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A Remark on the Spherical Bipartite Spin Glass

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Abstract
Auffinger and Chen (J Stat Phys 157:40–59, 2014) proved a variational formula for the free energy of the spherical bipartite spin glass in terms of a global minimum over the overlaps. We show that a different optimisation procedure leads to a saddle point, similar to the one achieved for models on the vertices of the hypercube.

Keywords Bipartite spin glasses · Spherical models · Variational principles

Mathematics Subject Classification 60K35 · 60G15 · 82B44

1 Introduction

Let $\sigma_N(dx)$ denote the uniform probability measure on $S^N := \{x \in \mathbb{R}^N : \|x\|_2^2 = N\}$, where $\|x\|_2$ is the Euclidean norm. For $x := (x_1, \dots x_{N_1}) \in \mathbb{R}^{N_1}$ and $y := (y_1, \dots, y_{N_2}) \in \mathbb{R}^{N_2}$ the bipartite spin glass is defined by the energy function

$$H_{N_1,N_2}(x, y; \xi) := -\frac{1}{\sqrt{N}} \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \xi_{ij} x_i y_j \, . \tag{1}$$

Here $\{\xi_{ij}\}_{i\in[N_1], j\in[N_2]}$ are $\mathcal{N}(0, 1)$ i.i.d. quenched r.vs. and we set $N := N_1 + N_2$. The object of interest of this note is the free energy

$$A_{N_1,N_2}(\beta, \xi) := \frac{1}{N} \log \int \sigma_{N_1}(dx)\sigma_{N_2}(dy) \exp(-\beta H_{N_1,N_2}(x, y; \xi) - b_1(x, 1) - b_2(y, 1)) \tag{2}$$

in the limit in which $N_1, N_2 \to \infty$ with $N_1/N \to \alpha \in (0, 1)$. Here $\beta \geqslant 0$ is the inverse temperature, $b_1, b_2 \in \mathbb{R}$ are external fields and $(\cdot, \cdot)$ denotes the Euclidean inner product. By concentration of Lipschitz functions of Gaussian random variables

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one reduces to study the average free energy $A_{N_1,N_2}(\beta) := E[A_{N_1,N_2}(\beta,\xi)]$, whose limit we denote by $A(\alpha,\beta)$.

Auffinger and Chen proved in [1] the following variational formula for $A(\alpha,\beta)$ for β small enough

$$A(\alpha,\beta) = \min_{q_1,q_2\in[0,1]^2} P(q_1,q_2) \tag{3}$$

$$P(q_1,q_2) = \frac{\beta^2\alpha(1-\alpha)}{2}(1-q_1q_2) + \frac{\alpha}{2}\left(b_1^2(1-q_1) + \frac{q_1}{1-q_1} + \log(1-q_1)\right) + \frac{1-\alpha}{2}\left(b_2^2(1-q_2) + \frac{q_2}{1-q_2} + \log(1-q_2)\right) \tag{4}$$

(the normalisation in (1) leads to different constants w.r.t. [1]). The above formula was successively proved to hold in the whole range of $\beta \geqslant 0$ in [2, 9]. Yet these proofs are indirect, as in both cases one obtains a formula for the free energy and then verifies a posteriori (analytically for [2] and numerically [14] for [9]) that it coincides with (3). We just mention that the results in [1] have been recently extended in [10, 11] for the complexity and in [5, 6] for the free energy.

The convex variational principle found by Auffinger and Chen appears to be in contrast with the min max characterisation given in [4, 7] for models on the vertices of the hypercube (see also [3] for the Hopfield model). The aim of this note is to show that the Auffinger and Chen formula can be equivalently expressed in terms of a min max.

One disadvantage of the spherical prior is that the associated moment generating function

$$\Gamma_N(h) := \frac{1}{N}\log\int \sigma_N(dx)e^{(h,x)}\,, \quad h\in\mathbb{R}^N\,, \tag{5}$$

is not easy to compute. If h is random with i.i.d. $\mathcal{N}(b,q)$ components it is convenient to set

$$\Gamma(b,q) := \lim_N E\Gamma_N(h)\,. \tag{6}$$

The so-called Crisanti–Sommers variational characterisation of it as $N\to\infty$ reads as follows.

Lemma 1 *Let $b\in\mathbb{R}$, $q>0$, $h\in\mathbb{R}^N$ with i.i.d $\mathcal{N}(b,\sqrt{q})$ components. Then*

$$\Gamma(b,q) = \frac{1}{2}\min_{r\in[0,1)}\left((b^2+q)(1-r) + \frac{r}{1-r} + \log(1-r)\right) \tag{7}$$

At the end of this note we give a simple proof of this statement, based on the method of [8, 9]. We first get a variational characterisation of the moment generating function of a Gaussian distribution (whose variance is Legendre conjugate to q) and then use concentration of measure.

A direct computation shows that the minimum of (7) is attained for

$$\frac{r}{(1-r)^2} = q + b^2\,. \tag{8}$$

A standard replica symmetric interpolation gives that for any $q_1, q_2 \in [0,1]$

$$A_{N_1,N_2}(\beta) = \frac{\beta^2\alpha(1-\alpha)}{2}(1-q_1)(1-q_2) + (1-\alpha)\Gamma(b_2, \beta^2\alpha q_1) + \alpha\Gamma(b_1, \beta^2(1-\alpha)q_2) + \mathrm{Error}_N(q_1,q_2)\,. \tag{9}$$

The last summand is an error term whose specific form is not important here. What matters is that by [1, Lemma 1] there is a choice of (q_1, q_2) (see below) for which this remainder goes to zero as $N \to \infty$ if β is small enough. Combining (7) and (8) we can rewrite the first line of (9) as

$$\begin{aligned}\mathrm{RS}(q_1,q_2) := {} & \frac{\beta^2\alpha(1-\alpha)}{2}(1-q_1)(1-q_2) \\ & + \frac{\beta^2\alpha(1-\alpha)}{2}\left(\left(q_2 + \frac{b_1^2}{\beta^2(1-\alpha)}\right)(1-r_1) + \left(q_1 + \frac{b_2^2}{\beta^2\alpha}\right)(1-r_2)\right) \\ & + \frac{\alpha}{2}\frac{r_1}{1-r_1} + \frac{\alpha}{2}\log(1-r_1) + \frac{1-\alpha}{2}\frac{r_2}{1-r_2} + \frac{1-\alpha}{2}\log(1-r_2)\,,\end{aligned} \tag{10}$$

under the condition

$$\frac{r_1}{(1-r_1)^2} = \beta^2(1-\alpha)q_2 + b_1^2\,, \qquad \frac{r_2}{(1-r_2)^2} = \beta^2\alpha q_1 + b_2^2\,. \tag{11}$$

Here we used that there is a sequence $o_N \to 0$ uniformly in q_1, q_2, β, α such that

$$\frac{\beta^2\alpha(1-\alpha)}{2}(1-q_1)(1-q_2) + (1-\alpha)\Gamma_N(\beta^2\alpha q_1) + \alpha\Gamma_N(\beta^2(1-\alpha)q_2) = \mathrm{RS}(q_1,q_2) + o_N\,. \tag{12}$$

Indeed (12) follows easily once we use Lemma 1 for the limit of the functions Γ_N and we note that (11) are the critical point equations related to the minimisation of (7).

The main observation of this note is that (10) under (11) is optimised as a min max.

Proposition 1 *Assume* $b_1^2 + b_2^2 > 0$. *The function* $\mathrm{RS}(q_1,q_2)$ *has a unique stationary point* $(\bar q_1, \bar q_2)$. *It solves*

$$\frac{q_2}{(1-q_2)^2} = \beta^2\alpha q_1 + b_1^2\,, \qquad \frac{q_1}{(1-q_1)^2} = \beta^2(1-\alpha)q_2 + b_2^2\,. \tag{13}$$

Moreover

$$\mathrm{RS}(\bar q_1, \bar q_2) = \min_{q_2\in[0,1]}\max_{q_1\in[0,1]} \mathrm{RS}(q_1,q_2)\,. \tag{14}$$

If $b_1 = b_2 = 0$ *and*

$$\beta^4\alpha(1-\alpha) < 1 \tag{15}$$

the origin is the unique solution of (13) *and*

$$\mathrm{RS}(0,0) = \min_{q_2\in[0,1]}\max_{q_1\in[0,1]} \mathrm{RS}(q_1,q_2)\,. \tag{16}$$

If $b_1 = b_2 = 0$ *and*

$$\beta^4\alpha(1-\alpha) > 1 \tag{17}$$

there is a unique $(\bar{q}_1, \bar{q}_2) \neq (0,0)$ *which solves* (13) *and such that* (14) *holds. Moreover*

$$\mathrm{RS}(0,0) = \max_{q_2\in[0,1]} \max_{q_1\in[0,1]} \mathrm{RS}(q_1,q_2)\,. \tag{18}$$

The crucial point of [1, Lemma 1] (for us) is that from the Latala argument [13, Sect. 1.4] it follows that the overlaps self-average as $N \to \infty$ at a point $(\tilde{q}_1, \tilde{q}_2)$ uniquely given by

$$\frac{\tilde{q}_1}{(1-\tilde{q}_1)^2} = \beta^2(1-\alpha)\tilde{q}_2 + b_1^2\,, \quad \frac{\tilde{q}_2}{(1-\tilde{q}_2)^2} = \beta^2\alpha\tilde{q}_1 + b_2^2\,, \tag{19}$$

which (see [12, Lemma 7]) are indeed asymptotically equivalent to

$$q_{1,N} := \frac{1}{N} E\left[\frac{\int \sigma_{N_1}(dy)\sigma_{N_1}(dy')(y,y')e^{\beta\sqrt{q_2}(y+y',h)}}{\left(\int \sigma_{N_1}(dy)e^{\beta\sqrt{q_2}(y,h)}\right)^2}\right], \tag{20}$$

$$q_{2,N} := \frac{1}{N} E\left[\frac{\int \sigma_{N_2}(dx)\sigma_{N_2}(dx')(x,x')e^{\beta\sqrt{q_1}(x+x',h)}}{\left(\int \sigma_{N_2}(dx)e^{\beta\sqrt{q_1}(x,h)}\right)^2}\right], \tag{21}$$

naturally arising from the replica symmetric interpolation (here h is random with i.i.d. $\mathcal{N}(0,1)$ entries). Comparing (11) and (19) readily implies that we can plug $(r_1, r_2) = (q_1, q_2)$ into (10) and obtain the convex function $P(q_1, q_2)$ of [1, Theorem 1], optimised by (19).

On the other hand, without using the Latala method one might still optimise (10) as a function of four variables, ignoring (11). Taking derivatives first in q_1, q_2, the critical point equations (24), (24) below select exactly $(q_1, q_2) = (r_1, r_2)$. This procedure is however unjustified a priori and this particular application of Latala's method legitimises the exchange in the order of the optimisation of the q and the r variables for small β, which a posteriori can be extended to all β [2, 9].

We stress that by itself the Latala method is not variational, it only gives the self-consistent equations for the critical points. It is the Crisanti–Sommers formula (7) which makes it implicitly variational. Such a variational representation is not necessary in other cases of interest, for instance for the bipartite SK model (namely Hamiltonian (1) with ± 1 spins), for which one simply has the log cosh. Indeed in this case a direct use of the Latala method yields the validity of the min max formula of [4] for β and $|b_1|, |b_2|$ small enough. The proof is essentially an exercise after [13, Proposition 1.4.8] and [1, Formula (9)] and will not be reproduced here in details. The replica symmetric sum-rule for the free energy (analogue of formula (9)) reads as

$$A_{N_1,N_2}(\beta) = \frac{\beta^2\alpha(1-\alpha)}{2}(1-q_1)(1-q_2) + (1-\alpha)E\log\cosh(b_2+\beta\sqrt{\alpha q_1}g)$$
$$+\alpha E\log\cosh(b_1+\beta\sqrt{(1-\alpha)q_2}g)$$
$$+\text{Error}_N(q_1,q_2)\,, \tag{22}$$

(here $g \sim \mathcal{N}(0,1)$) and the error term can be shown by the Latala method to vanish for small β, $|b_1|$, $|b_2|$, if $(q_1,q_2)=(\bar{q}_1,\bar{q}_2)$ are given by

$$\bar{q}_1 = E[\tanh(b_1+\beta\sqrt{(1-\alpha)\bar{q}_2}g)]\,, \qquad \bar{q}_2 = E[\tanh(b_2+\beta\sqrt{\alpha\bar{q}_1}g)]\,. \tag{23}$$

Therefore the free energy equals the first two lines on the r.h.s. of (22) evaluated in $(q_1,q_2)=(\bar{q}_1,\bar{q}_2)$, which is the value attained at the min max, as shown in [4, 7].

2 Proofs

Proof of Proposition 1 Assume first $b_1^2+b_2^2>0$. We differentiate (10) and by (11) we get

$$\partial_{q_1}\text{RS} = \frac{\beta^2\alpha(1-\alpha)}{2}(q_2-r_2(q_1)) \tag{24}$$
$$\partial_{q_2}\text{RS} = \frac{\beta^2\alpha(1-\alpha)}{2}(q_1-r_1(q_2))\,. \tag{25}$$

The functions r_1, r_2 write explicitly as

$$r_1(q_2) = \frac{\sqrt{1+4(\beta^2(1-\alpha)q_2+b_1^2)}-1}{\sqrt{1+4(\beta^2(1-\alpha)q_2+b_1^2)}+1} \tag{26}$$
$$r_2(q_1) = \frac{\sqrt{1+4(\beta^2\alpha q_1+b_2^2)}-1}{\sqrt{1+4(\beta^2\alpha q_1+b_2^2)}+1}\,. \tag{27}$$

We easily see that r_1, r_2 are increasing from $r_1(0), r_2(0) > 0$ (obviously computable by the formulas above) to 1 and concave. Moreover we record for later use that if $b_1=b_2=0$ we have

$$\frac{d}{dq_2}r_1(q_2)\Big|_{q_2=0} = \beta^2(1-\alpha)\,, \quad \frac{d}{dq_1}r_2(q_1)\Big|_{q_1=0} = \beta^2\alpha\,. \tag{28}$$

Now we take the derivative w.r.t. q_1 and note that the r.h.s. of (24) is decreasing as a function of q_1, thus $\partial^2_{q_1}\text{RS}<0$. Therefore by the implicit function theorem there is a unique function q_1 such that $q_2=r_2(q_1)$. As a function of q_2, q_1 is non-negative, increasing and convex and it is $q_1(r_2(0))=0$. We set

$$\mathrm{RS}_1(q_2) := \max_{q_1} \mathrm{RS}(q_1, q_2) = \mathrm{RS}(q_1(q_2), q_2) \tag{29}$$

and compute

$$\partial_{q_2} \mathrm{RS}_1(q_2) = \frac{\beta^2\alpha(1-\alpha)}{2} \left(q_1(q_2) - r_1(q_2)\right) . \tag{30}$$

By the properties of the functions q_1 and r_1 it is clear that there is a unique intersection point $\bar{q}_2$; moreover $q_1 \leqslant r_1$ for $q_2 \leqslant \bar{q}_2$ and otherwise $q_1 \geqslant r_1$. Therefore $\partial_{q_2} \mathrm{RS}_1(q_2)$ is increasing in a neighbourhood of $\bar{q}_2$ which allows us to conclude $\partial^2_{q_2} \mathrm{RS}_1 > 0$. This finishes the proof if $b_1^2 + b_2^2 > 0$.

If $b_1 = b_2 = 0$ the origin is always a stationary point. It is unique if

$$\left[\frac{d}{dq_1} r_2(q_1)\big|_{q_1=0}\right]^{-1} = \frac{d}{dq_2} q_1(q_2)\big|_{q_2=0} > \frac{d}{dq_2} r_1(q_2)\big|_{q_2=0} , \tag{31}$$

which, bearing in mind (28), amounts to ask (15).

Since r_2 is increasing around the origin, we have $\partial^2_{q_1} \mathrm{RS} < 0$ and by the implicit function theorem we define locally a function $q_1(q_2)$ increasing and positive, vanishing at the origin. We set

$$\mathrm{RS}_1(q_2) := \max_{q_1} \mathrm{RS}(q_1, q_2) = \mathrm{RS}(q_1(q_2), q_2) \tag{32}$$

and compute

$$\partial_{q_2} \mathrm{RS}_1(q_2) = \frac{\beta^2\alpha(1-\alpha)}{2} \left(q_1(q_2) - r_1(q_2)\right) . \tag{33}$$

By (31) we have $\partial^2_{q_2} \mathrm{RS}_1 \big|_{q_2=0} > 0$, whence we obtain (16).

If (17) holds, then

$$\frac{d}{dq_2} q_1(q_2)\big|_{q_2=0} < \frac{d}{dq_2} r_1(q_2)\big|_{q_2=0} , \tag{34}$$

which proceeding as before leads to (18).

However also in the case $b_1 = b_2 = 0$ we can repeat all the steps done in the case $b_1^2 + b_2^2 > 0$, showing the existence of a point $(\bar{q}_1, \bar{q}_2)$ in which a min max of RS is attained. If (31) (*i.e.* (15)) holds then it must be $(\bar{q}_1, \bar{q}_2) = (0, 0)$. If (17) holds, then (34) enforces

$$q_1(q_2) - r_1(q_2) \ \leqslant \ 0$$

in a neighbourhood of the origin (as $q_1(0) = r_1(0) = 0$), which implies that the critical point $(\bar{q}_1, \bar{q}_2)$ must fall elsewhere. □

Proof of Lemma 1 We will prove that for all $u \in \sqrt{q} S^N$

$$\Gamma^{(\sigma)}(q) := \lim_N \Gamma_N(u) = \frac{1}{2} \min_{r \in [0,1)} \left(q(1-r) + \frac{r}{1-r} + \log(1-r) \right) . \tag{35}$$

We show first that (35) implies the assertion. Let h be a random vector with i.i.d. $\mathcal{N}(0, q)$ entries. (As customary we write $X \simeq Y$ if there are constants $c, C > 0$ such that $cY \leqslant X \leqslant CY$). The classical estimates

$$\Gamma_N(h) \leqslant \frac{\|h\|_2}{\sqrt{N}}, \quad P\left(\left|\frac{\|h\|_2}{\sqrt{N}} - \sqrt{q}\right| \geqslant t\right) \simeq e^{-\frac{t^2 N}{2}} \tag{36}$$

permit us to write for all $t > 0$ (small)

$$\begin{aligned} |E[\Gamma_N] - \Gamma^{(\sigma)}(q)| &\leqslant |E[\Gamma_N 1_{\left\{\left|\frac{\|h\|}{\sqrt{N}}-\sqrt{q}\right|<t\right\}}] - \Gamma^{(\sigma)}(q)| + \left|E\left[\frac{\|h\|_2}{\sqrt{N}} 1_{\left\{\left|\frac{\|h\|}{\sqrt{N}}-\sqrt{q}\right| \geqslant t\right\}}\right]\right| \\ &\simeq \left|\Gamma_N(u^*) P\left(\left|\frac{\|h\|_2}{\sqrt{N}} - \sqrt{q}\right| < t\right) - \Gamma^{(\sigma)}(q)\right| + o(t) + e^{-t^2 N/2} \\ &\simeq \left|\Gamma_N(u^*) - \Gamma^{(\sigma)}(q)\right| + o(t) + e^{-t^2 N/2}, \end{aligned} \tag{37}$$

for some $u^* \in \sqrt{q} S^N$ and $o(t) \to 0$ as $t \to 0$. Since $t > 0$ is arbitrary we obtain

$$|E[\Gamma_N] - \Gamma^{(\sigma)}(q)| \leqslant \left|\Gamma_N(u^*) - \Gamma^{(\sigma)}(q)\right|.$$

It remains to show (35). Given $\varepsilon > 0$ we introduce the spherical shell

$$S^{N,\varepsilon} := S^N + \sqrt{\frac{\varepsilon}{N}} S^N$$

and the measure $\sigma_N^{(\varepsilon)}$ as the uniform probability on it. For any $\theta > 0$ we have

$$\begin{aligned} \int \sigma_N^{(\varepsilon)}(dx) e^{(u,x)} &\leqslant e^{\frac{\theta(N+\varepsilon)}{2}} \int \sigma_N^{(\varepsilon)}(dx) e^{-\frac{\theta}{2}\|x\|_2^2 + (u,x)} \\ &\leqslant e^{\frac{\theta(N+\varepsilon)}{2}} \frac{\sqrt{2\pi}^N}{\theta^{\frac{N}{2}} |S^{N,\varepsilon}|} \int e^{-\frac{\theta}{2}\|x\|_2^2 + (u,x)} \frac{dx}{\sqrt{2\pi}^N} \\ &= e^{\frac{\theta(N+\varepsilon)}{2} + \frac{qN}{2\theta}} \frac{\sqrt{2\pi}^N}{\theta^{\frac{N}{2}} |S^{N,\varepsilon}|}. \end{aligned} \tag{38}$$

Therefore for $C > 0$ large enough

$$\frac{1}{N} \log \int \sigma_N^{(\varepsilon)}(dx) e^{(u,x)} \leqslant \frac{\theta}{2} + \frac{q}{2\theta} - \frac{1}{2}(\log\theta + 1) + C\theta\frac{\varepsilon}{N}. \tag{39}$$

Since this inequality holds for all $\theta > 0$ and $\varepsilon > 0$ we have

$$\limsup_N \Gamma_N(u) \leqslant \inf_{\theta > 0} \left(\frac{q}{2\theta} + \frac{\theta - 1}{2} - \frac{1}{2}\log\theta\right). \tag{40}$$

We set for brevity

$$\Gamma_1(\theta) := \frac{q}{2\theta} + \frac{\theta - 1}{2} - \frac{1}{2}\log\theta$$

and notice that Γ_1 is uniformly convex in all the intervals $(0, \theta_0)$ for finite $\theta_0 > 0$.

For the reverse bound, again we let $\theta > 0$ and write

$$\int \sigma_N^{(\varepsilon)}(dx)e^{(u,x)} \geqslant e^{\frac{\theta}{2}N}\int_{\mathbb{R}^N}\frac{dx}{|S^{N,\varepsilon}|}e^{-\frac{\theta}{2}\|x\|_2^2+(u,x)} - e^{\frac{\theta}{2}N}\int_{(S^{N,\varepsilon})^c}\frac{dx}{|S^{N,\varepsilon}|}e^{-\frac{\theta}{2}\|x\|_2^2+(u,x)}\,. \tag{41}$$

The first summand on the r.h.s. can be written as before

$$e^{\frac{\theta}{2}N}\int_{\mathbb{R}^N}\frac{dx}{|S^{N,\varepsilon}|}e^{-\frac{\theta}{2}\|x\|_2^2+(u,x)} = e^{\frac{\theta N}{2}+\frac{qN}{2\theta}}\frac{\sqrt{2\pi}^N}{\theta^{\frac{N}{2}}|S^{N,\varepsilon}|}\,. \tag{42}$$

For the second summand we introduce $\eta \in (0, \frac{\theta}{2})$ and bound

$$e^{\frac{\theta}{2}N}\int_{\|x\|^2 \leqslant N-\varepsilon}\frac{dx}{|S^{N,\varepsilon}|}e^{-\frac{\theta}{2}\|x\|_2^2+(u,x)} \leqslant e^{\frac{\theta}{2}N+(N-\varepsilon)\frac{\eta}{2}+\frac{qN}{2(\theta+\eta)}}\frac{\sqrt{2\pi}^N}{\theta^{\frac{N}{2}}|S^{N,\varepsilon}|} \tag{43}$$

$$e^{\frac{\theta}{2}N}\int_{\|x\|^2 \geqslant N+\varepsilon}\frac{dx}{|S^{N,\varepsilon}|}e^{-\frac{\theta}{2}\|x\|_2^2+(u,x)} \leqslant e^{\frac{\theta}{2}N-(N+\varepsilon)\frac{\eta}{2}+\frac{qN}{2(\theta-\eta)}}\frac{\sqrt{2\pi}^N}{\theta^{\frac{N}{2}}|S^{N,\varepsilon}|}\,. \tag{44}$$

Thus

$$\liminf_N \frac{1}{N}\log\int \sigma_N^{(\varepsilon)}(dx)e^{(u,x)} \geqslant \max(\Gamma_1, \Gamma_2, \Gamma_3) \tag{45}$$

with

$$\Gamma_2(\eta, \theta) := \frac{q}{2(\theta-\eta)} + \frac{\eta(1-\frac{\varepsilon}{N})}{2} + \frac{\theta-1}{2} - \frac{1}{2}\log\theta\,,$$

$$\Gamma_3(\eta, \theta) := \frac{q}{2(\theta+\eta)} - \frac{\eta(1+\frac{\varepsilon}{N})}{2} + \frac{\theta-1}{2} - \frac{1}{2}\log\theta.$$

Now we define

$$\Delta_{12}(\eta, \theta) := \Gamma_1(\theta) - \Gamma_2(\eta, \theta)\,, \quad \Delta_{13}(\eta, \theta) := \Gamma_1(\theta) - \Gamma_3(\eta, \theta)\,, \tag{46}$$

and we seek $\bar{\theta} > 0$ for which $\Delta_{12}, \Delta_{13} \geqslant 0$ for sufficiently small η. Since $\Delta_{12}(0, \theta) = \Delta_{13}(0, \theta) = 0$ it suffices to study

$$\frac{d}{d\eta}\Delta_{12}\Big|_{\eta=0}\,, \quad \frac{d}{d\eta}\Delta_{13}\Big|_{\eta=0}\,. \tag{47}$$

A direct computation shows

$$\frac{d}{d\eta}\Delta_{12}\Big|_{\eta=0} = \frac{\varepsilon}{2N} - \partial_\theta \Gamma_1(\theta)\,, \tag{48}$$

$$\frac{d}{d\eta}\Delta_{13}\Big|_{\eta=0} = \frac{\varepsilon}{2N} + \partial_\theta \Gamma_1(\theta)\,. \tag{49}$$

Combining (47), (48) and (49) we see that plugging $\bar{\theta} = \arg\min \Gamma_1$ into (45) we arrive to

$$\liminf_N \Gamma_N(u) \geqslant \min_{\theta>0}\left(\frac{q}{2\theta} + \frac{\theta-1}{2} - \frac{1}{2}\log\theta\right). \tag{50}$$

Therefore (40) and (50) give

$$\lim_N \Gamma_N(u) = \min_{\theta>0}\left(\frac{q}{2\theta} + \frac{\theta-1}{2} - \frac{1}{2}\log\theta\right)$$

and changing variable $\theta = (1-r)^{-1}$ we obtain (35). □

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