β is low and hence the GC function approximates the conventional form where the GJT and Fare elasticities are strongly dependent upon the proportion each variable forms of GC. This seems odd given that all but the airport model have an inferior fit than the GJT-Fare model of Table 1 where there is independence from the GC effect.

When the revised GC function of Eq. (14) was investigated, there were either convergency problems or else the parameters were far from significant. We therefore resorted

Table 2 Generalised GC model results

	Urban non-London non-seasons		Urban non-London seasons		Inter-urban non-London non-seasons		Airports	
Eq^n	11 and 6[a]	11 and 14[a]	11 and 6	11 and 14	11 and 6	11 and 14	11 and 6	11 and 14
Model	I	II	III	IV	V	VI	VII	VIII
α	−169.20 (0.2)	−94.01 (0.3)	−2.21 (0.7)	−0.79 (3.8)	−15.47 (3.0)	−3.41 (9.6)	−0.15 (1.5)	−1.85 (2.5)
β	0.01 (0.1)	0.02 (0.3)	0.16 (1.5)	0.43 (3.4)	0.07 (4.7)	0.85 (79.9)	0.40 (6.5)	0.80 (14.5)
ν	5.34 (15.1)	5.19 (6.8)	5.08 (5.8)	2.40 (3.1)	7.14 (36.2)	1.22 (25.9)	5.62 (11.8)	1.20 (6.9)
GVA	1.70 (14.1)	1.72 (12.8)	n.a.	n.a.	1.62 (29.5)	6.48 (16.9)	1.71 (11.3)	4.36 (4.8)
RSS	216.70	216.64	1994.58	1993.26	459.69	452.27	60.67	60.46
Adj R^2	0.978	0.978	0.912	0.912	0.985	0.986	0.986	0.986
OBS	10,616		16,904		31,712		4512	

Other parameter estimates were very similar to Table 1 and are not here reported

[a]Model failed to converge and the results of the final 500th iteration are reported

to constraining both θ and γ to equal 1-β which reduces the number of parameters to be estimated yet still tests the extent to which the GJT and fare elasticities depend upon the proportion these variables form of GC. This improves model fit compared to the use of GC Eq. (6) for all four flow types and, with the exception of Model II, the β estimates are somewhat larger indicating a movement towards the properties of the GJT-Fare specification and a superior fit compared to the standard GC model. Models VI and VIII, which have the largest β estimates and most closely approximate the elasticity properties of the GJT-Fare models, also have a better fit than the latter, although it is surprising that Model IV, which has elasticity properties farther from those of the GJT-Fare approach, has a worse fit than the GJT-Fare model of Table 1.

A potential limitation of this generalisation of the GC approach is that there has to be strong variation in the GC elasticity with the level of GC for a weak impact of GC on the GJT and fare elasticities and such strong variation might not exist. A more appealing approach is to directly allow the GJT and fare elasticities to depend upon the proportion their variables form of GC to the extent empirically justified, and it is to this that we now turn.

Generalising the GJT-Fare approach towards the GC approach: results

Table 3 reports models based on Eqs. (17) and (20), and as would be expected these generalisations of the GJT-Fare approach achieve better goodness of fit than the corresponding standard GJT-Fare models of Table 1.

Two models of each are reported; the first allows the parameters that drive the elasticity variation (δ and ω) to differ whereas the second, on account of the large correlation that can be expected between the terms that specify the GC effect, constrains the two to be the same. The υ used in creating the reference level of GC are those estimated in the models reported in Table 1 which vary by distance but with income fixed at 2009 levels.

Table 3 Generalised GJT-Fare Model Results (towards the GC Approach)

	Urban Non-London Non-Seasons				Urban Non-London Seasons				Inter-Urban Non-London Non-Seasons				Airports			
Eq[n]	17	17	20	20	17	17	20	20	17	17	20	20	17	17	20	20
Model	Ia	Ib	IIa	IIb	IIIa	IIIb	IVa	IVb	Va	Vb	VIa	VIb	VIIa	VIIb	VIIIa	VIIIb
γ	−28.49 (0.2)	0.54 (5.6)	−0.73 (3.9)	−1.55 (15.1)	0.14 (2.6)	−1.59 (8.0)	−0.90 (2.7)	−1.36 (12.0)	−13.55 (1.0)	−17.78 (1.3)	−0.46 (5.3)	−0.89 (18.6)	0.27 (1.7)	−3.23 (2.1)	−1.38 (3.9)	−0.46 (3.0)
δ	0.04 (0.2)	–	−0.59 (1.4)	–	−1.51 (7.1)	–	0.00 (0.0)	–	0.08 (1.0)	–	−0.99 (5.6)	–	−1.10 (3.5)	–	1.43 (1.8)	–
θ	0.76 (6.5)	0.94 (6.7)	−2.39 (15.7)	−1.95 (15.3)	−1.35 (10.3)	−1.37 (10.1)	−1.15 (11.4)	−1.11 (11.3)	−22.66 (0.9)	−19.22 (1.3)	−1.18 (19.3)	−0.98 (18.9)	−1.56 (13.0)	−3.68 (1.9)	−0.25 (1.3)	−0.51 (2.9)
ω	−0.91 (11.1)	–	2.15 (8.1)	–	0.90 (6.0)	–	1.01 (6.0)	–	0.04 (0.8)	–	0.32 (2.8)	–	1.11 (5.4)	–	−1.16 (3.2)	–
$\delta = \omega$	–	−0.79 (5.7)	–	1.35 (6.2)	–	0.86 (6.0)	–	0.95 (5.9)	–	0.05 (1.2)	–	−0.07 (0.7)	–	0.29 (1.6)	–	−0.67 (2.1)
GVA	1.88 (15.8)	1.81 (15.4)	0.88 (7.9)	0.89 (8.0)	n.a.	n.a.	n.a.	n.a.	1.68 (30.6)	1.68 (30.6)	0.74 (14.5)	0.75 (14.5)	1.77 (11.6)	1.76 (11.5)	1.01 (7.0)	1.02 (7.0)
RSS	210.55	211.03	210.30	210.99	1976.33	1979.00	1979.91	1980.20	455.37	455.37	454.78	455.39	60.23	60.76	60.59	60.72
Adj R^2	0.979	0.978	0.979	0.979	0.912	0.912	0.912	0.912	0.986	0.986	0.986	0.986	0.986	0.986	0.986	0.986
OBS	10,616				16,904				31,712				4512			

Other parameter estimates were very similar to Table 1 and are not here reported

Table 4 Generalised GJT-Fare model results (towards the SC Approach)

	Urban Non-London Non-Seasons		Urban Non-London Seasons		Inter-Urban Non-London Non-Seasons		Airports	
Eq[n]	23 and 1	23 and 27	23 and 1	23 and 27	23 and 1	23 and 27	23 and 1	23 and 27
Model	I	II	III[a]	IV[a]	V[a]	VI[a]	VII[a]	VIII[a]
α	−0.96 (1.2)	−15.93 (1.5)	−84.07 (0.1)	−76.02 (0.2)	−77.36 (0.3)	−88.14 (1.1)	−70.70 (0.1)	−65.19 (0.3)
β	0.32 (2.7)	0.06 (1.7)	0.01 (0.1)	0.01 (0.2)	0.01 (0.3)	0.01 (1.2)	0.01 (0.1)	0.01 (0.3)
Fare	−1.18 (32.8)	−1.18 (32.6)	−0.57 (17.6)	−0.57 (17.6)	−1.01 (73.6)	−1.09 (73.8)	−0.87 (25.1)	−0.88 (25.3)
RSS	211.73	211.49	1984.80	1985.10	455.43	455.66	60.79	61.53
Adj R^2	0.979	0.979	0.912	0.912	0.986	0.986	0.986	0.986
OBS	10,616		16,904		31,712		4512	

Other parameter estimates were very similar to Table 1 and are not reported here

[a]Model failed to converge and the results of the final iteration are reported

Where δ and ω are separately estimated, in models Ia, IIa through to VIIIa, they tend to have opposite signs which in most cases are very much different or they are not significant. These results are a symptom of the very large correlations between the δ and ω estimates. We therefore concentrate on the models denoted Ib, IIb through to VIIIb where δ and ω are constrained to be the same.

Considering Eq. (17) first, Model Ib has a wrong sign effect for the constrained parameter whilst it is insignificant for Models Vb and VIIb. Only in Model IIIb is there a case for a movement towards the elasticity properties of the GC approach.

As for Eq. (20), with δ and ω constrained to be the same, two models (IIb and IVb) have incremental effects indicating that the GJT and fare elasticities would actually fall as their variables form a larger proportion of GC, whilst the incremental effect in Model VIb is insignificant. However, Model VIIIb for airport flows suggests that a movement towards GC is warranted.

The best fit model with δ and ω constrained is Eq. (17) for seasons and long distance flows and Eq. (20) for the remaining two flow types. For season tickets and airport flows there is support for a movement towards the GC approach.

Generalising the GJT-Fare approach towards the SC approach: results

Table 4 reports the models that allow the GJT-Fare approach to have implied timetable related elasticities that approximate those of the SC approach.

Models I, III, V and VII are based on the demand function of Eq. (23) along with the standard GJT function. In three cases, the model failed to converge and this is most likely due to the β parameter being very close to zero. Only in Model I is β not close to zero, and generally the results support a strong impact from GJT on the time, headway and interchange elasticities.

Adopting the more general GJT function of Eq. (27), the non-linear least squares estimation procedure cannot handle the zero interchange values and removes them as missing

Table 5 Generalised SC Model Results

	Urban non-London non-seasons		Urban non-London seasons		Inter-urban non-London non-seasons		Airports	
Eq^n	31	35	31	35	31	35	31	35
Model	I	II	III	IV	V	VI	VII	VIII
ϖ	−1.06 (3.3)	n.s.	−0.46 (5.5)	−0.46 (5.4)	0.41 (2.7)	−0.53 (22.1)	−0.32 (3.9)	−0.30 (3.7)
γ	0.76 (2.1)	−0.86 (11.4)	n.s.	n.s.	−0.82 (4.2)	n.s	n.s	n.s
ψ	−1.78 (7.3)	n.s.	−1.06 (7.1)	n.s.	−0.96 (16.1)	n.s	−0.93 (5.8)	n.s
δ	2.33 (7.7)	−0.66 (12.2)	1.07 (2.4)	−0.62 (7.5)	1.02 (7.4)	−0.59 (16.9)	0.74 (2.3)	−0.56 (7.5)
ϕ	−0.47 (7.4)	−0.48 (7.7)	−0.23 (3.6)	n.s.	−0.84 (12.1)	−0.11 (2.7)	−0.78 (3.3)	n.s
θ	n.s.	n.s.	n.s.	−1.50 (3.8)	0.82 (8.5)	−0.71 (7.3)	1.06 (2.8)	−0.75 (13.9)
Fare	−1.18 (32.8)	−1.18 (32.7)	−0.58 (17.5)	−0.57 (17.5)	−1.02 (73.3)	−1.02 (73.8)	−0.88 (25.2)	−0.88 (25.3)
RSS	210.73	211.60	1984.34	1984.55	455.38	455.36	60.72	60.73
Adj R^2	0.979	0.979	0.912	0.912	0.986	0.986	0.986	0.986
OBS	10,616		16,904		31,712		4512	

Other parameter estimates were very similar to Table 1 and are not here reported

observations. To overcome this, we added 0.05 to the interchange value. The estimated β parameter was not particularly sensitive to the figure added.[14] Unfortunately, the models could not converge, nor even yield remotely credible results at final iteration, when β, θ, ϖ, and ψ were freely estimated. Even when θ, ϖ and $\psi - 1$ were constrained to equal $1-\beta$ to simplify estimation yet still test the GJT influence, only Model II converged and then the estimated β is low. The issue again seems to be that the constrained parameter approximates zero.

The models here indicate that moving from the GJT-Fare approach towards the SC approach is not warranted, and indeed this is reflected in a worse fit than the standard GJT-Fare model of Table 1 for all but the urban non-season models. However, there remains a possibility that these results are in fact due to the lack of any relationship between the GJT elasticity and the level of GJT as forced in Eq. (23). The generalisation of the SC approach towards the GJT-Fare approach offers more flexibility and is now discussed.

[14] Indeed, when estimating on the reduced datasets after zero interchanges were removed as missing values, the β estimates were 0.18 (4.5) for Model II, 0.08 (0.1) for Model IV, 0.11 (10.9) for Model VI and 0.17 (2.9) for Model VIII, for respective sample sizes of 953, 3936, 16,470 and 3251. These β estimates are broadly in line with those reported in Table 4 where 0.05 was added to the interchange value in order to retain all observations.

Generalising the SC approach towards the GJT-Fare approach: results

The results of the estimations of the demand functions represented by Eqs. 31 and 35 that allow the SC approach's elasticities to take on the properties of the GJT-Fare approach are reported in Table 5. All eight models are statistically superior to the corresponding standard SC models of Table 1, with a large number of parameters statistically significant, suggesting that movement away from the latter is empirically justified.

There is a high degree of correspondence between the elasticity variation recovered by the two functions, as represented by the γ, δ and θ parameters, and we discuss the results for each flow type in turn.

Equations 31 and 35 are entirely consistent for urban non-season models in finding strong evidence that the elasticities to time and headway but not interchange are dependent upon the proportion these variables form of GJT. Strong consistency is also apparent for airport flows, where both models find the headway and interchange elasticities to increase with the proportion they form of GJT and for the time elasticity to be constant.

As for urban season ticket demand, both functions recovered a constant time elasticity and found the headway elasticity to increase in line with the proportion headway forms of GJT, although the results for interchange are contrasting.

Finally, with respect to the inter-urban flows, both equations find the headway and interchange elasticities to depend upon the proportion their variable forms of GJT. However, whereas Eq. (35) detects no variation in the time elasticity Eq. (31) indicates a wrong sign effect.

Equation (35) is the best fitting model for all flows except urban non-seasons. In summary, there is empirical support for some allowance of the SC elasticities to have properties that approximate the GJT-Fare approach.

As mentioned in Section "Generalising the SC approach towards the GJT-Fare approach", Eqs. (36–38) could instead specify the Π variables in terms of the proportions that time, headway and interchange form of GC. When this was done, the goodness of fit was worse for all four flow types. Adding in fare dependency upon the proportion it forms of GC led to an improved fit for all but airport flows but in each case the incremental effect did not have the expected sign.

Synthesis

Best fit demand functions

Table 6 provides the RSS for each of the models estimated along with the number of included parameters (k) other than the fixed effects which indicates the number of restrictions that enter the F test of Eq. (39). The best fit models are indicated in underlined bold. The standard models, whether GC, GJT-Fare or SC, never provide the best fit and a more flexible variant is preferable.

In two cases, covering the long distance and the airport flows, a generalisation of the GC approach towards the GJT-Fare approach provides the best fit whilst for season tickets it is a generalisation of the GJT-Fare approach towards the GC approach. A generalisation of the SC approach provides the best fit for the remaining urban flows.

Table 6 Comparative model fits

Model	Equations	Urban non-London non-seasons		Urban non-London seasons		Inter-urban non-London non-seasons		Airports	
		RSS	k	RSS	k	RSS	k	RSS	k
Base GC		216.62	8	1994.87	6	460.05	7	61.10	6
Base GJT-Fare		211.82	9	1984.77	7	455.40	9	60.78	8
Base SC		213.30	10	1986.80	8	457.83	9	61.10	8
Generalised GC	11 and 6	216.70	9	1994.58	7	459.69	8	60.67	7
Generalised GC	11 and 14	216.64	9	1993.26	7	**452.27**	8	**60.46**	*7*
Generalised GJT	17	211.03	9	**1979.00**	7	455.37	8	60.76	7
Generalised GJT	20	210.99	9	1980.20	7	455.39	8	60.72	7
Generalised GJT	23 and 1	211.73	9	1984.80	7	455.43	8	60.79	7
Generalised GJT	23 and 27	211.49	9	1985.10	7	455.66	8	61.53	7
Generalised SC	31	**210.73**	12	1984.34	9	455.38	12	60.72	10
Generalised SC	35	211.60	10	1984.55	8	455.36	10	60.73	8

k denotes the number of estimated parameters

These are interesting findings. If analysis is restricted to standard (base) models then the GC approach would be written-off, as it was by Wardman and Toner (2020). But it emerges that some influence from GC on elasticities is warranted in three out of the four flow types examined when more flexible functions are permitted.

It would be desirable for the different demand specifications to be telling a consistent story. Table 7 summarises, in broad terms, the direction in which the more flexible generalised models are taking the implied elasticities. The scenario with the best fit is again indicated in underlined bold.

With respect to the urban Non-London non-season ticket models, Scenarios B, C and D point to elasticities that take on the properties of the GJT-Fare approach, with the statistically superior model exhibiting a movement towards the GJT-Fare approach. Whilst the generalised GC approach in Scenario A indicates that the elasticity properties of the standard GC approach are preferred, we have previously recognised that this could be because the GC elasticity does not increase with the level of GC regardless of the appropriate elasticity properties.[15]

As for the season ticket models, Scenarios A and B indicate elasticity properties between the GC and GJT-Fare approaches. Scenario C remains at the 'upper bound' of the GJT-Fare approach whilst Scenario D moves towards it. The same can be said to apply for long distance flows.

Scenarios A and B of the airport models indicate movement to somewhere between the GJT-Fare and GC formulations, with Scenario C consistent in remaining at the 'upper bound' of the GJT-Fare approach and Scenario D providing some movement towards it.

There is a reasonable degree of consistency between the results of the different demand formulations in terms of appropriate elasticity properties.

[15] These flows exhibit the least variation in GC over time of the four flow types considered.

Table 7 Degree of model consistency

Scenario	Function	Urban non-London non-seasons	Urban non-London seasons	Inter-urban non-London non-seasons	Airports
A	Generalised GC	Remain GC	Towards GJT-Fare	**Towards GJT-Fare**	**Towards GJT-Fare**
B	Generalised GJT to GC	Remain GJT-Fare	**Strong to GC**	Remain GJT-Fare	Towards GC
C	Generalised GJT to SC	Remain GJT-Fare	Remain GJT-Fare	Remain-GJT Fare	Remain GJT-Fare
D	Generalised SC	**Towards GJT-Fare**	Towards GJT-Fare	Towards GJT-Fare	Towards GJT-Fare

Table 8 Implied GJT and fare elasticities

	Urban non-London non-seasons			Urban non-London seasons		
	Std GC	Gen SC (I)	Std GJT-Fare	Std GC	Gen GJT (IIIb)	Std GJT-Fare
1%	−0.46: −0.68	−0.56: −1.18	−0.98: −1.19	−0.30: −0.28	−0.45: −0.35	−0.91: −0.57
10%	−0.58: −0.93	−0.68: −1.18	−0.98: −1.19	−0.38: −0.42	−0.58: −0.53	−0.91: −0.57
25%	−0.67: −1.07	−0.70: −1.18	−0.98: −1.19	−0.43: −0.48	−0.65: −0.61	−0.91: −0.57
Mean	−0.81: −1.18	−0.73: −1.18	−0.98: −1.19	−0.50: −0.55	−0.73: −0.68	−0.91: −0.57
75%	−0.92: −1.32	−0.75: −1.18	−0.98: −1.19	−0.57: −0.62	−0.81: −0.76	−0.91: −0.57
90%	−1.06: −1.41	−0.80: −1.18	−0.98: −1.19	−0.63: −0.67	−0.90: −0.81	−0.91: −0.57
99%	−1.31: −1.53	−1.05: −1.18	−0.98: −1.19	−0.77: −0.75	−1.05: −0.94	−0.91: −0.57
	Inter-urban non-London non-seasons			Airports		
	Std GC	Gen GC (VI)	Std GJT-Fare	Std GC	Gen GC (VIII)	Std GJT-Fare
1%	−0.43: −0.53	−0.62: −0.81	−0.93: −1.02	−0.39: −0.51	−0.39: −0.66	−0.75: −0.87
10%	−0.60: −0.72	−0.68: −0.86	−0.93: −1.02	−0.56: −0.70	−0.52: −0.73	−0.75: −0.87
25%	−0.74: −0.84	−0.74: −0.89	−0.93: −1.02	−0.65: −0.79	−0.58: −0.78	−0.75: −0.87
Mean	−0.87: −0.98	−0.85: −0.96	−0.93: −1.02	−0.74: −0.87	−0.67:−0.83	−0.75: −0.87
75%	−1.01: −1.11	−0.95: −1.02	−0.93: −1.02	−0.82: −0.96	−0.74: −0.88	−0.75: −0.87
90%	−1.13: −1.25	−1.05: −1.09	−0.93: −1.02	−0.91: −1.05	−0.83: −0.93	−0.75: −0.87
99%	−1.32: −1.42	−1.18: −1.17	−0.93: −1.02	−1.10: −1.22	−0.95: −1.03	−0.75: −0.87

Figures are GJT elasticity and Fare elasticity. Each are provided for various percentiles and the mean

Comparing elasticities

This paper is primarily concerned with the elasticity properties of demand models and Table 8 provides the implied GJT and fare elasticities for the standard GC and GJT-Fare models along with those of the best fitting model.[16] The railway industry in Great Britain routinely forecasts changes in GJT and fare; the elasticities to the component parts of GJT are subsequently addressed.

The implied GJT and fare elasticities can vary considerably across different model forms, although encouragingly the mean values are broadly similar. The standard GC model imposes appreciable GJT and fare elasticity variation compared to the rail industry's GJT-Fare approach but without any empirical testing. Nonetheless, it can here be seen that the more flexible, preferred models exhibit elasticity variation more in line with the GC approach than the GJT-Fare approach. Although the preferred generalised GJT model for season ticket demand actually implies more variation in the implied GJT and fare elasticity than does the standard GC model, we note that the latter recovered a low GC elasticity compared to the sum of directly estimated GJT and fare elasticities and this will limit the implied GJT and fare elasticity variation.

The significance of the elasticity differences in Table 8 will depend upon the size of those differences, and the findings here are indicating that the differences can sometimes

[16] For the generalised SC model and urban non-seasons, the implied GJT elasticity is calculated as the sum of the time, headway and interchange elasticities, where the latter is here calculated as the proportionate change in demand after a proportionate change in interchange (ϕI).

be large. Given that comparing differences is easier when the means are the same, which is not always the case here, and that deriving confidence intervals for some of the more general functions is not straightforward, we note that the confidence intervals of the standard GJT-Fare model expressed as a proportion of the central estimate are less than ± 10% in six out of the eight cases and hence many of the preferred models' elasticities where they vary will be outside the confidence interval of the standard model. But ultimately, even if quite large differences in elasticities are generally not significant, reasons would have to be advanced for preferring a model that is statistically inferior, such as lesser modelling complexity, particularly when, as is the case here, the preferred models' parameters are estimated precisely.

In summary, the preferred model can imply somewhat different elasticities than the GJT-Fare approach used by the railway industry in Great Britain. The potential for discrepancies in GJT elasticities is noticeable on all but the airport flows although they are less for the fare elasticities where the largest differences are for season tickets.

Even though GJT elasticities are central to forecasting rail demand in Great Britain, and sometimes timetable changes involve all the aspects of GJT, there are instances when forecasts are required for timetable changes that do not cover all variables. Table 9 provides time elasticities, headway elasticities and interchange demand effects implied by the preferred and standard GJT-Fare models and directly estimated by the standard SC models. Again, the mean elasticities tend to be similar across the different functions.

There are inevitably some large differences between the directly estimated SC elasticities and the implied elasticities since the former are constant but the latter are not. However, the more important comparison is between the preferred, flexible model and the standard GJT-Fare approach.

It is not uncommon for service frequencies to be varied in isolation. There are some noticeable differences between the headway elasticities for the preferred and GJT-Fare models on urban Non-Season ticket flows but for other flows the large differences where they occur are restricted to the more extreme elasticities.

Turning to journey time changes, which can also occur in isolation, there is a high degree of correspondence between the preferred and standard models for urban non-season flows but some larger differences for the other flow types at the lower elasticity levels.

As far as interchange is concerned, and given that train time is specified as station-to-station, changes in the provision of through trains invariably leads to changes, often quite large, in journey times. Focussing first just on the interchange demand effects, there is generally a good degree of correspondence between the preferred and standard GJT-Fare models for seasons and airports. However, given the preferred model for urban non-seasons has a constant interchange effect there can be large differences here, which is also the case for long distance flows at larger demand effects.

When more realistically considering interchange variations alongside journey time variations, the large differences in interchange effects in urban non-season ticket models detracts from the close correspondence of the time elasticities whilst there would also be noticeable widening of the difference between the preferred and standard GJT-Fare models for long distance flows throughout and season ticket and airport flows at larger elasticity levels.

In summary, large differences in the forecasts of timetable related demand effects can result from imposing the elasticity variations of the standard GJT-Fare approach rather than identifying the preferred form of elasticity variation.

Table 9 Implied time, headway and interchange elasticities

	Urban non-London non-seasons			Urban non-London seasons		
	Std GJT-Fare	Gen SC (I)	Std SC	Std GJT-Fare	Gen GJT (IIIb)	Std SC
1%	−0.38: −0.12: −0.20	−0.37: −0.01: −0.47	−0.54: −0.19: −0.45	−0.26: −0.12: −0.11	−0.20: −0.09: −0.09	−0.51: −0.25: −0.19
10%	−0.49: −0.19: −0.30	−0.46: −0.03: −0.47	−0.54: −0.19: −0.45	−0.37: −0.20: −0.13	−0.28: −0.15: −0.12	−0.51: −0.25: −0.19
25%	−0.56: −0.25: −0.35	−0.52: −0.06: −0.47	−0.54: −0.19: −0.45	−0.46: −0.25: −0.16	−0.35: −0.19: −0.13	−0.51: −0.25: −0.19
Mean	−0.63: −0.33: −0.44	−0.57: −0.15: −0.47	−0.54: −0.19: −0.45	−0.54: −0.35: −0.19	−0.43: −0.29: −0.15	−0.51: −0.25: −0.19
75%	−0.72: −0.42: −0.51	−0.63: −0.19: −0.47	−0.54: −0.19: −0.45	−0.63: −0.44: −0.22	−0.51: −0.36: −0.17	−0.51: −0.25: −0.19
90%	−0.78: −0.49: −0.60	−0.67: −0.28: −0.47	−0.54: −0.19: −0.45	−0.67: −0.54: −0.26	−0.58: −0.46: −0.19	−0.51: −0.25: −0.19
99%	−0.85: −0.60: −0.78	−0.73: −0.47: −0.47	−0.54: −0.19: −0.45	−0.74: −0.65: −0.34	−0.71: −0.63: −0.24	−0.51: −0.25: −0.19
	Inter-urban non-London non-seasons			Airports		
	Std GJT-Fare	Gen GC (VI)	Std SC	Std GJT-Fare	Gen GC (VIII)	Std SC
1%	−0.38: −0.06: −0.25	−0.29: −0.07: −0.22	−0.55: −0.13: −0.41	−0.18: −0.10: −0.19	−0.10: −0.11: −0.17	−0.33: −0.17: −0.32
10%	−0.46: −0.10: −0.30	−0.37: −0.10: −0.28	−0.55: −0.13: −0.41	−0.27: −0.14: −0.23	−0.20: −0.14: −0.20	−0.33: −0.17: −0.32
25%	−0.50: −0.14: −0.34	−0.43: −0.13: −0.32	−0.55: −0.13: −0.41	−0.30: −0.17: −0.26	−0.25: −0.16: −0.23	−0.33: −0.17: −0.32
Mean	−0.58: −0.23: −0.46	−0.54: −0.20: −0.42	−0.55: −0.13: −0.41	−0.35: −0.27: −0.33	−0.31: −0.22: −0.29	−0.33: −0.17: −0.32
75%	−0.66: −0.30: −0.55	−0.62: −0.25: −0.48	−0.55: −0.13: −0.41	−0.38: −0.35: −0.39	−0.36: −0.27: −0.33	−0.33: −0.17: −0.32
90%	−0.73: −0.39: −0.66	−0.74: −0.30: −0.59	−0.55: −0.13: −0.41	−0.45: −0.43: −0.47	−0.45: −0.32: −0.38	−0.33: −0.17: −0.32
99%	−0.81: −0.52: −0.85	−0.95: −0.40: −0.76	−0.55: −0.13: −0.41	−0.53: −0.56: −0.67	−0.56: −0.37: −0.48	−0.33: −0.17: −0.32

Figures are time elasticity, headway elasticity and interchange demand effect. Each are provided for various percentiles and the mean

Conclusions

The main purpose of the research reported here was to extend that conducted by Wardman and Toner (2020) who recommended that more flexible functional forms should be explored where the elasticity properties would depart from those rigidly implied by the standard GC, GJT-Fare and SC approaches. We are not aware of such previous research.

The analysis has been based on large datasets that yield precise and credible parameter estimates, providing a firm basis for this research that generalises standard functions but also usefully adding to the existing evidence base, particularly for the rarely estimated GC and SC elasticities.

Whilst the standard GC approach, widely used in transport planning and appraisal, was discounted by Wardman and Toner (2020) as forcing unwarranted large variations in GJT and fare elasticities and achieving a worse fit than the GJT-Fare and SC models, and our initial standard models would here draw the same conclusion, we have found that the more flexible functions support variations in GJT, time, headway, interchange and fare elasticities that resemble the properties of GC models. Indeed, the standard models never provide the best fit.

The preferred generalised functions can imply elasticities to the key variables that are somewhat different to those of the GJT-Fare approach used by the rail industry in Great Britain and therefore challenge the longstanding recommendations of its Passenger Demand Forecasting Handbook. Rail demand analysis, of which there is a very considerable amount in Great Britain, should not restrict itself to the standard GJT-Fare approach, or indeed to any one specific approach across flows types, since more flexible functions provide a better fit to the data and “one size does not fit all”.

Further insights relate to the weights to be attached to the frequency and interchange components in creating the industry’s standard GJT measure and the results indicate that the current weights, based as they are on somewhat dated and SP-centric evidence, are in need of significant amendment. The evidence points to large reductions in the headway penalties of around two-thirds and one-third for urban non-season and season flows, with a reduction of around a quarter for inter-urban flows but a large increase of 82% for airport flows. As for the interchange penalties, these should be more than doubled on long distance and airport flows. It is worth noting that the currently used weights were never estimated specifically for airport flows. Indeed, without amending these frequency and interchange weights both the GC and GJT-Fare approaches would have been placed at a distinct disadvantage relative to the SC approach.

The railways, at least in Great Britain, are in the fortunate position of being able to investigate the functional form issues that are the subject of this paper since there is ample variation in price and time-related variables and reliable demand data is readily available. This is not generally the case for other modes, where only price varies and demand data is often less reliable, and hence there are concerns that the demand models and forecasting procedures used for those modes will not be based on the most appropriate functional forms. A way forward for other modes, and indeed of interest for rail demand analysis, would be to conduct similar analysis with disaggregate mode choice models. These could explore GC, GJT-Fare and variant functions, alongside the customary SC approach and generalisations upon it, but we are not aware that this has been conducted.

Although the econometric demand analysis reported here is focussed on rail travel in Great Britain, the functional form issues addressed and the findings obtained provide lessons more generally for demand analysis of other modes and in other countries. And it would also seem that greater attention should be paid to the investigation of more flexible functional forms with more general elasticity properties in conventional urban transportation models, that often make use of the GC formulation, and disaggregate choice models, that tend to be based on the SC approach.

The specific focus of this paper was on extending the Wardman and Toner (2020) research, exploring the extent to which elasticities dependent upon GC and GJT are justified. Of course, there are other sources of elasticity variation which have not been covered here and their investigation is encouraged. These include: the impacts of journey length, journey purpose mix and the competitive environment; possible dependencies of a variable's elasticity on the level of other variables, particularly the influence of income levels and local socio-demographic characteristics; whether there are inter-temporal variations in elasticities, and possible differences in elasticities according to the size and sign of variation.

Finally, the empirical findings here can be compared with the largely theoretical considerations of Bruzelius (1981) who concluded within the framework of conventional economic theory that "It is shown here that necessary and sufficient conditions for expressing travel demand in terms of generalised cost are that this cost, when measured in monetary units, is linear, and that the time variable is weighted by a constant marginal value of time. It is also shown that these conditions imply strong assumptions about the consumer's behaviour—e.g., that the willingness to pay to save time whilst travelling is not a function of real income". It was recognised that these are strong assumptions; the results presented here, based entirely on empirical investigation, challenge these conclusions.

In assessing whether these strong assumptions mean that GC should be viewed as a "useless toy", Bruzelius refers to the assertions by Searle (1978) that the GC approach works in practice and superior performing alternative models had not been demonstrated. On the first point, Bruzelius questions whether the VTTS that are used to formulate GC are appropriate for use in evaluation. The results here reaffirm those in Wardman and Toner (2020) in indicating that they are not and that demand consistent VTTS must be directly estimated. On the second point, Bruzelius goes on to state, "Therefore, the use of generalised cost has to be based on the second type of argument: travel demand functions in terms of generalised costs seem to be working better than other types of travel demand models". Unlike Wardman and Toner (2020), we have here found that elasticity properties resembling those of the GC approach can be empirically justified.

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Declarations

Competing interests The authors declare no competing interests.

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