Period Math Hung (2014) 68:185–192
DOI 10.1007/s10998-014-0029-0

The M-principal graph of a commutative ring

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Published online: 27 May 2014

Abstract Let R be a commutative ring and M be an R-module. In this paper, we introduce the M-principal graph of R, denoted by $M-PG(R)$. It is the graph whose vertex set is $R\backslash\{0\}$, and two distinct vertices x and y are adjacent if and only if $xM = yM$. In the special case that $M = R$, $M-PG(R)$ is denoted by $PG(R)$. The basic properties and possible structures of these two graphs are studied. Also, some relations between $PG(R)$ and $M - PG(R)$ are established.

Keywords M-principal graph · Commutative ring · Module · Clique number · Independence number

Mathematics Subject Classification 05C25 · 05C69 · 13A99 · 13C99

1 Introduction

There are many papers on assigning a graph to a ring R, see [1–4,9,10]. In this paper, we introduce the *M-principal graph* of R, denoted by $M - PG(R)$, where M is an R-module. Throughout the paper all rings are commutative with non-zero identity and all modules are non-zero unitary.

Let R be a ring and M be an R-module. The *annihilator* of M is denoted by $ann(M)$. The module M is called a *faithful* R-module if $ann(M) = 0$. Also, M is called an *simple* R-module if $M \neq 0$, and M has no submodules other than 0 and M. We denote the characteristic of R by char R. Also, $J(R)$ denotes the Jacobson radical of R and $U(R)$ denotes the group of units of R. A ring having just one maximal ideal is called a *local ring* and a ring having only

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finitely many maximal ideals is said to be a *semilocal ring*. The direct product of a family of rings $\{R_i \mid i \in I\}$ is denoted by $\prod_{i \in I} R_i$. As usual, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_n$ will denote the integers, real numbers, complex numbers and integers modulo n, respectively.

A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. The complete graph on n vertices is denoted by K_n. A graph G is called *regular* if each vertex has the same number of neighbors. An *empty graph* is one whose edge set is empty. Let G be a graph. The set of vertices and the set of edges of G are denoted by $V(G)$ and $E(G)$, respectively. "A subgraph H of G is said to be an *induced subgraph of* G if it is a subgraph i.e. $V(H) \subset V(G)$ and $E(H) \subset E(G)$ and it has exactly the same edges that appear in G over the vertices $V(H)$ i.e. $\forall u, v \in V(H)$ an edge $e = uv \in E(H)$ if and only if $e \in E(G)$."

Also a subgraph H of G is called a *spanning subgraph* if $V(H) = V(G)$. We say that G is *connected* if there is a path between any two distinct vertices of G. A *cycle* of G is a path such that the start and end vertices are the same. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). A *Hamiltonian cycle* is a spanning cycle in a graph. A graph G is called *Hamiltonian* if G has a Hamiltonian cycle. A *forest* is a graph with no cycles. A *clique* in G is a set of pairwise adjacent vertices and a set in G whose no two vertices are adjacent is called an *independent set*. The *clique number* and the *independence number* of G, denoted by $\omega(G)$ and $\alpha(G)$, are the largest orders of a clique and an independent set of G, respectively. Also, the *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors.

In this article, we introduce and investigate the M-principal graph of R, denoted by $M - PG(R)$, where R is a commutative ring and M is a non-zero R-module. If R is regarded as a module over itself, that is, $M = R$, then the M-principal graph of R is denoted by $PG(R)$. Also, we study some properties of $PG(R)$, in particular we consider the graph $PG(\mathbb{Z}_n)$ for each positive integer $n > 1$. Finally, some relations between $PG(R)$ and $M - PG(R)$ are established.

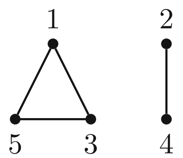
2 The M-principal graph of R

In this section, we introduce the M-principal graph of R and study its basic properties.

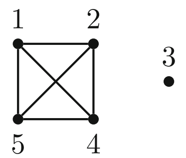
Definition 2.1 Let R be a ring and M be a non-zero R-module. The *M-principal graph* of R, denoted by $M - PG(R)$, is the (undirected) graph whose vertex set is $R \backslash \{0\}$ and two distinct vertices x and y are adjacent if and only if $xM = yM$.

It is clear that if M and N are isomorphic modules over R, then $M - PG(R)$ is the same as $N - PG(R)$, but the converse is not true in general. For instance, consider $\mathbb{R} - PG(\mathbb{Z})$ and $\mathbb{C} - PG(\mathbb{Z})$.

Example 2.2 Let $R = \mathbb{Z}_6$. So we have the following graphs.



$\mathbb{Z}_2$-$PG(\mathbb{Z}_6)$



$\mathbb{Z}_3$-$PG(\mathbb{Z}_6)$

Remark 2.3 Clearly, $M - PG(R)$ is a disjoint union of complete graphs. Hence $gr(M - PG(R)) \in \{3, \infty\}$ and also $|U(R)| \leq \omega(M - PG(R)) = \chi(M - PG(R))$.

By the previous remark, we have the next immediate result.

Corollary 2.4 *Let R be a ring with $|R| > 3$, and let M be a non-zero R-module. Then the following conditions are equivalent:*

(1) *$M - PG(R)$ is a complete graph.*
(2) *$M - PG(R)$ is a Hamiltonian graph.*
(3) *$M - PG(R)$ is a connected graph.*

Remark 2.5 Let x be a non-zero element of R. Then each coset $x + ann(M)$ is a clique in $M - PG(R)$ and so $|ann(M)| \leq \omega(M - PG(R))$. Also, $\alpha(M - PG(R)) \leq |R/ann(M)|$. Moreover, if $ann(M) \neq 0$, then $M - PG(R)$ is disconnected.

Theorem 2.6 *Let R be a ring and M be a non-zero R-module. Then M is a faithful simple R-module if and only if $M - PG(R)$ is a complete graph and M is a cyclic R-module.*

Proof Suppose that $M - PG(R)$ is a complete graph and there exists a non-zero element $m \in M$ such that $M = Rm$. If $0 \neq x \in ann(M)$, then x is not adjacent to 1, a contradiction. So $ann(M) = 0$. Let m' be a non-zero element of M. We have $m' = xm$ for some $x \in R$. Since $M - PG(R)$ is a complete graph, $M = xM = xRm = Rm'$. Thus M is simple. For the other direction, assume that $x \in R$ and $x \neq 0$. Since xM is a non-zero submodule of M and M is simple, we have $xM = M$. The proof is complete. □

Theorem 2.7 *Let R be a ring and M be a (non-zero) finitely generated R-module. If $M - PG(R)$ is a complete graph, then $J(R) = 0$.*

Proof By contradiction, assume that $x \in J(R)$ and $x \neq 0$. Since $M - PG(R)$ is a complete graph, $xM = M$. So $J(R)M = M$ and by Nakayama's Lemma [7, Proposition 2.6], we have $M = 0$, a contradiction. □

Theorem 2.8 *Let R be a ring and M be a non-zero R-module. If $M - PG(R)$ is empty, then M is a faithful R-module and we have $|U(R)| = 1$. Moreover, if R is an Artinian ring, then $R \cong (\mathbb{Z}_2)^k$ for some positive integer k.*

Proof By Remark 2.5, $ann(M) = 0$ and since $xM = M$ for every $x \in U(R)$, we have $|U(R)| = 1$ (so char $R = 2$ and $J(R) = 0$). If R is Artinian, then by [7, Theorem 8.7], $R \cong R_1 \times \cdots \times R_k$, where R_i is a local ring and k is a positive integer. Suppose that m_i is a maximal ideal of R_i, for each integer i, $1 \leq i \leq k$. Since $|R_i \backslash m_i| = |U(R_i)| = 1$, we have $|m_i| = 1$ and hence $R_i \cong \mathbb{Z}_2$. Thus $R \cong (\mathbb{Z}_2)^k$. □

Theorem 2.9 *Let R be an integral domain and M be an Artinian module. If R contains a non-zero element x such that $x^k \neq 1$ for each positive integer k, then $\omega(M - PG(R)) = \infty$.*

Proof Since M is Artinian, there exists a positive integer n, such that $x^k M = x^n M$ for each positive integer $k \geq n$. Now, $\{x^k \mid k \geq n\}$ is a clique in $M - PG(R)$ and hence $\omega(M - PG(R)) = \infty$. □

Let R be a commutative ring. An R-module M is said to be secondary (see [8, p. 42]), if $M \neq 0$ and, for each $x \in R$, $xM = M$ or $x \in \sqrt{ann(M)}$. ($\sqrt{ann(M)}$ denotes the radical of $ann(M)$.)

Theorem 2.10 *Let R be a ring and M be a secondary R-module. Then $|R\backslash P| \leq \omega(M - PG(R))$, where $P = \sqrt{ann(M)}$.*

Proof Since M is a secondary R-module, $xM = M$ for every $x \in R\backslash P$. So $R\backslash P$ is a clique in $M - PG(R)$ and hence $|R\backslash P| \leq \omega(M - PG(R))$. □

Theorem 2.11 *Let R be a ring and M be a non-zero R-module. If N is a proper submodule of M, then $M - PG(R)$ is a spanning subgraph of $M/N - PG(R)$.*

Proof Suppose that $x, y \in R$ and $xM = yM$. So $x(M/N) = (xM + N)/N = (yM + N)/N = y(M/N)$. Hence if x and y are adjacent vertices of $M - PG(R)$, then x is adjacent to y in $M/N - PG(R)$. Therefore $M - PG(R)$ is a spanning subgraph of $M/N - PG(R)$. □

Suppose that R is a commutative ring and M is a module over R. It is well known (see [7, p. 19]) that if I is an ideal of R such that $IM = 0$, M can be regarded as an R/I-module, as follows: if $\overline{x} \in R/I$ is represented by $x \in R$, define $\overline{x}m$ to be xm for every $m \in M$. So we can deduce the next theorem.

Theorem 2.12 *Let R be a ring and M be a non-zero R-module, and let I be an ideal of R such that $IM = 0$. Then $M - PG(R/I)$ is an induced subgraph of $M - PG(R)$. Moreover, if $I = ann(M) \neq 0$ and $M - PG(R/I)$ is a complete graph, then $M - PG(R)$ is a disjoint union of two complete graphs.*

Proof Suppose that $x, y \in R$. Clearly, $xM = yM$ if and only if $\overline{x}M = \overline{y}M$. So (by assigning $\overline{x}$ to x), $M - PG(R/I)$ is an induced subgraph of $M - PG(R)$. Now, assume that $M - PG(R/ann(M))$ is a complete graph and let $x, y \in R\backslash\{0\}$. If $x, y \in ann(M)$, then $xM = yM = 0$. Suppose that $x, y \in R\backslash ann(M)$. If $\overline{x} = \overline{y}$, then $x - y \in ann(M)$ implies that $xM = yM$. Otherwise, $\overline{x}$ and $\overline{y}$ are two distinct vertices of $M - PG(R/ann(M))$ and hence $\overline{x}$ is adjacent to $\overline{y}$. So x is adjacent to y. It is clear that if $x \in ann(M)$ and $y \in R\backslash ann(M)$, then $xM \neq yM$. Thus $M - PG(R)$ is a disjoint union of two complete graphs. □

Let S be a multiplicatively closed subset of a commutative ring R, and let M be an R-module. We denote the ring of fractions of R and the module of fractions of M (with respect to S) by R_S and M_S, respectively. Now, we have the following theorem.

Theorem 2.13 *Let R be a ring and M be a non-zero R-module. If $S = Reg(R)$, then $M - PG(R)$ is a subgraph of $M_S - PG(R_S)$. Moreover, if $M - PG(R)$ is complete, then $M_S - PG(R_S)$ is also a complete graph.*

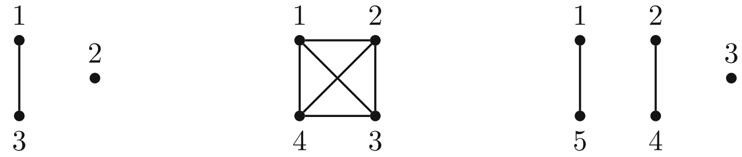
Proof Suppose that $x, y \in R$ and $xM = yM$. So $(x/1)M_S = (xM)_S = (yM)_S = (y/1)M_S$. Thus by assigning x to $x/1$, $M - PG(R)$ is a subgraph of $M_S - PG(R_S)$. Note that if $x \neq 0$, then $x/1 \neq 0$. Now, assume that $M - PG(R)$ is a complete graph and $x/r, y/t$ are two distinct vertices of $M_S - PG(R_S)$. Then tx and ry are two distinct vertices of $M - PG(R)$ and so $txM = ryM$. Thus $(x/r)M_S = (tx/tr)M_S = (txM)_S = (ryM)_S = (ry/tr)M_S = (y/t)M_S$ and hence x/r is adjacent to y/t. □

3 The principal graph of R

If R is regarded as a module over itself, that is, $M = R$, then the M-principal graph of R is denoted by $PG(R)$ (the principal graph of R). In this section we study some properties of $PG(R)$, in particular we consider the graph $PG(\mathbb{Z}_n)$ for each positive integer $n > 1$.

Note that for two rings R and S if $R \cong S$, then $PG(R) \cong PG(S)$, but the converse is not true. For instance, consider $PG(\mathbb{Z}_4)$ and $PG(\mathbb{Z}_2[x]/(x^2))$.

Example 3.1



$PG(\mathbb{Z}_4)$ $PG(\mathbb{Z}_5)$ $PG(\mathbb{Z}_6)$

Corollary 3.2 *Let R be a ring with $|R| > 3$. Then the following conditions are equivalent:*

(1) *R is a field.*
(2) *$PG(R)$ is a complete graph.*
(3) *$PG(R)$ is a Hamiltonian graph.*
(4) *$PG(R)$ is a connected graph.*

Notice that $\alpha(PG(R))$ is equal to the number of non-zero principal ideals of R. So we have the next theorem.

Theorem 3.3 *Let R be a ring. If $\alpha(PG(R))$ is finite, then R is Artinian.*

Proof R has only a finite number of distinct principal ideals, since $\alpha(PG(R))$ is finite. Now, suppose that I is a non-zero ideal of R. We have $I = \sum_{x \in I} xR$, so R contains only finitely many ideals and hence R is Artinian. □

Remark 3.4 Note that there exist some infinite rings which have only a finite number of ideals, for example the ring $F[x]/(x^2)$, where F is an infinite field.

Theorem 3.5 *Let R be a semilocal ring. If $\omega(PG(R))$ is finite, then R is finite.*

Proof Suppose that $m_1, \dots, m_k$ are all the maximal ideals of R. We have $U(R) = R \setminus \bigcup_{i=1}^{k} m_i$. Since $\omega(PG(R))$ is finite, $U(R)$ is finite and so by [4, Theorem 2], R is finite. □

Corollary 3.6 *Let R be a semilocal ring. If $PG(R)$ is empty, then $R \cong (\mathbb{Z}_2)^k$ for some positive integer k.*

Proof By the previous theorem, R is finite and so by Theorem 2.8, the result holds. □

Theorem 3.7 *Let R be an Artinian ring. If $PG(R)$ is a forest, then R is isomorphic to one of the rings:*

$$\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ (\mathbb{Z}_2)^k,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_3,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_4,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_2[x]/(x^2),$$

for some positive integer k.

Proof Since $gr(PG(R)) = \infty$, so $|U(R)| \leq 2$ and by [4, Lemma 1], R is isomorphic to one of the rings:

$$\mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ (\mathbb{Z}_2)^k,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_3,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_4,\ (\mathbb{Z}_2)^k \times \mathbb{Z}_2[x]/(x^2),$$

for some positive integer k. □

The strong product $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and any two distinct vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \boxtimes H$ if and only if ($u_1 = v_1$ and u_2 adj v_2) or (u_1 adj v_1 and $u_2 = v_2$) or (u_1 adj v_1 and u_2 adj v_2). Now, suppose that R_1 and R_2 are two rings. Let $x_1, y_1 \in R_1$ and $x_2, y_2 \in R_2$. We know that $(x_1, x_2)(R_1 \times R_2) = (y_1, y_2)(R_1 \times R_2)$ if and only if $x_1R_1 = y_1R_1$ and $x_2R_2 = y_2R_2$. So $PG(R_1 \times R_2) = PG(R_1) \boxtimes PG(R_2)$ and also $\omega(PG(R_1 \times R_2)) = \omega(PG(R_1)) \times \omega(PG(R_2))$.

Theorem 3.8 *Let R_1 and R_2 be two rings, and let $\alpha(PG(R_1)) = \alpha_1$ and $\alpha(PG(R_2)) = \alpha_2$. Then $\alpha(PG(R_1 \times R_2)) = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$.*

Proof It is obvious that $PG(R_1)$ is the union of α_1 disjoint complete graphs and similarly, $PG(R_2)$ is the union of α_2 disjoint complete graphs. Suppose that $A_1, \ldots, A_{\alpha_1}$ are all the components of $PG(R_1)$ and $B_1, \ldots, B_{\alpha_2}$ are all the components of $PG(R_2)$. Then $A_1 \times 0, \ldots, A_{\alpha_1} \times 0$ and $0 \times B_1, \ldots, 0 \times B_{\alpha_2}$ and $A_i \times B_j$, where $1 \leq i \leq \alpha_1$ and $1 \leq j \leq \alpha_2$ are all the components of $PG(R_1 \times R_2)$, each of which is a complete graph. Hence $\alpha(PG(R_1 \times R_2)) = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$. □

Let $R_1, \ldots, R_n$ be rings. By induction, one can easily prove that $PG(R_1 \times \cdots \times R_n) = PG(R_1) \boxtimes \cdots \boxtimes PG(R_n)$.

Now, we consider the graph $PG(\mathbb{Z}_n)$ for each positive integer $n > 1$. Let $d(n)$ denote the number of positive divisors of n. Clearly, $\alpha(PG(\mathbb{Z}_n)) = d(n) - 1$. Suppose that φ is the Euler phi function. By [6, Theorem 2.5], if d is a positive divisor of n, then $\varphi(d) \leq \varphi(n)$. So $\chi(PG(\mathbb{Z}_n)) = \omega(PG(\mathbb{Z}_n)) = \varphi(n)$.

The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest number of colors which can be assigned to the edges of G such that no two edges incident on the same vertex have the same color. By Vizing's Theorem, if G is a graph whose maximum vertex degree is Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$. Vizing's Theorem divides the graphs into two classes according to their chromatic index; graphs satisfying $\chi'(G) = \Delta$ are called class 1, those with $\chi'(G) = \Delta + 1$ are class 2.

We now state the following result which shows that the graph $PG(\mathbb{Z}_n)$ is class 1.

Theorem 3.9 *The principal graph $PG(\mathbb{Z}_n)$ is class 1, for each positive integer $n > 1$.*

Proof By [6, Theorem 2.5], $\varphi(d)$ is even for each positive integer $d \geq 3$. Thus each connected component of $PG(\mathbb{Z}_n)$ with two or more vertices is a complete graph which contains an even number of vertices. So by [5, Theorem 5.11], $PG(\mathbb{Z}_n)$ is class 1. □

Remark 3.10 By [6, Theorem 2.5], $\varphi(d) \geq 2$ for each positive integer $d \geq 3$. So if n is even, then $PG(\mathbb{Z}_n)$ contains exactly one isolated vertex and otherwise it contains no isolated vertex.

Theorem 3.11 *If $G = PG(\mathbb{Z}_n)$, then the following conditions are equivalent:*

(1) *G is a regular graph.*

(2) *n is a prime number.*
(3) *G is a complete graph.*
(4) *G is a Hamiltonian graph or $G \cong PG(\mathbb{Z}_2)$ or $G \cong PG(\mathbb{Z}_3)$.*
(5) *G is a connected graph.*

Proof Suppose that $PG(\mathbb{Z}_n)$ is a regular graph. So if $d \neq 1$ is a positive divisor of n, then $\varphi(d) = \varphi(n)$. Assume that p and q are prime divisors of n. Then $\varphi(p) = \varphi(q)$ implies that $p = q$ and thus n is a prime number (note that $\varphi(p^2) \neq \varphi(p)$). By Corollary 3.2, the proof is complete. □

Let $m, n > 1$ be positive integers. We know that $PG(\mathbb{Z}_m \times \mathbb{Z}_n) = PG(\mathbb{Z}_m) \boxtimes PG(\mathbb{Z}_n)$ and so $\omega(PG(\mathbb{Z}_m \times \mathbb{Z}_n)) = \varphi(m)\varphi(n)$. Then $\omega(PG(\mathbb{Z}_m \times \mathbb{Z}_n)) \leq \omega(PG(\mathbb{Z}_{mn}))$.

We close this article by considering some relations between $PG(R)$ and $M - PG(R)$, where R is a ring and M is a non-zero R-module.

Theorem 3.12 *Let R be a ring and M be a non-zero R-module. Then $PG(R/ann(M))$ is a subgraph of $M - PG(R)$. Moreover, $|ann(M)|\omega(PG(R/ann(M))) \leq \omega(M\text{-}PG(R))$.*

Proof Let $\overline{R} = R/ann(M)$ and suppose that $\overline{x}$ and $\overline{y}$ are two adjacent vertices of $PG(\overline{R})$. Now, assume that $m \in M$. Since $\overline{x}\overline{R} = \overline{y}\overline{R}$, there exists $r \in R$ such that $\overline{x} = \overline{yr}$. Hence $(x - yr)m = 0$ which implies that $xM \subseteq yM$. Similarly, $yM \subseteq xM$. So x and y are two adjacent vertices of $M - PG(R)$. Thus by assigning $\overline{x}$ to x, $PG(\overline{R})$ is a subgraph of $M - PG(R)$. By Remark 2.5, the last part is clear. □

Theorem 3.13 *Let R be a ring and M be a non-zero R-module. Then $PG(R)$ is a spanning subgraph of $M - PG(R)$. Furthermore, if M is a faithful cyclic R-module, then the graph $PG(R)$ is exactly the same as the graph $M - PG(R)$.*

Proof Suppose that x and y are two adjacent vertices of $PG(R)$. Clearly, $xR = yR$ implies that $xM = yM$. So x and y are two adjacent vertices of $M - PG(R)$. Thus $PG(R)$ is a spanning subgraph of $M - PG(R)$. Now, assume that $ann(M) = 0$ and $M = Rm$ for some $m \in M$. If x and y are two adjacent vertices of $M - PG(R)$, then $xM = yM$ and so there exists $r \in R$ such that $xm = yrm$. Hence $x - yr \in ann(M) = 0$ which implies that $xR \subseteq yR$. Similarly, $yR \subseteq xR$. Therefore x and y are two adjacent vertices of $PG(R)$. So $PG(R)$ is exactly the same as $M - PG(R)$. □

Theorem 3.14 *Let R be a ring. If $M = \prod_{i \in I} R_i$, where $R_i \cong R$ for each $i \in I$, then $PG(R)$ is exactly the same as $M - PG(R)$.*

Proof Suppose that $x, y \in R\backslash\{0\}$. By the previous theorem, $xR = yR$ implies that $x\prod_{i \in I} R_i = y\prod_{i \in I} R_i$. Now, assume that $x\prod_{i \in I} R_i = y\prod_{i \in I} R_i$. Let $e = (e_i)$, where $e_1 = 1$ and $e_i = 0$, for each $i \neq 1$. So there exists $r = (r_i) \in \prod_{i \in I} R_i$ such that $xe = y(r_i)$ and hence $x = yr_1$. Thus $xR \subseteq yR$. Similarly, $yR \subseteq xR$ and so $xR = yR$. □

Corollary 3.15 *Let R be a ring and F be a free R-module. Then $PG(R)$ is exactly the same as $F - PG(R)$.*

Proof Since F is isomorphic to a direct sum of copies of R, the result holds by the previous theorem. □

Acknowledgments The authors would like to express their deep gratitude to the referee for careful reading of the paper and useful comments.

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