Journal of Philosophical Logic (2022) 51:1215–1231
https://doi.org/10.1007/s10992-022-09693-1

EDITORIAL

Editorial Introduction: Substructural Logics and Metainferences

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Accepted: 22 November 2022 / Published online: 4 January 2023

Abstract

The concept of *substructural logic* was originally introduced in relation to limitations of Gentzen's structural rules of Contraction, Weakening and Exchange. Recent years have witnessed the development of substructural logics also challenging the Tarskian properties of Reflexivity and Transitivity of logical consequence. In this introduction we explain this recent development and two aspects in which it leads to a reassessment of the bounds of classical logic. On the one hand, standard ways of defining the notion of logical consequence in classical logic naturally induce substructural logics when admitting more than two truth values; on the other hand, these substructural logics give rise to hierarchies of *metainferences* that can be used to approximate classical logic at different levels.

1 Varieties of Substructural Logics

In the 1930s, Tarski and Gentzen isolated some abstract features that logical consequence relations ought to have, whether treated set-theoretically as in Tarski's approach, or proof-theoretically as in Gentzen's approach.

For Tarski [66], the defining characteristics of a logical consequence operation were *Reflexivity* (any sentence must be a consequence of itself; Axiom 2 in his paper), *Transitivity* (the consequences of the consequences of a set of sentences must equal the consequences of that set; Axiom 3), and *Finitariness* (aka. *Compactness*, namely for a sentence to follow from a set of premises, is to follow from a finite set thereof; Axiom 4). From his formulation of the latter, Tarski immediately derived another

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constraint, the *Monotonicity* of logical consequence (the fact that adding premises to a valid argument preserves its validity, see [66, Theorem 1a]).

Similarly, Gentzen in [27] highlighted basic properties of the way in which premises and conclusions of an argument ought to be combined, which he called *structural* properties, to distinguish them from *operational* properties governing the behavior of logical connectives. Among the former, he included *Weakening* of premises and conclusions (thereby generalizing Monotonicity, adding conclusions too would preserve validity), *Contraction* of premises and conclusions (repetition in premises and conclusions does not matter), *Exchange* of premises and conclusions (they can be freely permuted), and the *Cut* rule, relating directly to Transitivity of logical consequence (when a formula appears both as a conclusion and as a premise of two valid arguments, it can be suppressed and the premises and conclusions joined to form a valid argument).

Since the founding works of Tarski and Gentzen, a variety of nonclassical logics have emerged that have called these properties into question. One strand of such logics is linked to the limitation of the very properties marked as structural by Gentzen: famously, Linear logics reject Contraction [28], variants of the Lambek calculus reject Exchange [33, 40], and Relevant logics give up the Weakening rule [2, 48]. In seminal texts in which the term "substructural" makes its first appearance [20, 45, 59], the very notion of a substructural logic is thus tied to the drop of Contraction, Exchange, Weakening, and sometimes of related properties such as Associativity [34, 41].

However, the Tarskian properties of Reflexivity and Transitivity have generally been perceived as constitutive of the notion of logical consequence, and indeed they stay put in most substructural systems free of Contraction, Exchange, or Weakening [46, 48, 52]. In particular, they hold in Gentzen's calculus **LJ** for intuitionistic logic, which may itself be seen as a system of substructural logic (compared to his system **LK** for classical logic), since by allowing only single formulae in succedent position of a sequent, it forbids Contraction or Weakening on the right. As Gentzen proved, Cut is eliminable in those systems, but Cut is thereby an admissible rule. At first blush systems in which Cut it not even admissible appear problematic.

From the 1950s onwards, however, some attention was gradually given to logics that reject the Tarskian properties of Transitivity or Reflexivity, though earlier precursors can be identified in either case (see Paoli's discussion of nontransitivity in [48, p. 18], which traces the idea to Bolzano, and French's historical discussion of nonreflexivity in [25], who cites Swyneshead as a medieval forerunner). In the case of nontransitivity, one important source can be found in the work of J.Y. Girard precisely on the semantics of the Cut-free sequent calculus. As Girard put it [29, p. 161]:

> "the semantics of the Cut-free sequent calculus is necessarily of a different nature [from that of the full sequent calculus]: we need a concept of model in which the validity of $\vdash A$ and $A \vdash B$ *does not* entail the validity of $\vdash B$"

Girard used a three-valued semantics for that purpose, affording a notion of validity that permits to block Cut (more on it below). A separate source of inspiration concerning the rejection of Reflexivity and Transitivity comes from the Polish tradition and from a problem raised by R. Suszko regarding the adequate number of truth-values needed to represent any given non-classical logic [64, 65]. Suszko argued that a logic satisfying the Tarskian properties of Reflexivity, Transitivity, and Monotonicity is essentially two-valued. Malinowski later proved that a logic that is monotonic and transitive but that lacks reflexivity is essentially three-valued [36]. In the wake of Malinowski's work, Frankowski showed that the same holds if Reflexivity is retained but Transitivity dropped [24].

Independently of these results, philosophical work in logic has been produced arguing that long-standing paradoxes of vagueness and of truth can be solved by an appeal to substructural logics (see the recent special issue of *Synthese* on substructural approaches to paradox, [76]). Several authors have proposed that the key to solving semantic paradoxes (such as the Liar, or the Curry, or the sorites paradox), is by giving up some of the structural rules, including Transitivity [12, 55, 56, 69, 71, 74], Reflexivity [25, 44], Weakening [15, 16, 39], or Contraction [38, 61, 75]. Figure 1 gives an illustration of these various possible strategies by considering the following example, borrowed from [25], of a derivation of the Liar paradox in classical logic (the sentence λ is the Liar sentence, saying of itself that it is not true, $\neg T\langle\lambda\rangle$; the rules T-R and T-L express the intersubstitivity of ϕ with $T\langle\phi\rangle$). As the derivation shows, structural Reflexivity, Contraction, Cut, and then Weakening [16], are involved at some crucial steps, and each of those rules may therefore be challenged to block the derivation.

More generally, the last decade has witnessed an upsurge of philosophical interest in substructural logics, both from a theoretical and a more applied perspective. This special issue is a reflection of this development. In comparison to extant contributions on substructural logics, the focus of this special issue is twofold. First of all, the papers collected in this issue are all connected, whether directly or indirectly, to certain nontransitive and nonreflexive systems that can be obtained by shifting standards of truth for premises and conclusions. As will be explained shortly, these systems can be obtained by considering natural generalizations of Tarski's notion of consequence. But they also raise the problem of logical pluralism, by making very vivid the question of what makes the choice of a specific system better justified than another.

The second main emphasis of this special issue concerns the notion of *metainference*. When looking at the rules that Gentzen called structural, we can see that they correspond to relations between sequents. Whereas sequents encode *inferences* (aka.

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}{
\dfrac{\Rightarrow}{\Gamma \Rightarrow \Delta}\ \textbf{Weaken-L, -R}
}\ \textbf{Cut}
$$

Fig. 1 A sequent-calculus derivation of the Liar paradox, with structural rules boldfaced arguments), namely relations between formulae, rules such as Contraction, Weakening, Exchange, or Cut encode relations between sequents, hence *metainferences*. But as highlighted by [6], there is no reason to stop there: we can also consider relations between metainferences, relations between such relations, and so forth [6, 58]. This means that in the same way in which Contraction, Exchange, and Cut, reflect structural relations between sequents, one can consider higher-level structural rules of Meta-Contraction, Meta-Exchange, Meta-Cut, and so forth.

About half of the papers in this special issue are concerned with this hierarchy of metainferences, and the way in which the very notion of a structural rule can be lifted and articulated at higher levels. The other half is more specifically concerned with the implications of adopting a nonreflexive or a nontransitive logic. Both of these perspectives interact, however. In order to say more about those, and before giving an overview of the content of this issue, in what follows we first introduce more background regarding the definition of those systems and the notion of metainference. We start with a quick overview of the way in which nonreflexive and nontransitive logics arise from the adoption of a generalization of the classical definition of logical consequence in terms of mixed standards.

2 Logical Consequence and the Tarskian Properties

The semantic concept of logical consequence is commonly presented as the preservation of truth from premises to conclusion in an argument.[1] Based on reflections made by Carnap, Tarski in his classic paper on logical consequence introduced the concept slightly differently, in terms of the exclusion of falsity coming from the truth of the premises [67, p. 414]:

> "From an intuitive point of view, it can never happen that both the class K consists only of true sentences and the sentence X is false."

Tarski assumed bivalence, however, and True and False to be mutually exclusive and exhaustive notions. Thus, his definition can be used to justify the structural properties of Reflexivity, Monotonicity, and Transitivity of logical consequence. For instance, Reflexivity is justified by observing that the truth of a sentence rules out its being false. Monotonicity results from the fact that if B is not false in any model in which A is true, then it is not false in any model in which A and C are both true. Transitivity can be similarly justified: if the truth of A rules out the falsity of B, and the truth of B rules out the falsity of C, A cannot be true and C be false, because B would have to be true (in virtue of A being true, and the equivalence of B's non-falsity with its being true) and false (in virtue of C being false).

In two-valued logic, truth and non-falsity are coextensional and it makes no difference, therefore, to define an argument as valid provided [11, 14, 54, 72]:

[1] See for instance Wittgenstein's definition of logical consequence in the *Tractatus* [73, 5.11-5.121]. In 5.121 Wittgenstein writes:

> "The truth-grounds of q are contained in those of p: p follows from q".

- the *truth* of the premises entails the *truth* of the conclusion. [ss-validity]
- the *nonfalsity* of the premises entails the *nonfalsity* of the conclusion. [tt-validity]
- the *truth* of the premises entails the *nonfalsity* of the conclusion. [st-validity]
- the *nonfalsity* of the premises entails the *truth* of the conclusion. [ts-validity]

When more truth values are admitted between True and False, however, the previous definitions are no longer equivalent, and Tarski's informal definition of logical consequence can no longer straightforwardly justify Transitivity (see below). This is the case in three-valued semantics, where a sentence may be called *strictly* true when it takes the value True (=1), and *tolerantly* true when it takes a value other than False (=1 or 1/2) [12, 14]. Based on these definitions, we label ss, tt, st and ts the corresponding definitions of logical consequence (other names have been used, see below, but these labels afford a unified picture, see [11]).[2]

The difference between ss-consequence and tt-consequence can be seen in the difference between the logics **K3** (Strong Kleene logic) and **LP** (Priest-Asenjo's Logic of Paradox). Both systems involve the same set of connectives with the same truth tables (the Strong Kleene tables), but in **K3** logical validity is defined as the preservation of strict truth (namely of the designated value 1), whereas in **LP** logical validity is defined as the preservation of tolerant truth (namely of the designated values 1 and 1/2, see [50, 51]). As a result, the logics **K3** and **LP** validate different inferences, for instance **K3** does not have tautologies, and **LP** does not support Modus Ponens. Despite that, they both preserve the Tarskian properties of Reflexivity, Transitivity, and Monotonicity. The basic reason, highlighted by Suszko [65], is that both logics rest on a bipartition of the set of truth values into designated values, and anti-designated values, playing the same role as the values True and False in the two-valued case.

The situation is different when entailment is defined using distinct standards for premises and for conclusions. The definition of consequence in terms of the entailment of non-falsity from truth (st-entailment) is precisely the one adopted by Girard and by Frankowski to invalidate the Cut rule (st-consequence is another name for what Frankowski calls P-consequence, or plausible consequence, [24]). Consider a sentence B taking the value 1/2 in all models, a sentence A taking the value 1 in all models, and a sentence C taking the value 0 in all models. Then the truth of A entails the non-falsity of B; the truth of B vacuously entails the non-falsity C, but the truth of A does not entail the non-falsity of C. So Transitivity is lost with st-entailment. Conversely, ts-entailment invalidates Reflexivity of logical consequence, since the same B sentence fails to ts-entail itself. In fact, ts is another name for the notion of Q-consequence (or Quasi-consequence) put forward by Malinowski to establish the existence of an essentially three-valued logical system that abandons one of the Tarskian constraints [36].

The idea of using mixed standards of truth to define logical consequence was proposed independently by several logicians in the past decades. For instance, it makes a

[2] We write "ss", "tt", "st" and "ts" in small letters to refer to the definitions of entailment; we will use boldface and write "**SS**", "**TT**", "**ST**", "**TS**", to refer to the resulting logics in (three-valued) languages of a specific signature.

brief appearance in a paper by Belnap on trivalent conditional logics, in which Belnap states a definition of "implication$_1$", which he notes to correspond to nontransitive entailment, and which is a counterpart to the notion of st-entailment [7, p. 55].[3] The idea has also appeared since in other areas of many-valued logics, including in partial logics [43], in supervaluationist logics [8], in fuzzy logic [62], and more generally in logics of vagueness (see [12, 74]), where the idea is to solve the sorites paradox by observing that each step of a sorites argument can be safely assumed to be valid, but that the strength of the conclusion may decrease compared to the strength of the premises. Possibly, one of its oldest incarnations can be found in Strawson's definition of entailment for presuppositional sentences in [63, p. 175–177] (to which Belnap himself refers), which can be articulated as follows: Γ *Strawson-entails* A provided the truth of the premises in Γ, together with the truth of their presuppositions and the presuppositions of A, classically entails the truth of A (see [22, 60]).[4] This is a weakening of the classic notion of logical consequence, and it is nontransitive too (see [11] on the link between Strawson-entailment and st-entailment). For instance, "John loves his siblings" Strawson-entails "John loves his brother" (because the former presupposes that John has siblings, and the second that he has a brother, and the premise, with those two presuppositions, entails the conclusion). And "John loves his brother" Strawson-entails "John has a brother" (since it is presupposed). But "John loves his siblings" does not Strawson-entail "John has a brother" (John may have only sisters).

The first take-home message of this introduction is therefore that nonreflexive and nontransitive features of a logic emerge quite naturally from the definition of logical consequence standardly admitted for classical logic. This feature is surprising at first, since as already mentioned, the various definitions of logical consequence given above collapse in a two-valued setting. But they all instantiate the same intensional concept: they basically tell us that when accepting premises (for some standard of truth), one cannot reject some conclusions (for some possibly different standard). This view of logical consequence, incidentally, agrees with *bilateralist* conceptions of logic and inference (see [53, 55, 57]), which view acceptance and rejection as basic attitudes that constrain inferential and metainferential norms.

So far we have not said much about Monotonicity, but Monotonicity is another Tarskian property that was famously challenged, notably in relation to conditional reasoning and in the field of belief revision (see [1, 35]). The basic objection to Monotonicity is the idea that incorporating new information to an argument can undercut the connection between premises and conclusions. This shows in inductive arguments: "Tweety flies" is a plausible consequence of "Tweety is a bird", but it is not a plausible consequence of "Tweety is a bird" combined with "Tweety is an ostrich". Formally, there are different ways to expand the classical definition of

[3] In that same paper, Belnap also considers the equivalent of ss-entailment (which he calls t-implication), of tt-entailment (which he calls nf-implication), and of their intersection (which he calls tnf-implication).

[4] See also [37] for a treatment of presupposition using the framework of *bi-matrices* as a way of teasing apart the Tarskian constraints.

logical consequence to make it non-monotonic. One of them is to fix an order on valuations, and to say that in order for A to imply B, B must be true in the minimal valuations satisfying A. On that definition it can happen that $A \models C$, but $A, B \not\models C$, in case the minimal valuations satisfying A and B are not the minimal valuations satisfying A. Other templates have been used to characterize nonmonotonic consequence, we refer to [35] for an overview.

As it turns out, the property of Monotonicity admits a semantic characterization in terms of the notion of mixed consequence. It is easy to see that any mixed consequence relation has to be monotonic (adding premises and adding conclusions will not break the connection between given subsets thereof). In [10], it is shown that a logic is monotonic if and only if it has a truth-adequate intersective mixed semantics (Theorem 3.2'). Basically, this means that a logic is monotonic exactly if it can be represented in many-valued logic by an intersection of mixed consequence relations (see [9, 26, 31, 37] for closely related results). A basic example of an intersective mixed consequence relation that is not reducible to a mixed consequence relation is the relation $ss \cap tt$, which can be characterized in order-theoretic terms (A entails B if and only if every standard making A true makes B true, see [11, 21], and Cook's contribution in this issue).

Another aspect in which nonmonotonicity connects to problems raised in this special issue is the following: Makinson describes typical non-monotonic logics as *supraclassical* [35, pg. 10–12]. What Makinson means by that distinction is that at the *inferential* level, generally when A classically entails B, it is the case that A nonmonotonically entails B, a feature that he contrasts with other non-classical logics, such as intuitionistic logics (which is inferentially non-classical, for example by giving up the law of excluded middle).[5] But nonmonotonic logics depart from classical logic in their metainferential properties.

Similarly, nontransitive logics of truth or vagueness such as the systems **STT** and **STV** put forth in [14, 55] have also been described as supraclassical (viz. [30]), in the sense that they preserve all classical inferences and validate more inferences not admitted by classical logic (the Tarski biconditionals for truth, or the tolerance principle behind the sorites paradox). That is, the systems **STT** (for truth) and **STV** (for vagueness) are proved to be conservative extensions of classical logic (see [14, 54]). However, a system of logic can only be supraclassical at the inferential level if it is infraclassical at the metainferential level (it gives up some classical metainferences). This distinction between levels features quite centrally in several of the papers in this special issue, in particular to adjudicate the extent to which a logic can be considered to depart from classical logic. To present this issue more thoroughly, in the next section we first need to say more about the way in which classical logic can be characterized in terms of metainferences.

[5]Not all nonmonotonic consequence relations need be supraclassical, however. For example, it is possible to define entailment in such a way that $p \models p$, but $p, \neg p \not\models p$, by requiring that in order for Γ to entail B, Γ must classically entail B, and the premises of Γ be jointly satisfiable. Bolzano's definition of deducibility incorporates a compatibility constraint of this kind, see [49] for a recent presentation, with a comparison to Tennant's logic CR, [70].

3 Metainferences and the ST Hierarchy

While inferences encode logical relations between *formulas*, metainferences register relations between inferences themselves (i.e. *sequents*). However, this difference between metainferences and inferences can be transposed a level down onto a difference between inferences and formulae. That is, structural features of entailments, where no specific mention of any logical connective is made in the object-language, can be translated in terms of features of the corresponding conditional in an appropriate logic for the metalanguage.

For example, the failure of the structural property of Reflexivity:

$$\overline{A \Rightarrow A} \tag{Ref}$$

can be translated into a failure of Identity of the conditional in a corresponding logic:

$$\Rightarrow A \to A \tag{Id}$$

And the failure of the structural property of Transitivity

$$\frac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C} \tag{Transitivity}$$

can be internalized in a logic with a conditional that does not validate Hypothetical Syllogism:

$$A \to B, B \to C \Rightarrow A \to C \tag{HS}$$

The correspondence between $\to$ and $\Rightarrow$ is neatly expressed in the condition known as the Deduction Theorem, which connects the derivability of conditional formulae to inferential relations between antecedent and consequent:

$$\Gamma, A \Rightarrow B, \Delta \text{ iff } \Gamma \Rightarrow A \to B, \Delta \tag{DT}$$

In one direction DT states that if A entails B then the conditional "if A then B" holds. In the other direction it states that if the conditional "if A then B" holds, then A entails B. Under certain conditions, DT allows internalizing the links between the formulas and the inferences that a logic recommends accepting. In a nutshell, modifications in the structure of inferences or in some properties between inferences affect the derivability of formulas whose main connective is a conditional.

It is well known that the logics **LP** and **K3** fail one of the directions of the DT when the conditional is defined in the standard way using Strong Kleene negation and conjunction ($\neg(A \wedge \neg B)$) [14]. Thus, although $A \Rightarrow A$ is **K3**-valid, $\Rightarrow A \to A$ is not, and although $\Rightarrow A \to ((A \to B) \to B)$ is **LP**-valid, $A, A \to B \Rightarrow B$ is not. Hence, these logics break the relationship between valid conditional formulas and valid inferences. This imbalance is repaired in mixed standards logics such as **ST** and **TS**: both preserve the Deduction Theorem for the Strong Kleene conditional. This feature was originally used as a criterion to select **ST**, in particular, relative to other mixed consequence relations (see [12]), and it is moreover linked to the fact that **ST** and **TS** are self-dual consequence relations at the inferential level for negation (that is, $\Gamma \models \Delta$ iff $\neg(\Delta) \models \neg(\Gamma)$). Moreover, [10] show that the Strong Kleene conditional is the only three-valued conditional that satisfies the Deduction Theorem and the bidirectional Gentzen rules of **LK** when the formula $A \rightarrow B$ appears in premise position (that is, $\Gamma, A \rightarrow B \Rightarrow \Delta$ iff $\Gamma, B \Rightarrow \Delta$ and $\Gamma \Rightarrow A, \Delta$).

Despite that, although the mixed logics **ST** and **TS** maintain a systematic link between inferences and conditional formulae, they allow divergences between inferences and metainferences: the former do not internalize the latter. In particular, the logic **ST** validates the law known as Pseudo Modus Ponens,

$$\Rightarrow (A \wedge (A \rightarrow B)) \rightarrow B \qquad \text{(PMP)}$$

and also validates the inference Modus Ponens,

$$A, A \rightarrow B \Rightarrow B \qquad \text{(MP)}$$

but it invalidates the metainference Meta Modus Ponens:

$$\frac{\Rightarrow A \qquad \Rightarrow A \rightarrow B}{\Rightarrow B} \qquad \text{(MMP)}$$

This implies that structural properties of the entailment of **ST** cannot be internalized by the material conditional.

There is, however, a systematic correspondence between the metainferential properties of **ST** and the inferential properties of **LP**. In [5] Barrio, Rosenblatt and Tajer show that the set of valid metainferences of **ST** is *modulo* translation coextensive with the set of valid inferences of **LP**. Further developments partly rely on this result to argue that the logic **ST** is in relevant respects similar to **LP**, or even, that it may be seen as a disguised version of **LP**, as argued by [19]. In particular, giving up Cut in **ST** results in the loss of other metainferences, closely connected to Modus Ponens and Explosion in **LP** (Meta-Modus Ponens and Meta-Explosion). In a way, this means that **ST** is substructurally paraconsistent [3, 18].

Furthermore, these divergences can be extended to metainferences of any level. That is, a logic can validate DT, but invalidate Meta-DT at some further level. Likewise, a logic can preserve Meta-Meta-Cut, but invalidate Meta-Meta-Meta-Cut. In both cases, the failure of a metainference reflects an element of substructurality. By generalizing upon these observations, Barrio, Pailos and Szmuc in [6] introduced a hierarchy based on **ST**, in which each level preserves more of classical logic than **ST** itself, or indeed than the previous level (for more on the hierarchy and its generalizations, see [23, 47, 58]). They prove the following result, which constitutes the backdrop to several of the papers included in this special issue:

> **Hierarchy theorem** ([6])**:** For every natural number $n \geq 1$, there is a logic which agrees with classical logic on inferences, meta-inferences, ..., meta^n-inferences, but that disagrees with classical logic on meta^{n+1}-inferences.

For example, for the next level up **ST**, a metainference between inferences of the first level is valid in the metainferential logic **TS**/**ST** if and only if:

- If for the inferences that are the premises of the metainference, the *nonfalsity* of their premises entails the *truth* of their conclusions ($\Gamma_1 \Rightarrow_{TS} \Delta_1, ..., \Gamma_n \Rightarrow_{TS} \Delta_n$),

- then for the inferences that are the conclusions of the metainference, the *truth* of their premises entails the *nonfalsity* of their conclusions ($\Sigma_1 \Rightarrow_{ST} \Pi_1, ..., \Sigma_m \Rightarrow_{ST} \Pi_m$).

Now, **TS**/**ST** comes closer to classical logic than **ST**, since it satisfies PMP, MP, and Cut; however it is nonclassical, since it fails to satisfy Meta-Cut. To retrieve the latter, the next level is the logic (**ST**/**TS**)/(**TS**/**ST**), and so forth.

A central problem raised in particular by Scambler in [58], and picked up by several papers in this special issue, concerns whether classical logic can be identified as the system $\mathbf{ST}^{\omega}$ which is the limit of that sequence of mixed logics **ST**, **TS**/**ST**, etc (see also [47]). In the same way in which a hierarchy can be built from **ST**, a symmetric hierarchy can be built from **TS**. This **TS** hierarchy includes **TS**, **ST**/**TS**, etc. Scambler proves a result symmetric to the previous one, namely that for every natural number n, there is a logic in this **TS** hierarchy which has the same anti-valid meta^n-inferences as classical logic, but which fails a classical anti-valid meta^{n+1}-inference. The limit of that sequence of logics, $\mathbf{TS}^{\omega}$, coincides with classical logic on its anti-validities at all levels, but not on its validities. These results leave us with a puzzle, namely whether classical logic can be characterized in terms of a metainferential hierarchy.

Another important issue behind these results concerns the proper definition of validity for metainferences. It is important to note that all the results we mentioned are supported by the so-called *local* definition of metainferential validity. A metainference is *locally valid* if and only if for every model, either some conclusion inference is satisfied or some premise inference is not. A metainference is *globally valid* if and only if either some conclusion inference is valid or some premise inference is invalid. Local validity can be thought of as the preservation of satisfaction, while global validity can be thought of as the preservation of validity (see [19] for the distinction between the two definitions). For reasons explained in [6] and [4], the logics of metainferences are standardly presented utilizing the local definition and not the global notion. One reason is that when metainferences are defined with single conclusions, local validity is stronger than global validity (although with multiple conclusions this does not hold, see [17, 23]). In some conditions, however, both notions can collapse (see [68] for some results, and Da Ré et al.'s paper, this issue).

4 Overview of the Issue

This special issue includes thirteen papers, which address several of the problems we presented above.[6] Two main groups can be distinguished that echo the main topics we distinguished. One group of papers is concerned with the nature of metainferences

[6] A fourteenth paper entitled "Non-reflexivity and Revenge" and intended for this special issue, by **Murzi and Rossi**, was inadvertently published in a regular issue of the JPL [42]. This paper raises some objections against **TS**-like treatments of the semantic paradoxes, and would normally have been inserted right before Zardini's paper in this special issue.

and with the **ST** hierarchy. Another group of papers is concerned with the justification of mixed consequence relations, and with the relation between logical consequence and substructural logics.

Among the papers dealing with the strict-tolerant hierarchy, the papers by Ripley, by Ferguson and Ramírez-Cámara, by Golan, and by Fjellstad, are concerned specifically with the special position of **ST** in the metainferential hierarchy, and by the extent to which the "limit" logic $\mathbf{ST}^{\omega}$ coincides with classical logic. The papers by McAllister, Pailos, and Porter, deal with further issues regarding what makes the identity of a logic. The pieces by Golan, Fjellstad, and by Cobreros, La Rosa and Tranchini, deal specifically with the proof theory of metainferential logics.

In the second main group of papers, the papers by French and by Cook are concerned with the proof-theoretic and semantic foundations of three-valued and four-valued systems such as **FDE**, **LP**, **K3**, **ST**, **TS** or **RM3**. The papers by Da Ré, Szmuc and Teijeiro, and then by Incurvati and Schlöder, are concerned with competing definitions of validity for metainferences. Finally, Zardini's article presents a criticism of the **ST** treatment of truth, giving reasons to prefer noncontractive approaches.

4.1 ST and the ST Hierarchy

In "One step is enough", **Ripley** deals with the question of whether an advocate of **ST**, whose reasons rest partly on the fact that **ST** preserves more classical inferences than other nonclassical logics in its vicinity, ought thereby to endorse the stronger logics introduced by [6], or even $\mathbf{ST}^{\omega}$, whose validities coincide with those of classical logic at all levels. Ripley rejects that view and argues that there are no compelling reasons to go beyond **ST** in the hierarchy of meta-inferential logics. Ripley submits that the first level is enough and that it strikes the right balance between desirable classical features and the nonclassicality originally called for by applications to paradoxes of vagueness and truth.

In "Deep ST", **Ferguson and Ramírez-Cámara** deal with some limitations that they observe in the metainferential hierarchies put forward by Barrio, Pailos and Szmuc. One is that metainferences of level $n+1$ only accept metainferences of level n as operands. However, they argue that more flexibility between levels can be desirable and that a metainference can relate inferences of different levels. Another main limitation concerns the lack of a uniform tractable semantics for the evaluation of metainferences in the hierarchy. Their paper deals with both problems by establishing the logic **LPTT** (Priest's logic of transparent truth) as a basic semantic framework for all levels (compare with [19]). They use it to discuss whether the **ST**-theorist can abide by **ST** principles in the metatheory of **ST**, and they argue that it is possible.

Golan's paper "Metainferences from a Proof-Theoretic Perspective, and a Hierarchy of Validity Predicates" explores, from a proof-theoretic perspective, the **ST** hierarchy of classical logics introduced by [6]. Golan provides sound and complete sequent calculi for all levels, based on a generalization of the standard structural rules of the first level, and then goes on to investigate a hierarchy of validity predicates for these logics. According to Golan, since the logics in the hierarchy differ from one another on the rules, each such logic maintains its own distinct identity. At the first level, this implies in particular that **ST** should not be identified with either Classical Logic or with **LP**.

In the same spirit, **Fjellstad** in "Metainferential Reasoning on Strong Kleene Models" argues that all levels in the **ST** hierarchy are non-classical to the same extent, basically because although higher levels can recover forms of transitivity given up at the previous level, they inherit the dialetheist character of **STT** (which both accepts and rejects the Liar). This view goes against the view originally defended by Barrio et al. that the **ST** hierarchy contains closer and closer approximations to classical logic. Fjellstad furthermore argues that $\mathbf{ST}^{\omega}$ is not classical logic, but in fact just the original non-transitive logic **ST** in disguise, and draws further lessons about whether **ST** can be used as its own metatheory.

4.2 Identity Criteria for Logics

The next three papers are concerned with the results of Barrio et al. and Scambler regarding the problematic characterization of classical logic in terms of the **ST** and **TS** hierarchies.

Barrio et al.'s hierarchy theorem means that classical logic cannot be uniquely characterized at any finite level $\mathbf{ST}^{n}$ of the **ST** hierarchy. But the first transfinite logic $\mathbf{ST}^{\omega}$ recaptures all classical finitary meta-inferences. In "Classical Logic is not Uniquely Characterizable", **McAllister** extends the non-uniqueness results of Barrio et al. to transfinite levels, and submits that it is not possible to uniquely characterize classical logic when working within classical set theory. She argues that in order to characterize classical logic, one would need to consider ordinal-many levels (a view incidentally rejected by Ferguson and Ramírez-Cámara in the aforementioned piece), but in a way that is incompatible with a standard set-theoretic characterization of classical logic. McAllister shows that this negative result can be bypassed to some extent in a paraconsistent set theory, but in a way that views classical validity as an "indefinitely extensible concept".

In "Supervaluations and the Strict-Tolerant Hierarchy", using a super/sub-valuationist setting, **Porter** shows how to construct a logic that has exactly the validities and antivalidities of classical logic at every inferential level, but which still falls short of coinciding with classical logic. Porter uses this result to propose a stronger identity criterion for two logics, namely to have the same sets of jointly satisfiable inferences. Similarly, in "Empty Logics", **Pailos** too argues that validities and antivalidities are not enough to characterize a logic. Pailos's focus in his paper concerns a hierarchy of non-reflexive logics starting with **TS**, which is empty of inferences (but not of metainferences). This hierarchy has no valid metainferences, but contains antivalid inferences. However, in order to create a truly empty logic, Pailos argues that not just validities and antivalidities ought to be considered, but also contingencies (inferences that are neither valid, nor antivalid). Thus, what a logic accepts and what a logic rejects might not be the only dimensions to pay attention to in discussing the nature of meta-inferential hierarchies.

4.3 Proof Theory

Another focus of this issue is how to analyze the proof-theoretical nature of the different hierarchies involving metainferences. Different approaches are included here.

On the one hand, Golan shows that the metainferential behavior of each logic in the **ST** hierarchy is completely determined by higher level structural rules. So, according to Golan, there is no need to provide operational rules over and above the regular ones, i.e., those of level 1 that govern the sequents. A different approach is adopted by Fjellstad, as well as by **Cobreros, La Rosa and Tranchini** in "Higher-level Inferences in the Strong-Kleene Setting". Both present two labelled sequent calculi based on Girard's technique of labelled sequents, with *s* ant *t* serving as basing labels to mark strict vs. tolerant truth. The systems for the hierarchy are called **HST** and $\mathbf{G3SK}^{\omega}$ respectively. Both papers furthermore present "lowering" results, connecting metainferences of level $n+1$ to metainferences of level n in the style of [19].

A fourth paper that can be added in the group of papers dealing specifically with proof theory is **French**'s article "Metasequents and Tetravaluations", though it belongs as much to the next group of papers concerned with semantic foundations of logical consequence. Unlike the previous papers, French's paper does not deal with the extended **ST** hierarchy, but with the proof-theoretic connection between monotonic nonreflexive or nontransitive logics like **TS** and **ST** and the Tarskian logics **K3** and **LP**. French proposes a metasequent calculus, in which metasequents are operands that stand to sequents as sequents stand to formulae in ordinary sequent calculi. He formulates structural and operational rules for metasequents (including two kinds of structural rules, called outer vs. inner depending on whether they concern metasequents or sequents), and then he shows that such calculi are sound and complete for tetravaluations. Tetravaluations, following work by [31], provide a canonical framework for the semantic representation of monotonic logics, whether reflexive, transitive, both, or neither (see [9, 10, 26]). French's main result in his paper establishes a systematic correspondence between metasequents in (just) monotonic, monotonic reflexive, and monotonic transitive logics on the one hand, and sequents in **FDE**, **LP**, and **K3** on the other.

4.4 Validity and Consequence

Along with the previous papers, this issue features papers that are not concerned with the **ST** hierarchy, but with foundations of logical consequence in relation to substructural logics.

In this direction, **Cook** in "MTV Logics" introduces a novel framework for studying many-valued logics. The main effect of his Movable Truth Value approach is to unify a large number of many-valued logics under an order-theoretic definition of logical consequence (instead of taking mixed consequence and their intersection as basic, compare with [11]). Given a simple ordering on truth values in a many-valued logic, the MTV approach defines an ordering on pairs of truth values with an index representing whether they encode the position of a sentence in premise position or in conclusion position in an inference. The approach allows Cook both to redefine well-known logics but also to chart a number of novel many-valued logics with interesting substructural properties (including failures of Weakening).

In "Derivability and Metainferential Validity", **Da Ré, Szmuc, and Teijeiro** address the issue of what is the semantic counterpart of the proof-theoretic notion of derivability in the context of nontransitive and nonreflexive logics. They show that derivability and local validity don't coincide in general. Then they provide sufficient conditions under which local and global validity can be expected to coincide with derivability. They also investigate a definition of validity pioneered by Humberstone in [32]: *absolute global validity*, and they prove that this is the proper counterpart of derivability in full generality. Finally, they analyze the consequences of these developments on some nontransitive and nonreflexive systems, such as **ST** and **TS**.

In "Meta-inferences and Supervaluationism", **Incurvati and Schlöder** deal with the supervaluationist logic of vagueness and with some failures of classical metainferences in it (such as Conditional Introduction, Proof by Cases, or Contraposition) when the language incorporates a Definiteness operator. These failures are connected to the definition of logical consequence in supervaluationism, which is standardly defined as "global validity" (preservation of supertruth from premises to conclusion in an argument) rather than in terms of "local validity" (preservation of truth at a world; note that this global/local distinction in the case of supervaluationism is analogous to the global/local distinction discussed above for metainferences). They develop a proof system for supervaluationist logic which offers to vindicate the notion of global validity for supervaluationism. This system can be philosophically interpreted by analyzing truth as licensing assertion, falsity as licensing negative assertion, and lack of truth value as licensing rejection and weak assertion.

The closing piece in this series of papers is **Zardini**'s "The Final cut". In [74] Zardini had pioneered a nontransitive treatment of the paradoxes of vagueness, which inspired Cobreros, Égré, Ripley and van Rooij in their definition of **ST**. Despite that, as he explains in this paper, in the case of truth Zardini was reluctant to adopt the same nontransitive approach, unlike Ripley and associates in [54] and [13], who use **ST** as a common framework to handle paradoxes of vagueness and of truth (see Ripley's contribution, this issue). In his paper, Zardini formulates several strictures against the system **STT** of transparent truth (named $\mathbf{K3LP^T}$ in his paper). One objection made by Zardini concerns the fact **STT** handles the Liar as an explosive sentence (it entails any absurdity), another is that it predicts some instances of the T-schema to be contradictory (such as the one involving the Liar in premise position). Zardini extends these criticisms to the system augmenting **ST** with a validity predicate, and outlines reasons to favor a noncontractive substructural approach of the paradoxes of truth and validity instead.

5 Conclusions

Almost a century after the pioneering insights of Tarski and Gentzen, the high-quality contributions gathered in this special issue show that the foundations of logic remain a lively and active domain of philosophical and mathematical inquiry. The papers in this special issue also open up novel perspectives concerning the problem of logical pluralism, by showing that classical logic is richly connected to systems that may have been thought of as exotic or as more removed only a few decades ago. We hope that further debates and clarifications will ensue in response to the various results and puzzles set forth in this collection of papers on substructural logics and metainferences.

Acknowledgements We thank Reinhard Muskens and Quentin Blomet for comments on the first version of this text, as well as Bruno Da Ré, Bogdan Dicher, Luca Incurvati, Sergi Oms, Federico Pailos, and David Ripley for feedback on specific points. We are most grateful to the École normale supérieure and to the Philosophy Department of ENS for making possible the visit of Eduardo Barrio to Paris in June 2022, as well as to the INALCO and the CNRS (Second summer school "Conditionals in Paris"). Special thanks go to Dimitri El Murr, Patrick Caudal, and Ghanshyam Sharma for their support in relation to this collaboration. PE also acknowledges grants ANR-19-CE28-0019-01 (AMBISENSE) and ANR-17-EURE0017 (FRONTCOG).

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