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Ergodic theorem in variable Lebesgue spaces

Przemysław Górka[1]

Abstract We prove an Ergodic Theorem in variable exponent Lebesgue spaces, whenever the exponent is invariant under the transformation. Moreover, a counterexample is provided which shows that the norm convergence fails to hold for an arbitrary exponent.

Keywords Lebesgue spaces with variable exponents · Ergodic theorem · Probability measure

AMS Subject Classification 28D05 · 54C35

1 Introduction

Variable exponent Lebesgue and Sobolev spaces are natural extensions of classical constant exponent L^p-spaces. Such kind of theory finds many applications for example in nonlinear elastic mechanics [23], electrorheological fluids [20] or image restoration [18]. During the last decade Lebesgue and Sobolev spaces with variable exponents have been intensively studied; see for instance the surveys [5,21]. In particular, the Sobolev inequalities have been shown for variable exponent spaces on Euclidean spaces (see [4,7] and [15]) and on Riemannian manifolds (see [9] and [11]). Recently, the theory of variable exponent spaces has been extended to metric measure spaces, see e.g. [8,10,16,17,19]. Moreover, the theory of Lebesgue spaces with variable exponent on probability spaces exists as well, see e.g. [1]

In this article we investigate Birkhoff's Ergodic Theorem in the context of variable Lebesgue spaces. Let us mention that Ergodic theorems in spaces other than Lebesgue spaces have been studied in the past (see e.g. [2,13,14,22]).

We organize this paper as follows. In the next section we review some definitions and present the theory of variable exponent spaces. In the third section we present and prove the main result.

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2 Variable exponent Lebesgue spaces

In this section we recall some basic facts and notation about variable exponent Lebesgue spaces. Most of the properties of these spaces can be found in the book of Cruz-Uribe and Fiorenza [3] and in the book of Diening et al. [6].

Let (Ω, μ) be a σ-finite, complete measure space. By a variable exponent we shall mean a bounded measurable function $p : \Omega \to [1, \infty)$. We put

$$p^+ = \text{ess} \sup_{x \in \Omega} p(x), \quad p^- = \text{ess} \inf_{x \in \Omega} p(x).$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of those μ-measurable functions $f : \Omega \to \mathbb{R}$ for which semimodular

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} \, d\mu(x)$$

is finite. This is a Banach space with respect to the following Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\},$$

where $f \in L^{p(\cdot)}(\Omega)$. Variable Lebesgue space is a special case of the Musielak–Orlicz spaces. When the variable exponent p is constant, then $L^{p(\cdot)}(\Omega)$ is an ordinary Lebesgue space. It is needed very often to pass between norm and semimodular. An important property of the variable Lebesgue spaces is the so-called ball property: $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$ if and only if $\rho_{p(\cdot)}(f) \leq 1$. Moreover, the following inequality

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(f) + 1$$

holds (see e.g. [6]). Let us remark that, if $p^+ < \infty$, then convergence in norm is equivalent to convergence in semimodular.

3 Main result

In this note we would like to present the following observation.

Theorem 3.1 *Let (Ω, μ) be a probability space and $T : \Omega \to \Omega$ a measure preserving transformation. Moreover, let p be T-invariant variable exponent, such that $p^+ < \infty$.*

(i) If $f \in L^{p(\cdot)}(\Omega)$, then the limit

$$f_{av}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x))$$

exists for almost each point $x \in \Omega$ and $f_{av} \in L^{p(\cdot)}(\Omega)$.

(ii) For every $f \in L^{p(\cdot)}(\Omega)$, we have

$$\lim_{n \to \infty} \left\| f_{av} - \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j \right\|_{L^{p(\cdot)}(\Omega)} = 0, \tag{3.1}$$

$$f_{av}(x) = f_{av}(T(x)), \tag{3.2}$$

$$\int_\Omega f_{av} d\mu = \int_\Omega f \, d\mu. \tag{3.3}$$

Proof Let us start with the proof of (*i*). Since $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$, the existence of the limit $f_{av}(x)$ for almost every point of Ω follows from the standard Birkhoff's Theorem. Thus, by the Fatou Lemma we obtain

$$\int_\Omega |f_{av}(x)|^{p(x)} d\mu \le \int_\Omega \lim_{n\to\infty} \left(\frac{1}{n} \sum_{j=0}^{n-1} |f(T^j(x))| \right)^{p(x)} d\mu$$

$$\le \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{n} \sum_{j=0}^{n-1} |f(T^j(x))| \right)^{p(x)} d\mu$$

$$\le \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_\Omega |f(T^j(x))|^{p(x)} d\mu,$$

where in the last step we applied convexity and Jensen inequality. Now, since T is a measure preserving map and p is $T-$invariant we have

$$\int_\Omega |f(T(x))|^{p(x)} d\mu = \int_\Omega |f(T(x))|^{p(T(x))} d\mu = \int_\Omega |f(x)|^{p(x)} d\mu.$$

Thus, we get

$$\int_\Omega |f_{av}(x)|^{p(x)} d\mu \le \int_\Omega |f(x)|^{p(x)} d\mu. \tag{3.4}$$

Hence, $f_{av} \in L^{p(\cdot)}(\Omega)$.

Now, we turn our attention into (*ii*). Let us mention that (3.2) and (3.3) follows from Ergodic Theorem in classical Lebesgue spaces. In order to prove (3.1) we assume that $f \in L^\infty(\Omega)$. Thus,

$$\lim_{n\to\infty} \left| f_{av}(x) - \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) \right|^{p(x)} = 0, \quad \text{a.e.}$$

$$\|f_{av}\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}.$$

Subsequently, we get

$$\left| f_{av}(x) - \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) \right|^{p(x)} \le \left| \|f\|_{L^\infty(\Omega)} + \frac{1}{n} \sum_{j=0}^{n-1} \|f(T^j)\|_{L^\infty(\Omega)} \right|^{p(x)}$$

$$\le \left(2\|f\|_{L^\infty(\Omega)} + 1 \right)^{p^+}.$$

Hence, by dominated Lebesgue Theorem we have (3.1), provided $f \in L^\infty(\Omega)$. Now, let us take any $f \in L^{p(\cdot)}(\Omega)$ and $\epsilon > 0$, then there exists $g \in L^\infty$ such that

$$\rho_{p(\cdot)}(f - g) \le \epsilon.$$

By the previous step, there exists n_0, such that the following inequality

$$\rho_{p(\cdot)} \left(g_{av} - \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \right) \le \epsilon$$

holds for $n \geq n_0$. Let $q \geq 1$, then by convexity of the function $y \mapsto y^q$, we have for any nonnegative a, b, c the inequality $(a + b + c)^q \leq 3^{q-1}(a^q + b^q + c^q)$. Hence, we get

$$\rho_{p(\cdot)}\left(f_{\mathrm{av}} - \frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j\right) = \int_\Omega \left| f_{\mathrm{av}}(x) - \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x)) \right|^{p(x)} d\mu(x)$$

$$\leq 3^{p^+ - 1}\left(\rho_{p(\cdot)}((f-g)_{\mathrm{av}}) + \rho_{p(\cdot)}\left(g_{\mathrm{av}} - \frac{1}{n}\sum_{j=0}^{n-1} g \circ T^j\right)\right.$$

$$\left. + \rho_{p(\cdot)}\left(\frac{1}{n}\sum_{j=0}^{n-1}(f-g)\circ T^j\right)\right).$$

Thus, using (3.4) and convexity of $\rho_{p(\cdot)}$ we have

$$\rho_{p(\cdot)}\left(f_{\mathrm{av}} - \frac{1}{n}\sum_{j=0}^{n-1} f \circ T^j\right) \leq 3^{p^+}\epsilon.$$

This finishes the proof of the theorem. □

Finally, let us give the following example showing that the norm convergence fails to hold for an exponent $p(x)$ which is not T invariant. In fact, we shall give an example of $f \in L^{p(\cdot)}$ such that $f_{\mathrm{av}} \notin L^{p(\cdot)}$.

Example 3.1 Let (Ω, μ) be a probability space defined as follows: $\Omega = \mathbb{Z} \times \{0\} \cup \{0\} \times \mathbb{Z} \setminus (0,0)$, $\mu((m,0)) = \mu((0,m)) = \frac{1}{4}\frac{1}{2^{|m|}}$. Next, we define a measure preserving map $T : \Omega \to \Omega$ by formulas $T(m,0) = (0,m)$, $T(0,k) = (-k,0)$. Moreover, we put variable exponent $p(-m,0) = p(0,m) = p(0,-m) = 1$, $p(m,0) = 2$. Now, if we take f defined in the following manner $f(m,0) = 0$, $f(0, \pm m) = f(-m, 0) = (\sqrt{2})^m$, then one can see that $f \in L^{p(\cdot)}(\Omega)$ but $f_{\mathrm{av}} \notin L^{p(\cdot)}(\Omega)$.

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