Order
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Applications of Order Trees in Infinite Graphs

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Abstract
Traditionally, the trees studied in infinite graphs are trees of height at most ω, with each node adjacent to its parent and its children (and every branch of the tree inducing a path or a ray). However, there is also a method, systematically introduced by Brochet and Diestel, of turning arbitrary well-founded order trees T into graphs, in a way such that every T-branch induces a generalised path in the sense of Rado. This article contains an introduction to this method and then surveys four recent applications of order trees to infinite graphs, with relevance for well-quasi orderings, Hadwiger's conjecture, normal spanning trees and end-structure, the last two addressing long-standing open problems by Halin.

Keywords Order trees · Normal tree orders · Normal spanning trees · Well-quasi ordering · Minor antichains · Colouring number · End · Ray graph

1 Introduction

Ask someone working in graph theory or in order theory to explain the concept of a *tree*, and you get two quite different answers: to a graph theorist, a tree T is a minimally connected graph, i.e. a connected graph such that deleting any edge disconnects it. To an order theorist, a tree is a partial order $(T,\leq)$ with a least element (called the root) such that down-closures, i.e. subsets of the form $\lceil t \rceil = \{t' \in T : t' \leq t\}$ for nodes $t \in T$, are well-ordered under $\leq$.[1]

Of course, the second concept encompasses the first, as graph-theoretic trees T can be interpreted as order trees: After designating one vertex of T as its root r and letting $t \leq_r s$ if t lies on the unique path from r to s in T, the relation $\leq_r$ describes a tree order on T (whose exact nature of course depends on the choice of r). Irrespectively of the choice of the root, however, the resulting order trees $(T,\leq_r)$ are of the simplest kind: their down-closures $\lceil t \rceil$ are well-ordered simply because they are finite, corresponding to the vertex set of the path from t to the root r in T. Another way to phrase this is to say that $(T,\leq_r)$ contains no $\omega + 1$

[1] Some authors only require that down-closures are linearly ordered, and call our trees *well-founded* order trees.

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chains, or that all linearly ordered subsets of $(T,\leq_r)$ are of order type at most ω, or simply that graph-theoretic trees have *height* at most ω.

From an order theoretic viewpoint, the fact that trees of height ω are so ubiquitous in graph theory raises the question whether also general order trees have their uses in graph theory. The purpose of this paper is to argue that they do – and highlight one particular aspect of this by showing how one can encode the rich structure of order trees $(T,\leq)$ into, well, not graph-theoretic trees, but certain highly-connected subgraphs of the comparability graph of $(T,\leq)$, called T-graphs. This method has been suggested by Rado in 1978 for well-orders, and in its general form for arbitrary order trees by Diestel and Brochet in 1994.

In this article I want to advertise this method: first, by giving an introduction to this method along with a number of examples, and then illustrate its use by four quite recent applications of it to infinite graphs, with relevance for normal spanning trees, colouring number, well-quasi orderings, and end-structure of graphs, with the first and last item solving two long-standing open problems by Halin. The sections in this article are organised as follows:

§2 Generalised paths,
§3 T-graphs: Definition and first examples,
§4 Normal spanning trees, colouring number and forbidden minors,
§5 T-graphs and Hadwiger's conjecture,
§6 T-graphs and well-quasi orderings,
§7 T-graphs and Halin's end degree conjecture.

This survey-like article aims to convey an impression of the methods and proofs about T-graphs from [4, 9, 19, 24, 27] by explaining the common theme and pointing a number of connections and parallels. However, this article does not give all proof details, nor should it be considered an exhaustive survey on the topic of order trees in combinatorics (for example, we don't discuss two exciting recent applications [21, 30] of order trees to the study of uncountably chromatic graphs). Only one part is new, namely a short proof concerning Halin's result about Hadwiger's conjecture in infinite graphs, see Theorem 5.1.

For common notions in graph– and set theory see the textbooks by Diestel [6] and Jech [15]. We denote cardinals by λ,κ,σ and ordinals by i,j,k,ℓ.

2 Generalised Paths

A classic result by Erdős and Rado from 1978 [28] says that the vertex set of any countable complete graph where the edges have been coloured with finitely many, say r, colours, can be partitioned into at most r monochromatic paths or rays. In the same paper, Rado asked whether a similar result holds for all infinite complete graphs, even the uncountable ones. If we insist on the fixed quantity r and don't change the objects allowed in our decomposition, this is clearly impossible: rays and paths are just countable, so finitely many of them won't suffice to cover an uncountable vertex set.

Hence, Rado introduced the following notion of *generalised path* (also called a *long ray* in [4]), which is a graph P together with a well-order $\preccurlyeq$ on $V(P)$ such that the neighbours of v are cofinal in $\lceil v \rceil := \{w \in P : w \prec v\}$ for every vertex $v \in V(P)$, i.e. for every $v' \prec v$ there is a neighbour w of v with $v' \preceq w \prec v$ (cf. Figure 1). In particular, every successor element is adjacent to its predecessor in the well-order.

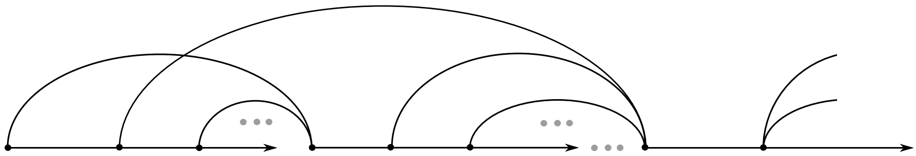


Fig. 1 A generalised path

Why is this definition appealing? A striking property of the ordinary finite path is that any interval is again connected. Hence, the term 'generalised path' is justified by the following:

(1) *A well-order $\preccurlyeq$ on P witnesses that P is a generalised path if and only if between any two vertices $w \preccurlyeq v$ on P there exists a strictly increasing finite path from w to v.*

Indeed, starting from the higher vertex $v = v_0$ and using the defining property of a generalised path, simply select the least down-neighbour of v above w and call it v_1. If $v_1 = w$, we are done. Otherwise select the least down-neighbour of v_1 above w and call it v_2. Continuing like this, we must eventually arrive at $v_n = w$, since there is no infinite decreasing sequence $v_0 \succ v_1 \succ \cdots$ in the well-order $\preccurlyeq$ on P. Conversely, if for fixed v any interval $[w,v]$ for $w \prec v$ is connected, then the neighbours of v must be cofinal in $\lceil \mathring{v} \rceil := \{w \in P : w \prec v\}$.

Rado's suggestion turned out to be correct, but it took another 40 years to prove it. First, Elekes, Soukup, Soukup and Szentmiklòssy [8] confirmed Rado's question for $\aleph_1$-sized complete graphs and two colours, and shortly after in 2017, D. Soukup [31] gave a complete affirmative answer to Rado's question:

Theorem 2.1 (D. Soukup, [31, Theorem 7.1]) Let r be a positive integer. Every r-edge-coloured complete graph of infinite order can be partitioned into monochromatic generalised paths of different colours.

In [31, Conjecture 8.1], D. Soukup conjectured a similar result for complete bipartite graphs, namely that every r-coloured infinite complete bipartite graph with bipartition classes of the same cardinality can be partitioned into $2r - 1$ monochromatic generalised paths. Also this conjecture is now a theorem:

Theorem 2.2 (Bürger & Pitz [5]) Let r be a positive integer. Every r-edge-coloured infinite complete balanced bipartite graph can be partitioned into $2r - 1$ monochromatic generalised paths.

Let's close this section with a fundamental fact about complete minors of uncountable generalised paths that will play a role throughout this survey:

(2) *Any uncountable generalised path contains a subdivision of an uncountable clique.*

To see why this is true, consider an uncountable generalised path P. Considering an initial segment only, we may assume that the order type of $(P,\preccurlyeq)$ is ω_1. From our earlier observation (1) it follows that all end-segments of $(P,\preccurlyeq)$ are connected and so must contain vertices of uncountable degree. Hence, P has uncountably many vertices of uncountable degree. Using some of these vertices as branch vertices, we can now build a subdivision of an uncountable complete graph recursively in ω_1 steps, at each step adding a new (unused) branch vertex and joining it to the previously selected branch vertices one by one by a monotone path between a sufficiently high pair of their neighbours.

It is straightforward to check that this argument generalises to show that for regular infinite cardinals κ, every κ-sized generalised path contains a subdivision of the clique of size κ.

3 *T*-graphs: Definition and First Examples

In this section we meet the main method of how to encode an order tree into a graph reflecting the tree structure, introduced systematically by Diestel and Brochet in [4]. We'll use the following notation. Let T be an order tree, which we imagine to grow upwards from the root. Write $\lceil t \rceil = \{t' \in T : t' \leq t\}$ and $\lfloor t \rfloor := \{t' \in T : t \leq t'\}$. A maximal chain in T is called a *branch* of T; note that every branch inherits a well-ordering from T. The *height* of T is the supremum of the order types of its branches. The *height* of a node $t \in T$ is the order type of $\lceil \mathring{t} \rceil := \lceil t \rceil \setminus \{t\}$. The set T^i of all nodes at height i is the i th *level* of T, and we write $T^{<i} := \bigcup\{T^j : j < i\}$.

The intuitive interpretation of a tree order as expressing height will also be used informally. For example, we say that t is *above* t' if $t > t'$, and call $\lceil X \rceil = \lceil X \rceil_T := \bigcup\{\lceil x \rceil : x \in X\}$ the *down-closure* of $X \subseteq T$. And we say that X is *down-closed*, or X is a *rooted subtree*, if $X = \lceil X \rceil$.

If $t < t'$, we write $[t, t'] = \{x : t \leq x \leq t'\}$, and call this set a (closed) *interval* in T. (Open and half-open intervals in T are defined analogously.) If $t < t'$ but there is no node between t and t', we call t' a *successor* of t and t the *predecessor* of t'; if t is not a successor of any node it is called a *limit*.

Are there useful graphs on order trees? One often-used variant is to consider the comparability graph, in which precisely the pairs of comparable vertices are connected by an edge, used e.g. in [10, 19, 20, 35]. However, the following concept, first systematically studied in [4], offers much more flexibility, as demonstrated for example in Theorem 4.1 and 7.3 below.

Definition 3.1 (Brochet & Diestel) Given an order tree $(T,\leq)$, a graph $G = (V,E)$ is a *T-graph* if $V = T$, the ends of any edge $e = tt'$ are comparable in T, and the neighbours of any $t \in T$ are cofinal in $\lceil \mathring{t} \rceil := \{t' \in T : t' < t\}$.

It is easy to see how this relates to our earlier definition of a generalised path: a graph G with vertex set T is a T graph if and only if the subgraphs induced by the branches of T are generalised paths, together covering (all edges of) the graph G.

Rephrasing this once again, we require that all the edges of G run 'parallel' to branches of T, but never 'across', and that all intervals $[t, t']$ for $t \leq t'$ in T induce a connected subgraph in G. In particular, if t, t' are incomparable, then the only way to walk from t to t' in G is to go down-and-up again, via a third element in T that is comparable to both t and t'. The following captures this intuition about separation properties of T-graphs; see [4, §2] for details.

(3) *For incomparable vertices t, t' in T, the set $\lceil t \rceil \cap \lceil t' \rceil$ separates t from t' in G.*
(4) *Every connected subgraph of G has a unique T-minimal element.*
(5) *If $T' \subset T$ is down-closed, the components of $G - T'$ are spanned by the sets $\lfloor t \rfloor$ for t minimal in $T - T'$.*

To see how these separation properties are used in practise, let us establish a converse to our earlier observation (2) that κ-sized generalised paths for regular κ contain K_κ-minors, where K_κ denotes the complete graph on κ vertices.[2] This type of argument provides a blueprint for how separation properties of T-graphs are usually applied.

(6) *Suppose G is a T-graph and that κ is a regular infinite cardinal. Then G contains a K_κ minor if and only if T contains a branch of size at least κ.*

The backwards implication follows from (2) and the fact that every branch in a T-graph induces a generalised path. Conversely, if G contains a K_κ minor for regular κ, then G also contains a subdivision of K_κ by a result of Jung [17]. Now if T did not contain a branch of size at least κ, then we'd find two branch vertices $t \neq t'$ of this subdivided K_κ that are incomparable in T. But then $X := \lceil t \rceil \cap \lceil t' \rceil$ would be a $t - t'$ separator of size less than κ in G by (3), a contradiction.

Let us close this section with a class of T-graphs arising from order trees of height $\omega + 1$ that will play an important role in Sections 6 and 7. Given a cardinal $\lambda \geq 2$, let $(T_\lambda, \leq)$ be the order tree where the nodes of T_λ are all sequences of elements of λ of length $\leq \omega$ including the empty sequence, and $t \leq t'$ if t is an initial segment of t'. Then T_λ is an order tree of height $\omega + 1$ in which every node of finite height has exactly λ successors and above every branch of $T_\lambda^{<\omega}$ there is exactly one node in T_λ^{ω}, represented by a countable sequence of ordinals in λ.

***Example 3.2* (Trees with tops)** Let $\lambda \geq 2$ be any cardinal and $(T_\lambda, \leq)$ be the λ-regular tree of height $\omega + 1$ as defined above. Then the subtree $T_\lambda(X) := T_\lambda^{<\omega} \cup X$ of T_λ is a *λ-tree with tops*, the tops themselves being the nodes in X.

We remark that λ-regular trees with tops have been used recently in [9, 23]. They themselves form a generalisation of the so-called *binary trees with tops*, studied in more detail in [3, 7, 34].

As indicated by the dashed edges in Fig. 2, one obtains a T-graph from a λ-regular tree with tops by connecting every top to infinitely many vertices on 'its' branch.

The T-graphs for λ-regular trees with tops are especially interesting when the number of tops is larger than the cardinal $\max\{\aleph_0, \lambda\}$ of the λ-regular tree. This is possible either in the case where $\lambda < 2^{\aleph_0}$, or in the case where λ has countable cofinality (where it follows from König's Theorem that T_λ has strictly more than λ many branches). In contrast, recall that under CH the $\aleph_1$-regular tree only has $\aleph_1$ many branches.

[2] Recall that H is a *minor* of G if there are disjoint connected vertex sets $\{V_h : h \in H\}$ in G such that G has a $V_h - V_{h'}$ edge whenever hh' is an edge in H.

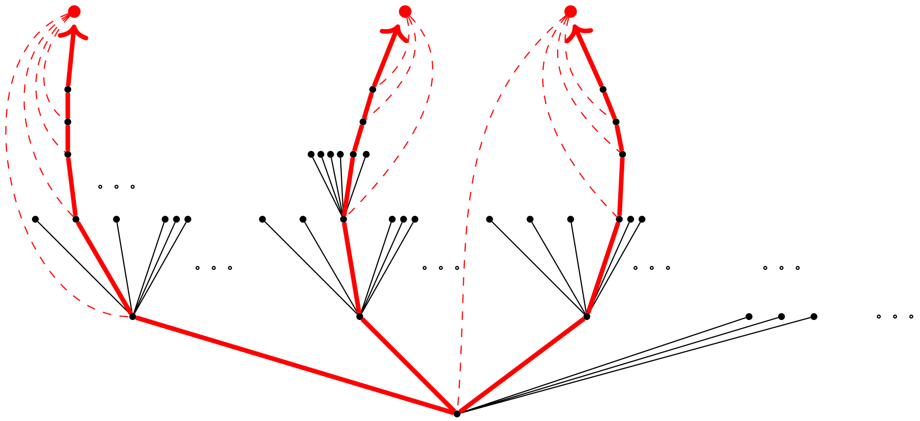


Fig. 2 An $\aleph_0$-regular tree with tops

4 Normal Spanning Trees, Colouring Number and Forbidden Minors

The definition of a T-graph tells us how to construct from a given order tree T a certain graph G. What about a converse to this procedure? Given an arbitrary connected graph G, must it come from an order tree, and can we tell what the order tree $(T,\leq)$ is?

To make this question precise, if a graph G is (isomorphic to) a T-graph for some order tree $(T,\leq)$, we say that $(T,\leq)$ is a *normal tree order* for G. The following problem from [4, §3] is the main open problem in the area:

Problem 1 Characterise which connected graphs admit a normal tree order.

To see that not all graphs do, consider an uncountable clique G where every edge has been subdivided once. By (6) any normal tree order for G would need to contain an uncountable branch. However, any limit node in that branch would require infinitely many lower neighbours of degree at least three, a contradiction.

It is, however, interesting to note that the uncountable clique itself, a minor of the subdivided clique, does have a normal tree order. This is not an accident: Brochet and Diestel have shown in [4, Theorem 4.2] that every connected graph G 'almost' has a normal tree order, if one is allowed to contract some small sets.

Theorem 4.1 (Brochet & Diestel) Every connected graph G admits a contraction minor that comes with a normal tree order $(T,\leq)$, such that its branch sets $(V_t)_{t\in T}$ in G are small in that $|V_t|$ is bounded by the cofinality of the height of t in T.

The earlier argument for the subdivided uncountable clique did split into two cases, ruling out the cases where there is normal tree order of height at most ω, and where there is normal tree order of height at least $\omega+1$. The first question which graphs have a normal tree order of height at most ω is in itself a fascinating subcase of Problem 1. When T has height at most ω, we say T is a *normal spanning tree* for G. In this case, T is a (rooted) spanning tree of G in the usual graph theoretic sense such that all cords of the tree run between vertices comparable in the tree order, see also [6, §1.5 and §8.2].

In contrast to Problem 1, we now have quite a good understanding which connected graphs have a normal spanning tree. First of all, every finite graph has one, even with arbitrarily prescribed root: Indeed, it is an easy exercise to show that these are precisely the depth-first search trees starting from the given root. By a result of Jung, also every countable graph contains a normal spanning tree with any arbitrarily chosen vertex as the root, see [16] or [6, 8.2.4]. Uncountable cliques, however, witness that uncountable graphs might not have normal spanning trees by (6). What is known is that

(7) *If G contains no subdivision of a countable clique, then G has a normal spanning tree.*
(8) *If G has a normal spanning tree, then it contains no subdivision of an uncountable clique.*

Indeed, (7) is a result by Halin from [13], with a short proof now available in [26]. Item (8) is just a corollary to (6). Further, we have the following characterisation for the existence of normal spanning trees due to Jung [16]. Here, a set of vertices U is *dispersed* in G if every ray in G can be separated from U by a finite set of vertices.

(9) *A connected graph has a normal spanning tree if and only if its vertex set is a countable union of dispersed sets.*

Item (9) implies in particular that having a normal spanning tree is a minor-closed property, which opens up the possibility of characterising the property of having a normal spanning tree by forbidden minors, and this is what we shall discuss next.

Definition 4.2 (Erdős & Hajnal) The *colouring number* $\mathrm{col}(G)$ is the least cardinal μ such that $V(G)$ has a well-order $\preccurlyeq$ such that every vertex has $< \mu$ neighbours preceding it in $\preccurlyeq$.

The connection between normal spanning trees and colouring number is as follows: Every graph with a normal spanning tree T has countable colouring number: simply well-order level by level, then all earlier neighbours of a vertex t belong to the finite down-closure of t. The naïve converse fails, however, as witnessed by uncountable cliques with all edges subdivided: if one first enumerates all real vertices followed by all subdivision vertices, we get a well-ordering in which every vertex has at most two neighbours preceding it. However, the property of having a normal spanning tree is preserved under taking connected minors, and this means that also all their minors have countable colouring number. In [14, Conjecture 7.6] from 1998 Halin conjectured a converse to this observation. This is now a theorem [27]:

Theorem 4.3 (Pitz, '20$^+$) A connected graph has a normal spanning tree if and only if every minor of it has countable colouring number.

As there is a forbidden subgraph characterisation for having colouring number $\leq \mu$ [2], this yields a forbidden minor characterisation for the property of having a normal spanning tree. For countable colouring number $\mu = \aleph_0$ these forbidden minors come in two structural types: First, the class of (λ,λ^+)-*graphs*, bipartite graphs (A,B) such that $|A| = \lambda$, $|B| = \lambda^+$ for some infinite cardinal λ, and every vertex in B has infinite degree. And second, the class of (κ,S)-*graphs*, graphs whose vertex set is a regular uncountable cardinal κ such that stationary many vertices $s \in S \subseteq \kappa$ have countably many neighbours that are cofinal below s.

Corollary 4.4 (Pitz) A graph G has a normal spanning tree if and only if it contains neither a (λ,λ^+)-graph nor a (κ,S)-graph as a minor.

A surprising consequence of Corollary 4.4 is that a graph of singular uncountable cardinality κ has a normal spanning tree as soon as all its minors of size strictly less than κ admit normal spanning trees. This is not the case when κ is regular [23, Theorem 5.1].

Finally, I remark that while (9) implies that having a normal spanning tree is preserved under taking connected minors, I do not currently know whether the same is true for the property of having a normal tree order.

5 *T*-graphs and Hadwiger's Conjecture

The purpose of this section is to discuss Halin's infinite version of Hadwiger's conjecture, and to see an example of a T-graph that describes an interesting boundary case.

Theorem 5.1 (Halin '67 12) Suppose G is a graph without a K_λ minor for some infinite cardinal λ. Then $\chi(G) \leq \mathrm{col}(G) \leq \lambda$.

Halin's original proof from [12] (see also [13, §8]) employs his theory of simplicial decompositions. The following short argument, inspired by the closure arguments from [25, 26], is new.

For distinct vertices v,w of G denote by $\kappa(v,w) = \kappa_G(v,w)$ the connectivity between v and w in G, i.e. the largest size of a family of independent $v - w$ paths. If v and w are non-adjacent, this is by Menger's theorem equivalent to the minimal size of a $v - w$ separator in G. The following basic observation by Halin from [12, (15)] gives a sufficient condition for a graph to contain subdivisions of large cliques on prescribed vertex sets.

(10) *Let U be an infinite set of vertices in G such that $\kappa(u,v) > |U|$ for all $u \neq v \in U$. Then there is a subdivision of an infinite clique of size $|U|$ with branch vertices U.*

Proof Proof of Theorem 5.1 By induction on $\sigma := |G|$. We may assume that $\sigma > \lambda$. We construct a continuous increasing transfinite sequence $(G_i \colon i < \sigma)$ of subgraphs with $|G_i| \leq |i| \cdot \lambda < \sigma$ and $G = \bigcup_{i<\sigma} G_i$ such that

($\star$) the end vertices of any G_i-path[3] in G have connectivity strictly larger than λ in G.

Indeed, enumerate $V(G) = \{v_i : i < \sigma\}$ and put $G_0 := \{v_0\}$. If $\ell < \sigma$ is a limit, let $G_\ell := \bigcup_{i<\ell} G_i$ and note that ($\star$) is preserved under increasing unions. To define G_{i+1} from G_i, we use a countable closure argument. Set $G_i^0 := G[G_i \cup v_{i+1}]$ and construct G_i^{n+1} from G_i^n by adding for every pair $v, w \in V(G_i^n)$ with $\kappa_G(v,w) \leq \lambda$ a maximal family of independent $v - w$ paths in G to G_i^n. Then $G_{i+1} := \bigcup_{n\in\mathbb{N}} G_i^n$ is as desired.

[3] A G_i-path is a path in G that meets G_i in precisely its end vertices.

Now every vertex $v \in G_{i+1} \setminus G_i$ satisfies $|N(v) \cap G_i| < \lambda$, as otherwise property ($\star$) together with (10) imply that G has a K_λ minor.

By induction assumption, we may well-order each $G_{i+1} \setminus G_i$ individually to witness $\mathrm{col}(G_{i+1} \setminus G_i) \leq \lambda$. Now concatenate all these well-orders to obtain a well-order $\preccurlyeq$ on G. Consider an arbitrary vertex v. If $v \in G_{i+1} \setminus G_i$, then all vertices preceding v in $\prec$ belong to G_{i+1}. By construction, v has fewer than λ neighbours in $G_{i+1} \setminus G_i$ preceding it in $\prec$, and fewer than λ neighbours in G_i altogether. Thus, $\preccurlyeq$ witnesses that $\mathrm{col}(G) \leq \lambda$. □

Corollary 5.2 (Halin) Every graph G with $\mathrm{col}(G) > \lambda \geq \aleph_0$ contains a subdivision of K_λ.

We remark that $\mathrm{col}(G) \geq \lambda^+$ does not imply the existence of a subdivided K_{λ^+} in G. Indeed, consider a complete bipartite graph $G = K_{\lambda,\lambda^+}$, which clearly cannot contain a K_{λ^+}; but it is an easy exercise to check that $\mathrm{col}(G) = \lambda^+$.

A harder question is whether $\chi(G) \geq \lambda^+$ implies the existence of a subdivided K_{λ^+} in G. It turns out that the answer to this question is also in the negative; for the counterexample, we shall make use of a T-graph.

Example 5.3 **(Komjáth '17 [19])** For every infinite cardinal κ there is a graph of cardinality 2^κ, chromatic number κ^+, with no K_{κ^+} minor.

Proof Sketch Our candidate for such a graph will be a T-graph G for a suitable order tree T. If G is not to contain a K_{κ^+} minor, we should choose a tree T without branches of size κ^+ by (6).

If G is to have large chromatic number, it can only help to add as many edges to our T-graph as possible, i.e. take G to be the comparability graph of T. Then independent sets of G correspond precisely to antichains of T, and so for G to have chromatic number at least κ^+ our tree should not be the union of κ many antichains (i.e. T should not be κ-*special*).

A tree with these properties is given by the tree consisting of all injective functions $\alpha \hookrightarrow \kappa$ for all $\alpha < \kappa^+$, ordered by extension (studied first by Galvin/Baumgartner [1, Section 4.1]). □

6 *T*-graphs and Well-quasi Orderings

A binary relation $\trianglelefteq$ on a set X is a *well-quasi-order* if it is reflexive and transitive, and for every sequence $x_1, x_2, \ldots \in X$ there is some $i < j$ such that $x_i \trianglelefteq x_j$. Kruskal proved that finite trees are well-quasi-ordered by topological embeddings, i.e. whenever $T_1, T_2, \ldots$ are finite trees, then there are $i < j$ such that T_j contains a subdivision of T_i, see [6, Theorem 12.2.1]. This was subsequently extended to all graph-theoretic trees by Nash-Williams [22]. In fact, this statement does also hold in the following, slightly stronger formulation: The class of order trees of height at most ω is well-quasi-ordered under order-embeddings that preserve meets. Galvin has observed that trees of height $\omega + 1$ are no longer well-quasi ordered under order embeddings (unpublished, quoted from [34]), but see also Theorems 6.2 and 6.4 below.

A graph H is a *minor* of G if there are disjoint connected vertex sets $\{V_h : h \in H\}$ in G such that G has a $V_h - V_{h'}$ edge whenever hh' is an edge in H. Write $H \preccurlyeq G$ if H is a minor of G.

Theorem 6.1 (Robertson & Seymour, '80s) Finite graphs are well-quasi ordered under the minor relation $\preccurlyeq$.

The main open problem in the field concerning infinite graphs is the following:

Problem 2 Are countable graphs well-quasi ordered by $\preccurlyeq$?

See [29] for additional information. Are all graphs well-quasi ordered under the minor relation? The answer is no – and the most transparent counterexamples are built once again from T-graphs.

Theorem 6.2 (Thomas '88 [34]) Graphs of size $2^{\aleph_0}$ are not well-quasi ordered by $\preccurlyeq$: There is a sequence $G_1, G_2, \ldots$ of binary trees with tops such that $G_i \not\preccurlyeq G_j$ whenever $i < j$.

Proof Sketch The binary trees with tops, cf. Example 3.2, used by Thomas are most conveniently phrased in topological terms. Indeed, the level of all tops of the binary tree is naturally isomorphic to the Cantor space. Thomas' strategy is to select continuum-sized subspaces $X_0, X_1, X_2, \ldots$ of the Cantor space such that there is no continuous 'almost' embedding from X_i into X_j for $i < j$ (X_i *almost embeds* into X_j if some co-countable subspace of X_i embeds into X_j.) Since there are only $2^{\aleph_0}$ many almost-continuous images of a space with a countable dense subset, this can be quite easily achieved by a transfinite recursion. Then the binary trees with tops $G_n := T_2(X_n)$ give the desired counterexample, as supposing $T_2(X_i) \preccurlyeq T_2(X_j)$ translates precisely to an almost embedding from X_i into X_j, an impossibility. □

Given that Thomas' counterexample are graphs of size $2^{\aleph_0}$, the reader interested in cardinal arithmetic might ask whether uncountable graphs smaller than size continuum are well-quasi ordered? The answer is no, and much more is true: for all uncountable cardinalities κ, the class of graphs of size κ always contains minor-antichains of maximal possible size.

Theorem 6.3 (Komjáth '95 [18]) For every uncountable cardinal κ there is a family $\{G_i : i < 2^\kappa\}$ of κ-sized graphs such that $G_i \not\preccurlyeq G_j$ whenever $i \neq j$.

A quite different construction for $\kappa = 2^{\aleph_0}$ assuming the continuum hypothesis has been obtained by Diestel and Leader, [7, §9]. But both constructions are somewhat hard to define. The following gives a more transparent construction for Komjáth's result in the regular case, which reinstates a pleasant similarity to Thomas's original strategy in that it uses T-graphs.

Theorem 6.4 (Pitz '20⁺) For every uncountable regular κ there is a family $\{G_i : i < 2^\kappa\}$ of κ-regular trees with tops such that $G_i \not\preccurlyeq G_j$ whenever $i \neq j$.

For a sketch of the construction, let $T = T_\kappa$ denote the κ-regular tree with all tops introduced in Example 3.2. Recall that a top s on level T^ω is formally represented by an infinite sequence $s : \mathbb{N} \to \kappa$. Let $\Lambda \subset \kappa$ denote the set of limit ordinals of countable cofinality. For every $\ell \in \Lambda$ pick an increasing cofinal sequence $s_\ell : \mathbb{N} \to \ell$, which we may interpret as a top of T (Fig. 3).

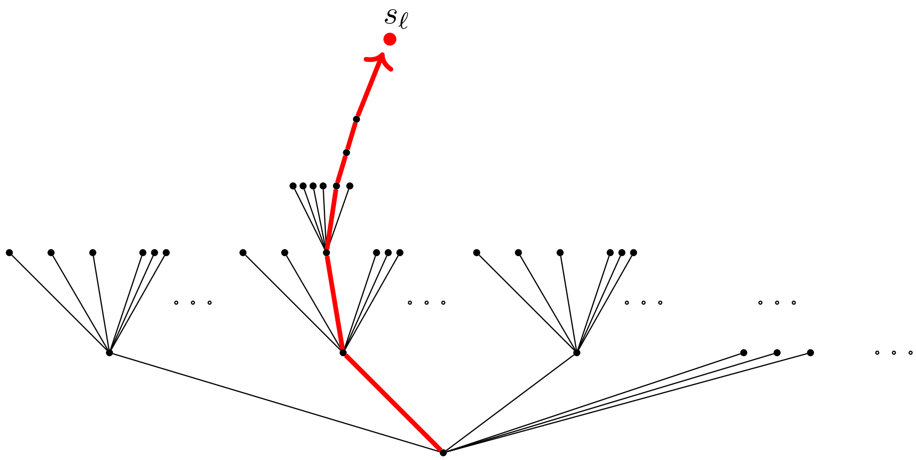


Fig. 3 Adding a top above the ray $s_\ell = (1,2,4,\ldots)$

Let $T(\Lambda)$ denote the κ-regular tree with all chosen tops s_ℓ for $\ell \in \Lambda$, and let G be any $T(\Lambda)$-graph. Let us choose a collection $\{S_i : i < 2^\kappa\}$ of stationary subsets of Λ such that $S_i \setminus S_j$ is stationary for every $i \neq j$, and let $G_i = G(S_i)$ be the subgraph of G induced by tree with tops $T_\kappa(S_i)$. We claim that the G_i are as desired.

The proof then proceeds as follows: Supposing for a contradiction that $G(S_i) \preccurlyeq G(S_j)$, then also $G(S_i \setminus S_j) \preccurlyeq G(S_j)$. Hence, to complete the proof, one has to show that if S,R are disjoint stationary subsets consisting of cofinality ω ordinals, then $G(S) \not\preccurlyeq G(R)$.

This verification relies on technical but routine arguments involving the combinatorics of stationary sets and Fodor's pressing down lemma; see [24] for the details. At the heart of the argument, however, lies the fact that this chosen collection of tops $\{s_\ell : \ell \in \Lambda\}$ behaves in the following curious way: On the one hand, the selected tops are evenly spread out across T^ω_κ in the sense that any $<\kappa$-sized subtree T' of $T^{<\omega}_\kappa$ contains strictly fewer than $|\Lambda| = \kappa$ many of the selected tops. One the other hand, the selected tops cluster after all: By Fodor's pressing down lemma, there is a stationary subset $\Lambda' \subset \Lambda$ such that $s_\ell(1)$ agree for all $\ell \in \Lambda'$, and iterating this observation, for any $n \in \mathbb{N}$ there are still stationary many elements of Λ whose sequences s_ℓ agree on the first n elements. This tension was first observed by A.H. Stone in his work on Borel isomorphisms [32, §5] and [33, §3.5]. This clustering property is also responsible for the fact that $G(\Lambda)$ does not contain a normal spanning tree [23, Theorem 5.1].

Finally, these trees with tops $T_\kappa(S_i)$ also form antichains in the category of order trees of height $\omega + 1$ ordered by injective order embeddings, giving another construction for Galvin's observation mentioned above.

7 *T*-graphs and Halin's End Degree Conjecture

An *end* of a graph G is an equivalence class of rays, where two rays of G are *equivalent* if there are infinitely many vertex-disjoint paths between them in G. The *degree* $\deg(\varepsilon)$ of an end ε is the maximum cardinality of a collection of pairwise disjoint rays in ε, see Halin [11].

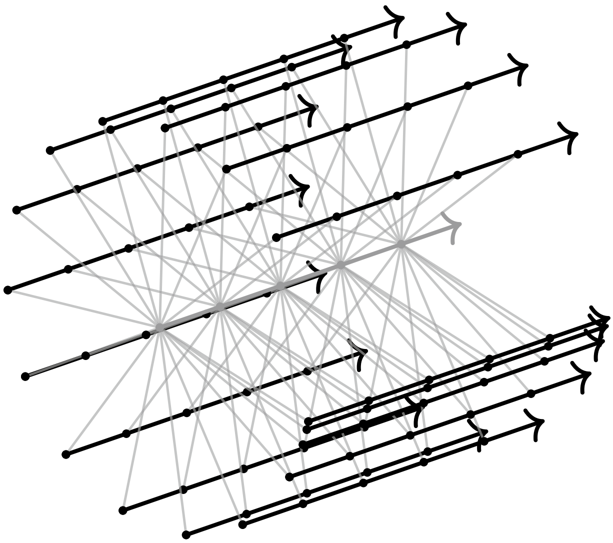
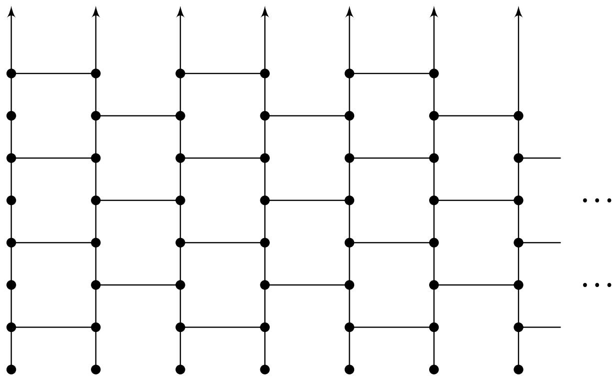


Fig. 4 The Cartesian product of a star and a ray

Fig. 5 The hexagonal half grid



A typical example of an end of countable degree is given by the *half-grid*, the graph on $\mathbb{N}^2$ in which two vertices (n,m) and (n', m') are adjacent if and only if $|n - n'| + |m - m'| = 1$. More generally, prototypes of ends of any prescribed degree can be obtained from the Cartesian product of a sufficiently large connected graph with a ray (see Fig. 4).

However, for many purposes a degree-witnessing collection $\mathcal{R} \subset \varepsilon$ on its own forgets significant information about the end, as it tells us nothing about how G links up the rays in $\mathcal{R}$; in fact $G[\bigcup \mathcal{R}]$ is usually disconnected. This raises the question of whether one can describe typical configurations in which G must link up the disjoint rays in some degree-witnessing subset of a pre-specified end.

That this is possible in the case of countable end-degree is a famous result by Halin: It says that every such end contains a modified version of the half-grid, namely the *hexagonal half-grid*, where one deletes every other rung from the half-grid as shown in Fig. 5.

Theorem 7.1 (Halin's grid theorem '65) Every graph with an end of infinite degree contains a subdivision of the hexagonal half-grid whose rays belong to that end.

To find a satisfying answer to this question for ends of arbitrary degree, however, is a longstanding open problem due to Halin. In fact, Halin put forward one conjecture how an answer for higher end degree could look like. Intuitively, this conjecture says that in every end of degree κ one finds a slightly modified Cartesian product of a connected graph with a ray, where the modifications model the shift from half-grid to hexagonal half-grid encountered in Theorem 7.1.

Given a set $\mathcal{R}$ of disjoint equivalent rays in a graph G, we call a graph H with vertex set $\mathcal{R}$ a *ray graph* in G if there exists a set $\mathcal{P}$ of independent $\mathcal{R}$-paths (independent paths with precisely their endvertices on rays from $\mathcal{R}$) in G such that for each edge RS of H there are infinitely many disjoint R–S paths in $\mathcal{P}$. Given an end ε in a graph G, a *ray graph for* ε is a connected ray graph in G on a degree-witnessing subset of ε.

Using this set-up, Halin conjectured the following in [14, Conjecture 6.1]:

Conjecture 7.2 (Halin's end degree conjecture) *Every graph has ray graphs for all its ends.*

For finite degree ends it is straightforward to answer this in the affirmative. For ends of countably infinite degree the answer is positive, too, by Halin's grid theorem. For uncountable end degree, however, the answer depends on the cardinality of the end degree in question.

Theorem 7.3 (Geschke, Kurkofka, Melcher, Pitz 20⁺ [9]) Halin's conjecture fails for end degrees $\deg(\epsilon) = \aleph_1$, holds for all end degrees $\aleph_2, \aleph_3, \ldots, \aleph_\omega$, fails again for $\deg(\epsilon) = \aleph_{\omega+1}$, and is undecidable for the next $\aleph_{\omega+n}$ for $n \in \mathbb{N}$, $n \geq 2$.

Informally, we may think of Halin's conjecture as saying that the only way to build an end of degree κ is to take a Cartesian product of a ray with some tree T with $|T| = \kappa$ (where formally, T represents the ray graph in question). If κ is regular, then every such tree contains a vertex of degree κ, and so in this case Halin's conjecture further simplifies as saying that the only way to build an end of degree κ is to take a Cartesian product of a ray with a star with κ many leaves.

However, for some end degrees such as $\aleph_1$ or $\aleph_{\omega+1}$, the rays of an end may not only be arranged like a connected graph, but may actually be arranged like an order tree (specifically, in the $\aleph_1$-case, they may be arranged like an Aronszajn tree). This is made precise as follows:

Suppose T is an order tree of height at most ω_1. We say that a family $\mathcal{R} = \{R_t : t \in T\}$ of disjoint rays in a graph G is *arranged like the order tree T* if there exists a family $\mathcal{P}$ of independent paths with precisely their endvertices on rays from $\mathcal{R}$ such that

(*i*) if t is a successor of s in T, there exist infinitely many disjoint $R_t - R_s$ paths in $\mathcal{P}$, and

(*ii*) if t is a limit node in T there are infinitely many disjoint paths $(P_n)_{n\in\mathbb{N}}$ from $\mathcal{P}$ where each P_n is an $R_t - R_{t_n}$ path such that the nodes t_n are cofinal below t in T.

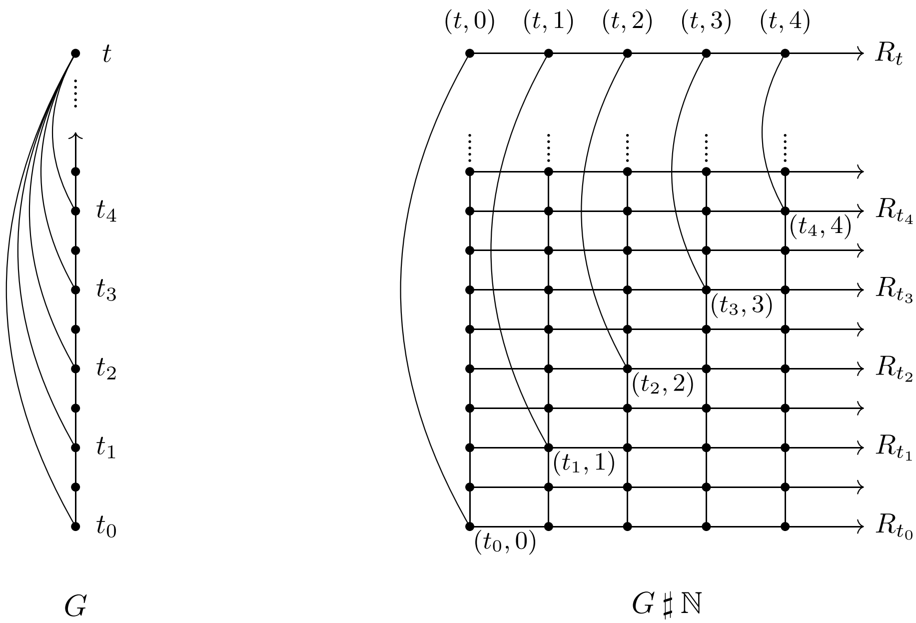


Fig. 6 The ray inflation of an $(\omega+1)$-graph G

Similar to how Cartesian products of a connected graph with a ray give prototypes of ray graphs, there is a canonical way to turn T-graphs into ray families which are arranged like T:

Definition 7.4 Let G be a T-graph where T be an order tree of height at most ω_1 such that for every limit node t of T, $N(t) \cap \lceil t \rceil$ has order type ω. The *ray-inflation* $G \sharp \mathbb{N}$ of G is the graph with vertex set $T \times \mathbb{N}$, and the following (Fig. 6) edges:

(*i*) For every $t \in T$ and $n \in \mathbb{N}$ we add the edge $(t,n)(t,n+1)$ (such that $R_t := \{t\} \times \mathbb{N}$ induces a ray).
(*ii*) If $t \in T$ is a successor with predecessor t', we add all edges $(t,n)(t',n)$ for all $n \in \mathbb{N}$.
(*iii*) If $t \in T$ is a limit with down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots$ in G we add the edges $(t,n)(t_n,n)$ for all $n \in \mathbb{N}$.

It is straightforward to check that the ray inflation $G \sharp \mathbb{N}$ has only one end, which has degree $|T|$. Our counterexamples to Halin's conjecture in the cases $\aleph_1$ and $\aleph_{\omega+1}$ are now obtained from ray inflations of suitable T-graphs, two instances of which look as follows:

(11) *Some ray family arranged like an Aronszajn tree refutes Halin's conjecture for* $\aleph_1$.
(12) *Some ray family arranged like an* $\aleph_\omega$*-regular tree with* $\aleph_{\omega+1}$*-many tops refutes Halin's conjecture for* $\aleph_{\omega+1}$.

Before explaining the idea behind (12) in detail, let me sketch the construction for (11): Recall that an Aronzsajn tree is a tree of size $\aleph_1$ with all levels and branches countable. The example for (11) is given by a ray inflation $G \sharp \mathbb{N}$ where G is a suitable T-graph for a suitable Aronzsajn tree T, relying on an idea of Diestel, Leader and Todorcevic [7]: Starting from a *special* Aronzsajn tree T, i.e. one that has an antichain partition $(U_n)_{n\in\mathbb{N}}$, we construct a T-graph as follows: Successors are connected to their predecessors, and given a limit $t \in T$, pick down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots <_T t$ with $t_i \in U_{n_i}$ recursively such that $t_{i-1} < t_i < t$ and each n_i is smallest possible. The resulting T-graph G has the property that for each t there is a finite set $S_t \subset \lceil \mathring{t} \rceil$ such that *every* $s >_T t$ satisfies $N(s) \cap \lceil t \rceil \subset S_t$. This is enough to show that $G \sharp \mathbb{N}$ contains no $\aleph_1$-star of rays, thus refuting Halin's conjecture. See [9, §6] for details.

The idea behind (12) is easier as long as one assumes the continuum hypothesis (the result also holds without the continuum hypothesis, but the proof becomes more complicated). So let T be an order tree corresponding to an $\aleph_\omega$-regular tree with $\aleph_{\omega+1}$ many tops (cf. Example 3.2), and let G be any T-graph. If Halin's conjecture were true for $\kappa = \aleph_{\omega+1}$, we would find in $H = G \sharp \mathbb{N}$ a configuration of rays arranged like a κ-star with say center ray R and leaf rays $(R_i : i < \kappa)$. Since $T^{<\omega} \times \mathbb{N}$ has size $< \kappa$, we may assume without loss of generality that each leaf ray R_i is a tail of a horizontal ray $R_{t(i)} \subset H$ for a top $t(i) \in T^\omega$, and that no path system from R_i to R uses an inner vertex from $T^{<\omega} \times \mathbb{N}$. Next, observe that there exists a countable subtree T' of T such that $R \subset T' \times \mathbb{N} \subset V(H)$. Since a countable tree can have at most $2^{\aleph_0}$ many tops, which is far smaller than κ assuming CH, we assume that no leaf ray R_i chooses a top $t(i)$ whose downclosure in included in T'. But now it follows from (5) that every leaf ray R_i is contained in some component $C_t := \lfloor t \rfloor \times \mathbb{N}$ for t a minimal element of $T^{<\omega} \setminus T'$. However, each R_i has only finitely many neighbours in $\lceil t \rceil \times \mathbb{N}$, contradicting that R_i sends an infinite path system to R avoiding $T^{<\omega} \times \mathbb{N}$.

Without CH, one has to pick a collection of tops of T corresponding to a *scale* in Shelah's *pcf* -theory, see [9, §7] for details.

I conclude with an open question. Halin's original conjecture for end degree $\aleph_1$ turned out to be false, because there is a second configuration: ray families of size $\aleph_1$ can not only be arranged like a connected graph of size $\aleph_1$, but also like an Aronszajn tree. I wonder whether these are now all possible configurations.

Problem 3 Is it true that every end of degree $\aleph_1$ contains a connected ray graph of size $\aleph_1$ or a ray family arranged like an Aronszajn tree?

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