

Stability of nonlinear time-varying digital 2-D Fornasini-Marchesini system

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Abstract Stability of a system described by the time-varying nonlinear 2-D Fornasini-Marchesini model is considered. There are given notions of stability of the system and theorems for stability and asymptotic stability which can be considered as the Lyapunov stability theorem extension for the system.

Keywords 2-D Fornasini-Marchesini system · Stability of 2-D systems · Nonlinear systems · Lyapunov function

1 Introduction

The Lyapunov stability theorem is frequently used in control theory. It enables ones to test stability of linear time-invariant and time-varying systems as well as nonlinear systems. Approach based on the Lyapunov theorem is also often used for analysis and design of robust control systems.

Two dimensional (2-D) systems have found many applications, for instance in analysis of systems described by the partial differential equations, in the design of digital filters, etc. However, one of the main problems in analysis and design of 2-D systems is stability. Whereas 2-D system may be viewed as a generalization of 1-D one, the extension of 1-D stability tests for 2-D systems is rather difficult. Until now there is no simple stability test for linear 2-D systems like for 1-D systems. Therefore, every new method for stability testing of 2-D systems can be useful.

In [Kurek \(1995\)](#) there is given stability condition for 2-D system described by the nonlinear Roesser model. The condition is similar to the Lyapunov one. Analogous conditions for linear Roesser model one can find in [El-Agizi and Fahmy \(1979\)](#), [Bliman \(2002\)](#). In [Kojima et al. \(2011\)](#) it is shown that asymptotic stability of linear 2-D system is equivalent to the

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existence of a vector Lyapunov functional satisfying certain positivity conditions together with its divergence along the system trajectories. In [Tatsuhi \(2001\)](#) it is, however, shown that application of 2-D Lyapunov matrix inequality is limited in application to robust stability of a system described by the Fornasini-Marchesini model. Alternatively, in [Zidong and Xiaohui \(2003\)](#) robust stability of the linear uncertain Fornasini-Marchesini model is considered using the LMI approach. Some recent results concerning stability of the nonlinear Fornasini-Marchesini model one can find in [Zhu and Hu \(2011\)](#).

In this note we consider the stability problem for nonlinear 2-D systems described by model similar to the Fornasini-Marchesini one. The stability and asymptotic stability notions are defined for the system. Next, sufficient conditions for stability and asymptotic stability are formulated similar to the second Lyapunov stability theorem. Presented results are illustrated by numerical example. Finally, concluding remarks are given. Obtained stability conditions are simply and can be used for testing stability of 2-D nonlinear as well linear systems.

2 Stability of 2-D system

There is a number of state-space models for linear digital 2-D systems, eg. [Roesser \(1975\)](#), [Fornasini and Marchesini \(1980\)](#), [Kurek \(1985\)](#). In this paper we will deal with digital time-varying nonlinear 2-D system described by a model similar to the 2-D Fornasini-Marchesini one

$$x(k + 1, t + 1) = f_{01}[x(k, t + 1), k, t + 1] + f_{10}[x(k + 1, t), k + 1, t] \tag{1}$$

where $x \in R^n$ is a state vector and f_{01} and f_{10} take values in R^n . The system we will call the time-varying nonlinear 2-D Fornasini-Marchesini system.

The state vector of 2-D system is finite dimensional but the solution to the system is calculated under infinite dimensional set of boundary conditions (BC). For instance, the BC set for system (1) can be defined as follows

$$x(k_0, t_0 + j) = x_{k_0}(t_0 + j) \text{ and } x(k_0 + i, t_0) = x_{t_0}(k_0 + i) \text{ for } i, j = 0, 1, \dots \tag{2}$$

or

$$x(k_0 + i, t_0 + j) = x_0(k_0 + i, t_0 + j) \text{ for } i + j = 0 \text{ and } i = \dots, -1, 0, 1, \dots \tag{3}$$

Since BC set (3) can be considered as a global state $\chi(h)$ of the 2-D system, $\chi(h) = \{x(k, t) \text{ for } k + t = h\}$, we define, for simplicity of the presentation, stability of system (1) assuming this BC set.

Then, denoting by $\|x\|$ a vector norm, the Lyapunov stability of 2-D system (1) can be defined analogously to 1-D systems.

Definition 1 A state $x_e \in R^n$ is an equilibrium state of system (1) if and only if for each integer numbers $k_0, t_0 < \infty$ the equality $\|x(k_0 + i, t_0 + j) - x_e\| = 0$ for $i + j = 0$ implies $\|x(k_0 + i, t_0 + j) - x_e\| = 0$ for $i + j > 0$.

Definition 2 An equilibrium state x_e of system (1) is stable if and only if for each real number $\varepsilon > 0$ and integer numbers $k_0, t_0 < \infty$ there is a real number $\delta(\varepsilon, k_0, t_0) > 0$ such that $\|x(k_0 + i, t_0 + j) - x_e\| \leq \delta(\varepsilon, k_0, t_0)$ for $i + j = 0$ implies $\|x(k_0 + i, t_0 + j) - x_e\| \leq \varepsilon$ for $i + j > 0$.

Remark It follows from the definition that $\|x(k, t) - x_e\| \leq \varepsilon$ for $k > k_0$ and $t > t_0$ for stable equilibrium state x_e independent on BC set, also for BC set (2).

Definition 3 An equilibrium state x_e of system (1) is asymptotically stable *if and only if* for each real number $\varepsilon > 0$ and integer numbers $k_0, t_0 < \infty$ there is a real number $\delta(\varepsilon, k_0, t_0) > 0$ such that $\|x(k_0+i, t_0+j) - x_e\| \leq \delta(\varepsilon, k_0, t_0)$ for $i+j = 0$ implies $\|x(k_0+i, t_0+j) - x_e\| \leq \varepsilon$ for $i + j > 0$ and $\|x(k_0 + i, t_0 + j) - x_e\| \rightarrow 0$ for $i + j \rightarrow \infty$.

Remarks

1. In short one can say that the equilibrium state x_e of system (1) is asymptotically stable *if and only if* it is stable and $\|x(k, t) - x_e\| \rightarrow 0$ for $k + t \rightarrow \infty$.
2. From the definition it follows that $x(k, t) \rightarrow x_e$ for $k, t \rightarrow \infty$ for asymptotically stable equilibrium state x_e independent on BC set, as well for BC set (2).

Definition 4 The equilibrium state x_e is stable (asymptotically stable) in the large *if and only if* it is stable (asymptotically stable) and $\delta(\varepsilon, k_0, t_0) \rightarrow \infty$ for $\varepsilon \rightarrow \infty$.

Definition 5 The equilibrium state x_e is uniformly stable (asymptotically stable) *if and only if* it is stable (asymptotically stable) and for each ε, k_0 and t_0 there exists $\delta(\varepsilon, k_0, t_0) = \delta(\varepsilon)$ independent on k_0 and t_0 .

Moreover, we say that an equilibrium state x_e is unstable if it is not stable.

Then, analyzing stability of the system it is easy to note that the system can have, analogously to 1-D systems, many stable, asymptotically stable or unstable equilibrium states but only one equilibrium state if the state is asymptotically stable in the large.

3 The main result

Based on Definition 1 one can prove the following Theorem (Kurek 1995).

Theorem 1 A state $x_e \in R^n$ is an equilibrium state of system (1) if and only if

$$x_e = f_{01}(x_e, k, t + 1) + f_{10}(x_e, k + 1, t) \quad \text{for all } k, t$$

Next one can prove the following theorems similar to the well known second stability theorem of Lyapunov for 1-D systems (Ogata 1967).

Theorem 2 Given system (1) with equilibrium state $x_e = 0$. The equilibrium state is uniformly stable if there exist real numbers $\xi > 0$ and $\rho \in [0, 1]$, and real scalar function $\varphi(x, k, t) : R^n \times R \times R \rightarrow R$, such that for $\|x\| \leq \xi$ and all k, t

- (a) $\varphi(x, k, t)$ is a positive definite function in x , i.e. $\varphi(x, k, t) = 0$ if $x = 0$ and $\varphi(x, k, t) > 0$ if $x \neq 0$,
- (b) $\varphi(x, k, t)$ has an upper bound $\alpha(\|x\|)$ such that $\varphi(x, k, t) \leq \alpha(\|x\|)$ where $\alpha(\cdot)$ is a nondecreasing continuous scalar function such that $\alpha(0) = 0$,
- (c) $\varphi(x, k, t)$ has a lower bound $\beta(\|x\|)$ such that $\varphi(x, k, t) \geq \beta(\|x\|)$ where $\beta(\cdot)$ is a nondecreasing continuous scalar function such that $\beta(0) = 0$ and $\beta(a) > 0$ for $a > 0$, and
- (d) the change $\Delta_\varphi(x, k, t; \rho)$ of function $\varphi(x, k, t)$ along trajectory x of system (1)

$$\begin{aligned} \Delta_\varphi(x, k, t; \rho) &= \varphi[x(k + 1, t + 1), k + 1, t + 1] \\ &\quad - \rho\varphi[x(k, t + 1), k, t + 1] - (1 - \rho)\varphi[x(k + 1, t), k + 1, t] \\ &= \varphi\{f_{01}[x(k, t + 1), k, t + 1] + f_{10}[x(k + 1, t), k + 1, t], k + 1, t + 1\} \\ &\quad - \rho\varphi[x(k, t + 1), k, t + 1] - (1 - \rho)\varphi[x(k + 1, t), k + 1, t] \quad (4) \end{aligned}$$

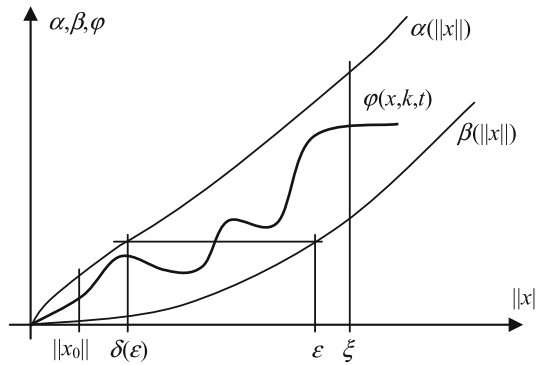


Fig. 1 Plots of functions α , β and φ

is negative semi-definite function for $\|x\| \leq \xi$, i.e.

$$\begin{aligned} \Delta_\varphi(x, k, t; \rho) &= \varphi[f_{01}(x_1, k, t + 1) + f_{10}(x_2, k + 1, t), k + 1, t + 1] \\ &\quad - \rho\varphi(x_1, k, t + 1) - (1 - \rho)\varphi(x_2, k + 1, t) \leq 0 \\ &\text{for } \|x_1\|, \|x_2\| \leq \xi \end{aligned} \tag{5}$$

The equilibrium state x_e is uniformly stable in the large if conditions (a), (b), (c) and (d) are satisfied for $\xi \rightarrow \infty$ and

$$\varphi(x, k, t) \rightarrow \infty \text{ if } \|x\| \rightarrow \infty \tag{6}$$

Proof Since $\alpha(\cdot)$ and $\beta(\cdot)$ are continuous functions for every $\varepsilon \in (0, \xi]$ one can find $\delta(\varepsilon) \in (0, \varepsilon]$ such that $\alpha[\delta(\varepsilon)] = \beta(\varepsilon)$, Fig. 1. Thus, for x_0 such that $\|x_0\| \leq \delta(\varepsilon)$ one has $\varphi(x_0, k, t) \leq \alpha(\|x_0\|)$.

Then, according to (4) one has

$$\begin{aligned} \Delta_\varphi(x, k, t; \rho) &= \varphi\{f_{01}[x(k, t + 1), k, t + 1] + f_{10}[x(k + 1, t), k + 1, t], k + 1, t + 1\} \\ &\quad - \rho\varphi[x(k, t + 1), k, t + 1] - (1 - \rho)\varphi[x(k + 1, t), k + 1, t] \\ &\geq \varphi[x(k + 1, t + 1), k + 1, t + 1] \\ &\quad - \max\{\varphi[x(k, t + 1), k, t + 1], \varphi[x(k + 1, t), k + 1, t]\} \end{aligned}$$

Hence, for every BC set (3) such that $\|x_0(k_0 + i, t_0 + j)\| \leq \|x_0\|$ according to (5) we have for $k + t + 1 \geq k_0 + t_0$

$$\begin{aligned} \varphi[x(k + 1, t + 1), k + 1, t + 1] - \max\{\varphi[x(k, t + 1), k, t + 1], \varphi[x(k + 1, t), k + 1, t]\} \\ \leq 0 \end{aligned}$$

and one finds

$$\varphi[x(k + 1, t + 1), k + 1, t + 1] \leq \max\{\varphi[x(k, t + 1), k, t + 1], \varphi[x(k + 1, t), k + 1, t]\} \tag{7}$$

Thus, we obtain

$$\begin{aligned} \varphi[x(k + 1, t + 1), k + 1, t + 1] &\leq \max_{i \in [k_0 - k - 1, t + 1 - t_0]} \varphi[x_0(k_0 + i, t_0 - i), k_0 + i, t_0 - i] \\ &\leq \max_{i+j=0} \varphi[x_0(k_0 + i, t_0 + j), k_0 + i, t_0 + j] \leq \alpha(\|x_0\|) \end{aligned}$$

This, however, implies $\|x(k + 1, t + 1)\| \leq \varepsilon$, Fig. 1. Clearly, for $\varepsilon \leq \xi$ there exists $\delta(\varepsilon) = \delta(\xi)$. Thus, for every ε there exists $\delta(\varepsilon) \leq \delta(\xi)$ such that $\|x(k + 1, t + 1)\| \leq \varepsilon$. It means, however, that the equilibrium state x_e is stable. Then, since functions $\alpha(\cdot)$ and $\beta(\cdot)$ depend neither on k nor on t also $\delta(\varepsilon)$ is independent on k and t . Thus, the equilibrium state x_e is uniformly stable.

Finally, if $\xi \rightarrow \infty$ and condition (6) is satisfied also $\alpha(\|x\|) \rightarrow \infty$ and there exists $\beta(\|x\|) \rightarrow \infty$. Thus, Fig. 1, also $\delta(\xi) \rightarrow \infty$ and the equilibrium state x_e is uniformly stable in the large. \square

Remarks

1. One should note that conditions (b) and (c) are satisfied if function $\varphi(x, k, t)$ is continuous in x, k and t since in this case $\varphi(x, k, t) \rightarrow 0$ only if $x \rightarrow 0$.
2. Any equilibrium state $x_e \neq 0$ can be shifted to the origin of the coordinates by translation of coordinates. Thus, the theorem gives a general result.
3. From proof of theorem it follows that except the stable equilibrium state $x_e = 0$ there could be also other equilibrium states x_s , stable or unstable, such that $\|x_s\| \leq \xi$ if theorem conditions are satisfied.
4. Function $\varphi(\cdot)$ satisfying conditions (a)–(c) can be named candidate of the Lyapunov function for 2-D system (1) and $\varphi(\cdot)$ satisfying all the conditions (a)–(d) can be named the Lyapunov function for 2-D system (1).
5. It is rather easy to find that instead of the change $\Delta_\varphi(x, k, t)$ in (4) one can use the following change of $\varphi(\cdot)$ in condition (d)

$$\begin{aligned} \bar{\Delta}_\varphi(x, k, t) &= \varphi[x(k + 1, t + 1), k + 1, t + 1] \\ &\quad - \max\{\varphi[x(k, t + 1), k, t + 1], \varphi[x(k + 1, t), k + 1, t]\} \end{aligned}$$

or more general for $\eta, \lambda \geq 0$ and $\eta + \lambda \leq 1$

$$\begin{aligned} \tilde{\Delta}_\varphi(x, k, t; \eta, \lambda) &= \varphi[x(k + 1, t + 1), k + 1, t + 1] \\ &\quad - \eta\varphi[x(k, t + 1), k, t + 1] - \lambda\varphi[x(k + 1, t), k + 1, t] \end{aligned}$$

Theorem 3 *Given system (1) with equilibrium state $x_e = 0$. The equilibrium state is uniformly asymptotically stable if there exist real numbers $\xi > 0$ and $\rho \in [0, 1]$, and function $\varphi(x, k, t)$ such that for $\|x\| \leq \xi$ and all k, t conditions (a), (b), (c) of Theorem 2 are satisfied and*

(d) the change $\Delta_\varphi(x, k, t; \rho)$ of function $\varphi(x, k, t)$ along trajectory x of system (1) has an upper bound $\gamma(\|x_1\|, \|x_2\|)$ such that

$$\begin{aligned} \Delta_\varphi(x_{12}, k, t; \rho) &= \varphi[f_{01}(x_1, k, t + 1) + f_{10}(x_2, k + 1, t), k + 1, t + 1] \\ &\quad - \rho\varphi(x_1, k, t + 1) - (1 - \rho)\varphi(x_2, k + 1, t) \\ &\leq \gamma(\|x_1\|, \|x_2\|) < 0 \\ &\text{for } x_1 \neq 0 \text{ or } x_2 \neq 0 \text{ and } \|x_1\|, \|x_2\| \leq \xi \end{aligned} \tag{8}$$

where $\gamma(\|x_1\|, \|x_2\|)$ is a nonincreasing continuous scalar function such that $\gamma(0, 0) = 0$ and $\gamma(\|x_1\|, \|x_2\|) \geq \gamma(\|x_1\| + a, \|x_2\| + b)$ for $a, b \geq 0$.

The equilibrium state x_e is uniformly asymptotically stable in the large if the conditions are satisfied for $\xi \rightarrow \infty$ and condition (6) is fulfilled.

Proof According to Theorem 2 the equilibrium state x_e is uniformly stable if conditions of the theorem are satisfied. Moreover, because of (8) it follows from Theorem 1 that there is only one equilibrium state $x_e = 0$ for $\|x\| \leq \xi$.

Next, from (8) and (7) one finds that function $\varphi[x(k, t), k, t]$ is a decreasing function for $k + t + 1 \geq k_0 + t_0$ and BC set (3) such that $\|x_0(k_0 + i, t_0 + j)\| \leq \xi$

$$\varphi[x(k + 1, t + 1), k + 1, t + 1] < \max\{\varphi[x(k, t + 1), k, t + 1], \varphi[x(k + 1, t), k + 1, t]\}$$

This implies

$$\max_{i+j=2} \varphi[x(k + i, t + j), k + i, t + j] < \max_{i+j=1} \varphi[x(k + i, t + j), k + i, t + j]$$

and since positive definite function $\varphi[x(k, t), k, t] \geq 0$ is a decreasing function for $k+t \rightarrow \infty$ there exists φ_0 such that

$$\max_{k+t=h} \varphi[x(k, t), k, t] \rightarrow \varphi_0 \quad \text{for } h \rightarrow \infty$$

Therefore, for BC set (3) such that $\|x_0(k_0 + i, t_0 + j)\| \leq \delta(\xi)$ one has for $h + 1 \geq k_0 + t_0$

$$\begin{aligned} \max_{k+t=h} \Delta_\varphi(x, k, t; \rho) &= \max_{k+t=h} \{\varphi[x(k + 1, t + 1), k + 1, t + 1] - \rho\varphi[x(k, t + 1), k, t + 1] \\ &\quad - (1 - \rho)\varphi[x(k + 1, t), k + 1, t]\} \xrightarrow{h \rightarrow \infty} \varphi_0 - \rho\varphi_0 - (1 - \rho)\varphi_0 = 0 \end{aligned}$$

However, according to (8) it is possible if and only if $x(k, t) \rightarrow 0$ and this implies that the equilibrium state is asymptotically stable. In this case also $\varphi[x(k, t), k, t] \rightarrow 0$. Thus, $\varphi_0 = 0$, too.

Finally, similarly as in the proof of Theorem 2, one easily finds that the equilibrium state x_e is uniformly asymptotically stable in the large if condition (6) is satisfied and $\xi \rightarrow \infty$. \square

Remarks

1. All remarks to Theorem 2 applies respectively to Theorem 3. Particularly, instead of condition (8) one can use the following one

$$\bar{\Delta}_\varphi(x_{12}, k, t) \leq \gamma(\|x_1\|, \|x_2\|) \quad \text{for } \|x_1\|, \|x_2\| \leq \xi$$

or, more general

$$\tilde{\Delta}_\varphi(x_{12}, k, t; \eta, \lambda) \leq \gamma(\|x_1\|, \|x_2\|) \quad \text{for } \|x_1\|, \|x_2\| \leq \xi$$

2. If $\Delta_\varphi(x, k, t; \rho)$ in (8) is negative definite function in x and it is also continuous function in x, k and t then condition (d) is satisfied since in this case $\Delta_\varphi(x_{12}, k, t; \rho) \rightarrow 0$ only if $x_1, x_2 \rightarrow 0$.
3. From Fig. 1 one can see that $\|x\| < \delta(\xi)$ is a subregion of attraction of the asymptotically stable equilibrium state x_e and there is only one equilibrium point x_e such that $\|x_e\| \leq \xi$.

One can note that the following corollaries simply follow from Theorem 2.

Corollary 1 *The equilibrium state x_e is unstable if there exists $\xi > 0$ such that the change $\Delta_\varphi(x, k, t; \rho)$ of function $\varphi(x, k, t)$ along trajectory x of system (1) is positive definite in x for all k and t , i.e.*

$$\Delta_\varphi(x_{12}, k, t; \rho) > 0 \text{ for } \|x_1\|, \|x_2\| \leq \xi \text{ and } x_1, x_2 \neq 0$$

Corollary 2 *The equilibrium state x_e can be stable or unstable if there exists $\xi > 0$ such that the change $\Delta_\varphi(x, k, t; \rho)$ of function $\varphi(x, k, t)$ along trajectory x of system (1) is neither negative nor positive definite function for $\|x\| < \xi$.*

Remark In this case, checking the stability, one has to design another function $\varphi(x, k, t)$ or apply different stability test.

In practice, for instance image filtering, we rather deal with linear 2-D systems. Unfortunately, in the contradiction to 1-D systems, there is no effective tests for its stability testing in spite of that the necessary and sufficient stability conditions for linear 2-D systems are well known. For the reason, we present simple examples which illustrate stability testing of linear 2-D system using the presented results.

Example 1 Given 1st order linear time invariant 2-D system described by the following equation

$$x(k + 1, t + 1) = 0.5x(k + 1, t) - 0.3x(k, t + 1)$$

It is easy to find that the system has an equilibrium state $x_e = 0$. Testing stability of the equilibrium state we choose candidate for the Lyapunov function $\varphi(x, k, t)$ as follows

$$\varphi(x, k, t) = \varphi(x) = x^2$$

Then, one finds

$$\begin{aligned} \varphi[x(k + 1, t + 1)] &= x^2(k + 1, t + 1) = [0.5x(k + 1, t) - 0.3x(k, t + 1)]^2 \\ &= 0.25x^2(k + 1, t) + 0.09x^2(k, t + 1) - 0.3x(k + 1, t)x(k, t + 1) \end{aligned}$$

Next, testing the stability condition (8) with $\rho = 0.4$ we have

$$\begin{aligned} \Delta_\varphi[x(k, t + 1), x(k + 1, t), k, t; \rho] &= \varphi[x(k + 1, t + 1)] \\ &\quad - 0.4\varphi[x(k, t + 1)] - 0.6\varphi[x(k + 1, t)] \\ &= 0.25x^2(k + 1, t) + 0.09x^2(k, t + 1) \\ &\quad - 0.3x(k + 1, t)x(k, t + 1) \\ &\quad - 0.4x^2(k, t + 1) - 0.6x^2(k + 1, t) \\ &= -0.35x^2(k + 1, t) - 0.31x^2(k, t + 1) \\ &\quad - 0.3x(k + 1, t)x(k, t + 1) \end{aligned}$$

Calculating the above function one finds that for $x(k + 1, t)x(k, t + 1) \geq 0$ we have Δ_φ negative. Next, for $x(k + 1, t)x(k, t + 1) < 0$ such that $0 \leq |x(k + 1, t)| \leq |x(k, t + 1)|$ one finds

$$\begin{aligned} \Delta_\varphi[x(k, t + 1), x(k + 1, t), k, t; \rho] &\leq -0.35x^2(k + 1, t) - 0.31x^2(k, t + 1) + 0.3x^2(k, t + 1) \\ &= -0.35x^2(k + 1, t) - 0.01x^2(k, t + 1) \end{aligned}$$

and for $0 \leq |x(k, t + 1)| \leq |x(k + 1, t)|$ we obtain

$$\begin{aligned} \Delta_\varphi[x(k, t + 1), x(k + 1, t), k, t; \rho] &\leq -0.35x^2(k + 1, t) - 0.31x^2(k, t + 1) + 0.3x^2(k + 1, t) \\ &= -0.05x^2(k + 1, t) - 0.31x^2(k, t + 1) \end{aligned}$$

Thus, we see that change of the function $\varphi(x, k, t)$ along the trajectory is decreasing negative function and tends to 0 as $k, t \rightarrow \infty$. This, according to Theorem 3, means that the equilibrium state $x_e = 0$ is asymptotically stable. Moreover, since it is satisfied for every x the equilibrium state is asymptotically stable in the large.

Remarks

1. Let us note that the proper choice of the Lyapunov candidate function can significantly improve stability test. For instance using the following function

$$\varphi(x, k, t) = \varphi(x) = |x|$$

one can analogously show that every 1st order system

$$x(k + 1, t + 1) = a_{10}x(k + 1, t) + a_{01}x(k, t + 1) \tag{9}$$

such that $|a_{10}| + |a_{01}| < 1$, has asymptotically stable in the large equilibrium state $x_e = 0$.

2. One can note that system (9) is unstable if $a_{10}, a_{01} > 0$ and $a_{10} + a_{01} > 1$. Indeed, in this case for $x(k + 1, t) = x(k, t + 1) = x_0$ one has $x(k + 1, t + 1) > x_0$.

Example 2 Given linear time-invariant 2-D Fornasini-Marchesini system

$$x(k + 1, t + 1) = A_{01}x(k, t + 1) + A_{10}x(k + 1, t)$$

where $x \in R^n$.

The equilibrium state of the system clearly is $x_e = 0$. Checking stability of the equilibrium state we choose the following positive definite function $\varphi(x, k, t)$

$$\varphi(x, k, t) = x^T P x \tag{10}$$

where $P \in R^{n \times n}$ is a symmetric positive definite matrix. Since the function is continuous conditions (b) and (c) of Theorem 2 are satisfied according to remark 1 to Theorem 2.

Then, one has

$$\begin{aligned} \varphi[x(k + 1, t + 1)] &= x(k + 1, t + 1)^T P x(k + 1, t + 1) \\ &= [A_{01}x(k, t + 1) + A_{10}x(k + 1, t)]^T P [A_{01}x(k, t + 1) \\ &\quad + A_{10}x(k + 1, t)] \\ &= \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix}^T \begin{bmatrix} A_{01}^T P A_{01} & A_{01}^T P A_{10} \\ A_{10}^T P A_{01} & A_{10}^T P A_{10} \end{bmatrix} \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \rho\varphi[x(k, t + 1)] + (1 - \rho)\varphi[x(k + 1, t)] &= \rho x^T(k, t + 1)Px(k, t + 1) \\ &\quad + (1 - \rho)x^T(k + 1, t)Px(k + 1, t) \\ &= \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix}^T \begin{bmatrix} \rho P & 0 \\ 0 & (1 - \rho)P \end{bmatrix} \\ &\quad \times \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix} \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \Delta_\varphi(x, k, t; a, b) &= \varphi[x(k + 1, t + 1)] - \rho\varphi[x(k, t + 1)] - (1 - \rho)\varphi[x(k + 1, t)] \\ &= \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix}^T Q \begin{bmatrix} x(k, t + 1) \\ x(k + 1, t) \end{bmatrix} \end{aligned} \tag{11}$$

where

$$Q = \begin{bmatrix} A_{01}^T P A_{01} & A_{01}^T P A_{10} \\ A_{10}^T P A_{01} & A_{10}^T P A_{10} \end{bmatrix} - \begin{bmatrix} \rho P & 0 \\ 0 & (1 - \rho)P \end{bmatrix}$$

Hence, the equilibrium state is asymptotically stable if the matrix Q is negative definite since in this case the quadratic form (11) according to remark 3 to Theorem 3 has an upper bound because it is continuous negative definite scalar function with maximum in $x = 0$.

Finally, since function $\varphi(x, k, t)$ in (10) satisfies conditions (a), (b) and (c) of Theorem 2 for all x for $\xi \rightarrow \infty$ as well since the change $\Delta_\varphi(x, k, t)$ is negative definite for all x if matrix Q is negative definite and condition (6) is satisfied then the equilibrium state x_e is asymptotically stable in the large.

Remark Matrix Q can be negative definite only if matrices

$$Q_{11} = A_{01}^T P A_{01} - \rho P \text{ and } Q_{22} = A_{10}^T P A_{10} - (1 - \rho)P$$

are negative definite. It is easy to see that the above condition can be satisfied only if $\rho > \max_i |\lambda_i|^2$ and $(1 - \rho) > \max_i |\mu_i|^2$ where λ_i and μ_i denote, respectively, eigenvalues of matrices A_{01} and A_{10} , $i = 1, \dots, n$. This means, however, that the stability condition can be satisfied only if $\max_i |\lambda_i|^2 + \max_i |\mu_i|^2 < 1$.

4 Concluding remarks

The stability notion for nonlinear parameter-varying digital 2-D systems was presented. Then, stability conditions were formulated for nonlinear time-varying digital 2-D systems similar to the Fornasini-Marchesini model. In particular, Example 2 gives simple sufficient stability condition for 2-D system described by the linear time-invariant Fornasini-Marchesini model.

The 2-D Roesser model can be easily presented as the Fornasini-Marchesini one. Thus, results obtained for stability of the Fornasini-Marchesini model can be also easily applied to the system described by the Roesser model. Then, it is easy to find that the presented stability conditions are neither simple consequence nor simple generalization of the results for the Roesser model for instance given in Kurek (1995), they are similar but different. This is a consequence of the fact that the conditions are only sufficient not necessary and sufficient.

It is, however, well known that there can be a lot of different only sufficient or only necessary stability conditions for the system.

The presented stability theorems are similar to the Lyapunov stability theorem for 1-D discrete-time systems and can be considered as a generalization. However, the presented theorems are not a simple consequence of the Lyapunov theorem since BC sets for 2-D systems are infinite dimensional, whereas they are finite dimensional for 1-D systems.

The presented results can be easily generalized on N-D systems.

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