

# Generalized uncertainty relations

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Received: 10 November 2015 / Revised: 10 October 2016 / Accepted: 11 October 2016  
Published online: 26 November 2016  
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**Abstract** The standard uncertainty relations (UR) in quantum mechanics are typically used for unbounded operators (like the canonical pair). This implies the need for the control of the domain problems. On the other hand, the use of (possibly bounded) functions of basic observables usually leads to more complex and less readily interpretable relations. In addition, UR may turn trivial for certain states if the commutator of observables is not proportional to a positive operator. In this letter we consider a generalization of standard UR resulting from the use of two, instead of one, vector states. The possibility to link these states to each other in various ways adds additional flexibility to UR, which may compensate some of the above-mentioned drawbacks. We discuss applications of the general scheme, leading not only to technical improvements, but also to interesting new insight.

**Keywords** Uncertainty relations · Quantum theory · Commutation relations

**Mathematics Subject Classification** 81Q10 · 47N50 · 47B15

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## 1 Introduction

In popular textbook terms, the quantum-mechanical UR states that if any three observables  $A, B, C$  satisfy the commutation relation

$$[A, B] = iC, \quad (1)$$

then in any normalized quantum state  $\psi$  there is

$$\Delta_\psi(A)\Delta_\psi(B) \geq \frac{1}{2}|\langle C \rangle_\psi|, \quad (2)$$

where  $\langle A \rangle_\psi = (\psi, A\psi)$  is the mean value of the probability distribution of  $A$  in the state  $\psi$  and  $\Delta_\psi(A) = \|(A - \langle A \rangle_\psi)\psi\|$  is the standard deviation of this distribution (see, e.g., [3]). This formulation follows the extension of the original Heisenberg relation [5] given by Robertson [13].

Quantum-mechanical observables are self-adjoint operators, which usually are not bounded (as is the case with the most prominent example of the canonical pair). Therefore, a more mathematically conscious formulation of UR has to take into account domain restrictions (see, e.g., [4]). By  $\mathcal{D}(A) \subseteq \mathcal{H}$  we denote the domain of an operator  $A$  acting in a Hilbert space  $\mathcal{H}$ . For any self-adjoint operators  $A, B$ , the relation (1) defines a symmetric operator  $C$  on the domain

$$\mathcal{D}(C) = \mathcal{D}(AB) \cap \mathcal{D}(BA). \quad (3)$$

It is now a simple mathematical theorem that the relation (2) is satisfied for any normalized vector  $\psi \in \mathcal{D}(C)$ .

The question that now arises is this: is  $\mathcal{D}(C)$  “sufficiently large” for the relation to be of use? In the worst possible case it could happen that  $\mathcal{D}(C)$  would not be dense in  $\mathcal{H}$ , which would leave outside the range of the relation the whole closed subspace  $\mathcal{D}(C)^\perp \subseteq \mathcal{H}$ . This case is of little use, so we assume from now on that  $\overline{\mathcal{D}(C)} = \mathcal{H}$  (bar denoting the closure). Even with this restriction, we are still left with a few open problems:

- (i) If  $\psi$  is not in  $\mathcal{D}(A)$ , then  $\Delta_\psi(A)$  may be regarded as infinite; the relation tells us then nothing on the spread of distribution of  $B$  in the state  $\psi$ .
- (ii) If  $\psi \in \mathcal{D}(A) \cap \mathcal{D}(B)$ , then the product of uncertainties is finite. It may happen that also  $C$  extends to this larger domain, but UR need not extend to this case.
- (iii) The restrictions of  $A$  and  $B$  to  $\mathcal{D}(C)$  need not determine self-adjoint operators  $A$  and  $B$  uniquely (i.e., in technical terms,  $\mathcal{D}(C)$  need not be a core for  $A$  and  $B$ ), so that the UR does not admit all states crucial for the determination of the observables  $A$  and  $B$  themselves.
- (iv) In general, if  $C$  is not strictly positive, it may have vanishing expectation value  $\langle C \rangle$  in the state  $\psi$  under consideration. The relation has no nontrivial content in this case.

Standard examples of the difficulties (ii) and (iii) occur for the ‘angle–angular momentum’ pair. Let  $\Phi, L$  be the operators in  $\mathcal{H} = L^2((0, 2\pi))$  defined by:

$$\begin{aligned} \mathcal{D}(\Phi) &= \mathcal{H}, \quad \mathcal{D}(L) = \{\psi \in \mathcal{H} \mid \psi' \in \mathcal{H}, \quad \psi(0) = \psi(2\pi)\} \\ (\Phi\psi)(\varphi) &= \varphi\psi(\varphi), \quad (L\psi)(\varphi) = -i\psi'(\varphi), \end{aligned}$$

(where  $\psi'$  is a measurable derivative function of  $\psi$ ). With these domains both operators are self-adjoint. The UR (2) on the domain (3) holds then with  $C\psi = \psi$  on

$$\mathcal{D}(C) = \{\psi \in \mathcal{H} \mid \psi' \in \mathcal{H}, \psi(0) = \psi(2\pi) = 0\},$$

so that on this domain, which may be shown to be dense:  $\overline{\mathcal{D}(C)} = \mathcal{H}$ , one has for normalized  $\psi$ :

$$\Delta_\psi(\Phi)\Delta_\psi(L) \geq \frac{1}{2}.$$

Both sides of this inequality are meaningful and finite for  $\psi \in \mathcal{D}(\Phi) \cap \mathcal{D}(L) = \mathcal{D}(L)$ , but the inequality does not extend to this larger domain: take any eigenstate of  $L$  to find  $0 \geq 1$ , which illustrates difficulty (ii). The explanation of this seeming paradox is that for  $L\psi = m\psi$  there is no sequence of vectors  $\psi_n \in \mathcal{D}(C)$  which converges to  $\psi$ , and at the same time satisfies  $\Delta_{\psi_n}(L) \rightarrow \Delta_\psi(L)$ .

Even more disturbing is the fact that the domain  $\mathcal{D}(C)$  is not sufficient to uniquely identify the self-adjoint operator  $L$  taking part in the above relation (i.e.,  $\mathcal{D}(C)$  is not a core for  $L$ ). To see this it is sufficient to note that for each complex  $\omega$  with  $|\omega| = 1$  one has a self-adjoint operator  $L_\omega$  defined as  $L$ , but on a different domain  $\mathcal{D}(L_\omega) = \{\psi \in \mathcal{H} \mid \psi' \in \mathcal{H}, \psi(2\pi) = \omega\psi(0)\}$  (see, e.g., [12]). Each of these operators may replace  $L$  in the above UR, with no change of  $C$  or  $\mathcal{D}(C)$ . This illustrates difficulty (iii).

Difficulty (iv) occurs in the well-known case of three-dimensional angular momentum operators; see below.

In the rest of this article we propose a simple extension of the minimization argument leading to UR. Our generalized UR are given in Sect. 2, Proposition 1. Applications of the general scheme to a few cases of particular physical interest are discussed in Sect. 3. Our use of the result of Proposition 1 is closely related to the mathematical and physical problem of the original formulation of UR: find optimal bounds on spreads of probability distributions defined by a given state for two incompatible observables.<sup>1</sup> We do not touch upon various other problems broadly related to ‘uncertainty’ in quantum mechanics, which recently draw considerable attention in physical literature (entropic uncertainty, error-disturbance problem, parameter estimation, etc.).

<sup>1</sup> A different generalization of UR has been recently proposed in [9].

## 2 Generalized uncertainty relation

We consider general normal operators. We recall that (see, e.g., [14]):

- (i)  $A$  is called normal if it is densely defined, closed and satisfies  $A^*A = AA^*$ .
- (ii) For normal  $A$  there is  $\mathcal{D}(A^*) = \mathcal{D}(A)$  and  $\|A^*\chi\| = \|A\chi\|$  for all vectors  $\chi \in \mathcal{D}(A)$ .
- (iii) All functions of self-adjoint operators are normal operators.
- (iv) All normal operators satisfy the spectral theorem,  $A = \int_{\sigma(A)} z dE_z^A$ , where  $\sigma(A)$  is the spectrum of  $A$ —a closed subset of  $\mathbb{C}$ , and  $E_\Omega^A$  is its spectral family. For each normalized vector  $\psi$ , the mapping  $\Omega \mapsto (\psi, E_\Omega^A \psi)$  is a probability measure on Borel sets  $\Omega \subseteq \sigma(A)$ , with the mean value  $\langle A \rangle_\psi = (\psi, A\psi) \in \mathbb{C}$  and the standard deviation  $\Delta_\psi(A) = \|(A - \langle A \rangle_\psi)\psi\|$ .

Let  $A$  and  $B$  be normal operators with the domains  $\mathcal{D}(A)$  and  $\mathcal{D}(B)$ , respectively. Then we define a sesquilinear form

$$q_{A,B}(\varphi, \chi) = (A^*\varphi, B\chi) - (B^*\varphi, A\chi)$$

with the domain  $\varphi, \chi \in \mathcal{D}(q_{A,B}) \equiv \mathcal{D}(A) \cap \mathcal{D}(B)$  (an example of a recent use of this ‘weak commutator’ may be found in [15]). In special case when  $\chi \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ , this weak commutator becomes the ordinary one,  $q_{A,B}(\varphi, \chi) = (\varphi, [A, B]\chi)$ . Moreover, for any complex numbers  $a, b$ , we denote  $A_a = A - a\mathbf{1}$ ,  $B_b = B - b\mathbf{1}$ , which are also normal operators.

**Proposition 1** (Generalized Uncertainty Relation). *For any normal operators  $A, B$  and unit vectors  $\varphi, \chi \in \mathcal{D}(q_{A,B})$ , the following inequality holds*

$$\begin{aligned} |q_{A,B}(\varphi, \chi)| &\leq \inf_{a,b \in \mathbb{C}} \left( \|A_a\varphi\| \|B_b\chi\| + \|B_b\varphi\| \|A_a\chi\| \right) \\ &= \inf_{\substack{\lambda_1, \lambda_2 \in (0, 1) \\ \lambda_1 + \lambda_2 = 1}} \left\{ \sqrt{\Delta_\varphi^2(A) + |\delta\langle A \rangle|^2 \lambda_1^2} \sqrt{\Delta_\chi^2(B) + |\delta\langle B \rangle|^2 \lambda_2^2} \right. \\ &\quad \left. + \sqrt{\Delta_\varphi^2(B) + |\delta\langle B \rangle|^2 \lambda_1^2} \sqrt{\Delta_\chi^2(A) + |\delta\langle A \rangle|^2 \lambda_2^2} \right\}, \end{aligned} \tag{4}$$

where  $\delta\langle A \rangle = \langle A \rangle_\varphi - \langle A \rangle_\chi$ ,  $\delta\langle B \rangle = \langle B \rangle_\varphi - \langle B \rangle_\chi$ .

*Proof* We first note that  $q_{A_a, B_b} = q_{A,B}$ . Therefore, the successive use of the triangle and the Schwarz inequalities (and property (ii) above) gives

$$|q_{A,B}(\varphi, \chi)| \leq \|A_a\varphi\| \|B_b\chi\| + \|B_b\varphi\| \|A_a\chi\|. \tag{5}$$

Thus, using the arbitrariness of  $a$  and  $b$ , we arrive at the first relation (inequality) in (4). The rhs of (5) is a real function  $F(a, \bar{a}, b, \bar{b})$ , nondecreasing for  $|a|$  or  $|b|$  sufficiently

large and tending to  $+\infty$ . Therefore, it reaches its infimum at one of its stationary points, which are the solutions of the set of equations  $\partial F/\partial \bar{a} = 0, \partial F/\partial \bar{b} = 0$ , i.e.,

$$\begin{aligned} \gamma_2(a - \langle A \rangle_\varphi) + \gamma_1(a - \langle A \rangle_\chi) &= 0, \\ \gamma_2(b - \langle B \rangle_\varphi) + \gamma_1(b - \langle B \rangle_\chi) &= 0, \\ \gamma_1 &= \|A_a\varphi\| \|B_b\varphi\|, \quad \gamma_2 = \|A_a\chi\| \|B_b\chi\|. \end{aligned}$$

Solving the first two of these equations for  $a$  and  $b$  in terms of  $\lambda_i \equiv \gamma_i/(\gamma_1 + \gamma_2)$  ( $i = 1, 2$ ), one obtains  $a = \lambda_2\langle A \rangle_\varphi + \lambda_1\langle A \rangle_\chi, b = \lambda_2\langle B \rangle_\varphi + \lambda_1\langle B \rangle_\chi$ . Setting these values into (5), one obtains the second relation (equality) in (4). The condition for stationary points is now reduced to the condition on  $\lambda_i$ :  $\gamma_1\lambda_2 = \gamma_2\lambda_1$ , i.e.,

$$\begin{aligned} & \left(\Delta_\varphi^2(A) + |\delta\langle A \rangle|^2\lambda_1^2\right)\left(\Delta_\varphi^2(B) + |\delta\langle B \rangle|^2\lambda_1^2\right)\lambda_2^2 \\ &= \left(\Delta_\chi^2(A) + |\delta\langle A \rangle|^2\lambda_2^2\right)\left(\Delta_\chi^2(B) + |\delta\langle B \rangle|^2\lambda_2^2\right)\lambda_1^2, \end{aligned} \tag{6}$$

where  $\lambda_1, \lambda_2 \in \langle 0, 1 \rangle, \lambda_1 + \lambda_2 = 1$ . This condition, when expressed in terms of one unknown, is an algebraic equation of the fifth order, which is not algebraically solvable in general. Thus, we postpone its solution to more special cases.  $\square$

The simplest special case of Proposition 1 is obtained for  $\varphi = \chi, \|\chi\| = 1$ . Relation (4) then immediately takes the form

$$\frac{1}{2}|q_{A,B}(\chi, \chi)| \leq \Delta_\chi(A)\Delta_\chi(B), \quad \chi \in \mathcal{D}(A) \cap \mathcal{D}(B). \tag{7}$$

The UR in this form was applied by Kraus [8] to the above-mentioned case of angle–angular momentum pair. Integrating by parts one finds that  $q_{L,\Phi}(\chi, \chi) = i(2\pi|\chi(2\pi)|^2 - 1)$ , hence

$$\frac{1}{2}|1 - 2\pi|\chi(2\pi)|^2| \leq \Delta_\chi(L)\Delta_\chi(\Phi), \quad \chi \in \mathcal{D}(L).$$

This yields correct  $0 \leq 0$  for  $L\chi = m\chi$ , but a drawback of this relation is that beside uncertainties it needs the value of  $\chi$  at a particular point.

Relation (7) reduces to the standard form (2) for  $\chi \in \mathcal{D}(C)$ .

The extended flexibility of the relation (4) relies on the possibility to assume more general relations between  $\varphi$  and  $\chi$ , than equality. Before discussing some special examples in the next section, we illustrate this with a simple proof of a property, which may also be proved without the use of Proposition 1, although with much more effort.

**Proposition 2** Let  $\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i, \chi = \sum_{i=1}^N \chi_i, \chi_i \in \mathcal{H}_i, \|\chi\| = 1$ .

If  $A\chi_i, B\chi_i \in \mathcal{H}_i$  for  $i = 1, \dots, N$  (in particular, if all  $\mathcal{H}_i$  are invariant under  $A$  and  $B$ ) then

$$\frac{1}{2} \sum_{i=1}^N |q_{A,B}(\chi_i, \chi_i)| \leq \Delta_\chi(A)\Delta_\chi(B).$$

*Proof* Set  $\varphi = \sum_{i=1}^N \frac{q_{A,B}(\chi_i, \chi_i)}{|q_{A,B}(\chi_i, \chi_i)|} \chi_i$  and use relation (4). The means and uncertainties of  $A$  and  $B$  in the states  $\chi$  and  $\varphi$  coincide, while  $\frac{1}{2}|q_{A,B}(\varphi, \chi)|$  is equal to the lhs of the above inequality.  $\square$

### 3 Applications of the generalized relation

We start with a few remarks on unitary operators and one-parameter groups. If  $V$  is a unitary operator on  $\mathcal{H}$ , then it is normal, with the spectrum on the unit circle, and with  $\Delta_\psi^2(V) = \| (V - \langle V \rangle_\psi) \psi \|^2 = 1 - |\langle V \rangle_\psi|^2 \leq 1$ . In this case we shall denote

$$\delta_\psi(V) \equiv \frac{\Delta_\psi(V)}{[1 - \Delta_\psi^2(V)]^{1/2}} = \frac{[1 - |\langle V \rangle_\psi|^2]^{1/2}}{|\langle V \rangle_\psi|}.$$

This parameter is an increasing function of the deviation, for small spread  $\delta_\psi(V) \approx \Delta_\psi(V)$ , while for  $\Delta_\psi(V) \rightarrow 1$  (maximal spread) it tends to infinity.

If  $V(s) = \exp[-isX]$  is a one-parameter unitary group with the self-adjoint generator  $X$ , then  $\langle V(s) \rangle_\psi = \int_{\sigma(X)} e^{-isx} d\mu_\psi(x)$ , where  $d\mu_\psi(x)$  is the spectral measure of  $X$  in the state  $\psi$ . Using this representation and its conjugate one finds

$$\Delta_\psi^2(V(s)) = 2 \int_{\sigma(X) \times \sigma(X)} \sin^2 \left[ \frac{1}{2}s(x - x') \right] d\mu_\psi(x) d\mu_\psi(x').$$

Therefore, if  $\psi \in \mathcal{D}(X)$ , the finite limit exists

$$\lim_{s \rightarrow 0} \frac{\Delta_\psi^2(V(s))}{s^2} = \frac{1}{2} \int (x - x')^2 d\mu_\psi(x) d\mu_\psi(x') = \Delta_\psi^2(X),$$

which is also equal to  $\lim_{s \rightarrow 0} s^{-2} \delta_\psi^2(V(s))$  (if  $\psi \notin \mathcal{D}(X)$  the limit is  $+\infty$ ).

#### 3.1 A Weyl pair

**Proposition 3** *Let  $U$  and  $W$  be unitary operators on  $\mathcal{H}$ , such that*

$$WU = \omega UW,$$

with some complex number  $\omega$ ,  $|\omega| = 1$ —we shall call any such system a Weyl pair. Then for each normalized vector  $\psi \in \mathcal{H}$  the following inequality holds

$$\frac{1}{2}|\omega - 1| \leq \delta_\psi(W) \delta_\psi(U). \tag{8}$$

*Proof* We consider relation (4) with the substitutions  $A = W$ ,  $B = U$ ,  $\varphi = U\psi$ ,  $\chi = W^*\psi$ . Then using the relation  $UW^* = \omega W^*U$  one finds  $q_{W,U}(\varphi, \chi) = \omega - 1$ , so the lhs of (4) is  $|\omega - 1|$ . In addition, algebraic relations give  $\langle W \rangle_\varphi = \omega \langle W \rangle_\psi$ ,  $\langle W \rangle_\chi = \langle W \rangle_\psi$ ,  $\langle U \rangle_\varphi = \langle U \rangle_\psi$ ,  $\langle U \rangle_\chi = \omega \langle U \rangle_\psi$ . We set these means into the stationary point condition (6). After some simple algebra this condition is reduced to the form

$$\left( [\delta_\psi(W)\delta_\psi(U)]^2 - [4\lambda_1\lambda_2\varepsilon^2]^2 \right) (\lambda_1 - \lambda_2) = 0,$$

where we have introduced  $\varepsilon = \frac{1}{2}|\omega - 1| \in \langle 0, 1 \rangle$ . For the stationary point  $\lambda_1 = \lambda_2 = \frac{1}{2}$  the relation (4) implies

$$\varepsilon \leq \sqrt{\Delta_\psi^2(W) + |\langle W \rangle_\psi|^2 \varepsilon^2} \sqrt{\Delta_\psi^2(U) + |\langle U \rangle_\psi|^2 \varepsilon^2}.$$

Solving this for  $\varepsilon$  one obtains relation (8). This stationary point is in fact the unique minimum point of the rhs of our UR (4) in all nontrivial cases, i.e., when  $\omega \neq 1$ . Indeed, for  $\lambda_1 \neq \lambda_2$  there is  $\lambda_1\lambda_2 < \frac{1}{4}$ , so using inequality (8) we find for  $\varepsilon > 0$ :

$$\delta_\psi(W)\delta_\psi(U) - 4\lambda_1\lambda_2\varepsilon^2 > \varepsilon(1 - \varepsilon) \geq 0,$$

which closes the proof. □

*Remark* Relation equivalent to our inequality (8) has been given earlier by Massar and Spindel [11], and their proof is to be found in the supplementary material to that reference. Our form of the inequality has more directly visible interpretation. In addition, our much simpler proof is an application of our general minimization scheme, showing that the inequality is in a certain sense optimal.

### 3.2 The canonical pair

The standard canonical pair operators  $X, P$  are uniquely (up to unitary equivalence) defined as being generators of irreducibly represented one-parameter groups  $W(\alpha) = \exp[-i\alpha X]$ ,  $U(\beta) = \exp[-i\beta P]$  which satisfy relation

$$W(\alpha)U(\beta) = \exp[-i\alpha\beta]U(\beta)W(\alpha).$$

Inequality (8) may be written in the form

$$\frac{|\sin(\frac{1}{2}\alpha\beta)|}{|\alpha\beta|} \leq \frac{\delta_\psi(W(\alpha))}{|\alpha|} \frac{\delta_\psi(U(\beta))}{|\beta|}.$$

For small  $\alpha, \beta$  quantities on the rhs approximate the standard deviations of  $X$  and  $P$ , but are finite for all  $\psi$ . If  $\psi$  is in the domain of one of the variables, say  $\psi \in \mathcal{D}(X)$ , then the limit in  $\alpha$  results in

$$\frac{1}{2} \leq \Delta_\psi(X) \frac{\delta_\psi(U(\beta))}{|\beta|},$$

and when  $\psi \in \mathcal{D}(X) \cap \mathcal{D}(P)$ , the usual UR is obtained for  $\beta \rightarrow 0$ , with the guarantee of its validity on this domain.

### 3.3 Angle–angular momentum pair

Similarly, the angle–angular momentum operators  $\Phi, L$  are uniquely (up to unitary equivalence) defined by irreducibly represented unitary operators  $W(n) = \exp[-in\Phi]$ ,  $n \in \mathbb{Z}$ , and  $U(\beta) = \exp[-i\beta L]$ ,  $\beta \in \mathbb{R}/\text{mod } 2\pi$ , which satisfy the relation

$$W(n)U(\beta) = \exp[-in\beta]U(\beta)W(n).$$

The UR now takes the form

$$\frac{|\sin(\frac{1}{2}n\beta)|}{|\beta|} \leq \delta_\psi(W(n)) \frac{\delta_\psi(U(\beta))}{|\beta|}.$$

For  $\psi \in \mathcal{D}(L)$  this implies

$$\frac{1}{2}|n| \leq \delta_\psi(W(n))\Delta_\psi(L).$$

(The latter relation for  $n = 1$  was earlier discussed by Hradil et al. [7]). In particular, if  $\psi$  is an eigenvector of  $L$ , then the spread reaches  $\Delta_\psi(W(n)) = 1$ , the maximal value.

### 3.4 Unitary transformation

**Proposition 4** *Let  $U$  be a unitary transformation,  $A$  a normal operator, and denote  $A_U = U^*AU$ . Then for each normalized  $\chi \in \mathcal{D}(A) \cap \mathcal{D}(A_U) = \mathcal{D}(A) \cap U^*\mathcal{D}(A)$  there is*

$$|\langle A_U \rangle_\chi - \langle A \rangle_\chi| \leq \delta_\chi(U) [\Delta_\chi(A_U) + \Delta_\chi(A)]. \tag{9}$$

*Proof* We set  $B = U$  and  $\varphi = U\chi$  in Proposition 1. Then the lhs of inequality (4) becomes  $|q_{A,U}(U\chi, \chi)| = |\langle A_U \rangle_\chi - \langle A \rangle_\chi| \equiv |\delta\langle A \rangle|$ . We also have  $\langle U \rangle_\varphi = \langle U \rangle_\chi$ ,  $\Delta_\varphi(U) = \Delta_\chi(U)$  and  $\Delta_\varphi(A) = \Delta_\chi(A_U)$ . Using these values in the stationary point condition (6), we find that its solution is given by  $\lambda_1 = \Delta_\chi(A_U)/(\Delta_\chi(A_U) + \Delta_\chi(A))$ ,  $\lambda_2 = 1 - \lambda_1$ . With these values, the rhs of inequality (4) is  $\Delta_\chi(U) ([\Delta_\chi(A_U) + \Delta_\chi(A)]^2 + |\delta\langle A \rangle|^2)^{1/2}$ . Solving now the inequality for  $|\delta\langle A \rangle|$  one obtains (9).  $\square$



### 3.5 Time evolution

For a quantum system in Heisenberg picture, with the time evolution operator  $U(t) = \exp[-itH]$ , with  $H$  the energy operator, consider Heisenberg normal variable  $A_t$ . Then  $A_{t_2} = (A_{t_1})_{U(t_2-t_1)}$  in the notation of the last subsection. Relation (9), for  $\chi \in \mathcal{D}(A_{t_2}) \cap \mathcal{D}(A_{t_1})$ , takes now the form

$$|\langle A_{t_2} \rangle_\chi - \langle A_{t_1} \rangle_\chi| \leq \delta_\chi(U(t_2 - t_1))[\Delta_\chi(A_{t_2}) + \Delta_\chi(A_{t_1})]. \tag{10}$$

If  $\chi \in \bigcap_{\tau \in (t-\varepsilon, t+\varepsilon)} \mathcal{D}(A_\tau) \cap \mathcal{D}(H)$  and such that  $\|A_\tau \chi - A_t \chi\| \rightarrow 0$  for  $\tau \rightarrow t$ , one can recover the well-known relation

$$\frac{1}{2} \left| \frac{d\langle A_t \rangle_\chi}{dt} \right| \leq \Delta_\chi(H) \Delta_\chi(A_t). \tag{11}$$

It is well known that there is no self-adjoint time operator which would be translated by time evolution as  $U(t)^* T U(t) = T + t$  (see, e.g., [2]).<sup>2</sup> Formula (11) may be interpreted as a substitute for time–energy relation, in the following sense [10]. Let  $d\langle A_t \rangle_\chi/dt \approx \text{const.}$  in some interval of  $t$ , then  $A$  may be rescaled so that in fact  $d\langle A_t \rangle_\chi/dt \approx 1$ . Then  $\langle A_t \rangle_\chi$  describes correctly the flow of time in this interval. The product of uncertainties is then bounded by 1/2 from below.

The full formula (10) tells us more. Suppose  $\langle A_t \rangle_\chi$  is an increasing function of time, say, for simplicity, proportional to  $t$ . For this quantity to be a good measure of time we demand that the deviation  $\Delta_\chi(A_t)$  stays bounded by a constant. This is then possible only if  $|\langle U(t) \rangle_\chi| = |\langle \chi, \chi_t \rangle|$  decreases at least as  $1/t$  with time—the Schrödinger evolution of the state has to bring it sufficiently fast away from the initial state.<sup>3</sup>

### 3.6 Angular momentum

Let  $\mathcal{H}$  be a representation space of the usual angular momentum operators  $J_i$ , either bosonic:  $\exp(i2\pi J_3) = \mathbf{1}$ , or fermionic:  $\exp(i2\pi J_3) = -\mathbf{1}$ . For  $\varphi, \chi \in \mathcal{D}(\sqrt{J^2})$  the weak commutator of  $J_i$  and  $J_j$  is equal to the strong one, that is  $q_{J_1, J_2}(\varphi, \chi) = i(\varphi, J_3 \chi)$  (and permuted relations). This is easily seen by differentiating the identity

$$\left( e^{-i\alpha J_1} \varphi, e^{i\beta J_2} \chi \right) - \left( e^{-i\beta J_2} \varphi, e^{i\alpha J_1} \chi \right) = \left( e^{-i\alpha J_1} \varphi, \left[ e^{i\beta J_2} - e^{i\beta(\cos \alpha J_2 + \sin \alpha J_3)} \right] \chi \right)$$

with respect to  $\beta$ , setting  $\beta = 0$ , and then performing the same operation with respect to  $\alpha$ . Therefore, for such vectors and any real numbers  $j_1, j_2$  we obtain

$$|(\varphi, J_3 \chi)| \leq \|(J_1 - j_1)\varphi\| \|(J_2 - j_2)\chi\| + \|(J_2 - j_2)\varphi\| \|(J_1 - j_1)\chi\|. \tag{12}$$

<sup>2</sup> We insist on self-adjointness; non-self-adjoint ‘time operators’ considered in the literature (a recent example is [6]) do not have spectral decompositions, thus the probabilistic interpretation does not apply in standard form.

<sup>3</sup> On other state decay estimates—in context of non-self-adjoint time operators—see [1].

There are now a few possible choices of  $\varphi$ . The standard choice  $\varphi = \chi$  and minimization with respect to  $j_1, j_2$  gives the standard relation

$$\frac{1}{2} |\langle J_3 \rangle_\chi| \leq \Delta_\chi(J_1) \Delta_\chi(J_2), \quad \chi \in \mathcal{D}(\sqrt{J^2}),$$

on the maximal possible domain. A weak point of this relation is that  $J_3$  is not positive, so the mean on the lhs may take arbitrarily small value for certain states, including zero.

Our second choice is  $\varphi = J_3 \chi / \|J_3 \chi\|$  for  $\chi \in \mathcal{D}(J^2)$ . We denote by  $S$  the positive operator such that  $J^2 = S(S + 1)$ , and choose  $j_i = \langle J_i \rangle_\chi$ . Then the lhs of (12) is  $\|J_3 \chi\|$ , and on the rhs we have  $\|(J_i - j_i)\varphi\| \leq \|J_i \varphi\| + |\langle J_i \rangle_\chi| \leq \|S\varphi\| + \langle S \rangle_\chi$ . Using this in (12) we find

$$\|J_3 \chi\| \leq \left( \frac{\|S J_3 \chi\|}{\|J_3 \chi\|} + \langle S \rangle_\chi \right) (\Delta(J_1) + \Delta(J_2)), \quad \chi \in \mathcal{D}(J^2). \tag{13}$$

In particular, if the spectrum of  $J^2$  is bounded, say<sup>4</sup>  $\|S\| = j$ , then it follows that

$$\|J_3 \chi\| \leq 2j (\Delta_\chi(J_1) + \Delta_\chi(J_2)), \tag{14}$$

but note that this form may become much weaker for some states and large  $j$ .

We go to our third choice of  $\varphi$ , explained below. We denote by  $P_m$  the projection operator onto the eigensubspace  $\text{Ker}(J_3 - m\mathbf{1})$ . Let  $\mu$  be a spectral value of  $J_3$  such that  $\mu - 1 \leq 0 \leq \mu$  (thus in fermionic case  $\mu = 1/2$ , while in bosonic case  $\mu = 0$  or  $\mu = 1$ ). We introduce further self-adjoint projection operators and an involution:

$$P = P_\mu + P_{\mu-1}, \quad E_+ = \sum_{m \geq \mu} P_m, \quad E_- = \mathbf{1} - E_+, \quad E = E_+ - E_-,$$

so that  $J_3 = E|J_3|$ . We choose now  $\varphi = E\chi$  and  $j_i = \langle J_i \rangle_\chi$ .

**Proposition 5** *In standing notation, for  $\chi \in \mathcal{D}(\sqrt{J^2})$ , there is*

$$(\chi, |J_3| \chi) \leq 2\Delta_\chi(J_1) \Delta_\chi(J_2) + \left\| \left[ J^2 + \frac{1}{4} \delta \right]^{1/2} P \chi \right\| (\Delta_\chi(J_1) + \Delta_\chi(J_2)), \tag{15}$$

where  $\delta = 0$  ( $\delta = 1$ ) in bosonic (fermionic) case, respectively.

*Proof* Expressing operators  $J_i$  ( $i = 1, 2$ ) in terms of  $J_\pm$  one shows that  $E J_i E - J_i = -2P J_i P \equiv W_i$ , and furthermore,  $W_i^2 = (J^2 + \mu(1 - \mu))P = (J^2 + \frac{1}{4}\delta)P$ , with

<sup>4</sup> Note that we do not assume that  $\mathcal{H}$  is an eigenspace of  $J^2$ .

$\delta$  defined in the thesis. It follows that  $|W_i| = [J^2 + \frac{1}{4}\delta]^{1/2}P$ . Setting now  $\varphi = E\chi$ ,  $j_i = \langle J_i \rangle_\chi$  we note that

$$\|(J_i - j_i)E\chi\| = \|(EJ_iE - j_i)\chi\| \leq \|(J_i - j_i)\chi\| + \|W_i\chi\|,$$

which gives the thesis when used in (12). □

In particular, if  $P\chi = 0$  (that is,  $P_0\chi = P_{+1}\chi = 0$  or  $P_0\chi = P_{-1}\chi = 0$  in bosonic case, or  $P_{-1/2}\chi = P_{+1/2}\chi = 0$  in fermionic case), then

$$\frac{1}{2}\langle |J_3| \rangle_\chi \leq \Delta_\chi(J_1)\Delta_\chi(J_2).$$

This fact follows also more directly from Proposition 2: set  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ ,  $\mathcal{H}_\pm = E_\pm\mathcal{H}$ ,  $\chi = \chi_+ + \chi_-$ ,  $\chi_\pm \in \mathcal{H}_\pm$ . Then  $\varphi = \chi_+ - \chi_-$  and the relation immediately follows.

*Discussion* In the present subsection, instead of minimizing the rhs of the inequality (12) with respect to  $j_1$  and  $j_2$  (in the spirit of Proposition 1), we have chosen specific values  $j_i = \langle J_i \rangle_\chi$ : minimization does not seem to be algebraically solvable, in general. In addition, some norm inequalities employed in the derivation could possibly be replaced by other arguments leading to more stringent conditions. Therefore, the inequalities (13) and (15) need not be optimal and may be improved in special cases.

We illustrate this in the special case of spin 1/2 operators. In this case  $j = 1/2$  and  $J_i^2 = (1/4)\mathbf{1}$  and one easily finds that inequalities (14) and (15) take in this case the form

$$\frac{1}{2} \leq \Delta_\chi(J_1) + \Delta_\chi(J_2) \quad \text{and} \quad \frac{1}{2} \leq 2\Delta_\chi(J_1)\Delta_\chi(J_2) + \Delta_\chi(J_1) + \Delta_\chi(J_2),$$

respectively. Both of these restrictions are weaker than the inequality

$$\frac{1}{4} \leq \Delta_\chi^2(J_1) + \Delta_\chi^2(J_2), \tag{16}$$

which is a consequence of the standard uncertainty relation for  $J_1$  and  $J_3$ , due to the sequence<sup>5</sup>  $\Delta_\chi(J_1)^2 \geq 4\Delta_\chi(J_3)^2\Delta_\chi(J_1)^2 \geq \langle J_2 \rangle_\chi^2 = (1/4) - \Delta_\chi(J_2)^2$ , or by the use of the explicit Pauli matrices representation.

However, in spin 1/2 case the minimization given by Proposition 1 may be effectively carried out. In this case the choice of vector  $\varphi$  coincides in the two cases leading to (14) and (15):  $\varphi = J_3\chi/\|J_3\chi\| = E\chi$ . Moreover, here  $EJ_iE = -J_i$ , so  $\langle J_i \rangle_\varphi = -\langle J_i \rangle_\chi$  and  $\Delta_\varphi(J_i) = \Delta_\chi(J_i)$ , and the relation (4) takes the form

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<sup>5</sup> We are grateful to the Referee for bringing this, and similar sequences for higher  $j$ , to our attention, which led us to the addition of the present discussion.

$$\frac{1}{2} \leq \inf_{\substack{\lambda_1, \lambda_2 \in (0, 1) \\ \lambda_1 + \lambda_2 = 1}} \left\{ \sqrt{\Delta_\chi^2(J_1) + 4\langle J_1 \rangle_\chi^2 \lambda_1^2} \sqrt{\Delta_\chi^2(J_2) + 4\langle J_2 \rangle_\chi^2 \lambda_2^2} \right. \\ \left. + \sqrt{\Delta_\chi^2(J_2) + 4\langle J_2 \rangle_\chi^2 \lambda_1^2} \sqrt{\Delta_\chi^2(J_1) + 4\langle J_1 \rangle_\chi^2 \lambda_2^2} \right\}.$$

Now take into account that here  $\Delta_\chi^2(J_i) = 1/4 - \langle J_i \rangle_\chi^2$ , set  $\lambda_1 = (1 + s)/2$ ,  $\lambda_2 = (1 - s)/2$ ,  $s \in \langle -1, 1 \rangle$ , and expand the rhs in  $s$ . The straightforward calculation gives

$$\frac{1}{2} \leq \inf_{s \in \langle -1, 1 \rangle} \left\{ \frac{1}{2} + (\langle J_1 \rangle_\chi^2 + \langle J_2 \rangle_\chi^2) \left[ 1 - 4(\langle J_1 \rangle_\chi^2 + \langle J_2 \rangle_\chi^2) \right] s^2 + O(s^4) \right\}.$$

Therefore, we find that  $1/4 \geq \langle J_1 \rangle_\chi^2 + \langle J_2 \rangle_\chi^2$ , which is equivalent to (16). Of course, this is not the simplest way to arrive at this result in this special case. However, in general the standard uncertainty relation alone does not seem to lead to analogous improvements of relations (13) or (15).

Finally, to illustrate further applications of our method we obtain another relation, which results from the combination of the standard one with the relation (13). Namely, for  $\chi \in \mathcal{D}(\sqrt{J^2})$  we have a sequence

$$\|J_2\chi\|^2 - \Delta_\chi^2(J_2) = \langle J_2 \rangle_\chi^2 \leq 4\Delta_\chi^2(J_3)\Delta_\chi^2(J_1) \leq 4\|J_3\chi\|^2\Delta_\chi^2(J_1)$$

and another one with indices 1 and 2 interchanged. Adding these inequalities side by side and using  $\|\sqrt{J^2}\chi\|^2 = \sum_k \|J_k\chi\|^2$  one obtains

$$\|\sqrt{J^2}\chi\|^2 \leq \|J_3\chi\|^2 + (4\|J_3\chi\|^2 + 1)(\Delta_\chi^2(J_1) + \Delta_\chi^2(J_2)), \quad \chi \in \mathcal{D}(\sqrt{J^2}). \tag{17}$$

We can now further estimate the rhs with the use of inequality (13), or (14) in the bounded spectrum case. We assume for simplicity that  $\|J^2\| = j(j + 1)$  (again, we do not need to assume that  $\mathcal{H}$  is an eigenspace of  $J^2$ ) and then we obtain

$$\langle J^2 \rangle_\chi \leq \Delta_\chi^2(J_1) + \Delta_\chi^2(J_2) + 4j^2(\Delta_\chi(J_1) + \Delta_\chi(J_2))^2(1 + 4\Delta_\chi^2(J_1) + 4\Delta_\chi^2(J_2)).$$

Estimating  $(\Delta_\chi(J_1) + \Delta_\chi(J_2))^2 \leq 2(\Delta_\chi^2(J_1) + \Delta_\chi^2(J_2))$  and solving the above inequality for  $\Delta_\chi^2(J_1) + \Delta_\chi^2(J_2)$  we find

$$\Delta_\chi^2(J_1) + \Delta_\chi^2(J_2) \geq \frac{2\langle J^2 \rangle_\chi}{\sqrt{(8j^2 + 1)^2 + 128j^2\langle J^2 \rangle_\chi} + 8j^2 + 1}.$$

The rhs is further bounded by  $2\langle J^2 \rangle_\chi/c(j)$ , where  $c(j)$  results from the replacement of  $\langle J^2 \rangle_\chi$  by  $j(j + 1)$  in the denominator. One shows that  $j(j + 1)/c(j)$  is a decreasing

function of  $j$  for  $j \geq 1/2$ , thus it is bounded from below by its limit value. In this way, we find

$$\Delta_{\chi}^2(J_1) + \Delta_{\chi}^2(J_2) \geq \frac{1}{4(\sqrt{3} + 1)} \frac{\langle J^2 \rangle_{\chi}}{j(j+1)}.$$

These relations may be compared with a three-observable relation<sup>6</sup>

$$\Delta_{\chi}^2(J_1) + \Delta_{\chi}^2(J_2) + \Delta_{\chi}^2(J_3) \geq \frac{\|\sqrt{J^2}\chi\|^2}{1 + 2\|\sqrt{J^2}\chi\|^2}, \quad \chi \in \mathcal{D}(\sqrt{J^2}),$$

obtained by summing relation (17) with its two cyclically permuted versions.

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<sup>6</sup> This is an improved/generalized version of a relation pointed out to us by Referee.