



Optimal strong convergence rates of some Euler-type timestepping schemes for the finite element discretization SPDEs driven by additive fractional Brownian motion and Poisson random measure

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Abstract

In this paper, we study the numerical approximation of a general second order semi-linear stochastic partial differential equation (SPDE) driven by a additive fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$ and Poisson random measure. Such equations are more realistic in modelling real world phenomena. To the best of our knowledge, numerical schemes for such SPDE have been lacked in the scientific literature. The approximation is done with the standard finite element method in space and three Euler-type timestepping methods in time. More precisely the well-known linear implicit method, an exponential integrator and the exponential Rosenbrock scheme are used for time discretization. In contract to the current literature in the field, our linear operator is not necessary self-adjoint and we have achieved optimal strong convergence rates for SPDE driven by fBm and Poisson measure. The results examine how the convergence orders depend on the regularity of the noise and the initial data and reveal that the full discretization attains the optimal convergence rates of order $\mathcal{O}(h^2 + \Delta t)$ for the exponential integrator and implicit schemes. Numerical experiments are provided to illustrate our theoretical results for the case of SPDE driven by the fBm noise.

Keywords Stochastic parabolic partial differential equations · Fractional Brownian motion · Finite element method · Errors estimate · Finite element methods · Timestepping methods · Poisson random measure

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1 Introduction

We analyse the strong numerical approximation of an SPDE defines in $\Lambda \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ with initial value and boundary conditions (Dirichlet, Neumann, Robin boundary conditions or mixed Dirichlet and Neumann). In Hilbert space, our model equation can be formulated as the following parabolic SPDE

$$\begin{cases} dX(t) + AX(t)dt = F(X(t))dt + \phi(t)dB^H(t) + \int_{\chi} z_0 \tilde{N}(dz, dt) \\ X(0) = X_0, \end{cases} \quad (1)$$

in Hilbert space $\mathcal{H} = L^2(\Lambda)$, with $z_0 \in \chi$, where χ is the mark set defined by $\chi := \mathcal{H} \setminus \{0\}$. Let $\mathcal{B}(\Gamma)$ be the smallest σ -algebra containing all open sets of Γ . Let $(\chi, \mathcal{B}(\chi), \nu)$ be a σ -finite measurable space and ν (with $\nu \neq 0$) a Lévy measure on $\mathcal{B}(\chi)$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\chi} \min(\|z\|^2, 1) \nu(dz) < \infty. \quad (2)$$

Let $N(dz, dt)$ be the \mathcal{H} -valued Poisson distributed σ -finite measure on the product σ -algebra $\mathcal{B}(\chi)$ and $\mathcal{B}(\mathbb{R}_+)$ with intensity $\nu(dz)dt$, where dt is the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+)$. In our model problem (1), $\tilde{N}(dz, dt)$ stands for the compensated Poisson random measure defined by

$$\tilde{N}(dz, dt) := N(dz, dt) - \nu(dz)dt. \quad (3)$$

We denote by $T > 0$, the final time, $F : \mathcal{H} \rightarrow \mathcal{H}$, ϕ are deterministic mappings that will be specified precisely later, X_0 is the initial data which is random, $-A$ is a linear operator, not necessary self-adjoint, unbounded and generator of an analytic semigroup $S(t) := e^{-tA}$, $t \geq 0$. Note that $B^H(t)$ is a \mathcal{H} -valued Q -cylindrical fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1]$ in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with the covariance operator $Q : \mathcal{H} \rightarrow \mathcal{H}$, which is positive definite and self-adjoint. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ is assumed to fulfil the usual condition (see [31, Definition 2.2.11]). It is well known [3] that the noise can be represented as

$$B^H(t) = \sum_{i \in \mathbb{N}^d} \beta_i^H(t) Q^{\frac{1}{2}} e_i = \sum_{i \in \mathbb{N}^d} \sqrt{q_i} \beta_i^H(t) e_i, \quad t \in [0, T]. \quad (4)$$

where q_i , e_i , $i \in \mathbb{N}^d$ are respectively the eigenvalues and eigenfunctions of the covariance operator Q , and β_i^H are mutually independent and identically distributed fractional Brownian motions (fBm).

In our study, we first study in details the following particular case where $z_0 = 0$, i.e. the SPDE is driven only by fBm

$$\begin{cases} dX(t) + AX(t)dt = F(X(t))dt + \phi(t)dB^H(t), \\ X(0) = X_0, \quad t \in [0, T]. \end{cases} \quad (5)$$

The self-similar and long-range dependence properties of the fBm make this process a suitable candidate to model many phenomena like financial markets (see, e.g., [4, 11, 22]) and traffic networks (see, e.g., [17, 39]). In most cases, SPDEs of type (5) do not have explicit solutions and therefore numerical algorithms are required for

their approximations. It is important to mention that if $H \neq \frac{1}{2}$ the process B^H is not a semi-martingale and the standard stochastic calculus techniques are therefore obsolete while studying SPDEs of type (5). Alternative approaches to the standard Itô calculus are therefore required in order to build a stochastic calculus framework for such fBm. In recent years, there have been various developments of stochastic calculus and stochastic differential equations with respect to the fBm especially for $H \in (\frac{1}{2}, 1]$ (see, for example, [2, 3, 24, 25]) and theory of SPDEs driven by fractional Brownian motion has been also studied. For example, stochastic partial differential equations in a Hilbert space with an infinite dimensional fractional Brownian motion are considered in [3, 6, 7, 18–20]. In contrast to standard Brownian ($H = 1/2$) where there are numerous literature on numerical algorithms for SPDEs, few works have been done for numerical methods for fBm for SPDEs of type (5). Indeed, standard explicit and linear implicit schemes have been investigated in the literature for SPDEs of type (5) (see [13, 14, 38]). The works in [13, 38] deal with self-adjoint operator and use the spectral Galerkin method for the spatial discretization. This is very restrictive as many concrete applications use non self-adjoint operators. Beside numerical algorithms used for spatial discretization and time discretization in [13, 38] are limited to few applications. Our goal in this work is to extend keys time stepping methods, which have been built for standard Brownian motion ($H = 1/2$). These extensions are extremely complicated due to the fact that the process B^H is not a semi-martingale. Our results will be based on many novel intermediate lemmas. Indeed, our schemes here are based on finite element method (or finite volume method) for spatial discretization so that we gain the flexibility of these methods to deal with complex boundary conditions and we can apply well-developed techniques such as upwinding to deal with advection. For time discretization, we will first update the implicit linear for finite element method and not necessarily self-adjoint. We also provide the strong convergence of the exponential scheme [21, 34] for ($H \in (\frac{1}{2}, 1]$). Note that this scheme is an explicit stable scheme, where the implementation is based on the computation of matrix exponential functions [21]. As the linear implicit and exponential scheme are stable only when the linear operator A is stronger than the nonlinear function F ,¹ we also provide the strong convergence of the stochastic exponential Rosenbrock scheme (SERS) [26] for ($H \in (\frac{1}{2}, 1]$), which is very stable when (5) is driven both by its linear or nonlinear parts.

However, the model (5) can be unsatisfactory and less realistic. For instance, in finance, the unpredictable nature of many events such as market crashes, announcements made by the central banks, changing credit risk, insurance in a changing risk, changing face of operational risk [5, 30] might have sudden and significant impacts on the stock price. As for standard Brownian motion, we can incorporate a non-Gaussian noise such as Lévy process or Poisson random measure to model such events. The corresponding equation is our model equation given in (1). In contrast to SPDE driven by fBm in (5) where at least few numerical schemes exist, numerical schemes for such SPDE of type (1) driven by fBm and Poisson measure have been lacked in scientific literature, to the best of our knowledge. In this work, we will

¹In this case the SPDE (5) is said to be driven by its linear part.

also fill the gap by extending the implicit scheme, the exponential scheme and the stochastic exponential Rosenbrock scheme to SPDE of type (1). For SPDE of type (5) and SPDE of type (1), our strong convergence results examine how the convergence orders depend on the regularity of the noise and the initial data and reveal that the full discretization attains optimal convergence rates of order $\mathcal{O}(h^2 + \Delta t)$ for the exponential integrator and implicit schemes.²

The paper is structured as follows. In Section 2, Mathematical setting for fBm is presented, along with the well posedness and regularities results of the mild solution of SPDE (5) driven by fBm. In Section 3, numerical schemes based on implicit scheme, stochastic exponential integrator and stochastic exponential Rosenbrock scheme for SPDE (5) driven by fBm are presented. In Section 4, the strong convergence proofs of schemes presented in Section 3 are provided. In Section 5, numerical schemes based on semi-implicit scheme, stochastic exponential integrator scheme and stochastic exponential Rosenbrock scheme are presented for SPDE (1) driven by fBm and Poisson measure, along with the extension of their strong convergence proofs. We end the paper in Section 6 with numerical experiments illustrating our theoretical results for SPDE (5) driven by fBm noise.

2 Mathematical setting

In this section, we review some standard results on fractional calculus and introduce notations, definitions and preliminaries results which will be needed throughout this paper.

Definition 1 [13, 23, 25, 38] The fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t), t \geq 0\}$ with the covariance function

$$R_H(t, s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}].$$

Notice that if $H = \frac{1}{2}$, the process is the standard Brownian motion.

Remark 1 [25, Remark 1.2.3] For $H = 1$, we set $\beta^H(t) = \beta^1(t) = t\xi$, where ξ is a standard normal random variable.

Throughout this paper the Hurst parameter H is assumed to be in the interval $(1/2, 1]$. Let $(K, \langle \cdot, \cdot \rangle_K, \|\cdot\|)$ be a separable Hilbert space. For $p \geq 2$ and for a Banach space U , we denote by $L^p(\Omega, U)$ the Banach space of p -integrable U -valued random variables. We denote by $L(U, K)$ the space of bounded linear mapping from U to K endowed with the usual operator norm $\|\cdot\|_{L(U, K)}$ and $\mathcal{L}_2(U, K) = HS(U, K)$ the space of Hilbert-Schmidt operators from U to K with

²Linear operator A self-adjoint for implicit scheme

$$\|l\|_{\mathcal{L}_2(U, K)} := \left(\sum_{i \in \mathbb{N}^d} \|l e_i\|^2 \right)^{\frac{1}{2}}, \quad l \in \mathcal{L}_2(U, K), \quad (6)$$

where $(e_i)_{i \in \mathbb{N}^d}$ is an orthonormal basis on U . The sum in (6) is independent of the choice of the orthonormal basis in U . For simplicity, we use the notation $L(U, U) =: L(U)$ and $\mathcal{L}_2(U, U) =: \mathcal{L}_2(U)$. It is well known that for all $l \in L(U, K)$ and $l_1 \in \mathcal{L}_2(U)$, $ll_1 \in \mathcal{L}_2(U, K)$ and

$$\|ll_1\|_{\mathcal{L}_2(U, K)} \leq \|l\|_{L(U, K)} \|l_1\|_{\mathcal{L}_2(U)}. \quad (7)$$

We denote by L_2^0 , the space of Hilbert-Schmidt operators from $Q^{\frac{1}{2}}(\mathcal{H})$ to \mathcal{H} by $L_2^0 := HS(Q^{\frac{1}{2}}(\mathcal{H}), \mathcal{H})$ with corresponding norm $\|\cdot\|_{L_2^0}$ by

$$\|l\|_{L_2^0} := \|l Q^{\frac{1}{2}}\|_{HS} = \left(\sum_{i \in \mathbb{N}^d} \|l Q^{\frac{1}{2}} e_i\|^2 \right)^{\frac{1}{2}}, \quad l \in L_2^0. \quad (8)$$

Now, let us introduce the Wiener integral with respect to the one-dimensional fBm β^H . Let $T > 0$ and the linear space \mathcal{E} denotes the collection of all \mathbb{R} -valued step functions defined on $[0, T]$, that is, $\phi \in \mathcal{E}$, if

$$\phi(t) = \sum_{i=1}^{n-1} \phi_i \mathbf{I}_{[t_i, t_{i+1})},$$

where $t \in [0, T]$, $\phi_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \dots < t_n = T$. For $\phi \in \mathcal{E}$, we define its Wiener integral with respect to β^H as

$$\int_0^T \phi(s) d\beta^H(s) = \sum_{i=1}^{n-1} \phi_i \left(\beta^H(t_{i+1}) - \beta^H(t_i) \right). \quad (9)$$

Let \mathcal{Z} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{I}_{[0, t]}, \mathbf{I}_{[0, s]} \rangle_{\mathcal{Z}} = R_H(t, s).$$

Then, the mapping

$$\phi = \sum_{i=1}^{n-1} \phi_i \mathbf{I}_{[t_i, t_{i+1})} \longmapsto \int_0^T \phi(s) d\beta^H(s)$$

is an isometry between \mathcal{E} and the linear space $\text{span}\{\beta_H, t \in [0, T]\}$, which can be extended to an isometry between \mathcal{Z} and the first Wiener chaos of the fBm $\overline{\text{span}}^{L^2(\Omega)}\{\beta_H, t \in [0, T]\}$ (see [3, 18]). The image of an element $\varphi \in \mathcal{Z}$ by this isometry is called the Wiener integral of φ with respect to β^H . Next introduce the square integrable Kernel

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$$

and its derivative

$$\frac{\partial}{\partial t} K_H(t, s) = c_H(t-s)^{H-3/2} \left(\frac{t}{s}\right)^{H-1/2}.$$

where $c_H = \left(\frac{H(2H-1)}{B(2H-2, H-1/2)}\right)^{1/2}$, with B denoting the Beta function and $t > s$. Let us define the operator K_H^* from \mathcal{E} to $L^2([0, T])$ by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(t) \frac{\partial}{\partial t} K_H(t, s) dt.$$

We easily have

$$(K_H^* \mathbf{I}_{[0,t]})(s) = K_H(s) \mathbf{I}_{[0,t]},$$

and

$$\begin{aligned} \langle K_H^* \mathbf{I}_{[0,t]}, K_H^* \mathbf{I}_{[0,s]} \rangle_{L^2([0,T])} &= R_H(t, s) \\ &= \langle \mathbf{I}_{[0,t]}, \mathbf{I}_{[0,s]} \rangle_{\mathcal{Z}}. \end{aligned}$$

Hence, the operator K_H^* is an isometry between \mathcal{E} and $L^2([0, T])$ which can be extended to \mathcal{Z} . By definition

$$\beta(t) = \beta^H \left(K_H^{*-1} \mathbf{I}_{[0,t]} \right), \quad t \in [0, T],$$

is a Brownian sheet, and in turn the fractional noise has a representation

$$\beta^H(t) = \int_0^t K_H(t, s) \beta(ds).$$

In addition, for any $\varphi \in \mathcal{Z}$

$$\int_0^T \varphi(s) d\beta^H(s) = \int_0^T (K_H^* \varphi(s)) d\beta(s) \quad (10)$$

if and only if $K_H^* \varphi \in L^2([0, T])$.

Also denoting $L_{\mathcal{Z}}^2([0, T]) = \{\varphi \in \mathcal{Z}, K_H^* \varphi \in L^2([0, T])\}$, since $H > 1/2$ we have the following embedding property.

Proposition 1 [7, 25] $L^2([0, T]) \subset L^{1/H}([0, T]) \subset L_{\mathcal{Z}}^2([0, T])$.

The next proposition shows the first and second moments of stochastic integral defined for function in $L_{\mathcal{Z}}^2([0, T])$.

Proposition 2 [6, 7] For $f \in L_{\mathcal{Z}}^2([0, T])$ we have

$$\mathbb{E} \left[\int_0^T f(s) d\beta^H(s) \right] = 0$$

and

$$\mathbb{E} \left| \int_0^T f(s) d\beta^H(s) \right|^2 = H(2H-1) \int_0^T \int_0^T f(u) f(v) |u-v|^{2H-2} du dv.$$

Moreover, we have the following lemma very important throughout this paper.

Lemma 1 [3, Lemma 1] *For any $\varphi \in L^{1/H}([0, T])$, the following inequality holds*

$$H(2H - 1) \int_0^T \int_0^T |\varphi(u)| |\varphi(v)| |u - v|^{2H-2} du dv \leq C_H \|\varphi\|_{L^{1/H}([0, T])}^2. \quad (11)$$

where $C_H > 0$ is a constant depending only on H .

Next, we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Definition 2 Let $f \in L^2([0, T], L_2^0)$. Then, its stochastic integral with respect to the fBm B^H is defined, for $t \geq 0$, as follows

$$\int_0^t f(s) dB^H(s) := \sum_{i \in \mathbb{N}^d} \int_0^t f(s) Q^{\frac{1}{2}} e_i d\beta_i^H(s). \quad (12)$$

As the sequence of random variables $\left(\int_0^T f(s) Q^{\frac{1}{2}} e_i d\beta_i^H(s), i \in \mathbb{N}^d \right)$ are mutually independent Gaussian random variable, by Propositions 1, 2 and Lemma 1, the mean of random variable (12) is also zero and we have the following result for its second moment.

Lemma 2 [3, 18] *Let $f \in L^2([0, T]; L_2^0)$ and $H \in (\frac{1}{2}, 1]$, then the following holds*

$$\mathbb{E} \left[\left\| \int_0^T f(s) dB^H(s) \right\|^2 \right] \leq C(H) \sum_{i \in \mathbb{N}^d} \left(\int_0^T \|f(s) Q^{\frac{1}{2}} e_i\|^{\frac{1}{H}} ds \right)^{2H} \quad (13)$$

$$\leq C(H, T) \int_0^T \|f(s)\|_{L_2^0}^2 ds. \quad (14)$$

In that follows, we will make some assumptions on F , ϕ , X_0 and A , which will allow us to ensure the existence and uniqueness of the mild solution X of (5) represented by (see, e.g., [38])

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)\phi(s)dB^H(s), \quad (15)$$

for $t \in [0, T]$. To ensure the existence and the uniqueness of solution for SPDE (5) and for the purpose of convergence analysis, we make the following assumptions.

Assumption 1 (Noise term) *We assume that for some constant $\beta \in (0, 1]$ and $\delta \in [\frac{2H+\beta-1}{2}, 1]$, the deterministic mapping $\phi : [0, T] \times \Lambda \rightarrow L_2^0$ satisfies*

$$\|A^{\frac{\beta-1}{2}} \phi(t)\|_{L_2^0} \leq C < \infty, \quad t \in [0, T] \quad (16)$$

$$\|A^{\frac{\beta-1}{2}} (\phi(t_2) - \phi(t_1))\|_{L_2^0} \leq C_T(t_2 - t_1)^\delta, \quad 0 \leq t_1 \leq t_2 \leq T. \quad (17)$$

Assumption 2 (Non linearity) For the deterministic mapping $F : \mathcal{H} \rightarrow \mathcal{H}$, we assume that there exists constant $L \in (0, \infty)$ such that

$$\|F(0)\| \leq L, \|F(u) - F(v)\| \leq L\|u - v\|, u, v \in \mathcal{H}. \quad (18)$$

As a consequence of (18) it holds that

$$\|F(v)\| \leq L(1 + \|v\|), v \in \mathcal{H}. \quad (19)$$

Assumption 3 (Initial value) We assume that $X_0 : \Omega \rightarrow \mathcal{H}$ is a $\mathcal{F}_0/\mathcal{B}(\mathcal{H})$ -measurable mapping and $X_0 \in L^2\left(\Omega, D\left(A^{\frac{2H+\beta-1}{2}}\right)\right)$.

In the Banach space $\mathcal{D}\left(A^{\frac{\alpha}{2}}\right)$, $\alpha \in \mathbb{R}$, we use notation $\left\|A^{\frac{\alpha}{2}} \cdot\right\| = \|\cdot\|_{\alpha}$ and we recall the following properties of the semigroup $S(t)$ generated by $-A$, which will be useful throughout this paper.

Proposition 3 (Smoothing properties of the semigroup) [29] Let $\alpha > 0$, $\delta \geq 0$ and $0 \leq \gamma \leq 1$, then there exists a constant $C > 0$ such that

$$\|A^{\delta} S(t)\|_{L(\mathcal{H})} \leq Ct^{-\delta}, \|A^{-\gamma}(I - S(t))\|_{L(\mathcal{H})} \leq Ct^{\gamma}, \quad t > 0 \quad (20)$$

$$\|D_t^l S(t)v\| \leq Ct^{-l-(\gamma-\alpha)/2}\|v\|_{\alpha}, \quad v \in D(A^{\alpha}), \quad (21)$$

where $l = 0, 1$ and $D^l = \frac{d^l}{dt^l}$. If $\delta > \gamma$ then $D(A^{\delta}) \supset D(A^{\gamma})$. Moreover, $A^{\delta} S(t) = S(t)A^{\delta}$ on $D(A^{\delta})$.

The next lemma (specially (24) and (25)) is an important result which plays a crucial role to obtain regularity results, very useful in this work.

Lemma 3 For any $0 \leq \rho \leq 1$, $0 \leq \gamma \leq 2$ and $0 \leq v \leq H$ with $H \in \left(\frac{1}{2}, 1\right]$, if the linear operator is given by (34), there exists a positive constant C such that for all $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2} \|A^{\rho/2} S(t_2 - r)\|_{L(\mathcal{H})}^2 dr \leq C(t_2 - t_1)^{1-\rho}, \quad (22)$$

$$\int_{t_1}^{t_2} \|A^{\gamma/2} S(t_2 - r)\|_{L(\mathcal{H})} dr \leq C(t_2 - t_1)^{1-\frac{\gamma}{2}}, \quad (23)$$

$$\int_{t_1}^{t_2} \|A^H S(t_2 - r)\|_{L(\mathcal{H})}^{\frac{1}{H}} dr \leq C, \quad (24)$$

$$\int_{t_1}^{t_2} \|A^v S(t_2 - r)\|_{L(\mathcal{H})}^{\frac{1}{H}} dr \leq C(t_2 - t_1)^{\frac{H-v}{H}}. \quad (25)$$

Proof See [27, Lemma 2.1] for the proof of (22) and (23). Concerning the proof of (24), the border case $H = \frac{1}{2}$ if obtained using (22) with $\rho = 1$ and the order border case $H = 1$ is also obtained using (23) with $\gamma = 2$. Hence the proof of (24) is then completed by interpolation theory. The proof of (25) for $0 \leq v \leq H$ is an immediate consequence of Proposition 3. The border case $v = H$ is proved by (24). This completes the proof of Lemma 3. \square

Remark 2 Proposition 3 and Lemma 3 also hold with a uniform constant C (independent of h) when A and $S(t)$ are replaced respectively by their discrete versions A_h and $S_h(t)$ defined in Section 3, see, e.g., [16, 21].

The well posedness result is given in the following theorem along with optimal regularity results in both space and time.

Theorem 1 *Assume that Assumptions 1–3 are satisfied, then there exists a unique mild solution given by (15) such that for all $t \in [0, T]$, $X(t) \in L^2\left(\Omega, D\left(A^{\frac{2H+\beta-1}{2}}\right)\right)$ with*

$$\|X(t)\|_{L^2(\Omega, \mathcal{H})} \leq C(1 + \|X_0\|_{L^2(\Omega, \mathcal{H})}), \quad (26)$$

$$\|F(X(t))\|_{L^2(\Omega, \mathcal{H})} \leq C(1 + \|X_0\|_{L^2(\Omega, \mathcal{H})}). \quad (27)$$

Moreover, if the linear operator is given by (34), the following optimal regularity results in space and time hold

$$\left\| A^{\frac{2H+\beta-1}{2}} X(t) \right\|_{L^2(\Omega, \mathcal{H})} \leq C \left(1 + \left\| A^{\frac{2H+\beta-1}{2}} X_0 \right\|_{L^2(\Omega, \mathcal{H})} \right), \quad t \in [0, T], \quad (28)$$

and for $0 \leq t_1 < t_2 \leq T$;

$$\begin{aligned} & \|X(t_2) - X(t_1)\|_{L^2(\Omega, \mathcal{H})} \\ & \leq C(t_2 - t_1)^{\frac{2H+\beta-1}{2}} \left(1 + \left\| A^{\frac{2H+\beta-1}{2}} X_0 \right\|_{L^2(\Omega, \mathcal{H})} \right), \end{aligned} \quad (29)$$

where $C = C(\beta, L, T, H)$ is a positive constant and β is the regularity parameter of Assumption 1.

Proof [38, Theorem 3.5] gives the result of existence and uniqueness of the mild solution X . For regularity in space, we adapt from [26, Theorem 2.1 (23), (24)] by just replacing β in their case by $2H + \beta - 1$. The difference will therefore be made at the level of the estimate of the stochastic integral

$$I^2 = \mathbb{E} \left\| \int_0^t A^{\frac{2H+\beta-1}{2}} S(t-s) \phi(s) dB^H(s) \right\|^2.$$

To reach our goal, we use triangle inequality, the estimate $(a + b)^2 \leq 2a^2 + 2b^2$, (13) and (14), Assumption 1, Proposition 3, Lemma 3 (24) to have

$$\begin{aligned}
 I^2 &= \mathbb{E} \left\| \int_0^t A^{\frac{2H+\beta-1}{2}} S(t-s) \phi(s) dB^H(s) \right\|^2 \\
 &\leq 2\mathbb{E} \left\| \int_0^t A^{\frac{2H+\beta-1}{2}} S(t-s) (\phi(t) - \phi(s)) dB^H(s) \right\|^2 \\
 &\quad + 2\mathbb{E} \left\| \int_0^t A^{\frac{2H+\beta-1}{2}} S(t-s) \phi(t) dB^H(s) \right\|^2 \\
 &\leq 2C \int_0^t \left\| A^{\frac{2H+\beta-1}{2}} S(t-s) (\phi(t) - \phi(s)) \right\|_{L_2^0}^2 ds \\
 &\quad + 2C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^t \left\| A^{\frac{2H+\beta-1}{2}} S(t-s) \phi(t) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq 2C \int_0^t \left\| A^H S(t-s) \right\|_{L(\mathcal{H})}^2 \left\| A^{\frac{\beta-1}{2}} (\phi(t) - \phi(s)) \right\|_{L_2^0}^2 ds \\
 &\quad + 2C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^t \left\| A^H S(t-s) \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A^{\frac{\beta-1}{2}} \phi(t) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C \int_0^t (t-s)^{-2H} (t-s)^{2\delta} ds + C_H \left(\sum_{i \in \mathbb{N}^d} \left\| A^{\frac{\beta-1}{2}} \phi(t) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\quad \times \left(\int_0^t \left\| A^H S(t-s) \right\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C \int_0^t (t-s)^{2\delta-2H} ds + C_H \left\| A^{\frac{\beta-1}{2}} \phi(t) \right\|_{L_2^0}^2 \left(\int_0^t \left\| A^H S(t-s) \right\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C t^{2\delta-2H+1} + C \left\| A^{\frac{\beta-1}{2}} \phi(t) \right\|_{L_2^0}^2 \leq C. \tag{30}
 \end{aligned}$$

For the proof of (29), triangle inequality yields

$$\begin{aligned}
 \|X(t_2) - X(t_1)\|_{L^2(\Omega, \mathcal{H})} &\leq \|(S(t_2 - t_1) - I) X(t_1)\|_{L^2(\Omega, \mathcal{H})} \\
 &\quad + \left\| \int_{t_1}^{t_2} S(t_2 - s) F(X(s)) ds \right\|_{L^2(\Omega, \mathcal{H})} \\
 &\quad + \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}.
 \end{aligned}$$

Using the stability property of the semigroup $S(t)$ (20) with $\gamma = \frac{2H+\beta-1}{2}$ and (28) allows to have

$$\begin{aligned}
 & \|X(t_2) - X(t_1)\|_{L^2(\Omega, \mathcal{H})} \\
 & \leq \left\| A^{-\frac{2H+\beta-1}{2}} (S(t_2 - t_1) - I) \right\|_{L(\mathcal{H})} \left\| A^{\frac{2H+\beta-1}{2}} X(t_1) \right\|_{L^2(\Omega, \mathcal{H})} \\
 & \quad + \int_{t_1}^{t_2} \|S_h(t_2 - s)\|_{L(\mathcal{H})} \|F(X(s))\|_{L^2(\Omega, \mathcal{H})} ds \\
 & \quad + \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\
 & \leq C(t_2 - t_1)^{\frac{2H+\beta-1}{2}} \left(1 + \|A^{\frac{2H+\beta-1}{2}} X_0\|_{L^2(\Omega, \mathcal{H})} \right) \\
 & \quad + C(t_2 - t_1) (1 + \|X_0\|_{L^2(\Omega, \mathcal{H})}) + \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\
 & \leq C(t_2 - t_1)^{\frac{2H+\beta-1}{2}} \left(1 + \|A^{\frac{2H+\beta-1}{2}} X_0\|_{L^2(\Omega, \mathcal{H})} + \|X_0\|_{L^2(\Omega, \mathcal{H})} \right) \\
 & \quad + \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\
 & \leq C(t_2 - t_1)^{\frac{2H+\beta-1}{2}} \left(1 + \|A^{\frac{2H+\beta-1}{2}} X_0\|_{L^2(\Omega, \mathcal{H})} \right) + II, \tag{31}
 \end{aligned}$$

because $D\left(A^{\frac{2H+\beta-1}{2}}\right)$ is continuously embedded in $L^2(\Omega)$ and

$$II := \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}.$$

For the estimate of II , using triangle inequality, the estimate $(a + b)^2 \leq 2a^2 + 2b^2$, (13) and (14), inserting an appropriate power of A , Proposition 3, Assumption 1, Lemma 3 (25) with $\nu = \frac{1-\beta}{2} \in [0, \frac{1}{2})$ (hence $0 \leq \nu < H$), we obtain

$$\begin{aligned}
II^2 &= \mathbb{E} \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(s) dB^H(s) \right\|^2 \\
&\leq 2\mathbb{E} \left\| \int_{t_1}^{t_2} S(t_2 - s) (\phi(t_2) - \phi(s)) dB^H(s) \right\|^2 \\
&\quad + 2\mathbb{E} \left\| \int_{t_1}^{t_2} S(t_2 - s) \phi(t_2) dB^H(s) \right\|^2 \\
&\leq 2C \int_{t_1}^{t_2} \|S(t_2 - s) (\phi(t_2) - \phi(s))\|_{L_2^0}^2 ds \\
&\quad + 2C_H \sum_{i \in \mathbb{N}^d} \left(\int_{t_1}^{t_2} \|S(t_2 - s) \phi(t_2) Q^{\frac{1}{2}} e_i\|^{\frac{1}{H}} ds \right)^{2H} \\
&\leq 2C \int_{t_1}^{t_2} \|S(t_2 - s) A^{\frac{1-\beta}{2}}\|_{L(\mathcal{H})}^2 \|A^{\frac{\beta-1}{2}} (\phi(t_2) - \phi(s))\|_{L_2^0}^2 ds \\
&\quad + 2C_H \sum_{i \in \mathbb{N}^d} \left(\int_{t_1}^{t_2} \|S(t_2 - s) A^{\frac{1-\beta}{2}}\|_{L(\mathcal{H})}^{\frac{1}{H}} \|A^{\frac{\beta-1}{2}} \phi(t_2) Q^{\frac{1}{2}} e_i\|^{\frac{1}{H}} ds \right)^{2H} \\
&\leq C \int_{t_1}^{t_2} (t_2 - s)^{\beta-1+2\delta} ds + 2C_H \left(\sum_{i \in \mathbb{N}^d} \|A^{\frac{\beta-1}{2}} \phi(t_2) Q^{\frac{1}{2}} e_i\|^2 \right) \\
&\quad \times \left(\int_{t_1}^{t_2} \|S(t_2 - s) A^{\frac{1-\beta}{2}}\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right)^{2H} \\
&\leq C(t_2 - t_1)^{\beta+2\delta} + C \|A^{\frac{\beta-1}{2}} \phi(t_2)\|_{L_2^0}^2 (t_2 - t_1)^{2H+\beta-1} \\
&\leq C(t_2 - t_1)^{2H+2\beta-1} + C(t_2 - t_1)^{2H+\beta-1} \leq C(t_2 - t_1)^{2H+\beta-1}. \quad (32)
\end{aligned}$$

Substituting (32) in (31) completes the proof of (29) and therefore that of Theorem 1. \square

3 Numerical schemes

Throughout this section, we assume that Λ is bounded and has smooth boundary or is a convex polygon of \mathbb{R}^d , $d \in \{1, 2, 3\}$. In the rest of this paper we consider the SPDE (5) to be of the following form

$$\begin{aligned}
dX(t, x) + [-\nabla \cdot (\mathbf{D} \nabla X(t, x)) + \mathbf{q} \cdot \nabla X(t, x)] dt &= f(x, X(t, x)) dt \\
&\quad + b(x, t) dB^H(t, x), \quad (33)
\end{aligned}$$

$x \in \Lambda$, $t \in [0, T]$, where the function $f : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously twice differentiable and the function $b : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz with respect to the second variable. In the abstract framework (5), the linear operator A takes the form

$$Au = - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(D_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^d q_i(x) \frac{\partial u}{\partial x_i}, \quad \mathbf{D} = (D_{i,j})_{1 \leq i,j \leq d}, \quad (34)$$

$\mathbf{q} = (q_i)_{1 \leq i \leq d}$, where $D_{ij} \in L^\infty(\Lambda)$, $q_i \in L^\infty(\Lambda)$. We assume that there exists a positive constant $c_1 > 0$ such that

$$\sum_{i,j=1}^d D_{ij}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Omega}. \quad (35)$$

The functions $F : \mathcal{H} \longrightarrow \mathcal{H}$ and $\phi : \mathbb{R} \longrightarrow HS(Q^{1/2}(\mathcal{H}), \mathcal{H})$ are defined by

$$(F(v))(x) = f(x, v(x)) \quad \text{and} \quad (\phi(t)(u))(x) = b(x, t).u(x), \quad (36)$$

for all $x \in \Lambda$, $v \in \mathcal{H}$, $u \in Q^{1/2}(\mathcal{H})$, with $\mathcal{H} = L^2(\Lambda)$. For an appropriate family of eigenfunctions (e_i) such that $\sup_{i \in \mathbb{N}^d} \left[\sup_{x \in \Lambda} \|e_i(x)\| \right] < \infty$, it is well known [12, Section 4] that the Nemytskii operator F related to f and the operator ϕ associated to b defined in (36) satisfy Assumption 1 and Assumption 2. As in [8, 21] we introduce two spaces \mathbb{H} and V , such that $\mathbb{H} \subset V$; the two spaces depend on the boundary conditions and the domain of the operator A . For Dirichlet (or first-type) boundary conditions we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \overline{C_c^\infty(\Lambda)}^{H^1(\Lambda)} = \{v \in H^1(\Lambda) : v = 0 \quad \text{on} \quad \partial\Lambda\}.$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take $V = H^1(\Lambda)$

$$\mathbb{H} = \{v \in H^2(\Lambda) : \partial v / \partial \nu_A + \alpha_0 v = 0, \quad \text{on} \quad \partial\Lambda\}, \quad \alpha_0 \in \mathbb{R},$$

where $\partial v / \partial \nu_A$ is the normal derivative of v and ν_A is the exterior pointing normal $n = (n_i)$ to the boundary of A , given by

$$\partial v / \partial \nu_A = \sum_{i,j=1}^d n_i(x) D_{ij}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial\Lambda.$$

Using the Green's formula and the boundary conditions, the corresponding bilinear form associated to A is given by

$$a(u, v) = \int_{\Lambda} \left(\sum_{i,j=1}^d D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_i} v \right) dx, \quad u, v \in V,$$

for Dirichlet and Neumann boundary conditions, and

$$a(u, v) = \int_{\Lambda} \left(\sum_{i,j=1}^d D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_i} v \right) dx + \int_{\partial\Lambda} \alpha_0 u v dx, \quad u, v \in V,$$

for Robin boundary conditions. Using the Gårding's inequality, it holds that there exist two constants c_0 and λ_0 such that

$$a(v, v) \geq \lambda_0 \|v\|_{H^1(\Lambda)}^2 - c_0 \|v\|^2, \quad v \in V. \quad (37)$$

By adding and subtracting $c_0 X dt$ in both sides of (5), we have a new linear operator still denoted by A , and the corresponding bilinear form is also still denoted by a . Therefore, the following coercivity property holds

$$a(v, v) \geq \lambda_0 \|v\|_1^2, \quad v \in V. \quad (38)$$

Note that the expression of the nonlinear term F has changed as we included the term $c_0 X$ in a new nonlinear term that we still denote by F . The coercivity property (38) implies that $-A$ is sectorial in $L^2(\Lambda)$, i.e. there exist C_1 , $\theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\|(\lambda I + A)^{-1}\|_{L^2(\Lambda)} \leq \frac{C_1}{|\lambda|}, \quad \lambda \in S_\theta, \quad (39)$$

where $S_\theta = \{\lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \rho > 0, 0 \leq |\phi| \leq \theta\}$ (see [10]). Then, $-A$ is the infinitesimal generator of a bounded analytic semigroup $S(t) = e^{-tA}$ on $L^2(\Lambda)$ such that

$$S(t) = e^{-tA} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{t\lambda} (\lambda I + A)^{-1} d\lambda, \quad t > 0, \quad (40)$$

where \mathcal{C} denotes a path that surrounds the spectrum of $-A$. The coercivity property (38) also implies that A is a positive operator and its fractional powers are well defined for any $\alpha > 0$, by

$$\begin{cases} A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA} dt, \\ A^\alpha = (A^{-\alpha})^{-1}, \end{cases} \quad (41)$$

where $\Gamma(\alpha)$ is the Gamma function (see [10]). Under condition (35), it is well known (see, e.g., [8]) that the linear operator $-A$ given by (34) generates an analytic semigroup $S(t) \equiv e^{-tA}$. Following [8, 21], we characterize the domain of the operator $A^{r/2}$ denoted by $\mathcal{D}(A^{r/2})$, $r \in \{1, 2\}$ with the following equivalence of norms, useful in our convergence proofs

$$\begin{aligned} \|v\|_{H^1(\Lambda)} &\equiv \|A^{r/2}v\| =: \|v\|_r, \quad \forall v \in \mathcal{D}(A^{r/2}), \\ \mathcal{D}(A^{r/2}) &= \mathbb{H} \cap H^r(\Lambda), \quad (\text{for Dirichlet boundary conditions}), \\ \mathcal{D}(A) &= \mathbb{H}, \quad \mathcal{D}(A^{1/2}) = H^1(\Lambda), \quad (\text{for Robin boundary conditions}). \end{aligned}$$

We consider the discretization of the spatial domain by a finite element triangulation [34, 37]. Let \mathcal{T}_h be a set of disjoint intervals of Ω (for $d = 1$), a triangulation of Ω (for $d = 2$) or a set of tetrahedra (for $d = 3$) with maximal length h satisfying the usual regularity assumptions.

Let $V_h \subset \mathcal{H}$ denote the space of continuous functions that are piecewise linear over triangulation \mathcal{T}_h . To discretise in space, we introduce the projection P_h from $L^2(\Omega)$ to V_h define for $u \in L^2(\Omega)$ by

$$\langle P_h u, \mathcal{X} \rangle = \langle u, \mathcal{X} \rangle, \quad \forall \mathcal{X} \in V_h. \quad (42)$$

The discrete operator $A_h : V_h \rightarrow V_h$ is defined by

$$\langle A_h \varphi, \mathcal{X} \rangle = -a(\varphi, \mathcal{X}), \quad \varphi, \mathcal{X} \in V_h, \quad (43)$$

where a is the corresponding bilinear form of A .

Like the operator A , the discrete operator $-A_h$ is also the generator of an analytic semigroup $S_h(t) := e^{-tA_h}$. The semidiscrete space version of problem (5) is to find $X^h(t) = X^h(\cdot, t)$ such that for $t \in [0, T]$

$$\begin{cases} dX^h(t) + A_h X^h(t)dt = P_h F(X(t))dt + P_h \phi(t)dB^H(t), \\ X^h(0) = P_h X_0, t \in [0, T]. \end{cases} \quad (44)$$

The mild solution of (44) can be represented as follows

$$\begin{aligned} X^h(t) &= S_h(t)X^h(0) + \int_0^t S_h(t-s)P_h F(X^h(s))ds \\ &\quad + \int_0^t S_h(t-s)P_h \phi(s)dB^H(s), \end{aligned} \quad (45)$$

and we have the following regularity results.

Lemma 4 Assume that Assumptions 1–3 are satisfied, then the unique mild solution $X^h(t)$ given by (45) satisfied

$$\left\| A^{\frac{2H+\beta-1}{2}} X^h(t) \right\|_{L^2(\Omega, \mathcal{H})} \leq C \left(1 + \left\| A^{\frac{2H+\beta-1}{2}} X_0 \right\|_{L^2(\Omega, \mathcal{H})} \right) t \in [0, T], \quad (46)$$

and for $0 \leq t_1 < t_2 \leq T$;

$$\|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, \mathcal{H})} \leq C(t_2 - t_1)^{\frac{2H+\beta-1}{2}} \left(1 + \left\| A^{\frac{2H+\beta-1}{2}} X_0 \right\|_{L^2(\Omega, \mathcal{H})} \right). \quad (47)$$

Proof Since the operators A_h and $S_h(t)$ satisfy the same properties as A and $S(t)$ (see Remark 2), then by using [34, (83)] and the boundedness of P_h in the proof of (28) and (29), we obtain the proof of the expression (46) and (47). The proof of Lemma 4 is thus completed. \square

Now applying the linear implicit Euler method [13, 37] to (44) gives the following fully discrete scheme

$$\begin{cases} X_0^h = P_h X_0 \\ X_{m+1}^h = S_{h, \Delta t} X_m^h + \Delta t S_{h, \Delta t} P_h F(X_m^h) \Delta t + S_{h, \Delta t} P_h \phi(t_m) \Delta B_m^H. \end{cases} \quad (48)$$

Furthermore applying the stochastic exponential integrator ([21], SETD1) and Rosenbrock scheme ([26], SERS) to (44) yields

$$\begin{cases} Y_0^h = P_h X_0 \\ Y_{m+1}^h = S_h(\Delta t) (Y_m^h + P_h \phi(t_m) \Delta B_m^H) + \Delta t \varphi_1(\Delta t A_h) P_h F(Y_m^h), \end{cases} \quad (49)$$

and

$$\begin{cases} Z_0^h = P_h X_0 \\ Z_{m+1}^h = e^{(-A_h + J_m^h) \Delta t} Z_m^h + (-A_h + J_m^h) \left(e^{(-A_h + J_m^h) \Delta t} - I \right) G_m^h(Z_m^h) \\ \quad + e^{(-A_h + J_m^h) \Delta t} P_h \phi(t_m) \Delta B_m^H, \end{cases}$$

where $\varphi_1(\Delta t A_h) = (\Delta t A_h)^{-1} (e^{\Delta t A_h} - I) = \frac{1}{\Delta t} \int_0^{\Delta t} e^{(\Delta t-s)A_h} ds$, J_m^h is the Frechet derivative of $P_h F$ at Z_m^h and $S_{h, \Delta t} := (I + \Delta t A_h)^{-1}$. The term G_m^h is the remainder

at Z_m^h and defines for all $\omega \in \Omega$ by

$$J_m^h := (P_h F)'(Y_m^h(\omega)) = P_h F'(Y_m^h(\omega)), \quad (50)$$

$$G_m^h(\omega)(Z_m^h) := P_h F(Z_m^h) - J_m^h(\omega)Z_m^h, \quad (51)$$

and

$$\Delta B_m^H := B_{m+1}^H - B_m^H = \sum_{i \in \mathbb{N}^d} \sqrt{q_i} (\beta_i^H(t_{m+1}) - \beta_i^H(t_m)) e_i.$$

Note that the exponential integrator scheme (49) is an explicit stable scheme when the SPDE (5) is driven by its linear part as the linear implicit method, while the stochastic exponential Rosenbrock scheme (SERS) (50) is very stable when (5) is driven by its linear or nonlinear part. When dealing with SERS, the strong convergence proof will make use of the following assumption, also used in [26, 27].

Assumption 4 For the deterministic mapping $F : \mathcal{H} \rightarrow \mathcal{H}$, we also assume that there exists constant $L \in (0, \infty)$ such that

$$\|F'(u)v\| \leq L\|v\|, \quad u, v \in \mathcal{H}. \quad (52)$$

3.1 Main result for SPDE driven by fBm

Theorem 2 Let $X(t_m)$ be the mild solution of (5) at time $t_m = m\Delta t$, $\Delta t \geq 0$ represented by (15). Let ζ_m^h be the numerical approximations through (48) and (50) ($\zeta_m^h = X_m^h$ for implicit scheme, $\zeta_m^h = Z_m^h$ for SERS). Under Assumptions 1–3 and (4) (essentially for SERS), $\beta \in (0, 1]$, then the following holds

$$\left(\mathbb{E}\|X(t_m) - Y_m^h\|^2\right)^{\frac{1}{2}} \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{\beta+2H-1}{2}}\right), \quad (53)$$

and

$$\left(\mathbb{E}\|X(t_m) - \zeta_m^h\|^2\right) \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{\beta+2H-1}{2}-\epsilon}\right), \quad (54)$$

where ϵ is a positive constant small enough.

4 Proofs of the main result for SPDE with fBm

We introduce the Riesz representation $R_h : V \rightarrow V_h$ defined by

$$\langle AR_h v, \mathcal{X} \rangle = \langle Av, \mathcal{X} \rangle = a(v, \mathcal{X}), \quad v \in V, \quad \forall \mathcal{X} \in V_h \quad (55)$$

under the regularity assumptions on the triangulation and in view of the V-ellipticity, it is well known ([8, 16]) that the following error bound holds:

$$\|R_h v - v\| + h\|R_h v - v\|_{H^1(\Omega)} \leq Ch^r \|v\|_{H^r(\Omega)}, \quad v \in V \cap H^r(\Omega), \quad (56)$$

for $r \in [1, 2]$. Let us consider the following deterministic linear problem:
Find $u \in V$ such that

$$\frac{du}{dt} + Au = 0, \quad u(0) = v, \quad t \in (0, T]. \quad (57)$$

The corresponding semidiscrete problem in space consists to finding $u^h \in V_h$ such that

$$\frac{du^h}{dt} + A_h u^h = 0, \quad u^h(0) = P_h v, \quad t \in (0, T]. \quad (58)$$

Let us define the following operator

$$G_h(t) := S(t) - S_h(t)P_h = e^{-At} - e^{-A_h t}P_h. \quad (59)$$

Then we have the following lemma

Lemma 5 *The following estimates hold for the semidiscrete approximation of (44). There exists a constant $C > 0$ such that*

(i) For $v \in D(A^{\gamma/2})$

$$\|u(t) - u^h(t)\| = \|G_h(t)v\| \leq Ch^r t^{-(r-\gamma)/2} \|v\|_{\gamma}, \quad r \in [0, 2], \quad \gamma \leq r \quad (60)$$

for any $t \in (0, T]$.

(ii) For $v \in D(A^{\frac{\gamma-1}{2}})$

$$\left(\int_0^t \|G_h(s)v\|^2 ds \right)^{\frac{1}{2}} \leq Ch^{\gamma} \|v\|_{\gamma-1}, \quad 0 \leq \gamma \leq 2, \quad t > 0. \quad (61)$$

(iii) For $v \in D(A^{-\frac{\rho}{2}})$

$$\left\| \int_0^t G_h(s)v ds \right\| \leq Ch^{2-\rho} \|v\|_{-\rho}, \quad 0 \leq \rho \leq 1, \quad t > 0. \quad (62)$$

(iv) For $v \in D(A^{\frac{\delta-1}{2}})$

$$\left(\int_0^t \|G_h(s)v\|^{\frac{1}{H}} ds \right)^{2H} \leq Ch^{2(2H+\delta-1)} \|v\|_{\delta-1}^2, \quad 0 \leq \delta \leq 1, \quad t > 0. \quad (63)$$

Proof See [34, Lemmas 3.1 and 3.2 (iv) and (v)] for the proofs of (i)–(iii). Let us prove (iv).

– For $H = \frac{1}{2}$, using (61) with $\gamma = \delta$, we obtain

$$\begin{aligned} \left(\int_0^t \|G_h(s)v\|^{\frac{1}{H}} ds \right)^{2H} &= \int_0^t \|G_h(s)v\|^2 ds \\ &\leq Ch^{2\delta} \|v\|_{\delta-1}^2 = Ch^{2(2H+\delta-1)} \|v\|_{\delta-1}^2. \end{aligned} \quad (64)$$

– For $H = 1$, using (62) with $\rho = 1 - \delta$, we obtain

$$\begin{aligned} \left(\int_0^t \|G_h(s)v\|^{\frac{1}{H}} ds \right)^{2H} &= \left(\int_0^t \|G_h(s)v\| ds \right)^2 \\ &\leq C \left(h^{2-(1-\delta)} \|v\|_{\delta-1} \right)^2 \\ &= Ch^{2+2\delta} \|v\|_{\delta-1}^2 = Ch^{2(2H+\delta-1)} \|v\|_{\delta-1}^2. \end{aligned} \quad (65)$$

Hence, the proof of (63) is thus completed by interpolation theory. \square

Lemma 6 (Space error) *Let Assumptions 1–3 be fulfilled, then the following error estimate holds for the mild solution (15) and the discrete problem (45) holds*

$$\|X(t) - X^h(t)\|_{L^2(\Omega, \mathcal{H})} \leq Ch^{2H+\beta-1}. \quad (66)$$

Proof Using triangle inequality, we have

$$\begin{aligned} e(t) &:= \|X(t) - X^h(t)\|_{L^2(\Omega, \mathcal{H})} \\ &= \|S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)\phi(s)dB^H(s) \\ &\quad - S_h(t)P_hX_0 - \int_0^t S_h(t-s)P_hF(X^h(s))ds \\ &\quad - \int_0^t S_h(t-s)P_h\phi(s)dB^H(s)\|_{L^2(\Omega, \mathcal{H})} \\ &\leq \|G_h(t)X_0\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^t S(t-s)F(X(s))ds - \int_0^t S_h(t-s)P_hF(X(s))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^t G_h(t-s)\phi(s)dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: e_0(t) + e_1(t) + e_2(t), \end{aligned}$$

with

$$\begin{aligned} e_0(t) &= \|G_h(t)X_0\|_{L^2(\Omega, \mathcal{H})}, \\ e_1(t) &= \left\| \int_0^t S(t-s)F(X(s))ds - \int_0^t S_h(t-s)P_hF(X^h(s))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ \text{and } e_2(t) &= \left\| \int_0^t G_h(t-s)\phi(s)dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}. \end{aligned}$$

Note that the deterministic error is already estimated, so we will mostly concentrate our study on the stochastic error. Indeed, Lemma 5 with $r = \gamma = 2H + \beta - 1$ yields

$$e_0(t) = \|G_h(t)X_0\|_{L^2(\Omega, V)} \leq Ch^{2H+\beta-1} \|A^{\frac{2H+\beta-1}{2}} X_0\|_{L^2(\Omega, \mathcal{H})}. \quad (67)$$

Using triangle inequality, the boundedness of $S_h(t-s)$ and P_h , Assumption 2 (more precisely (18)), we estimate the error $e_1(t)$ as follows

$$\begin{aligned} e_1(t) &= \left\| \int_0^t S(t-s)F(X(s))ds - \int_0^t S_h(t-s)P_h F(X(s))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\leq \left\| \int_0^t (S(t-s) - S_h(t-s)P_h) F(X(s))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^t S_h(t-s)P_h (F(X(s)) - F(X^h(s)))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\leq \left\| \int_0^t (S(t-s) - S_h(t-s)P_h) (F(X(t)) - F(X(s)))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^t (S(t-s) - S_h(t-s)P_h) F(X(t))ds \right\|_{L^2(\Omega, \mathcal{H})} + C \int_0^t e(s)ds. \end{aligned} \quad (68)$$

Applying Lemma 5 (i) with $r = 2H + \beta - 1$ and $\gamma = 0$, Assumption 2 (18), Theorem 1 (more precisely (29)) to the first term and Lemma 5 (iii) with $\rho = 0$, Theorem 1 (more precisely (26)) and Lemma 4 to the second term yields

$$\begin{aligned} e_1(t) &\leq Ch^{2H+\beta-1} \int_0^t (t-s)^{-\frac{2H+\beta-1}{2}} \|X(t) - X(s)\|_{L^2(\Omega, \mathcal{H})} ds \\ &\quad + Ch^2 \|F(X(t))\|_{L^2(\Omega, \mathcal{H})} + C \int_0^t e(s)ds \\ &\leq Ch^{2H+\beta-1} + C \int_0^t e(s)ds. \end{aligned} \quad (69)$$

For the estimation of $e_2(t)$, triangle inequality, the estimate $(a+b)^2 \leq 2a^2 + 2b^2$, (13) and (14), Lemma 5 (i) with $r = 2H + \beta - 1$, $\gamma = \beta - 1$, (iv) with $\delta = \beta$) and

Assumption 1 yields

$$\begin{aligned}
 e_2(t)^2 &= \mathbb{E} \left\| \int_0^t G_h(t-s) \phi(s) dB^H(s) \right\|^2 \\
 &\leq 2\mathbb{E} \left\| \int_0^t G_h(t-s) (\phi(t) - \phi(s)) dB^H(s) \right\|^2 \\
 &\quad + 2\mathbb{E} \left\| \int_0^t G_h(t-s) \phi(t) dB^H(s) \right\|^2 \\
 &\leq 2C \int_0^t \|G_h(t-s) (\phi(t) - \phi(s))\|_{L_2^0}^2 ds \\
 &\quad + 2C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^t \|G_h(t-s) \phi(t) Q^{\frac{1}{2}} e_i\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq Ch^{2(2H+\beta-1)} \int_0^t (t-s)^{-2H} \|A^{\frac{\beta-1}{2}} (\phi(t) - \phi(s))\|_{L_2^0}^2 ds \\
 &\quad + C_H \sum_{i \in \mathbb{N}^d} Ch^{2(2H+\beta-1)} \|A^{\frac{\beta-1}{2}} \phi(t) Q^{\frac{1}{2}} e_i\|^2 \\
 &\leq Ch^{2(2H+\beta-1)} \int_0^t (t-s)^{2\delta-2H} ds + Ch^{2(2H+\beta-1)} \|A^{\frac{\beta-1}{2}} \phi(t)\|_{L_2^0}^2 \\
 &\leq Ch^{2(2H+\beta-1)} t^{2\delta-2H+1} + Ch^{2(2H+\beta-1)} \\
 &\leq Ch^{2(2H+\beta-1)}. \tag{70}
 \end{aligned}$$

Combining the estimates (67), (69), (70) and applying Gronwall inequality ends the proof. \square

4.1 Proof of Theorem 2 for implicit scheme

It is important to mention that the estimates made in this section are inspired by the results in [37, (4.7)–(4.14), (4.25)–(4.29)], when the linear operator A is self-adjoint. For our case where A is not necessarily self-adjoint, let us present some preparatory results.

Lemma 7 *For any m, h and Δt the following estimates holds*

(i)

$$\|(I + \Delta t A_h)^{-m}\| \leq 1. \tag{71}$$

(ii) For all $u \in D(A^{\frac{\gamma-1}{2}})$, $0 \leq \gamma \leq 1$,

$$\|(S_{h,\Delta t}^m - S_h(t_m))P_h u\| \leq C \Delta t^{\frac{2H+\gamma-1}{2}} t_m^{-H} \|u\|_{\gamma-1}. \tag{72}$$

(iii) For all $u \in D(A^{\frac{\gamma-1}{2}})$, $0 \leq \gamma \leq 1$,

$$\|(S_{h,\Delta t}^m - S_h(t))P_h u\| \leq C \Delta t^{\frac{2H+\gamma-1}{2}} t^{-H} \|u\|_{\gamma-1}, t \in [t_{j-1}, t_j), \tag{73}$$

- for any $j = 1, 2, \dots, M$.
 (iv) If $u \in D(A^{\mu/2})$, $0 \leq \mu \leq 2$, then

$$\|(S_{h,\Delta t}^m - S_h(t_m))P_h u\| \leq C \Delta t^{\mu/2} \|u\|_{\mu}. \quad (74)$$

- (v) For all non smooth data $u \in \mathcal{H}$,

$$\|(S_{h,\Delta t}^m - S_h(t_m))P_h u\| \leq C \Delta t t_m^{-1} \|u\|. \quad (75)$$

Proof See [34, Lemma 3.3] for the proof of (i), (iv) and (v). For the proof of (ii), we use [34, Lemma 3.3 (iii) (88)] as follows

$$\begin{aligned} \left\| K_h(m) A_h^{\frac{1-\gamma}{2}} \right\|_{L(\mathcal{H})} &\leq C t_m^{-3/2+\gamma/2} \Delta t = C t_m^{-H} t_m^{\frac{2H+\gamma-3}{2}} \Delta t \\ &\leq C t_m^{-H} t_1^{\frac{2H+\gamma-3}{2}} \Delta t = C t_m^{-H} \Delta t^{\frac{2H+\gamma-1}{2}}. \end{aligned} \quad (76)$$

Hence substituting (76) in [34, (84)] completes the proof of (ii). Now for the proof of (iii), triangle inequality, Lemma 7 (ii), the property of discrete semigroup and [34, (83)] yields

$$\begin{aligned} &\|(S_{h,\Delta t}^m - S_h(t))P_h u\| \\ &\leq \|(S_{h,\Delta t}^m - S_h(t_j))P_h u\| + \|(S_h(t_j) - S_h(t))P_h u\| \\ &\leq \|S_h(t) A_h^H\|_{L(\mathcal{H})} \left\| A_h^{-\frac{2H+\gamma-1}{2}} (S_h(t_j - t) - I) \right\|_{L(\mathcal{H})} \|A_h^{\frac{\gamma-1}{2}} P_h u\| \\ &\quad + C \Delta t^{\frac{2H+\gamma-1}{2}} t_j^{-H} \|u\|_{\gamma-1} \\ &\leq C t^{-H} (t_j - t)^{\frac{2H+\gamma-1}{2}} \|A_h^{\frac{\gamma-1}{2}} u\| + C \Delta t^{\frac{2H+\gamma-1}{2}} t_j^{-H} \|u\|_{\gamma-1} \\ &\leq C \Delta t^{\frac{2H+\gamma-1}{2}} t^{-H} \|u\|_{\gamma-1}. \end{aligned} \quad (77)$$

The proof of Lemma 7 is thus completed. \square

Lemma 8 (i) For any $\rho \in [0, 1]$ and $u \in D(A^{-\rho/2})$ there exists a positive constant C such that

$$\left\| \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (S_{h,\Delta t}^j - S_h(s)) P_h u ds \right\| \leq C \Delta t^{\frac{2-\rho}{2}-\epsilon} \|u\|_{-\rho}. \quad (78)$$

- (ii) For any $\mu \in [0, 1]$ and $u \in D(A^{\frac{\mu-1}{2}})$ the following estimate holds

$$\left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s)) P_h u\|^{\frac{1}{H}} ds \right)^{2H} \leq C \Delta t^{2H+\mu-1-\epsilon} \|u\|_{\mu-1}^2. \quad (79)$$

Where ϵ is an arbitrary small number.

Proof See [34, Lemma 3.5] for the proof of (i). For the proof of (ii), we have

$$\begin{aligned}
 & \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s))P_h u\|^{\frac{1}{H}} ds \\
 &= \int_0^{\Delta t} \|(S_{h,\Delta t}^1 - S_h(s))P_h u\|^{\frac{1}{H}} ds + \sum_{j=2}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s))P_h u\|^{\frac{1}{H}} ds \\
 &=: K_1 + K_2.
 \end{aligned} \tag{80}$$

Using triangle inequality and the estimate $(a+b)^k \leq 2^{k-1}a^k + 2^{k-1}b^k$ (with $k \geq 1$ and $a, b \geq 0$) we obtain

$$\begin{aligned}
 K_1 &\leq 2^{\frac{1}{H}-1} \int_0^{\Delta t} \|(S_{h,\Delta t}^1 - S_h(t_1))P_h u\|^{\frac{1}{H}} ds \\
 &\quad + 2^{\frac{1}{H}-1} \int_0^{\Delta t} \|(S_h(t_1) - S_h(s))P_h u\|^{\frac{1}{H}} ds \\
 &=: 2^{\frac{1}{H}-1} K_{11} + 2^{\frac{1}{H}-1} K_{12}.
 \end{aligned} \tag{81}$$

By Lemma 7 (ii) with $\gamma = \mu$ we obtain

$$\begin{aligned}
 K_{11} &\leq C \int_0^{\Delta t} t_1^{-1} \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} ds \\
 &\leq C \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}}.
 \end{aligned} \tag{82}$$

By inserting an appropriate power of A_h , [34, (81)] and Remark 2 yields

$$\begin{aligned}
 & K_{12} \\
 &\leq \int_0^{\Delta t} \left\| (S_h(t_1 - s) - I) S_h(s) A_h^{\frac{1-\mu}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A_h^{\frac{\mu-1}{2}} P_h u \right\|^{\frac{1}{H}} ds \\
 &\leq C \int_0^{\Delta t} \left\| (S_h(t_1 - s) - I) A_h^{\frac{-2H-\mu+1}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| S_h(s) A_h^H \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A_h^{\frac{\mu-1}{2}} P_h u \right\|^{\frac{1}{H}} ds \\
 &\leq C \int_0^{\Delta t} (t_1 - s)^{\frac{2H+\mu-1}{2H}} \left\| S_h(s) A_h^H \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A_h^{\frac{\mu-1}{2}} u \right\|^{\frac{1}{H}} ds \\
 &\leq C \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} \left(\int_0^{\Delta t} \left\| S_h(s) A_h^H \right\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right) \\
 &\leq C \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} \left(\int_0^{\Delta t} \left\| S_h(\Delta t - s) A_h^H \right\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right) \\
 &\leq C \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}}.
 \end{aligned} \tag{83}$$

Substituting (82) and (83) in (81) yields

$$K_1 \leq C \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}}. \tag{84}$$

Concerning the estimate of K_2 , let $\epsilon > 0$ small enough, Lemma 7 (iii) with $\gamma = \mu$ yields

$$\begin{aligned}
 K_2 &= \sum_{j=2}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s)) P_h u\|^{\frac{1}{H}} ds \\
 &\leq C \sum_{j=2}^m \int_{t_{j-1}}^{t_j} \Delta t^{\frac{2H+\mu-1}{2H}} s^{-1} \|u\|_{\mu-1}^{\frac{1}{H}} ds \\
 &\leq C \left(\sum_{j=2}^m \int_{t_{j-1}}^{t_j} s^{-1+\frac{\epsilon}{2H}} s^{-\frac{\epsilon}{2H}} ds \right) \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} \\
 &\leq C \left(\sum_{j=2}^m \int_{t_{j-1}}^{t_j} s^{-1+\frac{\epsilon}{2H}} t_1^{-\frac{\epsilon}{2H}} ds \right) \Delta t^{\frac{2H+\mu-1}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} \\
 &\leq C \left(\sum_{j=2}^m \int_{t_{j-1}}^{t_j} s^{-1+\frac{\epsilon}{2H}} ds \right) \Delta t^{\frac{2H+\mu-1-\epsilon}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}} \\
 &\leq C \Delta t^{\frac{2H+\mu-1-\epsilon}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}}.
 \end{aligned} \tag{85}$$

Adding (84) and (85) yields

$$\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s)) P_h u\|^{\frac{1}{H}} ds \leq C \Delta t^{\frac{2H+\mu-1-\epsilon}{2H}} \|u\|_{\mu-1}^{\frac{1}{H}}, \tag{86}$$

hence

$$\left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \|(S_{h,\Delta t}^j - S_h(s)) P_h u\|^{\frac{1}{H}} ds \right)^{2H} \leq C \Delta t^{2H+\mu-1-\epsilon} \|u\|_{\mu-1}^2. \tag{87}$$

This completes the proof of Lemma 8. \square

With these two lemmas, we are now ready to prove our theorem for the implicit scheme. In fact, using the standard technique in the error analysis, we split the fully discrete error in two terms as

$$\begin{aligned}
 \|X(t_m) - X_m^h\|_{L^2(\Omega, \mathcal{H})} &\leq \|X(t_m) - X^h(t_m)\|_{L^2(\Omega, \mathcal{H})} + \|X^h(t_m) - X_m^h\|_{L^2(\Omega, \mathcal{H})} \\
 &=: err_0 + err_1.
 \end{aligned}$$

Note that the space error err_0 is estimated by Lemma 6. It remains to estimate the time error err_1 .

We recall that the exact solution at t_m of the semidiscrete problem (44) is given by

$$\begin{aligned}
 X^h(t_m) &= S_h(t_m) X_0^h + \int_0^{t_m} S_h(t_m - s) P_h F(X^h(s)) ds \\
 &\quad + \int_0^{t_m} S_h(t_m - s) P_h \phi(s) dB^H(s).
 \end{aligned} \tag{88}$$

We also recall that the numerical solution at t_m given by (48) can be rewritten as

$$\begin{aligned} X_m^h &= S_{h,\Delta t}^m X_0^h + \int_0^{t_m} S_{h,\Delta t}^{m-[s]^m} P_h F(X_{[s]^m}^h) ds \\ &\quad + \int_0^{t_m} S_{h,\Delta t}^{m-[s]^m} P_h \phi([s]) dB^H(s), \end{aligned} \quad (89)$$

where the notation $[t]$, $[t]^m$ are defined by

$$[t] := \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t \quad \text{and} \quad [t]^m := \left\lfloor \frac{t}{\Delta t} \right\rfloor. \quad (90)$$

It follows from (88) and (89) that

$$\begin{aligned} err_1 &\leq \left\| (S_h(t_m) - S_{h,\Delta t}^m) P_h X_0 \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s) P_h F(X^h(s)) - S_{h,\Delta t}^{m-[s]^m} P_h F(X_{[s]^m}^h) ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s) P_h \phi(s) - S_{h,\Delta t}^{m-[s]^m} P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: I_0 + I_1 + I_2. \end{aligned} \quad (91)$$

As we said at the beginning of this section, following closely the work done in [37, (4.7)–(4.14)] and replacing its preparatory results with Lemma 7 (i), (iv) with $\mu = 2H + \beta - 1$, (v), Lemma 8 (i) with $\rho = 0$, Remark 2 (20) with $\delta = \gamma = 1$, Assumptions 2-3, boundedness of $S_h(t_m - s)$ and P_h , the stability properties of a discrete semigroup $S_h(t)$, (18) and (47), we have

$$I_0 + I_1 \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t^{1-\epsilon} + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - X_i^h\|_{L^2(\Omega, \mathcal{H})}. \quad (92)$$

Note that in this work, we do not need to impose an assumption on F'' to increase the convergence rate as it is done in [37]. Indeed, thanks to (47) the following estimate is largely sufficient to reach a higher rate.

$$\begin{aligned} I_{11} &\leq \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \|S_h(t_m - s) P_h (F(X^h(s)) - F(X^h(t_i)))\|_{L^2(\Omega, \mathcal{H})} ds \\ &\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \|X^h(s) - X^h(t_i)\|_{L^2(\Omega, \mathcal{H})} ds \\ &\leq C \Delta t^{\frac{2H+\beta-1}{2}}. \end{aligned} \quad (93)$$

Let us focus now on the estimate I_2 , using triangle inequality and the estimate $(a + b)^2 \leq 2a^2 + 2b^2$ we split it in three terms

$$\begin{aligned}
 I_2^2 &= \left\| \int_0^{t_m} S_h(t_m - s) P_h \phi(s) - S_{h, \Delta t}^{m-[s]^m} P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq 2 \left\| \int_0^{t_m} S_h(t_m - s) P_h (\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 2 \left\| \int_0^{t_m} (S_h(t_m - s) - S_{h, \Delta t}^{m-[s]^m}) P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq 2 \left\| \int_0^{t_m} S_h(t_m - s) P_h (\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 4 \left\| \int_0^{t_m} (S_h(t_m - s) - S_{h, \Delta t}^{m-[s]^m}) P_h (\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 4 \left\| \int_0^{t_m} (S_h(t_m - s) - S_{h, \Delta t}^{m-[s]^m}) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &=: 2I_{21}^2 + 4I_{22}^2 + 4I_{23}^2.
 \end{aligned} \tag{94}$$

Firstly using (14), inserting an appropriate power of A_h , [34, (81)], Assumption 1 (more precisely (17)) and Remark 2 ((22) with $\rho = 1 - \beta$) we obtain

$$\begin{aligned}
 I_{21}^2 &= \left\| \int_0^{t_m} S_h(t_m - s) P_h (\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C \int_0^{t_m} \|S_h(t_m - s) P_h (\phi(s) - \phi([s]))\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_m} \left\| S_h(t_m - s) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \left\| A_h^{\frac{\beta-1}{2}} P_h (\phi(s) - \phi([s])) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_m} \left\| S_h(t_m - s) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \left\| A^{\frac{\beta-1}{2}} (\phi(s) - \phi([s])) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_m} (s - [s])^{2\delta} \left\| S_h(t_m - s) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 ds \\
 &\leq C \Delta t^{2\delta} \int_0^{t_m} \left\| S_h(t_m - s) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 ds \\
 &\leq C \Delta t^{2\delta} t_m^\beta \leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{95}$$

Secondly (14), the change of variable $j = m - k$ and $\varsigma = t_m - s$, Lemma 7 (iii) with $\gamma = \beta$ and Assumption 1 (more precisely (17)) yields

$$\begin{aligned}
 I_{22}^2 &= \left\| \int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h(\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C \int_0^{t_m} \left\| \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h(\phi([s]) - \phi(t_{m-1})) \right\|_{L_2^0}^2 ds \\
 &\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left\| \left(S_h(t_m - s) - S_{h,\Delta t}^{m-k} \right) P_h(\phi(t_k) - \phi(t_{m-1})) \right\|_{L_2^0}^2 ds \\
 &\leq C \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\| \left(S_h(\varsigma) - S_{h,\Delta t}^j \right) P_h(\phi(t_{m-j}) - \phi(t_{m-1})) \right\|_{L_2^0}^2 d\varsigma \\
 &\leq C \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \Delta t^{2H+\beta-1} \varsigma^{-2H} \left\| A^{\frac{\beta-1}{2}} (\phi(t_{m-j}) - \phi(t_{m-1})) \right\|_{L_2^0}^2 d\varsigma \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \varsigma^{-2H} t_{j-1}^{2\delta} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \varsigma^{2\delta-2H} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\int_0^{t_m} \varsigma^{2\delta-2H} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} t_m^{2\delta-2H+1} \leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{96}$$

Thirdly using (13), the change of variable $j = m - k$ and $\varsigma = t_m - s$, Lemma 8 (ii) with $\mu = \beta$ and Assumption 1 (more precisely (16)) we obtain

$$\begin{aligned}
 I_{23}^2 &= \left\| \int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{t_m} \left\| \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left\| \left(S_h(t_m - s) - S_{h,\Delta t}^{m-k} \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\| \left(S_h(\varsigma) - S_{h,\Delta t}^j \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} d\varsigma \right)^{2H} \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\Delta t^{2H+\beta-1-\epsilon} \|\phi(t_{m-1}) Q^{\frac{1}{2}} e_i\|_{\beta-1}^2 \right) \\
 &\leq C \Delta t^{2H+\beta-1-\epsilon} \|A^{\frac{\beta-1}{2}} \phi(t_{m-1})\|_{L_2^0}^2 \leq C \Delta t^{2H+\beta-1-\epsilon}.
 \end{aligned} \tag{97}$$

Hence, inserting (95)–(97) in (94) and taking the square-root give

$$I_2 \leq C \Delta t^{\frac{2H+\beta-1}{2}-\epsilon}. \quad (98)$$

Adding (92) and (98) we obtain

$$\begin{aligned} err_1 &= \|X^h(t_m) - X_m^h\|_{L^2(\Omega, \mathcal{H})} \\ &\leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t^{1-\epsilon} + C \Delta t^{\frac{2H+\beta-1}{2}-\epsilon} \\ &\quad + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - X_i^h\|_{L^2(\Omega, \mathcal{H})}. \end{aligned}$$

Applying the discrete version of the Gronwall inequality yields

$$err_1 \leq C \Delta t^{\frac{2H+\beta-1}{2}-\epsilon}. \quad (99)$$

Adding (66) and (99) completes the proof.

In what follows, we will present a corollary of Theorem 2 for the implicit Euler scheme where the linear operator A is assumed to be self-adjoint. The optimal strong convergence rate in time $\mathcal{O}(\Delta t)$ is reached.

Corollary 1 *Let $X(t_m)$ be the mild solution of (5) (A self-adjoint) at time $t_m = m \Delta t$, $\Delta t \geq 0$ represented by (15). Let X_m^h be the numerical approximation through (48). Under Assumptions 1–3, $\beta \in (0, 1]$, then the following holds*

$$\left(\mathbb{E} \|X(t_m) - X_m^h\|^2 \right)^{\frac{1}{2}} \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{2H+\beta-1}{2}} \right). \quad (100)$$

For the proof of this corollary, we need to update our preparatory results, more precisely Lemma 8 in the self-adjoint case. The result is presented in the following lemma:

Lemma 9 (i) *For any $\rho \in [0, 1]$ and $u \in D(A^{-\rho/2})$ there exists a positive constant C such that*

$$\left\| \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (S_{h, \Delta t}^j - S_h(s)) P_h u ds \right\| \leq C \Delta t^{\frac{2-\rho}{2}} \|u\|_{-\rho}. \quad (101)$$

(ii) *For any $\mu \in [0, 1]$ and $x \in D(A^{\frac{\mu-1}{2}})$ the following estimate holds*

$$\begin{aligned} &\sum_{i,j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle S_h(u, t_i) P_h x, S_h(v, t_j) P_h x \rangle \kappa(u, v) du dv \\ &\leq C \Delta t^{2H+\mu-1} \|x\|_{\mu-1}^2, \end{aligned} \quad (102)$$

and

$$\begin{aligned} & \sum_{i,j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \langle \mathcal{T}_h(i) P_h x, \mathcal{T}_h(j) P_h x \rangle \kappa(u, v) du dv \\ & \leq C \Delta t^{2H+\mu-1} \|x\|_{\mu-1}^2, \end{aligned} \quad (103)$$

where

$$\begin{aligned} S_h(u, t_i) &:= S_h(t_m - u) - S_h(t_m - t_i), \quad \mathcal{T}_h(i) := S_h(t_m - t_i) - S_{h,\Delta t}^{m-i}, \\ &\text{and } \kappa(u, v) := H(2H - 1)|u - v|^{2H-2}. \end{aligned} \quad (104)$$

Proof See [15, Proof of Lemma 4.4 (i)] for the proof of (i) and [38, Lemmas 4.8 and 4.9], [34, (83)] for the proof of (ii). \square

With this new lemma, we are now in position to prove our Corollary 1.

Proof of Corollary 1 Recall that the time error err_1 is defined as

$$\begin{aligned} err_1 &\leq \|(S_h(t_m) - S_{h,\Delta t}^m) P_h X_0\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s) P_h F(X^h(s)) - S_{h,\Delta t}^{m-[s]^m} P_h F(X_{[s]^m}^h) ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s) P_h \phi(s) - S_{h,\Delta t}^{m-[s]^m} P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: I_0 + I_1 + I_2. \end{aligned} \quad (105)$$

Following closely the work done in [37, (4.7)–(4.14)] and replacing its preparatory results with Lemma 7 (i), (iv) with $\mu = 2H + \beta - 1$, (v) with $\sigma = 0$, Lemma 9 (i) with $\rho = 0$, Remark 2 (20) with $\delta = \gamma = 1$, Assumptions 2–3, boundedness of $S_h(t_m - s)$ and P_h , the stability properties of a discrete semigroup $S_h(t)$, (18) and (47), we have

$$I_0 + I_1 \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - X_i^h\|_{L^2(\Omega, \mathcal{H})}. \quad (106)$$

Concerning the estimate I_2 , we also split it in three terms as in (94). The estimates I_{21}^2 and I_{22}^2 are still the same but we need to re-estimate I_{23}^2 . In this fact, since the sequence of random variables

$\left(\int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s), i \in \mathbb{N}^d \right)$ is mutually independent Gaussian random variable, using the estimate $(a + b)^2 \leq 2a^2 + 2b^2$,

notation (104), Assumption 1 (more precisely (16)), Lemma 9 (ii) with $\mu = \beta$, we obtain

$$\begin{aligned}
 I_{23}^2 &= \left\| \int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &= \left\| \sum_{i \in \mathbb{N}^d} \int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &= \sum_{i \in \mathbb{N}^d} \left\| \int_0^{t_m} \left(S_h(t_m - s) - S_{h,\Delta t}^{m-[s]^m} \right) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq 2 \sum_{i \in \mathbb{N}^d} \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 2 \sum_{i \in \mathbb{N}^d} \left\| \int_0^{t_m} (S_h(t_m - [s]) - S_{h,\Delta t}^{m-[s]^m}) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &= 2 \sum_{i \in \mathbb{N}^d} \left\| \int_0^{t_m} S_h(s, [s]) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 2 \sum_{i \in \mathbb{N}^d} \left\| \int_0^{t_m} \mathcal{T}_h([s]^m) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i d\beta_i^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &= 2 \sum_{i \in \mathbb{N}^d} \int_0^{t_m} \int_0^{t_m} \left\langle S_h(u, [u]) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i, S_h(v, [v]) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\rangle \kappa(u, v) du dv \\
 &\quad + 2 \sum_{i \in \mathbb{N}^d} \int_0^{t_m} \int_0^{t_m} \left\langle \mathcal{T}_h([u]^m) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i, \mathcal{T}_h([v]^m) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\rangle \kappa(u, v) du dv \\
 &= 2 \sum_{i \in \mathbb{N}^d} \left(\sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left\langle S_h(u, t_i) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i, S_h(v, t_j) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\rangle \kappa(u, v) du dv \right) \\
 &\quad + 2 \sum_{i \in \mathbb{N}^d} \left(\sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_i}^{t_{i+1}} \left\langle \mathcal{T}_h(i) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i, \mathcal{T}_h(j) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\rangle \kappa(u, v) du dv \right) \\
 &\leq C \sum_{i \in \mathbb{N}^d} \left(\Delta t^{2H+\beta-1} \|\phi(t_{m-1}) Q^{\frac{1}{2}} e_i\|_{\beta-1}^2 \right) \\
 &\leq C \Delta t^{2H+\beta-1} \|A^{\frac{\beta-1}{2}} \phi(t_{m-1})\|_{\ell_2^0}^2 \leq C \Delta t^{2H+\beta-1}, \tag{107}
 \end{aligned}$$

hence inserting (95), (96) and (107) in (94) and taking the square-root give

$$I_2 \leq C \Delta t^{\frac{2H+\beta-1}{2}}. \tag{108}$$

Adding (106) and (108) we obtain

$$\begin{aligned}
 err_1 &= \|X^h(t_m) - X_m^h\|_{L^2(\Omega, \mathcal{H})} \\
 &\leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - X_i^h\|_{L^2(\Omega, \mathcal{H})}.
 \end{aligned}$$

Applying the discrete version of the Gronwall inequality yields

$$err_1 \leq C \Delta t^{\frac{2H+\beta-1}{2}}. \quad (109)$$

Adding (66) and (109) completes the proof. \square

4.2 Proof of Theorem 2 for SETD1

As usual, splitting the fully discrete error in two terms yields

$$\begin{aligned} \|X(t_m) - Y_m^h\|_{L^2(\Omega, \mathcal{H})} &\leq \|X(t_m) - X^h(t_m)\|_{L^2(\Omega, \mathcal{H})} + \|X^h(t_m) - Y_m^h\|_{L^2(\Omega, \mathcal{H})} \\ &=: err_0 + err_2. \end{aligned}$$

Since the space error err_0 has been estimated by Lemma 6, we only need to estimate the time error err_2 . Remember that the exact solution at t_m is given by

$$\begin{aligned} X^h(t_m) &= S_h(t_m)X_0^h + \int_0^{t_m} S_h(t_m - s)P_h F(X^h(s))ds \\ &\quad + \int_0^{t_m} S_h(t_m - s)P_h \phi(s)dB^H(s) \end{aligned} \quad (110)$$

and we recall that the numerical solution at t_m given by (49) can be rewritten as

$$\begin{aligned} Y_m^h &= S_h(t_m)X_0^h + \int_0^{t_m} S_h(t_m - s)P_h F(X_{[s]^m}^h)ds \\ &\quad + \int_0^{t_m} S_h(t_m - [s])P_h \phi([s])dB^H(s), \end{aligned} \quad (111)$$

where the notations $[t]$ and $[t]^m$ are given by (90). By (110) and (111), we have

$$\begin{aligned} err_2 &\leq \left\| \int_0^{t_m} S_h(t_m - s)P_h (F(X^h(s)) - F(Y_{[s]^m}^h))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s)P_h \phi(s) - S_h(t_m - [s])P_h \phi([s])dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: I'_1 + I'_2. \end{aligned} \quad (112)$$

Applying the triangle inequality yields

$$\begin{aligned} I'_1 &\leq \left\| \int_0^{t_m} S_h(t_m - s)P_h (F(X^h(s)) - F(X^h([s])))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{t_m} S_h(t_m - s)P_h (F(X^h([s])) - F(Y_{[s]^m}^h))ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: I'_{11} + I'_{12}. \end{aligned} \quad (113)$$

Using the boundedness of P_h and $S_h(t_m - s)$, Lemma 3 and (47), we easily have

$$I'_{11} \leq C \Delta t^{\frac{2H+\beta-1}{2}}, \quad (114)$$

and

$$I'_{12} \leq C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - Y_i^h\|_{L^2(\Omega, \mathcal{H})}. \quad (115)$$

Adding (114) and (115), we obtain

$$I'_1 \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - Y_i^h\|_{L^2(\Omega, \mathcal{H})}. \quad (116)$$

We estimate at now I'_2 . Using triangle inequality and the estimate $(a + b)^2 \leq 2a^2 + 2b^2$, we split it in three terms

$$\begin{aligned} I_2'^2 &= \left\| \int_0^{t_m} S_h(t_m - s) P_h \phi(s) - S_h(t_m - [s]) P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\leq 2 \left\| \int_0^{t_m} S_h(t_m - s) P_h (\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\quad + 2 \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\leq 2 \left\| \int_0^{t_m} S_h(t_m - s) P_h (\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\quad + 4 \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h (\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\quad + 4 \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &=: 2I_{21}'^2 + 4I_{22}'^2 + 4I_{23}'^2. \end{aligned} \quad (117)$$

Thanks to (95) we have

$$I_{21}'^2 \leq C \Delta t^{2H+\beta-1}. \quad (118)$$

Thereafter, (14), the change of variable $j = m - k$ and $\varsigma = t_m - s$, inserting an appropriate power of A_h , [34, (81)] and Remark 2 (more precisely (20) with $\gamma = \frac{2H+\beta-1}{2}$, (21) and Assumption 1 (more precisely (17)) yields

$$\begin{aligned}
 I_{22}^2 &= \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h(\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C \int_0^{t_m} \|(S_h(t_m - s) - S_h(t_m - [s])) P_h(\phi([s]) - \phi(t_{m-1}))\|_{L_2^0}^2 ds \\
 &\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \|(S_h(t_m - s) - S_h(t_m - t_k)) P_h(\phi(t_k) - \phi(t_{m-1}))\|_{L_2^0}^2 ds \\
 &\leq C \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\| (S_h(\varsigma) - S_h(t_j)) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \\
 &\quad \times \left\| A_h^{\frac{\beta-1}{2}} P_h(\phi(t_{m-j}) - \phi(t_{m-1})) \right\|_{L_2^0}^2 d\varsigma \\
 &\leq C \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \left\| A_h^{-\frac{2H+\beta-1}{2}} (S_h(t_j - \varsigma) - I) \right\|_{L(\mathcal{H})}^2 \|A_h^H S_h(\varsigma)\|_{L(\mathcal{H})}^2 \\
 &\quad \times \left\| A_h^{\frac{\beta-1}{2}} (\phi(t_{m-j}) - \phi(t_{m-1})) \right\|_{L_2^0}^2 d\varsigma \\
 &\leq C \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \Delta t^{2H+\beta-1} \varsigma^{-2H} \left\| A_h^{\frac{\beta-1}{2}} (\phi(t_{m-j}) - \phi(t_{m-1})) \right\|_{L_2^0}^2 d\varsigma \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \varsigma^{-2H} t_{j-1}^{2\delta} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^m \int_{t_{j-1}}^{t_j} \varsigma^{2\delta-2H} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\int_0^{t_m} \varsigma^{2\delta-2H} d\varsigma \right) \\
 &\leq C \Delta t^{2H+\beta-1} t_m^{2\delta-2H+1} \leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{119}$$

Finally, using (13), inserting an appropriate power of A_h , [34, (81)] and Remark 2 (more precisely (20) with $\gamma = \frac{2H+\beta-1}{2}$, (21) and (24)), Assumption 1 (more precisely (16)) we obtain

$$\begin{aligned}
 I_{23}'^2 &\leq \left\| \int_0^{t_m} (S_h(t_m - s) - S_h(t_m - [s])) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{t_m} \left\| (S_h(t_m - s) - S_h(t_m - [s])) P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \left(\int_0^{t_m} \left\| (S_h(t_m - s) - S_h(t_m - [s])) A_h^{\frac{1-\beta}{2}} \right\|^{\frac{1}{H}}_{L(\mathcal{H})} ds \right)^{2H} \\
 &\quad \times \left(\sum_{i \in \mathbb{N}^d} \left\| A_h^{\frac{\beta-1}{2}} P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\leq C_H \left(\int_0^{t_m} \left\| A_h^{-\frac{2H+\beta-1}{2}} (S_h(s - [s]) - I) \right\|^{\frac{1}{H}}_{L(\mathcal{H})} \left\| A_h^H S_h(t_m - s) \right\|^{\frac{1}{H}}_{L(\mathcal{H})} ds \right)^{2H} \\
 &\quad \left(\sum_{i \in \mathbb{N}^d} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\leq C_H \left(\int_0^{t_m} \left\| A_h^H S_h(t_m - s) \right\|^{\frac{1}{H}}_{L(\mathcal{H})} (s - [s])^{\frac{2H+\beta-1}{2H}} ds \right)^{2H} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) \right\|_{L_2^0}^2 \\
 &\leq C_H \Delta t^{2H+\beta-1} \left(\int_0^{t_m} \left\| A_h^H S_h(t_m - s) \right\|^{\frac{1}{H}}_{L(\mathcal{H})} ds \right)^{2H} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) \right\|_{L_2^0}^2 \\
 &\leq C \Delta t^{2H+\beta-1} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) \right\|_{L_2^0}^2 \\
 &\leq C \Delta t^{2H+\beta-1}, \tag{120}
 \end{aligned}$$

hence inserting (118)–(120) in (117) and taking the square-root give

$$I_2' \leq C \Delta t^{\frac{2H+\beta-1}{2}}. \tag{121}$$

Adding (116) and (121) we obtain

$$err_2 \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t \sum_{i=0}^{m-1} \|X^h(t_i) - Y_i^h\|_{L^2(\Omega, \mathcal{H})}. \tag{122}$$

Using the discrete version of the Gronwall inequality yields

$$err_2 \leq C \Delta t^{\frac{2H+\beta-1}{2}}. \tag{123}$$

Combining (66) and (123) completes the proof.

4.3 Proof of Theorem 2 for SERS scheme

Before moving to the proof, we first present some preparatory results. Thanks to Assumption 4 and the works done in [26] we obtain

Lemma 10 [26, Lemma 5] *For all $M \in \mathbb{N}$ and all $\omega \in \Omega$, there is a positive constant C'_1 independent of $h, m, \Delta t$ and the sample ω such that*

$$\left\| e^{(-A_h + J_m^h(\omega))t} \right\|_{L(\mathcal{H})} \leq C'_1, \quad 0 \leq t \leq T.$$

Lemma 11 [26, Lemma 6] *The function $G_m^h(\omega)$ defined by (51) satisfies the global Lipschitz condition with a uniform constant $C' > 0$, independent of h, m and ω such that*

$$\left\| G_m^h(\omega)(u^h) - G_m^h(\omega)(v^h) \right\| \leq C' \|u^h - v^h\|, \quad \forall u^h, v^h \in V_h.$$

Lemma 12 [26, Lemma 9] *For all $\omega \in \Omega$, the stochastic perturbed semigroup $S_m^h(\omega)(t) := e^{(-A_h + J_m^h(\omega))t}$ satisfies the following properties*

(i) *For $\gamma_1, \gamma_2 \leq 1$ such that $0 \leq \gamma_1 + \gamma_2 \leq 1$,*

$$\left\| A_h^{-\gamma_1} S_m^h(\omega)(t) - I \right\|_{L(\mathcal{H})} \leq C t^{\gamma_1 + \gamma_2}, \quad t \in (0, T].$$

(ii) *For $\gamma_1 \geq 0$ we have*

$$\left\| S_m^h(\omega)(t) A_h^{\gamma_1} \right\|_{L(\mathcal{H})} \leq C t^{-\gamma_1}, \quad t \in (0, T].$$

(iii) *For $\gamma_1 \geq 0$ and $0 \leq \gamma_2 < 1$ such that $\gamma_2 - \gamma_1 \geq 0$, we have*

$$\left\| A_h^{-\gamma_1} S_m^h(\omega)(t) A_h^{\gamma_2} \right\|_{L(\mathcal{H})} \leq C t^{\gamma_1 - \gamma_2}, \quad t \in (0, T].$$

(iv) *For $\gamma_1, \gamma_2 > 0$ such that $0 \leq \gamma_1 - \gamma_2 \leq 1$, then the following estimate holds*

$$\left\| A_h^{-\gamma_1} (S_m^h(\omega)(t) - I) A_h^{\gamma_2} \right\|_{L(\mathcal{H})} \leq C t^{\gamma_1 - \gamma_2}, \quad t \in (0, T].$$

Lemma 13 [26, Lemma 10] *The stochastic perturbed semigroup $S_m^h(\omega)$ satisfies the following property*

$$\left\| e^{(-A_h + J_m^h(\omega))\Delta t} \dots e^{(-A_h + J_k^h(\omega))\Delta t} A_h^v \right\|_{L(\mathcal{H})} \leq C t_{m+1-k}^{-v}, \quad 0 \leq v < 1,$$

where C is a positive constant independent of $m, k, h, \Delta t$ and the sample ω .

We can now prove our theorem. As in the proof of the previous schemes, we split the fully discrete error in two terms as

$$\begin{aligned} \|X(t_m) - Z_m^h\|_{L^2(\Omega, \mathcal{H})} &\leq \|X(t_m) - X^h(t_m)\|_{L^2(\Omega, \mathcal{H})} + \|X^h(t_m) - Z_m^h\|_{L^2(\Omega, \mathcal{H})} \\ &=: \text{err}_0 + \text{err}_3. \end{aligned} \quad (124)$$

By Lemma 6 we have the estimate of the space error

$$\text{err}_0 \leq C h^{2H+\beta-1}.$$

Consider now the estimate of the time error err_3 . We recall that the semidiscrete problem (44) can be rewritten as

$$dX^h(t) = [-A_h X^h(t) + J_m^h X^h(t) + G_m^h(X^h(t))]\Delta t + P_h \phi(t) dB^H(t) \quad (125)$$

for all $t_m \leq t \leq t_{m+1}$ where J_m^h and G_m^h is given by (50) and (51). Hence, the exact solution at time t_m of the semidiscrete problem (125) is given by

$$\begin{aligned} X^h(t_m) &= e^{(-A_h + J_{m-1}^h)\Delta t} X^h(t_{m-1}) + \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} G_{m-1}^h(X^h(s)) ds \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h \phi(s) dB^H(s) \end{aligned} \quad (126)$$

and the numerical solution (50) can be rewritten as

$$\begin{aligned} Z_m^h &= e^{(-A_h + J_{m-1}^h)\Delta t} Z_{m-1}^h + \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} G_{m-1}^h(Z_{m-1}^h) ds \\ &\quad + \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)\Delta t} P_h \phi(t_{m-1}) dB^H(s). \end{aligned} \quad (127)$$

If $m = 1$ then from (126) and (127) we obtain

$$\begin{aligned} &\|X^h(t_1) - Z_1^h\|_{L^2(\Omega, \mathcal{H})} \\ &\leq \left\| \int_0^{\Delta t} e^{(-A_h + J_0^h)(\Delta t-s)} (G_0^h(X^h(s)) - G_0^h(Z_0^h)) ds \right\|_{L^2(\Omega, \mathcal{H})} \\ &\quad + \left\| \int_0^{\Delta t} e^{(-A_h + J_0^h)(\Delta t-s)} P_h \phi(s) - e^{(-A_h + J_0^h)\Delta t} P_h \phi(0) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})} \\ &=: I'' + II''. \end{aligned} \quad (128)$$

By [26, (93)] we have the estimate

$$I'' \leq C \Delta t. \quad (129)$$

Using triangle inequality and the estimate $(a + b)^2 \leq 2a^2 + 2b^2$ we split it into three terms

$$\begin{aligned} II''^2 &= \left\| \int_0^{t_m} e^{(-A_h + J_0^h)(\Delta t-s)} P_h \phi(s) - e^{(-A_h + J_0^h)\Delta t} P_h \phi(0) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\leq 2 \left\| \int_0^{t_m} e^{(-A_h + J_0^h)(\Delta t-s)} P_h (\phi(s) - \phi(0)) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\quad + 2 \left\| \int_0^{t_m} (e^{(-A_h + J_0^h)(\Delta t-s)} - e^{(-A_h + J_0^h)\Delta t}) P_h \phi(0) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &=: 2II_1''^2 + 2II_2''^2. \end{aligned} \quad (130)$$

For the estimate $II_1'^2$, using (14), inserting an appropriate power of A_h , [34, (81)], Assumption 1 (more precisely (17)) and Lemma 12 (ii) with $\gamma_1 = \frac{1-\beta}{2}$ we obtain

$$\begin{aligned}
 II_1'^2 &= \left\| \int_0^{\Delta t} e^{(-A_h + J_0^h)(\Delta t - s)} P_h (\phi(s) - \phi(0)) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C \int_0^{\Delta t} \left\| e^{(-A_h + J_0^h)(\Delta t - s)} P_h (\phi(s) - \phi(0)) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{\Delta t} \left\| e^{(-A_h + J_0^h)(\Delta t - s)} A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \left\| A_h^{\frac{\beta-1}{2}} P_h (\phi(s) - \phi(0)) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{\Delta t} (\Delta t - s)^{\beta-1} s^{2\delta} ds \\
 &\leq C \Delta t^{2H+\beta-1} \left(\int_0^{\Delta t} (\Delta t - s)^{\beta-1} ds \right) \\
 &\leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{131}$$

We denote by ϵ a positive constant small enough, using (13), inserting an appropriate power of A_h , Assumption 1, [34, (83)], Lemma 12 (ii) with $\gamma_1 = H - \frac{\epsilon}{2}$, (iv) with $\gamma_1 = H - \frac{\epsilon}{2}$ and $\gamma_2 = \frac{1-\beta}{2}$ if $0 \leq \beta < 1$ (or (i) with $\gamma_1 = H - \frac{\epsilon}{2}$ and $\gamma_2 = 0$ if $\beta = 1$) we have

$$\begin{aligned}
 II_2'^2 &= \mathbb{E} \left\| \int_0^{\Delta t} \left(e^{(-A_h + J_0^h)(\Delta t - s)} - e^{(-A_h + J_0^h)\Delta t} \right) P_h \phi(0) dB^H(s) \right\|^2 \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{\Delta t} \left\| \left(e^{(-A_h + J_0^h)(\Delta t - s)} - e^{(-A_h + J_0^h)\Delta t} \right) P_h \phi(0) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{\Delta t} \left\| e^{(-A_h + J_0^h)(\Delta t - s)} \left(e^{(-A_h + J_0^h)s} - I \right) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \right. \\
 &\quad \left. \left\| A_h^{\frac{\beta-1}{2}} P_h \phi(0) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \left(\int_0^{\Delta t} \left\| e^{(-A_h + J_0^h)(\Delta t - s)} A_h^{H-\frac{\epsilon}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A_h^{\frac{\epsilon}{2}-H} \left(e^{(-A_h + J_0^h)s} - I \right) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} ds \right)^{2H} \\
 &\quad \left(\sum_{i \in \mathbb{N}^d} \left\| A_h^{\frac{\beta-1}{2}} \phi(0) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\leq C_H \left(\int_0^{\Delta t} (\Delta t - s)^{\frac{\epsilon}{2H}-1} s^{\frac{2H+\beta-1-\epsilon}{2H}} ds \right)^{2H} \left\| A_h^{\frac{\beta-1}{2}} \phi(0) \right\|_{L_2^0}^2 \\
 &\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\int_0^{\Delta t} (\Delta t - s)^{\frac{\epsilon}{2H}-1} ds \right)^{2H} \\
 &\leq C \Delta t^{2H+\beta-1-\epsilon} \Delta t^\epsilon \\
 &\leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{132}$$

Hence, putting (132) and (132) in (130) and taking the square-root give

$$II'' \leq C \Delta t^{\frac{\beta+2H-1}{2}}. \quad (133)$$

Adding (129) and (133) yields

$$\|X^h(t_1) - Z_1^h\|_{L^2(\Omega, \mathcal{H})} \leq C \Delta t^{\frac{\beta+2H-1}{2}}. \quad (134)$$

For $m \geq 2$, we recall that the solution at t_m of the semidiscrete problem (125) is given by

$$\begin{aligned} & X^h(t_m) \\ &= e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{m-1}^h)\Delta t} X^h(0) \\ &+ \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} G_{m-1}^h(X^h(s)) ds \\ &+ \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h \phi(s) dB^H(s) \\ &+ \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} G_{[s]^m}^h(X^h(s)) ds \\ &+ \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} P_h \phi(s) dB^H(s). \end{aligned} \quad (135)$$

We recall also that the numerical solution at t_m given by (127) can be rewritten as

$$\begin{aligned} Z_m^h &= e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{m-1}^h)\Delta t} X^h(0) \\ &+ \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} G_{m-1}^h(Z_{m-1}^h) ds \\ &+ \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)\Delta t} P_h \phi(t_{m-1}) dB^H(s) \\ &+ \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} G_{[s]^m}^h(Z_{[s]^m}^h) ds \\ &+ \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)\Delta t} P_h \phi([s]) dB^H(s). \end{aligned} \quad (136)$$

Using (135), (136) and the triangle inequality, we have

$$\|X^h(t_m) - Z_m^h\|_{L^2(\Omega, \mathcal{H})} \leq III'' + IV'' + V'' + VI'' \quad (137)$$

where

$$\begin{aligned} III'' &= \left\| \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} [G_{m-1}^h(X^h(s)) - G_{m-1}^h(Z_{m-1}^h)] ds \right\|_{L^2(\Omega, \mathcal{H})}, \\ IV'' &= \left\| \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h \phi(s) - e^{(-A_h + J_{m-1}^h)\Delta t} P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}, \\ V'' &= \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} \right. \\ &\quad \left. [G_{[s]^m}^h(X^h(s)) - G_{[s]^m}^h(Z_{[s]^m}^h)] ds \right\|_{L^2(\Omega, \mathcal{H})}, \end{aligned}$$

and

$$\begin{aligned} VI'' &= \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} P_h \phi(s) \right. \\ &\quad \left. - e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)\Delta t} P_h \phi([s]) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}. \end{aligned}$$

By Lemma 10, triangle inequality and Lemma 4 (47), we easily have

$$III'' \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t \|X^h(t_{m-1}) - Z_{m-1}^h\|_{L^2(\Omega, \mathcal{H})}. \quad (138)$$

In a similar way, using Lemma 13 with $v = 0$, Lemma 10, Lemma 11, triangle inequality and Lemma 4 (47), we obtain

$$V'' \leq C \Delta t^{\frac{2H+\beta-1}{2}} + C \Delta t \sum_{k=0}^{m-2} \|X^h(t_k) - Z_k^h\|_{L^2(\Omega, \mathcal{H})}. \quad (139)$$

For the estimate IV'' , using triangle inequality and the estimate $(a+b)^2 \leq 2a^2 + 2b^2$ we split it in two terms

$$\begin{aligned} IV''^2 &= \left\| \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h \phi(s) - e^{(-A_h + J_{m-1}^h)\Delta t} P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\leq 2 \left\| \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h (\phi(s) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\quad + 2 \left\| \int_{t_{m-1}}^{t_m} \left[e^{(-A_h + J_{m-1}^h)(t_m-s)} - e^{(-A_h + J_{m-1}^h)\Delta t} \right] P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &=: 2IV_1''^2 + 2IV_2''^2. \end{aligned} \quad (140)$$

Firstly, using (14), inserting an appropriate power of A_h , [34, (81)], Assumption 1 (more precisely (17)) and Lemma 12 (ii) with $\gamma_1 = \frac{1-\beta}{2}$ we obtain

$$\begin{aligned} IV_1''^2 &= \left\| \int_{t_{m-1}}^{t_m} e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h (\phi(s) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\ &\leq C \int_{t_{m-1}}^{t_m} \left\| e^{(-A_h + J_{m-1}^h)(t_m-s)} P_h (\phi(s) - \phi(t_{m-1})) \right\|_{L_2^0}^2 ds \\ &\leq C \int_{t_{m-1}}^{t_m} \left\| e^{(-A_h + J_{m-1}^h)(t_m-s)} A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \left\| A_h^{\frac{\beta-1}{2}} P_h (\phi(s) - \phi(t_{m-1})) \right\|_{L_2^0}^2 ds \\ &\leq C \int_{t_{m-1}}^{t_m} (t_m - s)^{\beta-1} \left\| A_h^{\frac{\beta-1}{2}} (\phi(s) - \phi(t_{m-1})) \right\|_{L_2^0}^2 ds \\ &\leq C \int_{t_{m-1}}^{t_m} (t_m - s)^{\beta-1} (s - t_{m-1})^{2\delta} ds \\ &\leq C \Delta t^{2H+\beta-1} \left(\int_0^{\Delta t} (t_m - s)^{\beta-1} ds \right) \\ &\leq C \Delta t^{2H+\beta-1}. \end{aligned} \quad (141)$$

Secondly by (13), inserting an appropriate power of A_h , Assumption 1, [34, (81)], Lemma 12 (ii) with $\gamma_1 = H - \frac{\epsilon}{2}$, (iv) with $\gamma_1 = H - \frac{\epsilon}{2}$ and $\gamma_2 = \frac{1-\beta}{2}$, we obtain

$$\begin{aligned}
 IV''/2 &= \mathbb{E} \left\| \int_{t_{m-1}}^{t_m} \left(e^{(-A_h + J_{m-1}^h)(t_m-s)} - e^{(-A_h + J_{m-1}^h)\Delta t} \right) P_h \phi(t_{m-1}) dB^H(s) \right\|^2 \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_{t_{m-1}}^{t_m} \left\| \left[e^{(-A_h + J_{m-1}^h)(t_m-s)} - e^{(-A_h + J_{m-1}^h)\Delta t} \right] P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq C_H \left(\int_{t_{m-1}}^{t_m} \left\| e^{(-A_h + J_{m-1}^h)(t_m-s)} \left(e^{(-A_h + J_{m-1}^h)(s-t_{m-1})} - I \right) A_h^{\frac{1-\beta}{2}} \right\|^{\frac{1}{H}}_{L(\mathcal{H})} ds \right)^{2H} \\
 &\quad \left(\sum_{i \in \mathbb{N}^d} \left\| A^{\frac{\beta-1}{2}} \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\leq C_H \left(\int_{t_{m-1}}^{t_m} \left\| e^{(-A_h + J_{m-1}^h)(t_m-s)} A_h^{H-\frac{\epsilon}{2}} \right\|^{\frac{1}{H}}_{L(\mathcal{H})} \right. \\
 &\quad \left. \left\| A_h^{-H+\frac{\epsilon}{2}} \left(e^{(-A_h + J_{m-1}^h)(s-t_{m-1})} - I \right) A_h^{\frac{1-\beta}{2}} \right\|^{\frac{1}{H}}_{L(\mathcal{H})} ds \right)^{2H} \\
 &\quad \left(\sum_{i \in \mathbb{N}^d} \left\| A^{\frac{\beta-1}{2}} \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|^2 \right) \\
 &\leq C_H \left(\int_{t_{m-1}}^{t_m} (t_m-s)^{-1+\frac{\epsilon}{2H}} (s-t_{m-1})^{\frac{2H+\beta-1-\epsilon}{2H}} ds \right)^{2H} \left\| A^{\frac{\beta-1}{2}} \phi(t_{m-1}) \right\|_{L_2^0}^2 \\
 &\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\int_{t_{m-1}}^{t_m} (t_m-s)^{-1+\frac{\epsilon}{2H}} ds \right)^{2H} \\
 &\leq C \Delta t^{2H+\beta-1-\epsilon} \Delta t^\epsilon \\
 &\leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{142}$$

Hence, substituting (141) and (142) in (140) and taking the square-root give

$$IV'' \leq C \Delta t^{\frac{2H+\beta-1}{2}}. \tag{143}$$

For estimate VI'' , using triangle inequality and the estimate $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$ we split it in two terms as

$$\begin{aligned}
 VI''/2 &\leq 3 \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} \right. \\
 &\quad \left. P_h(\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 3 \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} \right. \\
 &\quad \left. \left(e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} - e^{(-A_h + J_{[s]^m}^h)\Delta t} \right) P_h(\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\quad + 3 \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^m+1}^h)\Delta t} \right. \\
 &\quad \left. \left(e^{(-A_h + J_{[s]^m}^h)([s]+\Delta t-s)} - e^{(-A_h + J_{[s]^m}^h)\Delta t} \right) \right. \\
 &\quad \left. \times P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &=: 3VI_1''/2 + 3VI_2''/2 + 3VI_3''/2.
 \end{aligned} \tag{144}$$

Let ϵ be a sufficient small number. At first, using (14), inserting an appropriate power of A_h , [34, (81)], Assumption 1 (more precisely (17)), Lemma 13 (ii) with $\nu = H - \frac{\epsilon}{2}$, Lemma 12 (iii) with $\gamma_1 = \gamma_2 = \frac{1-\beta}{2}$, the variable change $j = m - k - 1$ and [26, (169)] we have

$$\begin{aligned}
 VI_1'^2 &= \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} \right. \\
 &\quad \left. P_h(\phi(s) - \phi([s])) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C \int_0^{t_{m-1}} \left\| e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} \right. \\
 &\quad \left. P_h(\phi(s) - \phi([s])) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_{m-1}} \left\| e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \\
 &\quad \times \left\| A_h^{\frac{\beta-1}{2}} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \left\| A_h^{\frac{\beta-1}{2}} P_h(\phi(s) - \phi([s])) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_{m-1}} t_{m-[s]^{m-1}}^{-1+\beta} \left\| A^{\frac{\beta-1}{2}} (\phi(s) - \phi([s])) \right\|_{L_2^0}^2 ds \\
 &\leq C \int_0^{t_{m-1}} t_{m-[s]^{m-1}}^{-1+\beta} (s - [s])^{2\delta} ds \\
 &\leq C \sum_{k=0}^{m-2} \int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\beta} \Delta t^{2\delta} ds \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^{m-1} t_j^{-1+\beta} \Delta t \right) \\
 &\leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{145}$$

Using (14), inserting an appropriate power of A_h , Lemma 13 with $\nu = H$, Lemma 12 (iii) with $\gamma_1 = \gamma_2 = H$, (iv) with $\gamma_1 = H$ and $\gamma_2 = \frac{1-\beta}{2}$ if $0 < \beta < 1$ (or (i) with $\gamma_1 = H$ and $\gamma_2 = 0$ if $\beta = 1$), [34, (81)], Assumption 1 (more precisely (17)) the variable change $j = m - k - 1$ and [26, (169)] yields

$$\begin{aligned}
 VI_2'^2 &= \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} \right. \\
 &\quad \left. \left(e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} - e^{(-A_h + J_{[s]^m}^h)\Delta t} \right) \right. \\
 &\quad \left. P_h(\phi([s]) - \phi(t_{m-1})) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
 &\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{t_{m-1}} e^{(-A_h + J_{m-1}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} A_h^H \right\|_{L(\mathcal{H})}^2 \\
 &\quad \left\| A_h^{-H} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} A_h^H \right\|_{L(\mathcal{H})}^2 \\
 &\quad \left\| A_h^{-H} \left(e^{(-A_h + J_{[s]^m}^h)(s - [s])} - I \right) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^2 \\
 &\quad \left\| A_h^{\frac{\beta-1}{2}} P_h(\phi([s]) - \phi(t_{m-1})) Q^{\frac{1}{2}} e_i \right\|^2 ds \Bigg) \\
 &\leq C_H \left(\int_0^{t_{m-1}} t_{m-[s]^m-1}^{-2H} (s - [s])^{2H+\beta-1} \left\| A^{\frac{\beta-1}{2}} (\phi(t_{m-1}) - \phi([s])) \right\|_{L_2^0}^2 ds \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\int_0^{t_{m-1}} t_{m-[s]^m-1}^{-2H} (t_{m-1} - [s])^{2\delta} ds \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{k=0}^{m-2} \int_{t_k}^{t_{k+1}} t_{m-k-1}^{-2H} (t_{m-1} - t_k)^{2H+\beta-1} ds \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{k=0}^{m-2} t_{m-k-1}^{-1+\beta} \Delta t \right) \\
 &\leq C \Delta t^{2H+\beta-1} \left(\sum_{j=1}^{m-1} t_j^{-1+\beta} \Delta t \right) \\
 &\leq C \Delta t^{2H+\beta-1}.
 \end{aligned} \tag{146}$$

Afterwards, (13), inserting an appropriate power of A_h , Lemma 13 with $v = H - \frac{\epsilon}{2}$, Lemma 12 (iii) with $\gamma_1 = \gamma_2 = H - \frac{\epsilon}{2}$, (iv) with $\gamma_1 = H - \frac{\epsilon}{2}$ and $\gamma_2 = \frac{1-\beta}{2}$ if $0 < \beta < 1$ (or (i) with $\gamma_1 = H - \frac{\epsilon}{2}$ and $\gamma_2 = 0$ if $\beta = 1$), [34, (81)], Assumption 1 (more precisely (16)) the variable change $j = m - k - 1$ and [26, (169)] yields

$$\begin{aligned}
VI_3'^2 &= \left\| \int_0^{t_{m-1}} e^{(-A_h + J_{[s]^{m-1}}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} \right. \\
&\quad \left. \left(e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} - e^{(-A_h + J_{[s]^m}^h)\Delta t} \right) P_h \phi(t_{m-1}) dB^H(s) \right\|_{L^2(\Omega, \mathcal{H})}^2 \\
&\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{t_{m-1}} \left\| e^{(-A_h + J_{[s]^{m-1}}^h)\Delta t} \dots e^{(-A_h + J_{[s]^{m+1}}^h)\Delta t} A_h^{H-\frac{\epsilon}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \right. \\
&\quad \left\| A_h^{\frac{\epsilon}{2}-H} e^{(-A_h + J_{[s]^m}^h)([s] + \Delta t - s)} A_h^{H-\frac{\epsilon}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \\
&\quad \left\| A_h^{\frac{\epsilon}{2}-H} \left(e^{(-A_h + J_{[s]^m}^h)(s - [s])} - I \right) A_h^{\frac{1-\beta}{2}} \right\|_{L(\mathcal{H})}^{\frac{1}{H}} \left\| A_h^{\frac{\beta-1}{2}} P_h \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|_{L^2(\mathcal{H})}^{\frac{1}{H}} ds \Big)^{2H} \\
&\leq C_H \sum_{i \in \mathbb{N}^d} \left(\int_0^{t_{m-1}} t_{m-[s]^{m-1}}^{-1+\frac{\epsilon}{2H}} (s - [s])^{\frac{2H+\beta-1-\epsilon}{2H}} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|_{L^2(\mathcal{H})}^{\frac{1}{H}} ds \right)^{2H} \\
&\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\int_0^{t_{m-1}} t_{m-[s]^{m-1}}^{-1+\frac{\epsilon}{2H}} ds \right)^{2H} \left(\sum_{i \in \mathbb{N}^d} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) Q^{\frac{1}{2}} e_i \right\|_{L^2(\mathcal{H})}^2 \right) \\
&\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\sum_{k=0}^{m-2} \int_{t_k}^{t_{k+1}} t_{m-k-1}^{-1+\frac{\epsilon}{2H}} ds \right)^{2H} \left\| A_h^{\frac{\beta-1}{2}} \phi(t_{m-1}) \right\|_{L^2_0}^2 \\
&\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\sum_{k=0}^{m-2} t_{m-k-1}^{-1+\frac{\epsilon}{2H}} \Delta t \right)^{2H} \\
&\leq C \Delta t^{2H+\beta-1-\epsilon} \left(\sum_{j=1}^{m-1} t_j^{-1+\frac{\epsilon}{2H}} \Delta t \right)^{2H} \\
&\leq C \Delta t^{2H+\beta-1-\epsilon}.
\end{aligned} \tag{147}$$

Hence, inserting (145), (146) and (147) in (144) and taking the square-root give

$$VI'' \leq C \Delta t^{\frac{2H+\beta-1-\epsilon}{2}}. \tag{148}$$

Adding (134), (138), (139), (143), (148) and applying Gronwall's lemma yields

$$\|X^h(t_m) - Z_m^h\|_{L^2(\Omega, \mathcal{H})} \leq C \Delta t^{\frac{2H+\beta-1-\epsilon}{2}}. \tag{149}$$

Finally, adding (66) and (149) completes the proof.

5 Extension to SPDE driven simultaneously by fBm and Poisson random measure

5.1 Numerical schemes

Here the goal is to show how the previous results can be extended to the following SPDE driven simultaneously by fBm and Poisson random measure. The corresponding model equation is given

by

$$dX(t, x) = [\nabla \cdot (\mathbf{D} \nabla X(t, x)) - \mathbf{q} \cdot \nabla X(t, x) + f(x, X(t, x))] dt + b(x, t) dB^H(t, x) + \int_{\chi} z_0 \tilde{N}(dz, dt), \quad x \in \Lambda, \quad X(0) = X_0, \quad z_0 \in \chi \quad t \in [0, T], \quad (150)$$

In the Hilbert space $\mathcal{H} = L^2(\Lambda)$, (150) is equivalent to (1) where the linear operator A and the nonlinear function F are defined as in (34) and (36). The well posedness result for $H = 1/2$ presented in [1] can easily be extended to (1) for $H \in [1/2, 1]$ by combining with [38]. The corresponding exponential Euler (SETD1) scheme in integral form is therefore given

$$Y_{m+1}^h = S_h(\Delta t) Y_m^h + \int_{t_m}^{t_{m+1}} S_h(t_{m+1} - s) P_h F(Y_m^h) ds + \int_{t_m}^{t_{m+1}} S_h(\Delta t) P_h \phi(t_m) dB^H(s) + \int_{t_m}^{t_{m+1}} \int_{\chi} S_h(\Delta t) P_h z_0 \tilde{N}(dz, ds), \quad (151)$$

with $Y_0^h = P_h X_0$. In the same way, the semi-implicit scheme is given by

$$X_{m+1}^h = S_{h,\Delta t} Y_m^h + \int_{t_m}^{t_{m+1}} S_{h,\Delta t} P_h F(X_m^h) ds + \int_{t_m}^{t_{m+1}} S_{h,\Delta t} P_h \phi(t_m) dB^H(s) + \int_{t_m}^{t_{m+1}} \int_{\chi} S_{h,\Delta t} P_h z_0 \tilde{N}(dz, ds), \quad (152)$$

while the stochastic exponential Rosenbrock scheme (SERS) is given by

$$Z_{m+1}^h = S_h^m(\Delta t) Z_m^h + \int_{t_m}^{t_{m+1}} S_h^m(t_{m+1} - s) P_h F(Z_m^h) ds + \int_{t_m}^{t_{m+1}} S_h^m(\Delta t) P_h \phi(t_m) dB^H(s) + \int_{t_m}^{t_{m+1}} \int_{\chi} S_h^m(\Delta t) P_h z_0 \tilde{N}(dz, ds), \quad (153)$$

with $Z_0^h = P_h X_0$, where

$$S_h^m(t) := e^{(-A_h + J_m^h)(t)}. \quad (154)$$

To obtain the optimal order in time, as in [28], we need the following assumption in Poisson measure noise.

Assumption 5 The covariance operator $Q : \mathcal{H} \rightarrow \mathcal{H}$ and the jump coefficient satisfy the following estimate

$$\|A^{\frac{\eta-1}{2}} z_0\| < \infty. \quad (155)$$

where $\eta = 2H + \beta - 1$ with $\beta \in (0, 1]$ as in Assumptions 3 and 1.

Remark 3 All the regularity results in space and time, both for continuous (1) (or semidiscrete equation) important to achieve optimal convergence orders can easily be extended from our results on fBm in Theorem 1 and Lemma 4 by just following [28, Proposition 3.1].

5.2 Convergence results for SPDE with fBm and Poisson measure noise

The convergence result is exactly as for fBm when Assumption 5 is used

Theorem 3 Let $X(t_m)$ be the mild solution of (1) at time $t_m = m \Delta t$, $\Delta t \geq 0$. Let ζ_m^h be the numerical approximations through (152) and (127) ($\zeta_m^h = X_m^h$ for implicit scheme, $\zeta_m^h = Z_m^h$ for

SERS) and Y_m^h the numerical approximation through the SETD1 given in (151). If Assumptions 1–4 and Assumption 5 hold with $\beta \in (0, 1]$, then

$$\left(\mathbb{E}\|X(t_m) - Y_m^h\|^2\right)^{\frac{1}{2}} \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{2H+\beta-1}{2}}\right), \quad (156)$$

and

$$\left(\mathbb{E}\|X(t_m) - \zeta_m^h\|^2\right)^{\frac{1}{2}} \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{2H+\beta-1-\epsilon}{2}}\right), \quad (157)$$

where ϵ is a positive constant small enough.

Corollary 2 Let $X(t_m)$ be the mild solution of (1) (A self-adjoint) at time $t_m = m\Delta t$, $\Delta t \geq 0$. Let X_m^h be the numerical approximations through (152). If Assumptions 1–3 and Assumption 5 hold with $\beta \in (0, 1]$, then

$$\left(\mathbb{E}\|X(t_m) - X_m^h\|^2\right)^{\frac{1}{2}} \leq C \left(h^{2H+\beta-1} + \Delta t^{\frac{2H+\beta-1}{2}}\right). \quad (158)$$

5.3 Proof of convergence results for SPDE with fBm and Poisson measure noise

As in [28], the proofs are based on Burkholder-Davis-Gundy Inequality where the fBm version is given in [32, Theorem 1.2]. Under Assumptions 1–3 and 5, The regularity result in time is

$$\|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, \mathcal{H})} \leq C(t_2 - t_1)^{\frac{\min(2H+\beta-1, 1)}{2}}, \quad 0 \leq t_1 < t_2 \leq T. \quad (159)$$

Where $C = C(\beta, L, T, H)$ is a positive constant and β is the regularity parameter of Assumption 1. In all our schemes, the error can be splitted in space error err_0 and the time error err_1 . The space error err_0 can be estimated as in Lemma 6 using results from the proof of [28, Theorem 4.1]. More precisely in the estimation of the poisson term in their case, we replace $\beta \in [0, 2]$ by $\eta = 2H + \beta - 1$ defined in Assumption 5. The time error err_1 will be here splitted in three terms. More precisely the deterministic I_1 related to the nonlinear function F , the fBm term I_2 and the Poisson term I_3 . The estimation of I_1 is done with the aid of Assumption 4 and (159) similarly as the work done in [28, (77)–(88)] for implicit and exponential schemes. As in the proof of [26, Theorem 10], we use the Taylor expansion in Banach space (see [28, (77)]) to estimate I_1 for SERS. The fBm term I_2 is done exactly as in the previous section for the scheme without Poisson. By replacing $\beta \in [0, 2]$ by $\eta = 2H + \beta - 1$ defined in Assumption 5, the estimation of the Poisson term I_3 is done as in [28, Theorem 5.2] for implicit and SETD1 schemes using Burkholder-Davis-Gundy Inequality, the work in [26, Theorem 10] and preparatory results Lemmas 10–13 for SERS.

6 Numerical simulations

In opposite to the standard Brownian motion where the simulation is obvious, the simulation of fBm is not obvious and is an important research field in numerical analysis. Keys methods for simulations of fBm are Cholesky method [13] and the circulant method [32], which will be used in this work to generate the fBm. Here we consider the stochastic advection diffusion reaction SPDE (5)–(33) with constant diagonal diffusion tensor $\mathbf{D} = 10^{-2}\mathbf{I}_2 = (D_{i,j})$ in (34), and mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0, L_1] \times [0, L_2]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x, y) : x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. This example can be seen as an engineering application where our SPDE (5)–(33) models the uncertainties in chemical reaction rates for stochastic transport of pollutants in porous media due by diffusion-advection-reaction with long-range dependence. If we include the Poisson noise term, our (1)–(150) should model the uncertainties in chemical reaction rates

for stochastic transport with long long-range dependence and some rare high chemical reaction rate, which can occur in the fracture networks. If we include the dispersion tensor as in [9], our (5)–(33) models also the sub-grid fluctuations in transport velocities and uncertainties in the reaction rates with long long-range dependence. In all these cases, the unknown function X is the concentration of the pollutants.

The eigenfunctions $\{e_{i,j}\} = \{e_i^{(1)} \otimes e_j^{(2)}\}_{i,j \geq 0}$ of the covariance operator Q are taken to be the same as for Laplace operator $-\Delta$ with homogeneous boundary condition, so we have

$$e_0^{(l)}(x) = \sqrt{\frac{1}{L_l}}, \quad e_i^{(l)}(x) = \sqrt{\frac{2}{L_l}} \cos\left(\frac{i\pi}{L_l}x\right), \quad i \in \mathbb{N}$$

where $l \in \{1, 2\}$, $x \in \Lambda$. In the noise representation (4), we have used

$$\lambda_{i,j} = (i^2 + j^2)^{-(\beta+\delta)}, \quad \beta > 0, \quad (160)$$

for some small $\delta > 0$. We have used $b(x, t) = 2$ in (33), so ϕ in Assumption 1 is obviously satisfied for $\beta = (0, 1]$. In our simulations, we have used $\delta = 0.001$. The function f used in (36) to be $f(x, z) = \frac{z}{1+z}$ for all $(x, z) \in \Lambda \times \mathbb{R}$. Therefore the corresponding Nemytskii operator F

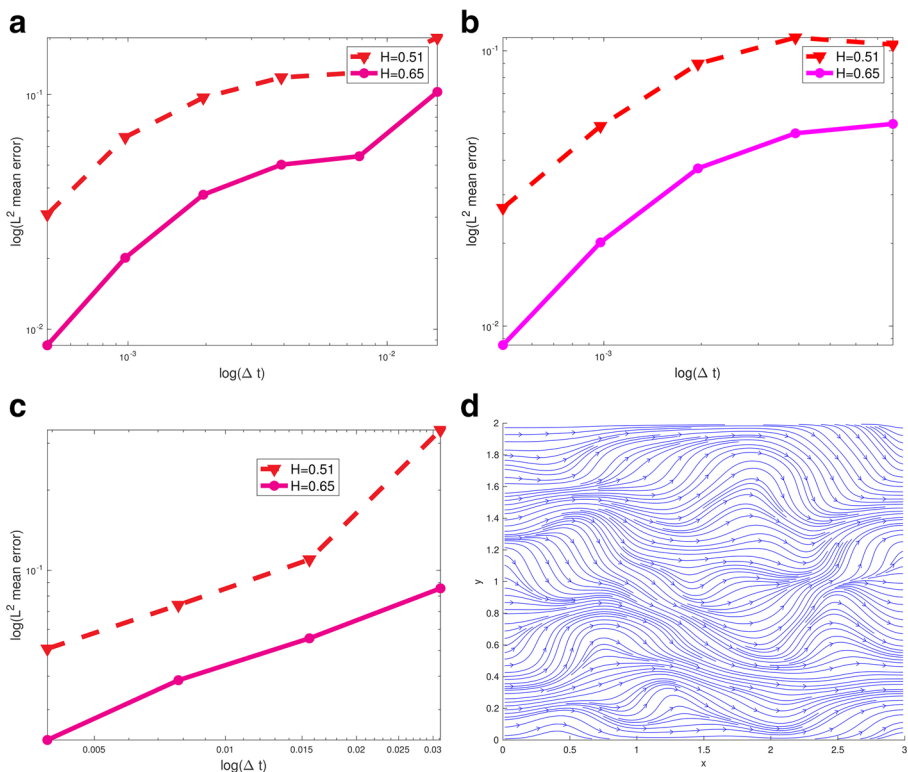


Fig. 1 Convergence in the root mean square L^2 norm at $T = 1$ as a function of Δt for implicit scheme (a), exponential scheme (b) and exponential Rosenbrock scheme (c). We have used here 50 realizations. The streamline of the velocity field \mathbf{q} is given in (d)

defined by (36) obviously satisfies Assumption 2. We obtain the Darcy velocity field $\mathbf{q} = (q_i)$ by solving the following system

$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q} = -\mathbf{k} \nabla p, \quad (161)$$

with Dirichlet boundary conditions on $\Gamma_D^1 = \{0, L_1\} \times [0, L_2]$ and Neumann boundary conditions on $\Gamma_N^1 = (0, L_1) \times \{0, L_2\}$ such that

$$p = \begin{cases} 1 & \text{in } \{0\} \times [0, L_2] \\ 0 & \text{in } \{L_1\} \times [0, L_2] \end{cases}$$

and $-\mathbf{k} \nabla p(\mathbf{x}, t) \cdot \mathbf{n} = 0$ in Γ_N^1 . Note that \mathbf{k} is the permeability tensor and p the pressure. We use a random permeability field as in [36, Figure 6]. The streamline of the velocity field \mathbf{q} are given in Fig. 1d. To deal with high Péclet number, we discretise in space using finite volume method, viewed as a finite element method (see [33]). We take $L_1 = 3$ and $L_2 = 2$ and our reference solutions samples are numerical solutions using at time step of $\Delta t = 1/4096$. The errors are computed at the final time $T = 1$. The initial solution is $X_0 = 0$, so we can therefore expect high-order convergence, which depends only on the noise term and H .

Figure 1a is the errors graph for the implicit scheme with different values of H . We have observed that the order of convergence is 0.48 in time for $H = 0.51$ and $\beta = 1$, 0.6476 for $H = 0.65$ and $\beta = 1$.

Figure 1b is the errors graph for the exponential scheme with two values of H . We have observed the order of convergence is 0.5012 in time for $H = 0.51$ and $\beta = 1$, 0.6653 for $H = 0.65$ and $\beta = 1$.

Figure 1c is the errors graph for the exponential Rosenbrock scheme with two values of H . We have observed the order of convergence is 0.5562 in time for $H = 0.51$ and $\beta = 1$, 0.6197 for $H = 0.65$ and $\beta = 1$.

As we can observe, our numerical orders in time are close to our theoretical results in Theorem 2 even if we have only used 50 samples in our Monte Carlo simulations.

Figure 2 shows two samples of the solution for $H = 0.65$ and $H = 0.51$. Here we have fixed $\beta = 1$ and same Gaussian random numbers have been to generate our fBm samples. As we can observe, the parameter H has significant influence on the sample of the numerical solution. This is independent of our timestepping methods.

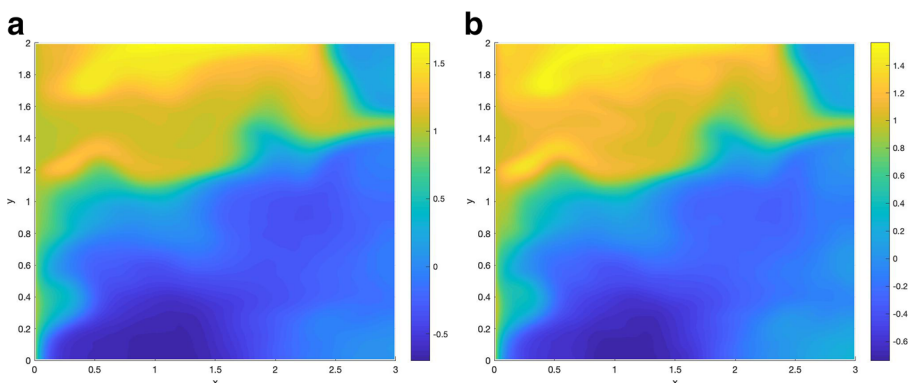


Fig. 2 Numerical samples solution using exponential scheme with $H = 0.65$ in (a) and $H = 0.51$ in (b)

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