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A construction of left equalizer simple medial semigroups

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Abstract

A semigroup S is said to be left equalizer simple if, for all elements a, b, x, y of S, $xa = xb$ implies $ya = yb$. It is known that left equalizer simplicity is a necessary condition for a semigroup to be embedded in a left simple semigroup. A semigroup satisfying the identity $axyb = ayxb$ is called a medial semigroup. In this paper we show how to construct left equalizer simple medial semigroups, especially, medial semigroups which can be embedded in idempotent-free left simple semigroups.

Keywords Semigroup · Medial semigroup · Left simple semigroup · Left equalizer simple semigroup

Mathematics Subject Classification 20M10 · 20M30

1 Introduction and motivation

In [5, Theorem 1], P.M. Cohn gave necessary and sufficient conditions for a semigroup to be embedded in a left simple semigroup. Conditions differ essentially according to whether or not the semigroup contains an idempotent element (i.e. an element e satisfying $e^2 = e$). Cohn proved (see also the dual of [4, Theorem 8.19]) that a semigroup S is embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and satisfies the condition: for all $a, b, x, y \in S$, $xa = xb$ implies $ya = yb$. Using the terminology of [12], a semigroup S satisfying this condition is called a left equalizer simple semigroup.

A semigroup is called a medial semigroup ([2]) if it satisfies the identity $axyb = ayxb$. Medial semigroups are well studied by many authors, see, for example, papers [1], [2], [6], [10], [17], and books [9], [14]. In the present paper, medial semigroups are also the subject of investigations. Using results of [12], we show how to construct left equalizer simple medial semigroups, especially medial semigroups which can be embedded in idempotent-free left simple semigroups.

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2 Preliminaries

For the notions and notations not defined but used in this paper, we refer the reader to books [3], [4] and [9].

By [12, Definition 2.1], a semigroup S is said to be left equalizer simple if, for every elements $a, b, x, y \in S$, $xa = xb$ implies $ya = yb$. By [5, Theorem 1], if a semigroup S is embedded in a left simple semigroup, then S is left equalizer simple.

In [12], we defined a construction, and showed how to get left equalizer simple semigroups by applying that construction. This construction plays an important role in our present investigation. Thus we cite it here.

Construction 2.1 [[12, Construction 1]] Let T be a left cancellative semigroup. For each $t \in T$, associate a nonempty set S_t to t such that $S_t \cap S_r = \emptyset$ for every $t, r \in T$ with $t \neq r$. For an arbitrary couple $(t, r) \in T \times T$ with $r \in tT$, let $\varphi_{t,r}$ be a mapping of S_t into S_r acting on the right. For all $t \in T$, $r \in tT$, $q \in rT \subseteq tT$, assume

$$\varphi_{t,r}\varphi_{r,q} = \varphi_{t,q}.$$

On the set $S = \bigcup_{t \in T} S_t$ define an operation $*$ as follows: for arbitrary $a \in S_t$ and $b \in S_x$, let

$$a * b = a\varphi_{t,tx}.$$

If $a \in S_t$, $b \in S_x$, $c \in S_y$ are arbitrary elements then

$$\begin{aligned} a * (b * c) &= a * b\varphi_{x,xy} = a\varphi_{t,t(xy)} \\ &= a\varphi_{t,tx}\varphi_{tx,t(xy)} = a\varphi_{t,tx} * c = (a * b) * c. \end{aligned}$$

Thus $(S; *)$ is a semigroup.

The semigroup $(S; *)$ defined in Construction 2.1 is said to be a right regular extension of the left cancellative semigroup T. This extension is said to be injective if the mappings $\varphi_{t,r}$ are injective for every $t \in T$ and $r \in tT$.

Using the terminology of a right regular extension, [12, Theorem 2.2] has the following form.

Proposition 2.2 [[12, Theorem 2.2]] A semigroup is left equalizer simple if and only if it is a right regular extension of a left cancellative semigroup. □

For a semigroup S, let θ_S^{right} denote the kernel of the right regular representation of S, that is

$$\theta_S^{\text{right}} = \{(a, b) \in S \times S : (\forall x \in S)\ xa = xb\}.$$

By [11, Theorem 2], the θ_S^{right}-classes of the semigroup $S = (S; *)$ defined in Construction 2.1 are the sets S_t $(t \in T)$, and $S/\theta_S^{\text{right}} \cong T$. Thus an element $e \in T$ is an idempotent element if and only if S_e is a subsemigroup of $(S; *)$.

Proposition 2.3 *Let $(S; *)$ be a right regular extension of a left cancellative semigroup T, and let e be an idempotent element of T. Then the subsemigroup S_e of $(S; *)$ is a retract ideal extension of a left zero semigroup by a zero semigroup.*

Proof Since S_e is a θ_S^{right}-class of $S = (S; *)$, the equality

$$x * a = x * b$$

holds for every $x, a, b \in S_e$. Let $a \in S_e$ be an arbitrary element. Then

$$a^2 = a * a = a * a^2 = a^3,$$

and so

$$(a^2)^2 = a * a^3 = a * a^2 = a^3 = a^2,$$

that is, a^2 is an idempotent element. Let $E(S_e)$ denote the set of all idempotent elements of S_e. For arbitrary $a, b \in S_e$, we have

$$a * b = a^2 \in E(S_e).$$

Thus $E(S_e)$ is an ideal of S_e, and the Rees factor semigroup $S_e/E(S_e)$ is a zero semigroup. For arbitrary $f_1, f_2 \in E(S_e)$,

$$f_1 * f_2 = f_1 * f_1 = f_1.$$

Hence $E(S_e)$ is a left zero semigroup. For arbitrary $a, x \in S_e$, we have

$$a\varphi_{e,e} = a * x \in E(S_e).$$

Thus $\varphi_{e,e}$ is a mapping of S_e into $E(S_e)$. Let $a, b, x \in S_e$ be arbitrary elements. Then

$$(a * b)\varphi_{e,e} = (a * b) * x = a * (b * x) = a * (x * b * x) = (a * x) * (b * x) = (a\varphi_{e,e}) * (b\varphi_{e,e}).$$

Hence $\varphi_{e,e}$ is a homomorphism of S_e into $E(S_e)$. For every $f \in E(S_e)$, we have

$$f = f * f = f\varphi_{e,e},$$

that is, $\varphi_{e,e}$ leaves the elements of $E(S_e)$ fixed. Thus $\varphi_{e,e}$ is a retract homomorphism of S_e onto the ideal $E(S_e)$. Hence S_e is a retract ideal extension of the left zero semigroup $E(S_e)$ by the zero semigroup $S_e/E(S_e)$. □

Applying also Proposition 2.3, we get the following corollary which will be used in the proof of Corollary 3.6.

Corollary 2.4 *A right regular extension* $(S; *)$ *of a left cancellative semigroup* T *is idempotent-free if and only if* T *is idempotent-free.*

3 Left equalizer simple medial semigroups

In our investigation, two subclasses of semigroups play an important role. These are the class of left commutative semigroups and the class of right commutative semigroups. Using the terminology of [14], a semigroup is said to be left commutative if it satisfies the identity $xyb = yxb$. The notion of a right commutative semigroup is defined analogously. We note that left (resp., right) commutative semigroups are also called left (resp., right) pseudo commutative semigroups (see, for example, [15], [18]). It is clear that every left commutative semigroup and every right commutative semigroup is medial.

Left commutative and right commutative semigroups appear in several papers, see, for example, [7], [8], [13], [15], [16], [18] and books [9], [14].

The next two lemmas are obvious consequences of the definitions.

Lemma 3.1 *On an arbitrary semigroup S, the following conditions are equivalent.*

(1) S is medial.
(2) The factor semigroup $S/\theta_S^{\text{right}}$ is left commutative.

Lemma 3.2 *On an arbitrary semigroup S, the following conditions are equivalent.*

(1) S is left commutative.
(2) The factor semigroup S/θ_S^{left} is commutative.

In the next proposition, we show that the right cancellativity is a sufficient condition for a right commutative semigroup to be left equalizer simple. This fact and its dual will be used in the proof of Theorem 3.5.

Proposition 3.3 *Every right cancellative right commutative semigroup is left equalizer simple.*

Proof Let a and b be arbitrary elements of a right cancellative right commutative semigroup S. Assume $xa = xb$ for some $x \in S$. Then, for an arbitrary $y \in S$,

$$yax = yxa = yxb = ybx.$$

Since S is right cancellative, we obtain that

$$ya = yb.$$

Thus S is left equalizer simple. □

In the next proposition, we give a necessary and sufficient condition for a left equalizer simple semigroup to be right cancellative. The dual of this proposition will be used in the proof of Theorem 3.5.

Proposition 3.4 *Let $(S; *)$ be a right regular extension of a left cancellative semigroup T. Then $(S; *)$ is right cancellative if and only if T is also right cancellative and the extension is injective.*

Proof Assume that the left cancellative semigroup T is also right cancellative and the mappings $\varphi_{t,r}$ $(t \in T, r \in tT)$ are injective. To show that $(S; *)$ is right cancellative, assume

$$a * s = b * s$$

for elements $a \in S_x$, $b \in S_y$ and $s \in S_t$. Then

$$a\varphi_{x,xt} = b\varphi_{y,yt}$$

from which it follows that

$$xt = yt.$$

As T is right cancellative, we get

$$x = y.$$

Thus

$$a\varphi_{x,xt} = b\varphi_{x,xt},$$

from which we get $a = b$ by the injectivity of $\varphi_{x,xt}$. Thus $(S; *)$ is right cancellative.

Conversely, assume that $(S; *)$ is right cancellative. First we show that the mappings $\varphi_{t,r}$ $(t \in T, r \in tT)$ are injective. Let $t \in T$, $r \in tT$ be arbitrary elements. Assume

$$a\varphi_{t,r} = b\varphi_{t,r}$$

for $a, b \in S_t$. Since T is left cancellative, then there is exactly one $y \in T$ such that $r = ty$. Then, for an arbitrary $s \in S_y$, we have

$$a * s = a\varphi_{t,r} = b\varphi_{t,r} = b * s.$$

From the right cancellativity of S it follows that $a = b$. Therefore, $\varphi_{t,r}$ is injective.

To show that T is right cancellative, assume

$$xt = yt$$

for $x, y, t \in T$. Then, for arbitrary $a \in S_x$, $b \in S_y$, $s \in S_t$, we have

$$a * s = a\varphi_{x,xt} \quad \text{and} \quad b * s = b\varphi_{y,yt}.$$

As $xt = yt$, we get

$$a * s,\ b * s \in S_{xt}.$$

As S_{xt} is a θ_S^{right}-class, we have

$$s * a * s = s * b * s.$$

By the assumption that S is right cancellative, we get

$$s * a = s * b,$$

and so

$$s\varphi_{t,tx} = s\varphi_{t,ty}.$$

From this it follows that $tx = ty$. As T is left cancellative, we get $x = y$. Hence T is right cancellative. □

The next theorem is the main result of the paper.

Theorem 3.5 *A semigroup is left equalizer simple and medial if and only if it is a right regular extension of a semigroup which is an injective left regular extension of a commutative cancellative semigroup.*

Proof Let S be a left equalizer simple medial semigroup. By [12, Theorem 2.2] and Lemma 3.1, S is a right regular extension of the left cancellative, left commutative semigroup $T = S/\theta_S^{\text{right}}$. By the dual of Proposition 3.3, T is right equalizer simple. By Lemma 3.2 and the dual of Proposition 2.2, T is a left regular extension of the commutative, cancellative semigroup $U \cong T/\theta_T^{\text{left}}$. Since T is left cancellative, then the dual of Proposition 3.4 implies that this left regular extension is injective. Hence S is a right regular extension of the semigroup T which is an injective left regular extension of the commutative cancellative semigroup U.

Assume that a semigroup S is a right regular extension of a semigroup T which is an injective left regular extension of a commutative cancellative semigroup U. First of all, we note that this assumption is correct, because the semigroup T is left cancellative by the dual of Proposition 3.4. Thus the semigroup S is left equalizer simple by Proposition 2.2. Since U is a commutative semigroup and $U \cong T/\theta_T^{\text{left}}$, then the semigroup T is left commutative by Lemma 3.2. As $T \cong S/\theta_S^{\text{right}}$, we get that the semigroup S is medial by Lemma 3.1. □

Corollary 3.6 *A semigroup is a medial semigroup which is embedded in an idempotent-free left simple semigroup if and only if it is a right regular extension of a semigroup which is an injective left regular extension of an idempotent-free commutative cancellative semigroup.*

Proof It is obvious by [5, Theorem 1], Theorem 3.5, Corollary 2.4 and its dual. □

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