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Packing rectangles into a large square

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Abstract If $c > 5$ and if x is sufficiently large, then any collection of rectangles of sides of length not greater than 1 with total area smaller than $x^2 - cx^{5/6}$ can be packed into a square of side length x.

Keywords Packing · Rectangle · Square

Mathematics Subject Classification 52C15

1 Introduction

First publications related to packing of rectangles or squares appeared over fifty years ago. In 1957 Kosiński [6] proved, among others, that any sequence of rectangles of total area V and with sides of length not greater than D can be packed into a rectangle of side lengths $3D$ and $(V + D^2)/D$. This result was improved in [4,7,8]. Other problems related to this subject were outlined in the sixties of the last century by L. Moser [9]. He asked, for example, "Can every set of rectangles of total area 1 and maximal side 1 be accommodated in a square of area 2?" (the answer is positive [5]) or "What is the smallest number A such that any set of squares of total area 1 can be packed into some rectangle of area A?" (some bounds are given in [3,10,11]). The question of packing of equal squares into a square as small as possible was posed in [2].

Let I_x be a square of side length x. We say that a collection $R_1, R_2, \ldots$ of rectangles can be *packed* into I_x, if it is possible to apply translations and rotations to the sets R_i so that the resulting translated and rotated rectangles are contained in I_x and have mutually disjoint interiors. Denote by $s(x)$ the greatest number such that any collection of rectangles of sides of length not greater than 1 with total area smaller than $s(x)$ can be packed into I_x.

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Groemer [4] proved that $s(x) \geq (x-1)^2$ provided $x \geq 3$. By Remark 3 of [5] we know that $s(x) \geq x^2 - 2x + 2$ for $x \geq 2$. The aim of this note is to show that $s(x) \geq x^2 - O(x^{5/6})$. It is an open question whether the exponent 5/6 may be lessened in the above-presented estimation.

If all rectangles are unit squares, then $s_{unit}(x) \geq x^2 - O\left(x^{(3+\sqrt{2})/7} \log x\right)$ (see [1]). Also in this case we do not know whether the exponent $(3+\sqrt{2})/7$ may be lessened. On the other hand, by [12] we know that $s_{unit}(x)$ is smaller than $x^2 - 10^{-100}\sqrt{x|x - \lfloor x + 1/2 \rfloor|}$ provided $x(x - \lfloor x \rfloor) > 1/6$.

2 Preliminaries

Let $\mathcal{R}$ be a finite collection of rectangles $R_1, R_2, \ldots, R_z$ of sides of length not greater than 1. Denote by w_i the width and by h_i the height of R_i. Furthermore, assume that $w_i \leq h_i$ for any $i = 1, \ldots, z$ and that $h_1 \geq h_2 \geq \cdots \geq h_z$.

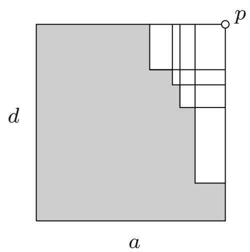
Let S be a rectangle of width $a \geq 1$ and height $d \geq 1$. Denote by p a vertex of S. Moreover, let $S_1, \ldots, S_r$ be a collection of rectangles S_i of width $v_i < a$ and height smaller than d such that $p \in S_i$ and $S_i \subset S$ for $i = 1, \ldots, r$. Then $S \setminus \bigcup_{i=1}^{r} S_i$ is called a $\sigma - polygon$ of *base* a, *top* $a - \max(v_1, \ldots, v_r)$ and *height* d (see Fig. 1). The rectangle S is also called a σ-polygon.

Lemma 2.1 *Let* $a \geq 1$, $d \geq 1$ *and assume that the total area of rectangles in* $\mathcal{R}$ *is not smaller than* $(a+1)(d+1)$. *There exist integers* $j_1 < \cdots < j_k$ *such that the following conditions are fulfilled:*

- $a \leq b_i < a + 1$, *where* $b_i = w_{j_{i-1}} + \cdots + w_{j_i - 1}$ *for* $i = 1, \ldots, k$ $(j_0 = 1)$;
- $d \leq h_1 + h_{j_1} + \cdots + h_{j_{k-1}} < d + 1$;
- *the rectangles* $R_1, \ldots, R_{j_k - 1}$ *can be packed into the union* $\bigcup_{i=1}^{k} L_i$ *of rectangles (with mutually disjoint interiors)* L_i *of sides of length* b_i *and* $h_{j_{i-1}}$;
- *the area of the uncovered part of* $\bigcup_{i=1}^{k} L_i$ *is smaller than* $a + 1$.

Proof Denote by j_1 the smallest integer such that $w_1 + w_2 + \cdots + w_{j_1 - 1} \geq a$. Moreover, denote by j_2 the smallest integer satisfying $w_{j_1} + \cdots + w_{j_2 - 1} \geq a$ and so on. Let k be the smallest integer such that $h_1 + h_{j_1} + \cdots + h_{j_{k-1}} \geq d$. Clearly, the rectangles $R_1, \ldots, R_{j_k - 1}$ can be packed into the union $\bigcup_{i=1}^{k} L_i$ of rectangles (with mutually disjoint interiors) L_i of sides of length b_i and $h_{j_{i-1}}$ (see Fig. 2). The area of the uncovered part in each L_i does not exceed $b_i(h_{j_{i-1}} - h_{j_i - 1})$. Consequently, the area of the uncovered part of $\bigcup_{i=1}^{k} L_i$ (the waste in this packing) does not exceed

Fig. 1 σ-polygon



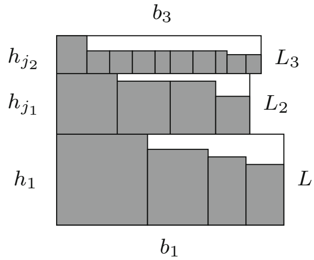


Fig. 2 $L_1, L_2, \ldots$

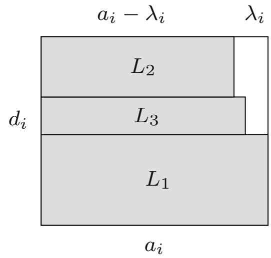


Fig. 3 A_i

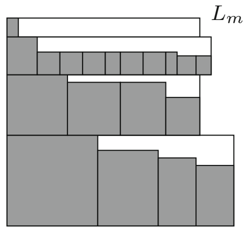


Fig. 4 L_m

$$
\begin{aligned}
\omega_0 &= (h_1 - h_{j_1-1})b_1 + (h_{j_1} - h_{j_2-1})b_2 + \cdots + (h_{j_{k-1}} - h_{j_k-1})b_k \\
&\le (h_1 - h_{j_1} + h_{j_1} - h_{j_2} + \cdots + h_{j_{k-1}} - h_{j_k}) \cdot \max(b_1, \ldots, b_k) \\
&= (h_1 - h_{j_k}) \cdot \max(b_1, \ldots, b_k) \\
&< \max(b_1, \ldots, b_k) \\
&< a + 1.
\end{aligned}
$$

□

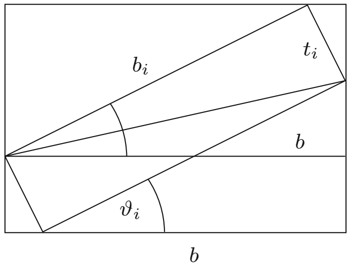
By the proof of Lemma 2.1 we deduce the following two results (see Figs. 2, 3, 4).

Lemma 2.2 *Let* $a \ge 1$, $d \ge 1$ *and let* n *be a positive integer. Assume that the total area of rectangles in* $\mathcal{R}$ *is not smaller than* $n(a+1)(d+1)$. *There is an integer* k *and there are* n *mutually disjoint* σ*-polygons* A_i *(for* $i = 1, \ldots, n$*) of base* a_i, *top* $a_i - \lambda_i$ *and height* d_i, *where* $a \le a_i < a + 1$, $d \le d_i < d + 1$ *and* $\sum_{i=1}^{n} \lambda_i < 1$ *such that the following conditions are fulfilled:*

- *the rectangles* $R_1, \ldots, R_k$ *can be packed into* $\bigcup_{i=1}^{n} A_i$;
- *the area of the uncovered part of* $\bigcup_{i=1}^{n} A_i$ *is smaller than* $a + 1$.

Lemma 2.3 *Let* $a \ge 1$. *There is an integer* m *and there are* m *rectangles* L_i *(with mutually disjoint interiors) of height not greater than* 1 *and width* b_i, *where* $a \le b_i < a + 1$, *such that:*

Fig. 5 ϑ_i



– *the rectangles from $\mathcal{R}$ can be packed into $\bigcup_{i=1}^m L_i$;*
– *the area of the uncovered part of $\bigcup_{i=1}^m L_i$ is smaller than $2a + 1$.*

In the following lemma we will describe how to efficiently pack rectangles L_i.

Lemma 2.4 *Let B be a σ-polygon of base b, top $b - \lambda$ and height h, where $h \geq b \geq 27$ and $0 \leq \lambda < 1$. Furthermore, let L_i (for $i = 1, \ldots, m$) be a rectangle of width b_i and height t_i, where*

$$b \leq b_1 \leq \cdots \leq b_m < b + 1$$

and where $t_i \leq 1$ for $i = 1, \ldots, m$. Put $\mu = b_m - b$ and

$$v(b, h, \lambda, \mu) = (b^2 + h + 1)b^{-1/2}\left[(2\,\lambda + 2\mu)^{1/2} + 3b^{-1/4}\right].$$

If b is sufficiently large and if

$$\sum_{i=1}^{m} area(L_i) \leq area(B) - v(b, h, \lambda, \mu),$$

then $L_1, \ldots, L_m$ can be packed into B.

Proof Assume that $b \geq 27$ and that the sum of the areas of rectangles $L_1, \ldots, L_m$ is not greater than $area(B) - v(b, h, \lambda, \mu)$.

Put

$$\vartheta_i = \arctan \frac{t_i}{b_i} + \arccos \frac{b}{\sqrt{b_i^2 + t_i^2}}$$

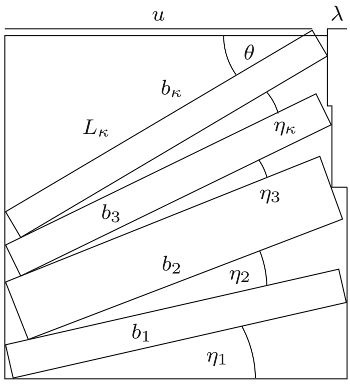
(see Fig. 5). Without loss of generality we can assume that $\vartheta_1 \leq \cdots \leq \vartheta_m$.

We pack the rectangles $L_1, L_2, \ldots$ into B as in Fig. 6. Contrary to the statement suppose that the rectangles cannot be packed. We show that this leads to a contradiction. Let L_κ be the first rectangle which cannot be packed into B.

By $b_\kappa \leq b + \mu < b + 1$ and $u > b - \lambda - 1 > b - 2$ in Fig. 6 we have

$$\theta \leq \tan\theta = \frac{\sqrt{b_\kappa^2 - u^2}}{u} < \frac{\sqrt{(b+1)^2 - (b-2)^2}}{b - 2} < \frac{\sqrt{6b}}{b - 2} < \sqrt{\frac{7}{b}}.$$

Fig. 6 $L_i \subset B$



We need a more precise estimation. Since $t_\kappa \leq 1$ and $\sin\theta \leq \theta < \sqrt{7/b}$, it follows that $u = b - \lambda - t_\kappa \sin\theta > b - \lambda - \sqrt{7/b}$. By $b_\kappa \leq b + \mu$ we obtain

$$\begin{aligned}\tan\theta &= \frac{\sqrt{b_\kappa^2 - u^2}}{u}\\ &< \frac{\sqrt{(b+\mu)^2 - (b-\lambda-\sqrt{7/b})^2}}{b-\lambda-\sqrt{7/b}}\\ &= \sqrt{\frac{2b(\lambda+\mu)+\mu^2-\lambda^2-7/b+2(b-\lambda)\sqrt{7/b}}{b^2-2\lambda b+\lambda^2+7/b-2(b-\lambda)\sqrt{7/b}}}\\ &< \sqrt{\frac{2b(\lambda+\mu)+1+6\sqrt{b}}{b^2-2\lambda b-6\sqrt{b}}}.\end{aligned}$$

It is easy to check that

$$\frac{2b(\lambda+\mu)+1+6\sqrt{b}}{b^2-2\lambda b-6\sqrt{b}} < 2(\lambda+\mu)b^{-1} + 7b^{-3/2},$$

for sufficiently large b.

Since $\sqrt{\alpha_1+\alpha_2} \leq \sqrt{\alpha_1} + \sqrt{\alpha_2}$ for non-negative values α_1 and α_2, it follows that

$$\theta \leq \tan\theta < f(b,\lambda,\mu),$$

where

$$f(b,\lambda,\mu) = (2\lambda+2\mu)^{1/2}b^{-1/2} + 7^{1/2}b^{-3/4}.$$

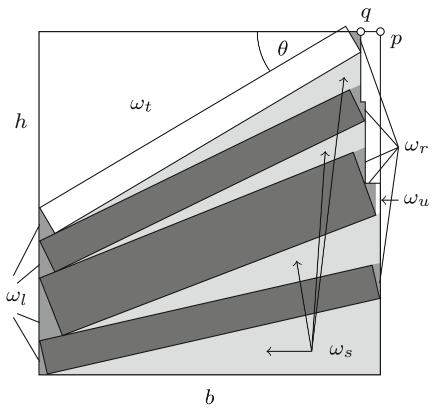
The uncovered dark shaded part on the left side of B in Fig. 7 consists of a number of triangles. The total length of the left sides of the triangles is smaller than $h+1$. The height of each such triangle (the height parallel to the bottom of B) is not greater than $\sin\theta$. Consequently, the uncovered dark shaded part on the left side of B in Fig. 7 is of the area

$$\omega_l < \frac{1}{2}(h+1)\sin\theta < \frac{1}{2}(h+1)\theta < \frac{1}{2}(h+1)f(b,\lambda,\mu).$$

Similarly we estimate the area ω_r of the uncovered dark shaded part on the right side of B:

$$\omega_r \leq \omega_l < \frac{1}{2}(h+1)f(b,\lambda,\mu).$$

Fig. 7 Wasted area



Since the distance between p and q is equal to λ and the height of each L_i is not greater than 1, it follows that the non-shaded uncovered part on the right side of B in Fig. 7 is of the area

$$\omega_u \leq \lambda < 1.$$

Denote by ω_s^+ the area of a right triangle of legs of length b and $b \tan\theta$. Moreover, denote by ω_s the area of the light shaded uncovered part of B in Fig. 7. By $\eta_1 + \eta_2 + \cdots + \eta_\kappa = \theta$ (see Fig. 6) we deduce that

$$\omega_s \leq \omega_s^+ = \frac{1}{2} b^2 \tan\theta < \frac{1}{2} b^2 f(b, \lambda, \mu).$$

The uncovered non-shaded part on the top of B in Fig. 7 is of the area

$$\omega_t < \frac{1}{2} b^2 \tan\theta + area(L_\kappa) < \frac{1}{2} b^2 f(b, \lambda, \mu) + b + 1.$$

Consequently, the area of the uncovered part of B does not exceed

$$\begin{aligned}\omega &= \omega_l + \omega_r + \omega_u + \omega_s + \omega_t \\ &< (b^2 + h + 1) f(b, \lambda, \mu) + b + 2 \\ &= (b^2 + h + 1)\left[(2\lambda + 2\mu)^{1/2} b^{-1/2} + 7^{1/2} b^{-3/4}\right] + b + 2 \\ &< (b^2 + h + 1)\left[(2\lambda + 2\mu)^{1/2} b^{-1/2} + 3b^{-3/4}\right],\end{aligned}$$

for sufficiently large b. This implies that

$$\sum_{i=1}^{\kappa-1} area(L_i) \geq area(B) - \omega > area(B) - v(b, h, \lambda, \mu),$$

which is a contradiction. □

3 Packing into a large square

In the main packing method I_x will be partitioned into a number of σ-polygons. Next, rectangles from $\mathcal{R}$ will be packed into adequate σ-polygons.

Theorem 3.1 *Let $\epsilon > 0$. Any collection of rectangles of sides of length not greater than 1 with total area smaller than $x^2 - (5+\epsilon)x^{5/6}$ can be packed into I_x, for sufficiently large x.*

Proof Assume that $\epsilon > 0$ and that $x > (5+\epsilon)^{6/7}$. Consider a collection $\mathcal{C}$ of rectangles P_i of sides of length not greater than 1 with total area smaller than $x^2-(5+\epsilon)x^{5/6}$. If $\mathcal{C}$ is finite, then put $\mathcal{R} = \mathcal{C}$ and denote by z the number of rectangles in $\mathcal{R}$. Otherwise, we can assume that $area(P_1) \geq area(P_2) \geq \ldots$. There is an integer z such that $\sum_{i=z}^{\infty} area(P_i) < \frac{1}{2}$. By [5] we know that rectangles $R_z, R_{z+1}, \ldots$ can be packed into I_1. Let $\mathcal{R}$ be a collection of rectangles R_i, where $R_1 = I_1$ and $R_i = P_{i-1}$ for $i = 2, \ldots, z$.

We show that rectangles from $\mathcal{R}$ can be packed into I_x provided x is sufficiently large. Clearly,

$$\sum_{i=1}^{z} area(R_i) < x^2 - (5+\epsilon)x^{5/6} + 1.$$

We can assume that the width w_i of R_i is not greater than its height h_i for $i = 1, \ldots, z$ and that $h_1 \geq \cdots \geq h_z$. Put

$$n = \lfloor x^{1/6} \rfloor,$$
$$a = x/n - x^{1/2}$$

and

$$d = x - x^{1/2}.$$

It is easy to verify that $n(a+1)(d+1) < (x-1)^2$. If $\rho < n(a+1)(d+1)$, then, by [4], all rectangles from $\mathcal{R}$ can be packed into I_x. Otherwise, by Lemma 2.2 we deduce that there is an integer k_1 and there are n mutually disjoint σ-polygons A_i of base a_i, top $a_i - \lambda_i$ and height d_i, where

$$a \leq a_i < a+1, \quad d \leq d_i < d+1$$

(for $i = 1, \ldots, n$) and where $\sum_{i=1}^{n} \lambda_i < 1$ such that $R_1, \ldots, R_{k_1}$ can be packed into $\bigcup_{i=1}^{n} A_i$ and that the waste in this packing (i.e., the area of the uncovered part of $\bigcup_{i=1}^{n} A_i$) is at most

$$\omega_1 = a + 1 = x/n - x^{1/2} + 1.$$

Clearly,

$$\sum_{i=1}^{k_1} area(R_i) \geq \sum_{i=1}^{n} area(A_i) - \omega_1.$$

We lose no generality in assuming that $d_1 \geq d_2 \geq \cdots \geq d_n$.

I_x will be divided into: n polygons A_i and $n+1$ other σ-polygons. Then $R_1, \ldots, R_{k_1}$ will be packed into $\bigcup_{i=1}^{n} A_i$. The remaining rectangles from $\mathcal{R}$ will be first packed into larger rectangles L_i or L'_i. Next, L_i and L'_i will be packed into $I_x \setminus \bigcup_{i=1}^{n} A_i$.

We apply Lemma 2.3 for packing $R_{k_1+1}, \ldots, R_z$. There is an integer m and there are rectangles L_i (for $i = 1, \ldots, m$) of width b_i, where

$$x^{1/2} \leq b_i < x^{1/2} + 1$$

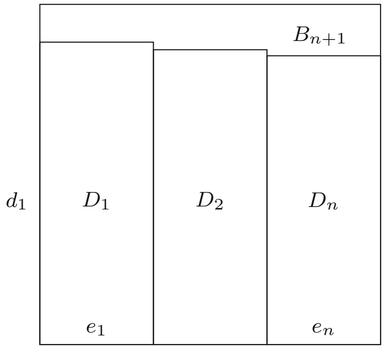


Fig. 8 Partition of I_x

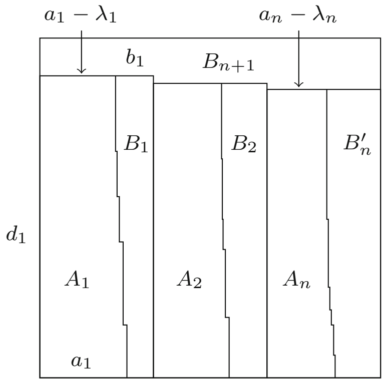


Fig. 9 Partition of D_i

and height not greater than 1 such that $R_{k_1+1}, \ldots, R_z$ can be packed into $\bigcup_{i=1}^{m} L_i$ and that the waste in this packing is no more than

$$\omega_2 = 2x^{1/2} + 1.$$

There is no loss of generality in assuming that $b_1 \leq b_2 \leq \cdots \leq b_m$.

We divide I_x into: n rectangles D_i (for $i = 1, \ldots, n$) of width e_i and height d_i and one σ-polygon B_{n+1} of base $x - d_n$ and height x (as in Fig. 8). Now we will describe how to choose proper values $e_1, \ldots, e_n$. This action depends on the width of some rectangles L_i.

Put $e_1 = a_1 - \lambda_1 + b_1$. Clearly, D_1 can be divided into the σ-polygon A_1 and a σ-polygon B_1 of base b_1, top $b_1 - \lambda_1$ and height d_1 (see Fig. 9). Denote by m_1 the greatest integer such that

$$\sum_{i=1}^{m_1} area(L_i) \leq area(B_1) - v(b_1, d_1, \lambda_i, b_{m_1} - b_1).$$

By Lemma 2.4 we know that $L_1, \ldots, L_{m_1}$ can be packed into B_1, for sufficiently large x. Obviously,

$$\sum_{i=1}^{m_1+1} area(L_i) > area(B_1) - v(b_1, d_1, \lambda_i, b_{m_1+1} - b_1).$$

Consequently,

$$\sum_{i=1}^{m_1} area(L_i) > area(B_1) - (x^{1/2}+1) - v(b_1, d_1, \lambda_i, b_{m_1+1} - b_1).$$

We proceed in a similar way for $i = 2, \dots, n-1$. Put

$$e_i = a_i - \lambda_i + b_{m_{i-1}+1}$$

for $i = 2, \dots, n-1$. Each D_i is divided into the σ-polygon A_i and a σ-polygon B_i of base $b_{m_{i-1}+1}$, top $b_{m_{i-1}+1} - \lambda_i$ and height d_i. Denote by m_i the greatest integer such that

$$\sum_{i=m_{i-1}+1}^{m_i} area(L_i) \le area(B_i) - v(b_{m_{i-1}+1}, d_i, \lambda_i, b_{m_i} - b_{m_{i-1}+1}),$$

By Lemma 2.4 we know that $L_{m_{i-1}+1}, \dots, L_{m_i}$ can be packed into B_i provided x is sufficiently large. Moreover,

$$\sum_{i=m_{i-1}+1}^{m_i} area(L_i) > area(B_i) - (x^{1/2}+1) - v(b_{m_{i-1}+1}, d_i, \lambda_i, b_{m_i+1} - b_{m_{i-1}+1}).$$

Clearly, if $m_i = m$ for some integer i, then all rectangles from $\mathcal{R}$ were packed into I_x. Denote by m_n the greatest integer such that

$$\sum_{i=m_{n-1}+1}^{m_n} area(L_i) \le area(B_{n+1}) - v(x - d_n, x, 1, 1).$$

By Lemma 2.4 we know that $L_{m_{n-1}+1}, \dots, L_{m_n}$ can be packed into B_{n+1}. Moreover,

$$\sum_{i=m_{n-1}+1}^{m_n} area(L_i) > area(B_{n+1}) - (x^{1/2}+1) - v(x - d_n, x, 1, 1).$$

Finally, put

$$e_n = x - \sum_{i=1}^{n-1} e_i.$$

The rectangle D_n is divided into the σ-polygon A_n and a σ-polygon B_n' of height d_n and base

$$b' = e_n - a_n + \lambda_n = x - \sum_{i=1}^{n} a_n + \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} b_{m_{i-1}+1}$$

($m_0 = 0$). Since

$$\sum_{i=1}^{n} \lambda_i < 1,$$

$$x - nx^{1/2} = na \le \sum_{i=1}^{n} a_i < n(a+1) = x - nx^{1/2} + n$$

and

$$(n-1)x^{1/2} \leq \sum_{i=1}^{n-1} b_{m_{i-1}+1} < (n-1)(x^{1/2}+1),$$

it follows that

$$x^{1/2} - 2n + 1 \leq b' < x^{1/2} + 1.$$

Denote by R_{k_2} the last rectangle packed in L_{m_n}. The rectangles $R_1, \dots, R_{k_1}$ were packed into $A_1 \cup \dots \cup A_n$. The rectangles $R_{k_1+1}, \dots, R_{k_2}$ were packed into $B_1 \cup \dots \cup B_{n-1} \cup B_{n+1}$. The remaining rectangles $R_{k_2+1}, \dots, R_z$ will be packed into B'_n. Unfortunately, it is possible that $L_{m_n+1}, \dots, L_m$ are too large to apply Lemma 2.4 and therefore we need to repack $R_{k_2+1}, \dots, R_z$ into other larger rectangles. We apply Lemma 2.3. There is an integer l and there are rectangles L'_i (for $i = 1, \dots, l$) of width b'_i, where

$$b' \leq b'_i < b' + 1$$

and height not greater than 1 such that $R_{k_2+1}, \dots, R_z$ can be packed into $\bigcup_{i=1}^{l} L'_i$ and that the waste in this packing is no more than

$$\omega_3 = 2b' + 1 < 2x^{1/2} + 3.$$

It remains to check that

$$\sum_{i=1}^{l} area(L'_i) \leq area(B'_n) - v(b', d_n, \lambda_n, 1), \tag{*}$$

for sufficiently large x (then, by Lemma 2.4, $L'_1, \dots, L'_l$ and, consequently, the rectangles $R_{k_2+1}, \dots, R_z$ can be packed into B'_n).

Put $\mu_1 = b_{m_1+1} - b_1$ and $\mu_i = b_{m_i+1} - b_{m_{i-1}+1}$ for $i = 2, \dots, n-1$. Obviously, $\sum_{i=1}^{n-1} \mu_i < 1$. By

$$\sum_{i=1}^{k_1} area(R_i) \geq \sum_{i=1}^{n} area(A_i) - \omega_1,$$

$$\sum_{i=k_1+1}^{k_2} area(R_i) \geq \sum_{i=1}^{m_n} area(L_i) - \omega_2$$

and

$$\sum_{i=k_2+1}^{z} area(R_i) \geq \sum_{i=1}^{l} area(L'_i) - \omega_3,$$

we have

$$\begin{aligned}\sum_{i=1}^{z} area(R_i) \geq \sum_{i=1}^{n} area(A_i) - \omega_1 + \sum_{i=1}^{n-1} area(B_i)\\ -(n-1)(x^{1/2}+1) - \sum_{i=1}^{n-1} v(b_{m_{i-1}+1}, d_i, \lambda_i, \mu_i)\\ +area(B_{n+1}) - (x^{1/2}+1) - v(x-d_n, x, 1, 1) - \omega_2\\ +\sum_{i=1}^{l} area(L'_i) - \omega_3.\end{aligned}$$

Consequently,

$$\begin{aligned}x^2 - (5+\epsilon)x^{5/6} + 1 > x^2 - area(B'_n) - \omega_1 - \omega_2 - \omega_3 - n(x^{1/2}+1)\\ -\sum_{i=1}^{n-1} v(b_{m_{i-1}+1}, d_i, \lambda_i, \mu_i) - v(x-d_n, x, 1, 1)\\ +\sum_{i=1}^{l} area(L'_i).\end{aligned}$$

To prove the inequality $(*)$ we show that

$$\begin{aligned}\zeta &= \omega_1 + \omega_2 + \omega_3 + 1 + n(x^{1/2}+1)\\ &+v(x-d_n, x, 1, 1) + v(b', d_n, \lambda_n, 1)\\ &+\sum_{i=1}^{n-1} v(b_{m_{i-1}+1}, d_i, \lambda_i, \mu_i) < (5+\epsilon)x^{5/6}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}\omega_1 + \omega_2 + \omega_3 + 1 + n(x^{1/2}+1) \leq x/n + 3x^{1/2} + 6 + n(x^{1/2}+1) < (1+\epsilon/4)x^{5/6},\\ v(x-d_n, x, 1, 1) \leq v(x^{1/2}, x, 1, 1) = (2x+1)(2x^{-1/4} + 3x^{-3/8}) < \epsilon x^{5/6}/4\end{aligned}$$

and

$$v(b', d_n, \lambda_n, 1) < \epsilon x^{5/6}/4,$$

for sufficiently large x. This implies that

$$\begin{aligned}\zeta &< (1+3\epsilon/4)x^{5/6} + \sum_{i=1}^{n-1} \frac{b_{m_{i-1}+1}^2 + x + 1}{b_{m_{i-1}+1}^{1/2}} \cdot \left[(2\lambda_i + 2\mu_i)^{1/2} + 3b_{m_{i-1}+1}^{-1/4}\right]\\ &< (1+3\epsilon/4)x^{5/6} + \frac{(x^{1/2}+1)^2 + x + 1}{(x^{1/2})^{1/2}} \cdot \sum_{i=1}^{n-1}\left[(2\lambda_i + 2\mu_i)^{1/2} + 3x^{-1/8}\right].\end{aligned}$$

The arithmetic mean of a list of non-negative real numbers is smaller than or equal to the quadratic mean of the same list. Therefore

$$\alpha_1^{1/2} + \cdots + \alpha_{n-1}^{1/2} \leq \left[(n-1)(\alpha_1 + \cdots + \alpha_{n-1})\right]^{1/2}$$

for non-negative numbers $\alpha_1, \ldots, \alpha_{n-1}$. Since $\sum_{i=1}^{n-1}(\lambda_i + \mu_i) < 2$, it follows that

$$\sum_{i=1}^{n-1}(2\lambda_i + 2\mu_i)^{1/2} < 2(n-1)^{1/2}.$$

Hence

$$\begin{aligned}\zeta &< (1+3\epsilon/4)x^{5/6} + \frac{2x + 2x^{1/2} + 2}{x^{1/4}} \cdot \left[2(n-1)^{1/2} + 3(n-1)x^{-1/8}\right] \\ &< (1+3\epsilon/4)x^{5/6} + (2x^{3/4} + 2x^{1/4} + 2x^{-1/4}) \cdot \left[2(x^{1/6})^{1/2} + 3x^{1/6}x^{-1/8}\right] \\ &= (5+3\epsilon/4)x^{5/6} + 6x^{19/24} + 4x^{1/3} + 6x^{7/24} + 4x^{-1/6} + 6x^{-5/24}.\end{aligned}$$

Consequently,

$$\zeta < (5+\epsilon)x^{5/6},$$

for sufficiently large x. □

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