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Turán-Type Results for Complete h-Partite Graphs in Comparability and Incomparability Graphs

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Abstract We consider an h-partite version of Dilworth's theorem with multiple partial orders. Let P be a finite set, and let $<_1, \dots, <_r$ be partial orders on P. Let $G(P, <_1, \dots, <_r)$ be the graph whose vertices are the elements of P, and $x, y \in P$ are joined by an edge if $x <_i y$ or $y <_i x$ holds for some $1 \leq i \leq r$. We show that if the edge density of $G(P, <_1, \dots, <_r)$ is strictly larger than $1-1/(2h-2)^r$, then P contains h disjoint sets $A_1, \dots, A_h$ such that $A_1 <_j \dots <_j A_h$ holds for some $1 \leq j \leq r$, and $|A_1| = \dots = |A_h| = \Omega(|P|)$. Also, we show that if the complement of $G(P, <)$ has edge density strictly larger than $1-1/(3h-3)$, then P contains h disjoint sets $A_1, \dots, A_h$ such that the elements of A_i are incomparable with the elements of A_j for $1 \leq i < j \leq h$, and $|A_1| = \dots = |A_h| = |P|^{1-o(1)}$. Finally, we prove that if the edge density of the complement of $G(P, <_1, <_2)$ is α, then there are disjoint sets $A, B \subset P$ such that any element of A is incomparable with any element of B in both $<_1$ and $<_2$, and $|A| = |B| > n^{1-\gamma(\alpha)}$, where $\gamma(\alpha) \to 0$ as $\alpha \to 1$. We provide a few applications of these results in combinatorial geometry, as well.

Keywords Poset · Dilworth · Bipartite graph · Turan problem

1 Introduction

Let k and n be positive integers. A weak version of the widely used Dilworth's theorem [2] states that every partially ordered set with n elements either contains a chain of size k or an antichain of size $\lceil n/k \rceil$. Applying Dilworth's theorem multiple times, one can easily deduce the following result. Let P be an n element set, and let $<_1, \dots, <_r$ be partial orders on P. There exists $H \subset P$ such that $|H| \geq \sqrt[r+1]{n}$, and H is either a $<_i$-chain for some $1 \leq i \leq r$ or any two elements of H are incomparable in any of the partial orders $<_1, \dots, <_r$.

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Bipartite versions of Dilworth's theorem have been considered in a series of papers by Fox, Pach and Tóth. Before we state their results, we introduce some notation.

Let $<_1, ..., <_r$ be partial orders on a set P. If $a, b \in P$, write $a \perp_i b$ if a and b are incomparable in $<_i$. Also, write $a \perp b$ if $a \perp_i b$ holds for $i = 1, ..., r$. If $A, B \subset P$ and $1 \leq i \leq r$, let $A <_i B$ if for every $a \in A$ and $b \in B$ we have $a <_i b$. Define $A \perp_i B$ and $A \perp B$ analogously.

In [4], Fox proved the following theorem for a single partial order.

Theorem 1 *([4]) There exists n_0 such that for all $n > n_0$ and for all partially ordered sets $(P, <)$ on n elements, there exist $A, B \subset P$ such that A and B are disjoint,*

$$|A| = |B| > \frac{n}{4 \log_2 n},$$

and either $A < B$ or $A \perp B$.

In [5], Fox and Pach generalized this result for multiple partial orders.

Theorem 2 *([5]) Let r be a fixed positive integer and let $<_1, ..., <_r$ be partial orders on the n element set P. There exist $A, B \subset P$ such that A and B are disjoint,*

$$|A| = |B| > \frac{n}{2^{(1+o(1))(\log_2 \log_2 n)^r}},$$

and either $A <_i B$ holds for some $1 \leq i \leq r$ or $A \perp B$.

In [6], Fox, Pach and Tóth proved a Turán-type version of these results. Before we state it we introduce some further notation. If $<_1, ..., <_r$ are partial orders on the set P, let $G(P, <_1, ..., <_r)$ be the graph whose vertex set is P and in which two elements $a, b \in P$ are joined by an edge if $a <_i b$ or $b <_i a$ holds for some $1 \leq i \leq r$. Call this graph the *r-comparability graph* of $(P, <_1, ..., <_r)$, and call the complement of $G(P, <_1, ..., <_r)$ the *r-incomparability graph* of $(P, <_1, ..., <_r)$. Similarly, the directed comparability graph of $(P, <_1, ..., <_r)$ is $\overrightarrow{G}(P, <_1, ..., <_r)$, in which $\overrightarrow{xy}$ is an edge if $x <_i y$ for some $1 \leq i \leq r$. We note that it is allowed to have both $\overrightarrow{xy}$ and $\overrightarrow{yx}$ in the directed edge set.

For positive integers h, r, n, m, define $f^C_{r,h}(n, m)$ and $f^I_{r,h}(n, m)$ as follows. Let $f^C_{r,h}(n, m)$ be the maximal s such that if P is an n element set with partial orders $<_1, ..., <_r$, and $G(P, <_1, ..., <_r)$ has exactly m edges, then there exist $1 \leq i \leq r$ and $A_1, ..., A_h \subset P$ pairwise disjoint subsets such that $|A_1| = ... = |A_h| = s$, and $A_1 <_i ... <_i A_h$.

Similarly, let $f^I_{r,h}(n, m)$ be the maximal s such that if P is an n element set with partial orders $<_1, ..., <_r$, and the incomparability graph of $(P, <_1, ..., <_r)$ has exactly m edges, then there exist $A_1, ..., A_h \subset P$ pairwise disjoint subsets such that $|A_1| = ... = |A_h| = s$, and $A_j \perp A_l$ for all $1 \leq j < l \leq h$.

Here is the promised theorem by Fox, Pach and Tóth [6].

Theorem 3 *([6])*

(i) *For every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that*

$$f^C_{1,2}\left(n, \left(\frac{1}{4} - \epsilon\right) n^2\right) < c(\epsilon) \log n.$$

(ii) *For every* $\epsilon > 0$,

$$f^C_{1,2}\left(n, \left(\frac{1}{4} + \epsilon\right) n^2\right) > \frac{\epsilon n}{2}.$$

(iii) *There is a constant* $c_2 > 0$ *such that for every* $0 < \lambda < 1/2$,

$$f^I_{1,2}(n, \lambda n^2) > \frac{c_2 \lambda n}{\log n \log 1/\lambda}.$$

The aim of this paper is to generalize the previous theorem and to understand the behavior of the functions $f^C_{r,h}$ and $f^I_{r,h}$. Let us note a few things about Theorem 3. The functions $f^I_{1,2}$ and $f^C_{1,2}$ behave quite differently. As we can see, $f^C_{1,2}(n, m)$ has a large jump at $m/n^2 = 1/4$, and for $m/n^2 > 1/4$ the function $f^C_{1,2}(n, m)$ is linear in n. We show that $f^C_{r,h}$ has a similar behavior.

However, as we shall see, $f^I_{1,h}$ also jumps at some value of m/n^2 for $h > 2$.

Our paper is organized as follows. In the next section, we prove bounds on $f^C_{r,h}$ for arbitrary r, h positive integers. We show that if $\alpha = 1/2 - 1/2(2h-2)^r$, the function $f^C_{r,h}(n, m)$ jumps at the point $m/n^2 = \alpha$. If m/n^2 is strictly below the threshold α, then $f^C_{r,h}(n, m)$ is $O(\log n)$, while above this point $f^C_{r,h}(n, m)$ becomes linear in n.

An h-partite graph is balanced if its classes have the same size. In Section 3, we investigate the largest balanced h-partite graph of the 1-incomparability graph. We show that $f^I_{1,h}$ also jumps. If $m/n^2 < 1/2 - 1/2(h-1)$, then $f^I_{1,h}(n, m) = 0$. However, for

$$\frac{m}{n^2} > \frac{1}{2} - \frac{1}{18(h-1)} + \epsilon,$$

we have $f^I_{1,h}(n, m) = n^{1-o(1)}$.

In Section 4, we investigate the largest balanced bipartite graph of the 2-incomparability graph. As we shall see, $f^I_{2,2}$ behaves quite differently as $f^I_{1,2}$. We show that $f^I_{2,2}(n, m)$ is approximately n^α for some α satisfying $\alpha \to 1$ as $m/n^2 \to 1/2$.

In the last section, we provide applications of these results for two problems in combinatorial geometry.

Before we start, we introduce some of the standard notation we use. As usual, $[n]$ denotes the set $\{1, ..., n\}$. If G is a graph, $V(G)$ is the vertex set of of G, $E(G)$ is the edge, $e(G) = |E(G)|$ is the number of edges, and $d(G) = e(G)/\binom{|V(G)|}{2}$ is the edge density of G. If $X, Y \subset V(G)$, $G[X]$ is the subgraph of G induced on X, and $G[X, Y]$ is the induced bipartite subgraph of G with vertex classes X and Y. Also, K_s is the complete graph on s vertices and $K_{s,t}$ is complete bipartite graph with vertex classes having sizes s and t.

A linear extension of a partial order $<$ is a total order $<^*$ such that $x < y$ implies $x <^* y$. Also, the dual of $<$ is $<^d$, where $<^d$ is defined such that $x <^d y$ if $y < x$.

To avoid clutters, we omit floors and ceilings whenever they are not crucial.

2 The r-Comparability Graph

In this section, generalizing part (i) and (ii) of Theorem 3, we prove the following result about the behaviour of $f^C_{r,h}$.

Theorem 4 *Let h, r, n be positive integers and $0 < \epsilon < 1/2(2h-2)^r$.*

(i) *We have*

$$f^C_{r,h}\left(n, \left(\frac{1}{2} - \frac{1}{2(2h-2)^r} - \epsilon\right)n^2\right) < 2\epsilon^{-1}(2h-2)^r \log n.$$

(ii) *There exists a constant $c(r, h, \epsilon) > 0$ such that*

$$f^C_{r,h}\left(n, \left(\frac{1}{2} - \frac{1}{2(2h-2)^r} + \epsilon\right)n^2\right) > c(r, h, \epsilon)n. \qquad (*)$$

Also, for $h = 2$, we have

$$f^C_{r,2}\left(n, \left(\frac{1}{2} - \frac{1}{2^{r+1}} + \epsilon\right)n^2\right) > \frac{\epsilon n}{r2^{r+1}}. \qquad (**)$$

Proof of (i). Let $G = (A, B, E)$ be a bipartite graph with

$$|A| = |B| = \frac{n}{(2h-2)^r},$$

and $|E| > |A||B|(1-\epsilon)$ such that G does not contain $K_{t,t}$ with $t > 2\epsilon^{-1}\log n$. A random bipartite graph, where the edges are chosen with probability $(1-\epsilon/2)$, has this property with high probability, see [1].

Define $(P, <_1, ..., <_r)$ as follows. Let $\{P_{\overline{t}}\}_{\overline{t}\in[2h-2]^r}$ be a partition of the n-element set P into $(2h-2)^r$ equal sized parts, and let $f_{\overline{t}} : P_{\overline{t}} \to A$ and $g_{\overline{t}} : P_{\overline{t}} \to B$ be arbitrary bijections. Let $\overline{t} = (t_1, ..., t_r)$ and $\overline{u} = (u_1, ..., u_r)$ be two different elements of $[2h-2]^r$ and suppose that the first coordinate they differ in is the q-th coordinate. Without loss of generality, $t_q < u_q$. If $t_q + 1 < u_q$, let $x <_q y$ for all $x \in P_{\overline{t}}$ and $y \in P_{\overline{u}}$. If $t_q + 1 = u_q$, let $x <_q y$ if $f_{\overline{t}}(x)g_{\overline{u}}(y) \in E$.

One can easily check that the relations $<_1, ..., <_r$ we have defined are partial orders. Also, $G(P, <_1, ..., <_r)$ contains at least

$$\binom{(2h-2)^r}{2}\frac{(1-\epsilon)n^2}{(2h-2)^{2r}} > \left(\frac{1}{2} - \frac{1}{2(2h-2)^r} - \epsilon\right)n^2$$

edges.

Suppose that $A_1, ..., A_h$ are disjoint subsets of P such that $|A_1| = ... = |A_h| = t$ and $A_1 <_q ... <_q A_h$ with some $q \in [r]$. Then there exist $\overline{t}_1, ..., \overline{t}_h$ such that for $i = 1, ..., h$, we have

$$|P_{\overline{t}_i} \cap A_i| > \frac{t}{(2h-2)^r}.$$

Also, the q-th coordinates of $\overline{t}_1, ..., \overline{t}_h$ are strictly monotone increasing. Hence, there exists $1 \leq j < h$ such that the difference between the q-th coordinate of $\overline{t}_j$ and $\overline{t}_{j+1}$ is 1. But then $f_{\overline{t}_j}(A_j \cap P_{\overline{t}_j})$ and $g_{\overline{t}_{j+1}}(A_{j+1} \cap P_{\overline{t}_{j+1}})$ span a complete bipartite graph in G, so $t/(2h-2)^r < 2\epsilon^{-1}\log n$. Hence,

$$f^C_{r,h}\left(n, \left(\frac{1}{2} - \frac{1}{2(2h-2)^r} - \epsilon\right)n^2\right) < 2(2h-2)^r\epsilon^{-1}\log n.$$

□

In the rest of this section, we shall prove part (ii) of the theorem. We are going to deduce part (ii) from a Turán-type result for multicolored directed graphs. But first, we need some definitions.

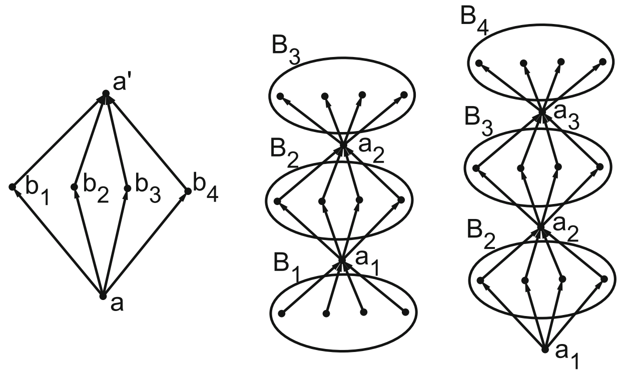

Fig. 1 Diamond, spiral and rooted spiral

A directed graph $D = (V, E)$ is a *k-diamond*, if $V = \{a, a', b_1, ..., b_k\}$ and

$$E = \{\overrightarrow{ab_i} : i = 1, ..., k\} \cup \{\overrightarrow{b_i a'} : i = 1, ..., k\}.$$

Call the vertex a the *bottom* of D and a' the *top* of D.

The directed graph $S = (V', E')$ is an *h-part spiral*, if its vertex set can be partitioned as $V' = \{a_1, ..., a_{h-1}\} \cup B_1 \cup ... \cup B_h$ such that $|B_1| = ... = |B_h|$ and

$$E' = \{\overrightarrow{ba_i} : i = 1, ..., h-1; b \in B_i\} \cup \{\overrightarrow{a_i b'} : i = 1, ..., h-1; b' \in B_{i+1}\}.$$

Call $|B_1|$ the *width* of the spiral and $B_1, ..., B_h$ the *classes* of the spiral.

Also, a directed graph $R = (V'', E'')$ is an *h-part rooted spiral*, if its vertex set can be partitioned as $V'' = \{a_1, ..., a_h\} \cup B_2 \cup ... \cup B_{h+1}$ such that $|B_1| = ... = |B_{h+1}|$ and

$$E'' = \{\overrightarrow{a_i b} : i = 1, ..., h; b \in B_{i+1}\} \cup \{\overrightarrow{ba_j} : j = 2, ..., h; b \in B_j\}.$$

Call a_1 the *root* and $|B_1|$ the *width* of the rooted spiral (Fig. 1).

It is clear that if the directed comparability graph of a partially ordered set $(P, <)$ contains an h-part rooted spiral with classes $B_1, ..., B_h$, then $B_1 < ... < B_h$. Hence, it is enough to find an h-part spiral with large width in the directed comparability graph. To prove such a result, we need the following lemma first.

Lemma 5 *Let $\epsilon > 0$ and q, n be positive integers. Let $G = (V, E)$ be a directed graph with $|V| = n$, $|E| > (1/2 - 1/2^{q+1} + \epsilon)n^2$. Let $\chi : E \to [q]$ be a q coloring of the edges. Then G contains a monochromatic k-diamond with*

$$k > \frac{\epsilon^2 n}{q^2 2^{2q+2}}.$$

Proof Let $\lambda = \epsilon / q2^{q+1}$. For $W \subset V$, $x \in V$ and $i = 1, ..., q$, let

$$U_i^W(x) = \{y \in W : \overrightarrow{xy} \in E, \chi(\overrightarrow{xy}) = i\},$$

and let

$$D_i^W(x) = \{z \in W : \overrightarrow{zx} \in E, \chi(\overrightarrow{zx}) = i\}.$$

For simplicity, write $U_i^V(x) = U_i(x)$ and $D_i^V(x) = D_i(x)$. Also, for all $H \subset [q]$, let

$$V_H = \{x \in V(G) : |U_i(x)| > \lambda n \Leftrightarrow i \in H\}.$$

The sets $\{V_H\}_{H\subset[q]}$ partition V into 2^q parts. The number of edges connecting two different parts in this partition is at most

$$\sum_{H_1,H_2\subset[q];H_1\neq H_2} |V_{H_1}||V_{H_2}| \leq \binom{2^q}{2}\left(\frac{n}{2^q}\right)^2 = \left(\frac{1}{2}-\frac{1}{2^{q+1}}\right)n^2.$$

Hence, there exists $F \subset [q]$ such that $G[V_F]$ contains at least $\epsilon n^2/2^q$ edges. Let E' be the set of edges in $G[V_F]$ whose color is in F. Note that for every $x \in V_F$ there are at most $q\lambda n$ edges e containing x such that $\chi(e) \notin F$. Thus,

$$|E'| > \left(\frac{\epsilon}{2^q} - q\lambda\right)n^2 = \frac{\epsilon n^2}{2^{q+1}}.$$

But then there exists $p \in F$ such that $G[V_F]$ contains at least $\epsilon n^2/q2^{q+1}$ edges of color p. So there exists $a \in V_F$ with

$$|U_p^{V_F}(a)| > \frac{\epsilon n}{q2^{q+1}}.$$

Let $A = U_p^{V_F}(a)$. There are at least

$$\lambda n|A| > \frac{\epsilon^2 n^2}{q^2 2^{2(q+1)}}$$

edges of color p connecting an element of A with an element of V, as every element of A has at least λn edges of color p containing it. Hence, there exists $a' \in V$ with

$$|D_p^A(a')| > \frac{\epsilon^2 n}{q^2 2^{2(q+1)}}.$$

Then the vertex set $\{a, a'\} \cup D_p^A(a')$ spans a p-colored k-diamond with

$$k > \frac{\epsilon^2 n}{q^2 2^{2(q+1)}}.$$

□

Now we are ready to prove our key result about spirals.

Theorem 6 *Let r, h be positive integers and $\epsilon > 0$. There exists $c(r, h, \epsilon) > 0$ with the following property. Let $G = (V, E)$ be a directed graph with $|V| = n$ and*

$$|E| > \left(\frac{1}{2} - \frac{1}{2(2h-2)^r} + \epsilon\right)n^2,$$

and let $\chi : E \to [r]$ be an r-coloring of the edges of G. Then G contains a monochromatic h-part spiral of width at least $c(r, h, \epsilon)n$.

Proof Let λ be the unique solution of the quadratic equation

$$\frac{\sqrt{\epsilon/h^r}(\epsilon/h^r - r\lambda)^2}{r^2 2^{2r+2}} = \lambda$$

satisfying $\lambda < \epsilon/h^r$. We shall prove that G contains an h-part spiral of width at least λn.

Suppose to the contrary that G does not contain an h-part spiral of width at least λn. For $W \subset V$, $x \in V$ and $i \in [r]$, define $U_i^W(x)$ and $D_i^W(x)$ as in the previous proof. For $x \in V$ and $i \in [r]$, let $l_i(x)$ be the largest l such that G contains an l-part rooted spiral with root x and width λn in color i. Note that if there exists $x \in V$ and $i \in [r]$ with $l_i(x) \geq h$, we are done as an h-part rooted spiral of width λn trivially contains an h-part spiral of width λn. Hence, we can suppose that $0 \leq l_i(x) < h$.

For $\bar{t} = (t_1, ..., t_r) \in \{0, ..., h-1\}^r$, define

$$V_{\bar{t}} = \{x \in V : l_i(x) = t_i, i \in [r]\}.$$

The sets $\{V_{\bar{t}}\}_{\bar{t}\in\{0,...,h-1\}^r}$ partition V into h^r parts. Let $n_{\bar{t}} = |V_{\bar{t}}|$. Also, let

$$I(\bar{t}) = \{i \in [r] : t_i \notin \{0, h-1\}\},$$

and $\epsilon' = \epsilon/h^r$. We show that $G[V_{\bar{t}}]$ contains at most

$$\left(\frac{1}{2} - \frac{1}{2^{|I(\bar{t})|+1}}\right) n_{\bar{t}}^2 + \epsilon' n^2$$

edges.

Suppose that $G[V_{\bar{t}}]$ has more than

$$\left(\frac{1}{2} - \frac{1}{2^{|I(\bar{t})|+1}}\right) n_{\bar{t}}^2 + \epsilon' n^2$$

edges. First of all, this forces n_t to be at least $\sqrt{\epsilon'}n$, as $G[V_{\bar{t}}]$ has more than $\epsilon' n^2$ edges. If $t_i = 0$ for some i, then the number of edges of color i in $G[V_{\bar{t}}]$ is at most λn^2. Otherwise, there exists $x \in G[V_{\bar{t}}]$ with $|U_i(x)| > \lambda n$, and $x \cup U_i(x)$ spans a 1-part rooted spiral of width λn, contradicting $t_i = 0$.

Similarly, if $t_i = h-1$ for some i, then the number of edges of color i in $G[V_{\bar{t}}]$ is also at most λn^2, otherwise there exist $x \in G[V_{\bar{t}}]$ with $|D_i(x)| > \lambda n$. Taking the union of $D_i(x)$ and an $(h-1)$-part rooted spiral with root x and width λn, we get an h-part spiral of width λn.

Hence, the number of edges in $G[V_{\bar{t}}]$ with color in $I(\bar{t})$ is at least

$$\left(\frac{1}{2} - \frac{1}{2^{|I(\bar{t})|+1}}\right) n_{\bar{t}}^2 + (\epsilon' - r\lambda)n^2 > \left(\frac{1}{2} - \frac{1}{2^{|I(\bar{t})|+1}} + \epsilon' - r\lambda\right) n_{\bar{t}}^2.$$

Applying Lemma 5 with $q = |I(\bar{t})|$, we get that there exists a monochromatic k-diamond in $G[V_{\bar{t}}]$ with color in $p \in I(\bar{t})$, where

$$k > \frac{(\epsilon' - r\lambda)^2 n_t}{q^2 2^{2q+2}} > \frac{(\epsilon' - r\lambda)^2 \sqrt{\epsilon'} n}{r^2 2^{2r+2}} = \lambda n.$$

Let $a, a', b_1, ..., b_k \in V_{\bar{t}}$ be the vertices of this k-diamond, where the vertex a is the bottom and a' is the top of the diamond. Let S be a t_p-part rooted spiral with root a' and width λn, then taking the union of this k-diamond and S, we get a p colored $t_p + 1$-part rooted spiral with root a and width λn, contradicting $l_p(a) = t_p$.

So far, we showed that the graph induced on $V_{\bar{t}}$ can contain at most

$$\left(\frac{1}{2} - \frac{1}{2^{|I(\bar{t})|+1}}\right) n_{\bar{t}}^2 + \epsilon' n^2$$

edges. Hence, the complement of G contains at least

$$-\epsilon n^2 + \sum_{\bar{t}\in\{0,...,h-1\}^r} \frac{n_{\bar{t}}^2}{2^{|I(\bar{t})|+1}}$$

edges. Using the Cauchy-Schwarz inequality, we have

$$\sum_{\bar{t}\in\{0,\dots,h-1\}^r} \frac{n_{\bar{t}}^2}{2^{|I(\bar{t})|+1}} \geq \left(\sum_{\bar{t}\in\{0,\dots,h-1\}^r} n_{\bar{t}}\right)^2 \left(\sum_{\bar{t}\in\{0,\dots,h-1\}^r} 2^{|I(\bar{t})|+1}\right)^{-1} =$$

$$= \frac{n^2}{2(2h-2)^r}.$$

Hence, G contains less than

$$\left(\frac{1}{2} - \frac{1}{2(2h-2)^r} + \epsilon\right) n^2$$

edges, which is a contradiction. □

Solving the quadratic equation in the beginning of the proof yields

$$c(r,h,\epsilon) = \Omega\left(\frac{\epsilon^{5/2}}{r^2 2^{2r} h^{5r/2}}\right).$$

However, in the case $h = 2$, we can get a better bound. In this special case, while we repeat the previous proof, we do not need to use Lemma 5 at any point. We can deduce the following result.

Proposition 7 *Let r be a positive integer and $\epsilon > 0$. Let $G = (V, E)$ be a directed graph with $|V| = n$ and $|E| > (1/2 - 1/2^{r+1} + \epsilon)n^2$. Any r coloring of the edges of G contains a monochromatic 2-part spiral of width at least $\epsilon n/r2^{r+1}$.*

Proof We shall proceed similarly as in the previous proof and in the proof of Lemma 5. Let $\lambda = \epsilon/r2^{r+1}$. For any $H \subset [r]$ let

$$V_H = \{x \in V : |U_i(x)| \geq \lambda n \Leftrightarrow i \in H\}.$$

The set system $\{V_H\}_{H\subset[r]}$ partitions V into 2^r parts. Thus, the number of edges connecting two different parts is at most $(1/2 - 1/2^{r+1})n^2$. Hence, there exists $H_0 \subset [r]$ such that $e(G[V_{H_0}]) > \epsilon n^2/2^r$. Let f be the number of edges of $G[V_{H_0}]$ whose color is not in H_0. Then

$$f < (r - |H_0|)|V_{H_0}|\lambda n < r\lambda n^2.$$

Hence, the number of edges of $G[V_{H_0}]$ whose color is in H_0 is at least

$$\left(\frac{\epsilon}{2^r} - r\lambda\right) n^2 = r\lambda n^2.$$

But then, there exists $i \in H_0$ and $v \in V_{H_0}$ such that

$$|D_i(v)| \geq |D_i^{V_{H_0}}(v)| > \lambda n.$$

Setting $B_1 = D_i(v)$, $a_1 = v$ and $B_2 = U_i(x)$, the set $\{a_1\} \cup B_1 \cup B_2$ spans a 2-spiral of width λn of color i in G. □

After these preparations, the proof of Theorem 4 is immediate.

Proof of Theorem 4, part (ii). Let $<_1, \dots, <_r$ be partial orders on the n element set P. Define the directed graph $G = (P, E)$ and the coloring $\chi : E \to [r]$ as follows: if $x, y \in P$ are comparable in at least one of the partial orders $<_1, \dots, <_r$, then choose one of them, say $<_i$. Without loss of generality, $x <_i y$. Let $\overrightarrow{xy} \in E$ and $\chi(\overrightarrow{xy}) = i$. By Theorem 6, there exists a color p such that the directed graph G contains a p-colored h-part spiral of width $c(r, h, \epsilon)n$, let its vertex set be $\{a_1, ..., a_{h-1}\} \cup B_1 \cup ... \cup B_h$. But then $B_1 <_p ... <_p B_h$ and $|B_1| = ... = |B_h| > c(r, h, \epsilon)n$. Hence, (*) is proved.

In case $h = 2$, we repeat the proof of (*), but we use Proposition 7 instead of Theorem 6. This yields

$$f^C_{r,2}\left(n, \left(\frac{1}{2} - \frac{1}{2^{r+1}} + \epsilon\right)n^2\right) > \frac{\epsilon n}{r2^{r+1}}.$$

□

3 Balanced Complete h-Partite Subgraph in the Incomparability Graph

In this section, we prove a result about large balanced complete h-partite subgraphs in the incomparability graph of $(P, <)$. Note that if P is the disjoint union of $h-1$ chains, each of size $n/(h-1)$, then there is no K_h in the incomparability graph of $(P, <)$. Hence, the incomparability graph of $(P, <)$ needs to have density at least $1 - 1/(h-1)$ if we hope to find a large balanced complete h-partite graph in it. Our next result shows that if we are slightly above this density, we do find a large balanced complete h-partite graph in the incomparability graph.

Theorem 8 *Let $h \geq 2$ be a positive integer and let $s = \lceil \log_2 h \rceil$.*

(i) *For $m < (1/2 - 1/2(h-1))n^2$, we have $f^I_{1,h}(n, m) = 0$.*
(ii) *For every $\epsilon > 0$, there exists $c(h, \epsilon) > 0$ such that*

$$f^I_{1,h}\left(n, \left(\frac{1}{2} - \frac{1}{18(h-1)} + \epsilon\right)n^2\right) > \frac{c(h, \epsilon)n}{(\log n)^s}.$$

In the proof, we shall use the following easy corollary of Theorem 3 and Theorem 4.

Proposition 9 *Let h, n be positive integers. Let s be the smallest integer such that $h \leq 2^s$. There exist $c(h) > 0$ with the following property. Let $<$ be a partial order on the n element set P. If n is sufficiently large, then either*

(i) *there exist $A_1, ..., A_h \subset P$ disjoint sets such that*

$$|A_1| = ... = |A_s| > \frac{c(h)n}{(\log n)^s},$$

and $A_i \perp A_j$ for $1 \leq i < j \leq h$;
(ii) *or there exist $B_1, B_2, B_3 \subset P$ disjoint sets such that*

$$|B_1| = |B_2| = |B_3| > \frac{c(h)n}{(\log n)^s},$$

and $B_1 < B_2 < B_3$.

Proof Let $c = c(1, 3, 1/16)$, where $c(r, h, \epsilon)$ is the constant defined in Theorem 4. If the comparability graph of a poset $(Q, <)$, with $|Q| = m$ has more than $7m^2/16$ edges, then by Theorem 4 there exists $B_1, B_2, B_3 \subset Q$ satisfying $|B_1| = |B_2| = |B_3| > cm$ and $B_1 < B_2 < B_3$. Hence, we can suppose that the comparability graph of P does not contain a subgraph of size at least $n/(\log n)^s$ with edge density larger than $7/8$, otherwise (ii) holds if $c(h) < c$. But then, applying Theorem 3, every subgraph of size $n' > n/(\log n)^s$ contains two sets, A and A' such that $|A| = |A'| > c_0 n'/(\log n')$ with a suitable constant $c_0 > 0$, and $A \perp A'$.

For $k = 0, ..., s$ and $i = 1, ..., 2^k$, we shall define the sets $X_{k,1}, ..., X_{k,2^k} \subset P$ with the following properties: $X_{0,1} = P$; $|X_{k,1}| = ... = |X_{k,2^k}| > c_0^k n/(\log n)^k$, and $X_{k,i} \perp X_{k,j}$ for $1 \le i < j \le 2^k$. Suppose that $X_{k,1}, ..., X_{k,2^k}$ are already defined satisfying those properties. We define $X_{k+1,1}, ..., X_{k+1,2^{k+1}}$ as follows. As $|X_{k,i}| > c_0^k n/(\log n)^k > n/(\log n)^s$ if n is sufficiently large, there exist $X_{k+1,2i-1}, X_{k+1,2i} \subset X_{k,i}$ such that

$$|X_{k+1,2i-1}| = |X_{k+1,2i}| > \frac{c_0|X_{k,i}|}{\log |X_{k,i}|} > \frac{c_0^{k+1} n}{(\log n)^{k+1}},$$

and $X_{k+1,2i-1} \perp X_{k+1,2i}$. Then $X_{k+1,1}, ..., X_{k+1,2^{k+1}}$ also satisfy the properties. Set $A_i = X_{s,i}$ for $i = 1, ..., h$. Then (i) holds. □

Proof of Theorem 8. We shall prove part (ii) of the theorem. Let $(P, <)$ be a partially ordered set on n elements such that

$$e(G(P, <)) < \left(\frac{1}{18(h-1)} - \epsilon\right) n^2.$$

Let $k = \lceil 2\epsilon^{-1} \rceil$. Let $<'$ be any linear extension of $<$, and let $x_1 <' ... <' x_n$ be the enumeration of the elements of P by $<'$. Partition P into k equal $<'$ intervals $P_1, ..., P_k$. Namely, for $i = 1, ..., k$, let $P_i = \{x_{(i-1)n/k+1}, ..., x_{in/k}\}$.

Let $c_0 = c(h)$ be the constant defined in Proposition 9, and set $c(h, \epsilon) = c_0\epsilon/k$. Also, let $z = c(h, \epsilon)n/(\log n)^s$. Suppose that P does not contain $A_1, ..., A_h$ disjoint sets such that

$$|A_1| = ... = |A_h| > z,$$

and $A_i \perp A_j$ for $1 \le i < j \le h$. By Proposition 9, every subset of P of size at least $\epsilon n/k$ contains three sets B_1, B_2, B_3 of size z such that $B_1 < B_2 < B_3$. Let

$$m = \frac{(1-\epsilon)n}{3kz}. \tag{1}$$

Picking greedily, for $i = 1, ..., k$, we can find $3m$ disjoint sets

$$\{B_{i,j,t}\}_{j=1,...,m;t=1,2,3}$$

in P_i, such that $|B_{i,j,t}| = z$ and $B_{i,j,1} < B_{i,j,2} < B_{i,j,3}$.

Define a new graph $H = ([k] \times [m], E)$ as follows: join (i, j) and (i', j') by an edge if $i = i'$ or there is an edge in $G(P, <)$ between $B_{i,j,2}$ and $B_{i',j',2}$.

Suppose H has d edges. If (i, j) and (i', j') are joined by an edge, where $i < i'$, then $G(P, <)$ contains every edge between $B_{i,j,1}$ and $B_{i',j',3}$. This is true as there exists $x \in B_{i,j,2}$ and $y \in B_{i',j',2}$ with $x < y$, so for any $x' \in B_{i,j,1}$ and $y' \in B_{i',j',3}$, we have $x' < x < y < y'$. The number of edges of H of the form $\{(i, j), (i, j')\}$ is $k\binom{m}{2}$. Hence, the number of edges $\{(i, j), (i', j')\}$ of H with $i \ne i'$ correspond to at least

$$\left(d - k\binom{m}{2}\right) z^2$$

edges in $G(P, <)$. But $G(P, <)$ has at most $(1/18(h-1) - \epsilon)n^2$ edges, so

$$dz^2 - kz^2\binom{m}{2} < \left(\frac{1}{18(h-1)} - \epsilon\right) n^2.$$

Here, $kz^2\binom{m}{2} < n^2/18k < \epsilon n^2/2$. Hence, we have

$$dz^2 < \left(\frac{1}{18(h-1)} - \frac{\epsilon}{2}\right)n^2.$$

Thus, using Eq. 1, we get

$$d < 9k^2m^2\left(\frac{1}{18h} - \frac{\epsilon}{2}\right)n^2(1-\epsilon)^{-2} < \left(\frac{1}{2(h-1)} - \epsilon\right)(km)^2.$$

Applying Turán's theorem [13] to H there is a complete graph on h vertices in the complement of H. Let the vertices of this K_h be $(i_1, j_1), ..., (i_h, j_h)$. For $l = 1, ..., h$, let $A_l = B_{i_l, j_l, 2}$. Then $|A_1| = ... = |A_h| = c(h, \epsilon)n/(\log n)^s$, and $A_l \perp A_{l'}$ for $1 \le l < l' \le h$, which is a contradiction. □

Slightly modifying the proof above, one can show that we can replace $1/2 - 1/18(h-1)$ in (ii) with $1/2 - 1/8(h-1)$. However, we conjecture that $1/2 - 1/2(h-1)$ is the sharp threshold.

Conjecture 10 *Let h be a positive integer, $\epsilon > 0$. There exists $c(h, \epsilon) > 0$ such that*

$$f\left(n, \left(\frac{1}{2} - \frac{1}{2(h-1)} + \epsilon\right)n^2\right) > \frac{c(h, \epsilon)n}{(\log n)^s}$$

holds.

4 Balanced Complete Bipartite Graph in the 2-Incomparability Graph

In this section, we investigate the size of the largest balanced complete bipartite graph in the 2-incomparability graph of $(P, <_1, <_2)$.

Fix a positive integer h. By our previous results, if the edge density of the incomparability graph of $(P, <)$ exceeds some threshold strictly less than 1, we have a complete balanced h-partite graph of size $n^{1-o(1)}$ in the incomparability graph. However, as we shall see, this is no longer true for the 2-incomparability graph, or in general, for the r-incomparability graph, where $r \ge 2$.

However, we show that if the incomparability graph of $(P, <_1, <_2)$ has edge density $(1 - \epsilon + o(1))$, there is a complete balanced bipartite graph of size $n^{\beta(\epsilon)}$, where $\beta(\epsilon) \to 1$ as $\epsilon \to 0$. This is still much larger than the size of the largest balanced complete bipartite graph of a random graph, whose edges are chosen with probability $1 - \epsilon$. With high probability, such a graph has edge density $(1 - \epsilon + o(1))$, and its largest balanced bipartite graph has size $O(\epsilon^{-1} \log n)$.

We prove the following result.

Theorem 11 (i) *For every $0 < \epsilon < 1$ and positive integer $k \ge 2$, we have*

$$f^I_{2,2}\left(n, \left(\frac{1}{2} - \frac{1}{2k} - \epsilon\right)n^2\right) < 2\epsilon^{-1}kn^{1-1/(k-1)}\log n.$$

(ii) *For every $\delta > 0$, if n is a sufficiently large positive integer, there exists $\gamma(\delta) > 0$ such that*

$$f^I_{2,2}\left(n, \left(\frac{1}{2} - \gamma(\delta)\right)n^2\right) > n^{1-\delta}.$$

The proof of part (i) is a probabilistic construction. We shall only briefly sketch the idea, the reader can find more about random graphs in [1].

Proof of (i). Our task is to construct partial orders $<_1, <_2$ on an n element set P, such that the complement of $G(P, <_1, <_2)$ does not contain a large complete bipartite graph.

For any positive integer N, let $G_N = (X_N, Y_N, E_N)$ be a bipartite graph with the following properties:

(1) $|X_N| = |Y_N| = N$;
(2) for every $x \in X_N \cup Y_N$ we have $deg(x) < \epsilon N^{1/(k-1)}$;
(3) the complement of G does not contain a $K_{t,t}$ with

$$t > 2\epsilon^{-1} N^{1-1/(k-1)} \log n;$$

(4) G_N has a complete matching M_N.

If the edges of G are chosen independently with probability $\epsilon N^{1/(k-1)-1}/2$, then with positive probability G satisfies conditions (2),(3) and (4).

Let $A_1, ..., A_k$ be disjoint sets of size n/k, and let $P = A_1 \cup ... \cup A_k$. Let $<_1$ be any partial order such that $A_1, ..., A_k$ are $<_1$-chains, and $A_i \perp_1 A_j$ for $1 \le i < j \le k$.

Now define $<_2$ as follows: for $i = 1, ..., k$, let $f_i : A_i \to X_{n/k}$ and $g_i : A_i \to Y_{n/k}$ be arbitrary bijections. Define the relation $<_2^*$ such that for any $a \in A_i$ and $b \in A_{i+1}$, where $1 \le i \le k-1$, we have $a <_2^* b$ if $f_i(a)g_{i+1}(b) \in E_{n/k}$. Let $<_2$ be the partial order induced by the relation $<_2^*$.

First of all, we shall bound the number of edges of $G(P, <_1, <_2)$ from above. Note that

$$e(G(P, <_1)) = k\binom{n/k}{2} < \frac{n^2}{2k}.$$

Also, $e(G(P, <_2)) < \epsilon n^2$. This is true as for every $1 \le i < j \le k$ and $x \in A_i$, $y \in A_j$, we have $x < y$ iff there exists a sequence $x_0, ..., x_{j-i}$ such that $x_0 = a$, $x_{j-i} = y$, $x_l \in X_{i+l}$ for $l = 1, ..., j-i-1$, and $f_{i+l'}(x_{l'})g_{i+l'+1}(x_{l'+1}) \in E(G_{n/k})$ for $l' = 0, ..., j-i-1$. As every vertex in $G_{n/k}$ has degree less than $\epsilon N^{1/(k-1)}$, the number of such sequences with given x_0 is at most

$$\epsilon^{|i-j|}\left(\frac{n}{k}\right)^{|i-j|/(k-1)} < \frac{\epsilon n}{k}.$$

Hence, for every $x \in P$ there are at most ϵn elements $y \in P$ such that $x <_2 y$. Thus,

$$e(G(P, <_2)) < \epsilon n^2.$$

We deduce that $e(G(P, <_1, <_2)) < (1/2k + \epsilon)n^2$.

Also, let $X, Y \subset P$ be disjoint sets such that $X \perp Y$ and $|X| = |Y|$. Then, there exist positive integers t and u such that $1 \le t, u \le k$, $|X \cap A_t| \ge |X|/k$ and $|Y \cap A_u| \ge |Y|/k$. We cannot have $t = u$, otherwise, there exist $x \in X \cap A_t$ and $y \in Y \cap A_t$ with $x <_1 y$ or $y <_1 x$, contradicting $X \perp Y$. Hence, $t \ne u$. Without loss of generality, suppose that $t < u$.

Let H be the bipartite subgraph of $G(P, <_2)$ induced on $A_t \cup A_u$. We show that H contains a subgraph isomorphic to $G_{n/k}$. Let $x \in A_t$ arbitrary, and let $a_0(x), ..., a_{u-t-1}(x)$ be the unique sequence such that $a_0(x) = x$, $a_l(x) \in A_{t+l}$ for $l = 1, ..., u-t-1$, and $f_{t+l'}(a_{l'}(x))g_{t+l'+1}(a_{l'+1}(x)) \in M_{n/k}$. As $M_{n/k}$ is a complete matching, every $a_l : A_t \to A_{t+l}$ is a bijection. Also, the subgraph of $G(P, <_2)$ induced on $A_{u-1} \cup A_u$ is isomorphic to $G_{n/k}$. If $x' \in A_{u-1}$ and $x'' \in A_u$ with $x' <_2 x''$, then $a_{u-1}^{-1}(x') <_2 x''$. Hence, the subgraph of $G(P, <_2)$ induced on $A_t \cup A_u$ contains a subgraph isomorphic to $G_{n/k}$.

Thus, the complement of H does not contain $K_{t,t}$ with

$$t > 2\epsilon^{-1}N^{1-1/(k-1)}\log n,$$

so

$$|X| = |Y| < 2kN^{1-1/(k-1)}\log n < 2\epsilon^{-1}kn^{1-1/(k-1)}\log n.$$

□

Our next aim is to prepare the proof of part (ii) of Theorem 11. It turns out, our proof would be simpler if $<_1$ and $<_2$ had a common linear extension, which is not the case in general. However, the next lemma shows that we can find a constant number of large subsets in our poset such that between these subsets $<_1$ and $<_2$ behave as if they had a common linear extension.

Lemma 12 *Let $r, h \geq 2$ be positive integers. There exists $c(r,h) > 0$ with the following property. Let $<_1^0, \dots, <_r^0$ be partial orders on the n element set P, and for $s = 1, \dots, r$, let $<_s^1$ be the dual of $<_s^0$. There exist $A_1, \dots, A_h \subset P$ pairwise disjoint sets and $\alpha_1, \dots, \alpha_r \in \{0, 1\}$ such that*

(i) *$|A_1| = \dots = |A_h| > c(r,h)n$;*
(ii) *if $x \in A_i$ and $y \in A_j$ with $1 \leq i < j \leq h$, and x and y are comparable in $<_s$, then $x <_s^{\alpha_s} y$.*

Proof For $s = 1, \dots, r$, let $<_s'$ be a linear extension of $<_s^0$. It is enough to prove our lemma for $<_1', \dots, <_r'$ instead of $<_1^0, \dots, <_r^0$. We shall deduce Lemma 12 from the following claim. □

Claim 13 *Let p and r be positive integers. There exists $c'(p,r) > 0$ with the following property. Let $<_1, \dots, <_r$ be total orders on the n element set P. There exist $B_1, \dots, B_p \subset P$ pairwise disjoint subsets such that*

(i) *$|B_1| = \dots = |B_p| > c'(p,r)n$;*
(ii) *for $s = 1, \dots, r$ and $1 \leq i < j \leq r$, we have either $B_i <_s B_j$ or $B_j <_s B_i$.*

Proof We shall proceed by induction on r. In case $r = 1$, the statement is trivial with $c'(p,1) = 1/p$. Let $r \geq 2$ and suppose the statement holds for $r-1$ instead of r. Let $C_1, \dots, C_p \subset P$ be disjoint sets such that

$$|C_1| = \dots = |C_p| > c'(p, r-1)n,$$

and for every $1 \leq i < j \leq p$ and $s = 1, \dots, r-1$, we have $C_i <_s C_j$ or $C_j <_s C_i$. Let $P' = \bigcup_{i=1}^{p} C_i$, and for $x \in P'$, let $\tau(x)$ be the position of x in the order $<_r$ in P'. For $i = 1, \dots, p$, let

$$D_j = \left\{ x \in P : \frac{(j-1)|P'|}{p} < \tau(x) \leq \frac{j|P'|}{p} \right\}.$$

We have $D_i <_r D_j$ for any $1 \leq i < j \leq p$. Our $B_1, \dots, B_p$ are going to be suitable subsets of $C_1, \dots, C_p$ and $D_1, \dots, D_p$.

Let S, T be two disjoint copies of $[p]$, and define the bipartite graph $G = (S, T, E)$ as follows: for $i \in S$ and $j \in T$, let $ij \in E$ if

$$|C_i \cap D_j| > \frac{|P'|}{p^2(p+1)}.$$

We show that G has a complete matching. By Hall's theorem [7], we only need to check if Hall's condition holds. Let $X \subset [p]$ be arbitrary and let $\Gamma(X)$ denote the set of neighbours of X in G. Let $U = \bigcup_{i \in X} D_i$, then

$$|U| = \frac{|X||P'|}{p}.$$

Also, the elements of $\Gamma(X)$ cover at most $|\Gamma(X)||P'|/p$ elements in U, while the elements not in $\Gamma(X)$ cover at most $p|X|(|P'|/p^2(p+1)) = |P'||X|/p(p+1)$ elements in U. Hence,

$$\frac{|X||P'|}{p} \leq \frac{|\Gamma(X)||P'|}{p} + \frac{|P'||X|}{p(p+1)}.$$

Thus, we have

$$|X|\left(1 - \frac{1}{(p+1)}\right) \leq |\Gamma(X)|.$$

But $|X|$ and $|\Gamma(X)|$ are integers not larger than p. Hence, $|X| \leq |\Gamma(X)|$ also holds. So, Hall's condition is satisfied and there exists a a complete matching in G. Let the edge set of such a matching be $\{ix_i : i \in S\}$. Setting $B_i = C_i \cap D_{x_i}$ and $c'(p, r) = c'(p, r-1)/p^2(p+1)$, we have both (i) and (ii) satisfied. □

Let $p = (h-1)^{2^{r-1}} + 1$ and let $B_1, ..., B_p \subset P$ be disjoint sets such that $|B_1| = ... = |B_p| > c'(p, r)n$, and for $1 \leq i < j \leq p$ and $1 \leq s \leq r$, we have either $B_i <_s B_j$ or $B_j <_s B_i$. Define the partial orders $\{\prec_{\overline{v}}\}_{\overline{v} \in \{0,1\}^{r-1}}$ on $[p]$ as follows: for $i, j \in [p]$ and $\overline{v} \in \{0, 1\}^{r-1}$, let $i \prec_{\overline{v}} j$ if $B_i <_r B_j$, and for $s = 1, ..., r-1$, we have $B_i <_s B_j$ in case $v_s = 0$, and $B_j <_s B_i$ in case $v_s = 1$. Then any two different elements of $[p]$ are comparable in at least one of the partial orders $\{\prec_{\overline{v}}\}_{\overline{v} \in \{0,1\}^{r-1}}$. Hence, by repeated applications of Dilworth's theorem, there exist $\overline{w} \in \{0, 1\}^{r-1}$ and $C \subset [p]$ such that

$$|C| \geq \lceil p^{1/2^{r-1}} \rceil = h,$$

and C is a $\prec_{\overline{w}}$ chain. Let $i_1 \prec_{\overline{w}} ... \prec_{\overline{w}} i_h$ be h elements of this chain, and for $j = 1, ..., h$, let $A_j = B_{i_j}$. Also, for $s = 1, ..., r$, let $\alpha_i = w_i$. Finally, let $c(r, h) = c'(r, p)$. Then the conditions of the theorem are satisfied. □

Before we start the proof of part (ii) of Theorem 11, we still need the following two lemmas.

Lemma 14 *Let $A_0, .., A_k$ be pairwise disjoint sets of size m, and let*

$$P = \bigcup_{i=1}^{k} A_i.$$

Let $<$ be a partial order on P such that whenever $x < y$ for some $x \in A_i$ and $y \in A_j$, then $i < j$. Suppose that $G(P, <)[A_0, A_k]$ has less than $m^2/4$ edges. There exist $0 \leq l \leq k-1$ and $X \subset A_l$, $Y \subset A_{l+1}$ such that $|X|, |Y| > m^{1-1/k}$, and $X \perp Y$.

Proof For any $X \subset P$ and $i = 1, ..., k$, let

$$U_i(X) = \{y \in A_i : \exists x \in X, x < y\}.$$

Let $B = \{x \in A_0 : |U_k(\{x\})| < m/2\}$. Then $|B| > m/2$, otherwise $G(P, <)[A_0, A_k]$ has more than $m^2/4$ edges. Suppose that there is no $l \in \{0, ..., k-1\}$ and subsets $X \subset A_l$, $Y \subset A_{l+1}$ such that $|X| = |Y| > m^{1-1/k}$, and $X \perp Y$.

We show that we can find a decreasing sequence of sets $B \supseteq B_1 \supseteq .... \supseteq B_k$ with the following properties: $|B_i| = 2^{k-i}m^{1-i/k}$, and $|U_i(B_i)| > m/2$. Note that B_k is a one element set. Hence, writing x for that one element, we have

$$|U_k(\{x\})| > \frac{m}{2},$$

contradicting $x \in B$, finishing our proof.

We shall define our sets $B_1, ..., B_k$ recursively. Let B_1 be any subset of B of size $2^{k-1}m^{1-1/k}$. If $|U_1(B_1)| \leq m/2$, then choosing $X = B_1$ and $Y = A_1 \setminus U_1(B_1)$, we have $X \perp Y$ and $|X|, |Y| > m^{1-1/k}$. Hence, we have $|U_1(B_1)| \leq m/2$. □

Suppose that B_i is already defined satisfying $|B_i| = 2^{k-i}m^{1-i/k}$ and $|U_i(B_i)| > m/2$.

Claim 15 *For any positive integer $t \leq |B_i|$, we can choose a set $C \subset B_i$ such that $|C| = t$ and $|U_i(C)| \geq |U_i(B)|t/|B_i|$.*

Proof Let $x_1, ..., x_p$ be the elements of B_i. Let $S_1, ..., S_p$ be a partition of $U(B_i)$ such that $S_j \subset U_i(\{x_j\})$ for $j = 1, ..., p$. Without the loss of generality, $|S_1| \geq ... \geq |S_p|$. Set $C = \{x_1, ..., x_t\}$, then

$$|U_i(C)| \geq |S_1| + ... + |S_t| \geq \frac{|U_i(C)|t}{|B_i|}.$$

□

Setting $t = 2^{k-i-1}m^{1-(i+1)/k}$, we get a set C such that

$$|C| = 2^{k-i-1}m^{1-(i+1)/k},$$

and $|U_i(C)| \geq m^{1-1/k}$. If $|U_{i+1}(C)| \leq m/2$, then set $X = C$ and $Y = A_{i+1} \setminus U_{i+1}(C)$. Then, we have $X \perp Y$ and $|X|, |Y| > m^{1-1/k}$, which is a contradiction. Hence, $|U_{i+1}(C)| > m/2$, and $B_{i+1} = C$ satisfies our conditions. □

We also need the following easy corollary of Theorem 2, which we shall state without proof.

Proposition 16 *Let $<_1, <_2$ be partial orders on the n element set P. At least one of the following holds:*

(i) *there exist $A_1, A_2 \subset P$ such that $|A_1| = |A_2| > n^{1-o(1)}$, and $A_1 \perp A_2$;*
(ii) *there exist $B_1, B_2, B_3 \subset P$ such that $|B_1| = |B_2| = |B_3| > n^{1-o(1)}$, and $B_1 <_1 B_2 <_1 B_3$ or $B_1 <_2 B_2 <_2 B_3$*

Proof of Theorem 11, (ii). We have to prove that there exists a constant $\gamma(\delta)$ such that if P is a set with n elements, and $<_1, <_2$ are partial orders on P satisfying $e(G(P, <_1, <_2)) < \gamma(\delta)n^2$, then P contains two disjoint subsets A, B of size at least $n^{1-\gamma}$ such that $A \perp B$. For simplicity, let $G_1 = G(P, <_1)$ and $G_2 = G(P, <_2)$.

Suppose that P does not contain two disjoint subsets A, B of size at least $n^{1-\delta}$ such that $A \perp B$. Let $k = \lceil 2/\delta \rceil$ and $h = 128k$, and let $c_1 = c(2, h)$, where $c(r, h)$ is the constant defined in Lemma 12. Then there exist $L_1, ..., L_h \subset P$ pairwise disjoint sets with the following properties: $|L_1| = ... = |L_h| = c_1 n$; replacing $<_2$ with its dual if necessary, if $x \in L_i$ and $y \in L_j$ for some $1 \leq i < j \leq h$, and x, y are comparable in $<_1$ or $<_2$, then $x <_1 y$ or $x <_2 y$, respectively.

Let $m = n^{1-\delta/2}$. By Proposition 16, if n is sufficiently large, every subset of P of size at least $c_1 n/2$ contains three disjoint subsets B_1, B_2, B_3 of size m such that $B_1 <_1 B_2 <_1 B_3$ or $B_1 <_2 B_2 <_2 B_3$. Hence, we can cover at least half of L_i with disjoint triples of subsets such that each set has size m and each triple spans a balanced complete 3-partite graph in G_1 or in G_2.

More precisely, let $s = c_1 n/2m$. Then, for $i = 1, ..., h$, there is a system of disjoint sets $\{B_{i,j,l}\}_{j=1,...,s; l=1,2,3}$ such that $B_{i,j,l} \subset L_i$, $|B_{i,j,l}| = m$, and $B_{i,j,1} <_1 B_{i,j,2} <_1 B_{i,j,3}$ or $B_{i,j,1} <_2 B_{i,j,2} <_2 B_{i,j,3}$. Call the pair $(i, j) \in [h] \times [s]$ type 1, if $B_{i,j,1} <_1 B_{i,j,2} <_1 B_{i,j,3}$, and call it type 2 otherwise. Without the loss of generality, we can suppose that there are at least $sh/2$ type 1 pairs in $[h] \times [s]$, and let S be the set of such pairs.

Let $H = (S, E)$ be the complete graph on S, and let w be a weight function defined on E as follows. Let $(i, j), (i', j') \in S$, and let f be the edge joining (i, j) and (i', j'). If $i = i'$, or there exist $x \in B_{i,j,2}$ and $y \in B_{i',j',2}$ such that $x <_1 y$ or $y <_1 x$, then let $w(f) = 1$; otherwise, let

$$w(f) = \frac{e(G_2[B_{i,j,2}, B_{i',j',2}])}{m^2}.$$

Note that if there exist $x \in B_{i,j,2}$ and $y \in B_{i',j',2}$ such that $x <_1 y$, then $B_{i,j,1} <_1 B_{i',j',3}$. Hence, there are at least m^2 edges between $B_{i,j,1} \cup B_{i,j,2} \cup B_{i,j,3}$ and $B_{i',j',1} \cup B_{i',j',2} \cup B_{i',j',3}$ in G_1. Thus, if $i \neq i'$, there are at least $w(f)m^2$ edges between $B_{i,j,1} \cup B_{i,j,2} \cup B_{i,j,3}$ and $B_{i',j',1} \cup B_{i',j',2} \cup B_{i',j',3}$. Also, the number of edges $\{(i, j), (i', j')\}$ in H, where $i = i'$, is at most

$$h\binom{s}{2} < hs^2.$$

Let $w(E) = \sum_{f \in E} w(f)$. Then the number of edges of $G(P, <_1, <_2)$ is at least

$$(w(E) - hs^2)m^2. \quad (2)$$

Let t be the number of edges $f \in E$ such that $w(f) \leq 1/4$. We show that

$$t \leq |S|^2 \left(\frac{1}{2} - \frac{1}{2k}\right).$$

Suppose that $t > |S|^2(1/2 - 1/2k)$. Consider the graph H' with vertex set S, and edge set $E' = \{f \in E : w(f) \leq 1/4\}$. By Turán's theorem [13], there exists $T \subset S$ of size $k+1$ such that $H'[T]$ is a complete graph. Let $(i_0, j_0), ..., (i_k, j_k)$ be the elements of T and suppose that $i_0 < ... < i_k$. First, note that $A_{i_l,j_l,2} \perp_1 A_{i_{l'},j_{l'},2}$ for all $0 \leq l < l' \leq k$, as the weight of the edge $\{(i_l, j_l), (i_{l'}, j_{l'})\}$ is less than 1.

Set $A_l = B_{i_l,j_l,2}$ for $l = 0, ..., k$. Then $e(G_2[A_0, A_k]) < m^2/4$. Hence, by Lemma 14, there exist $0 \leq l \leq k-1$ and $X \in A_l$, $Y \in A_{l+1}$ such that $|X| = |Y| = m^{1-1/k}$, and $X \perp_2 Y$. But then $X \perp Y$, and

$$m^{1-1/k} > n^{(1-\delta/2)^2} > n^{1-\delta},$$

contradiction. Thus, we must have

$$t \leq |S|^2 \left(\frac{1}{2} - \frac{1}{2k} \right).$$

Then

$$w(E) = \sum_{f \in E} w(f) > \frac{|E| - t}{4} >$$

$$> \frac{1}{4} \left(\binom{|S|}{2} - |S|^2 \left(\frac{1}{2} - \frac{1}{2k} \right) \right) = \frac{|S|^2}{8k} - \frac{|S|}{8} > \frac{|S|^2}{16k},$$

where the last inequality holds if n is sufficiently large. Plugging this result in Eq. 2, we get the following lower bound on the number of edges of $G(P, <_1, <_2)$:

$$e(G(P, <_1, <_2)) > (w(E) - hs^2)m^2 > \left(\frac{|S|^2}{16k} - hs^2 \right) m^2 >$$

$$\left(\frac{h^2 s^2}{64k} - hs^2 \right) m^2 = \frac{h^2 s^2 m^2}{128k} > 256 c_1^2 n^2 \delta^{-1}.$$

Thus, setting $\gamma(\delta) = 256 c_1^2 \delta^{-1}$ finishes the proof of the theorem. □

We remark that if $<_1$ and $<_2$ have a common linear extension, which is often the case in applications, then we do not need to use Lemma 12 in the previous proof and we can simply write $1/h$ instead of c_1. Then we get the bound $\gamma(\delta) = \delta/256$, which almost matches the constant of part (i) in Theorem 11. However, we conjecture that an even stronger bound holds in general.

Conjecture 17 *Let k be a positive integer. If $1 - 1/k \leq \alpha < 1 - 1/(k+1)$, we have*

$$f^I_{2,2}(n, \alpha n^2/2) = n^{1-1/k+o_\alpha(1)},$$

where $o_\alpha(1)$ is some function of n satisfying $o_\alpha(1) \to 0$ as $n \to \infty$, with α fixed.

We also conjecture that $f^I_{r,h}(n, m)$ has a similar growth as $f_{2,2}(n, m)$ for $r \geq 3$ or $r = 2$ and $h \geq 3$, but we cannot even quantify a precise conjecture for these cases.

5 Applications

Partial orders naturally arise in some geometric problems. The intersection graph of a set system $\mathcal{C}$ is the graph $G = G(\mathcal{C}, E)$, where $A, B \in \mathcal{C}$ forms an edge if $A \cap B \neq \emptyset$. The intersection graph of convex sets in the plane was investigated in a series of paper. Larman et. al. [8], and Pach and Törőcsik [10] showed that the intersection graph of convex sets is a 4-incomparability graph. Hence, by an immediate application of Dilworth's theorem yields that amongst n convex sets there are always at least $n^{1/5}$ such that they are pairwise disjoint, or any two of them intersects. Also, as it was noted in [5], Theorem 2 implies a bipartite version of this theorem, namely that if $\mathcal{C}$ is a family of n convex sets, then there are $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ of size $n^{1-o(1)}$ such that for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $A \cap B = \emptyset$, or for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $A \cap B \neq \emptyset$. In [6], this result was improved, showing that we can find two linear sized families $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ with the same property.

Call a set in $\mathbb{R}^2$ vertically convex, if every vertical line intersects the set in an interval. The intersection graph of connected, vertically convex sets is also a 4-incomparability graph.

Hence, Theorem 2 also implies the existence of a complete bipartite graph of size $n^{1-o(1)}$ either in the intersection graph or its complement. However, we can no longer guarantee a linear sized complete bipartite graph in the intersection graph or in its complement. In [9], it is shown that for any $\epsilon > 0$, there is a collection of n continuous functions on [0, 1] such that the largest bipartite graph in the intersection graph has size $O(n/\log n)$, and the largest complete bipartite graph in its complement has size $O(n^{\epsilon})$.

Nevertheless, Theorem 4 immediately implies that if the intersection graph of vertically convex sets is sparse enough, then we have a linear sized complete bipartite graph in its complement.

Theorem 18 *Let $\epsilon > 0$ and let $\mathcal{C}$ be a collection of n connected, vertically convex sets in the plane. If the number of unordered pairs $\{A, B\} \in \mathcal{C}^{(2)}$ with $A \cap B \neq \emptyset$ is less than $n^2(1/32 - \epsilon)$, then there are $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ such that*

$$|\mathcal{A}| = |\mathcal{B}| > \frac{\epsilon n}{128},$$

and for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ we have $A \cap B = \emptyset$.

Proof As we aim for the self-containment of the paper, we shall define the 4-partial orders on C, whose incomparability graph is the intersection graph. For any $C \in \mathcal{C}$, let

$$l(C) = \inf\{x \in \mathbb{R} : \exists y : (x, y) \in C\}$$

and let

$$r(C) = \sup\{x \in \mathbb{R} : \exists y : (x, y) \in C\}.$$

Define the relations $\prec_1, \prec_2, \prec_3$ on $\mathcal{C}$ as follows:
$C \prec_1 D$, if $l(C) \leq l(D)$ and $r(C) \leq l(D)$;
$C \prec_2 D$, if $l(C) \leq l(D)$ and $r(D) \leq r(C)$;
$C \prec_3 D$, if for every vertical line l which intersects both C and D, the interval $l \cap C$ is below $l \cap D$.

Note that $\prec_1, \prec_2, \prec_3$ are not partial orders, as it is possible that $C \prec_i D$ and $D \prec_i C$ both hold.

However, define the relations $<_1, <_2, <_3, <_4$ on $\mathcal{C}$ as follows:
$C <_1 D$, if $C \prec_1 D$ and $C \prec_3 D$;
$C <_2 D$, if $C \prec_1 D$ and $D \prec_3 C$;
$C <_3 D$, if $C \prec_2 D$ and $C \prec_3 D$;
$C <_4 D$, if $C \prec_2 D$ and $D \prec_3 C$.

One can easily check that $<_1, <_2, <_3, <_4$ are partial orders on $\mathcal{C}$, and C and D are comparable in some $<_i$ if and only if C and D are disjoint.

Now, if there are less than $(1/32 - \epsilon)n^2$ unordered pairs $\{A, B\} \in \mathcal{C}^{(2)}$ such that A and B intersect, then $G(P, <_1, <_2, <_3, <_4)$ has more than

$$\left(\frac{1}{2} - \frac{1}{32} + \epsilon\right) n^2$$

edges. Hence, by Theorem 4, there exists $i \in [4]$ and $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ such that $|\mathcal{A}| = |\mathcal{B}| > \epsilon n/128$ and $\mathcal{A} <_i \mathcal{B}$. But then for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $A \cap B = 0$. □

We note that these results have no analogue in higher dimensions: Tietze [11] proved that any graph can be realized as the intersection graph of 3-dimensional convex sets.

We also show an application of Theorem 4 for a variant of a classical problem of Erdős: let $\alpha < \pi$ be a positive real. Let $g_d(\alpha)$ be the smallest integer m such that in any configuration of m points in the d-dimensional space there is an angle larger than α. Erdős and Szekeres [3] proved that

$$g_2\left(\pi - \frac{\pi}{r} + \epsilon\right) = 2^r + 1,$$

if $r \geq 2$ is an integer and $\epsilon > 0$ is sufficiently small. They also proved that

$$2^{(1/\beta)^{d-1}} < g_d(\pi - \beta) < 2^{(4/\beta)^{d-1}}$$

for any $0 < \beta < \pi$.

The author of this paper [12] considered the following generalization of this problem: given $0 < \alpha < \pi$ and positive integers m and d, what is the maximal s such that any configuration of m points in the d-dimensional space contains s points, where every triangle has an angle larger than α. It was proved that any configuration of $t^r + 1$ points in the plane contains $t + 1$ points, where every triangle has an angle larger than $\pi - \pi/r$.

We prove a tripartite version of this result.

Theorem 19 *Let $0 < \alpha < \pi$ and let d be a positive integer. There exists a constant $c(\alpha, d) > 0$ with the following property. Suppose that n is a sufficiently large positive integer and S is a configuration of n points in the d-dimensional space. There exist three pairwise disjoint sets $A, B, C \subset S$ such that*

$$|A| = |B| = |C| > c(\alpha, d)n,$$

and for every $X \in A$, $Y \in B$, $Z \in C$, the angle $XYZ\angle$ is larger than α.

Proof Let $s = \lceil 1/(\pi - \alpha) \rceil$, and let V be a finite set of unit vectors with the property that for any $w \in \mathbb{R}^d$, there exists $v \in V$ such that the angle of v and w is less than $(\pi - \alpha)/2$. Such V trivially exists. For each $v \in V$, define the relation $<_v$ on $\mathbb{R}^d$ as follows: if $X, Y \in \mathbb{R}^d$, then $X <_v Y$ if the angle of v and $\overrightarrow{XY}$ is less than $(\pi - \alpha)/2$. Then $<_v$ is a partial order: $X <_v Y$ is equivalent to the inequality

$$\langle v, \overrightarrow{XY} \rangle > |\overrightarrow{XY}| \sin(\alpha/2).$$

Hence, if $X <_v Y$ and $Y <_v Z$, then

$$\langle v, \overrightarrow{XZ} \rangle = \langle v, \overrightarrow{XY} \rangle + \langle v, \overrightarrow{YZ} \rangle >$$
$$> (|\overrightarrow{XY}| + |\overrightarrow{YZ}|) \sin(\alpha/2) \geq |\overrightarrow{XZ}| \sin(\alpha/2),$$

so $X <_v Z$.

Also, if $X <_v Y <_v Z$ holds for some $X, Y, Z \in \mathbb{R}^d$, then by elementary geometry, the angle $XYZ\angle$ is larger than α.

Let $S \subset \mathbb{R}^d$, $|S| = n$. Then $G(S, \{<_v\}_{v \in V})$ is the complete graph on n vertices, because we choose V such that for any $X, Y \in \mathbb{R}^d$, there exists $v \in V$ with $X <_v Y$. Let

$$c(\alpha, d) = c(|V|, 3, 2^{-2|V|-2}),$$

where $c(r, h, \epsilon)$ is the constant defined in Theorem 4. If n is sufficiently large, then

$$\binom{n}{2} > \left(\frac{1}{2} - \frac{1}{2^{2|V|+2}}\right) n^2.$$

Hence, by Theorem 4, there exist $v \in V$ and $A, B, C \subset S$ such that A, B, C are pairwise disjoint, $|A| = |B| = |C| > c(\alpha, d)n$, and $A <_v B <_v C$. But then, for any $X \in A$, $Y \in B$ and $Z \in C$, we also have $XYZ\angle > \alpha$. □

Consider the case $d = 2$ and $\alpha < \pi - \pi/r$. In the proof above, we can choose V to be a $2r$ element set, so using the bound in the remark after Theorem 6, we can show that $c(\alpha, 2) > e^{-cr}$ with some constant $c > 0$. However, we conjecture that an even stronger bound holds.

Conjecture 20 *Let $0 < \alpha < \pi - \pi/r$ and let n be a positive integer. Let S be a set of n points in the plane. There exist $A, B, C \subset S$ disjoint subsets such that $|A| = |B| = |C| = \Omega(n/r)$, and for every $X \in A$, $Y \in B$, $Z \in C$, we have $XYZ\angle > \alpha$.*

Taking S to be the $[\sqrt{n}] \times [\sqrt{n}]$ square grid, one can easily show that the dependence on r in the conjecture cannot be improved.

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