



Subminimal Negation on the Australian Plan

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Abstract

Frame semantics for negation on the Australian Plan accommodates many different negations, but it falls short on accommodating subminimal negation when the language contains conjunction and disjunction. In this paper, I will present a multi-relational frame semantics –multi-incompatibility frame semantics– that can accommodate subminimal negation. I will first argue that multi-incompatibility frames are in accordance with the philosophical motivations behind negation on the Australian Plan, namely its modal and exclusion-expressing nature. Then, I will prove the soundness and completeness results of a subminimal logic that consists of the multi-incompatibility semantics and a proof system with operational rules that characterize subminimal negation, conjunction and disjunction. Lastly, I will prove some key correspondence theorems that relate frame conditions to certain principles that are associated with stronger negations, which will give rise to a new kite of negations that includes subminimal negation.

Keywords Negation · Subminimal negation · Compatibility semantics · Multi-relational frames · Non-classical logics · Modal logics

1 Introduction

There are two different plans for the semantics of negation in relevant logics: namely the American Plan and the Australian Plan. On the American Plan (Belnap-Dunn semantics), negation is characterized truth functionally in a four-valued¹ semantics “not A is true just in case A is false” [1, 11], whereas on the Australian Plan (Routley-Meyer semantics), negation is construed as a modal operator [20, 26, 27]. Even though Routley-Meyer semantics is deemed to be unintuitive (or not a semantics at all, see [5]), Berto and Restall present a frame semantics for the Australian negation

¹These four values are taken to be true, false, neither true nor false, and both true and false.

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that can provide an intuitive reading of the Routley-Meyer semantics [3].² In their paper, negation on the Australian Plan not only refers to negation in relevant logics but also refers to many other modal negations (more specifically negations of the impossibility conception of negation), and it is motivated on philosophical grounds. They argue that negation on the Australian Plan is based on two ideas: it is **modal**—the truth value of a negated sentence ($\neg A$) at a point depends on the truth value of A in other points— and it **expresses exclusion**, it rules things out. They claim that we explain both of these aspects of negation by grounding it in the incompatibility relation, where the grounding relation should be understood as an explanatory relation rather than a definitional reduction [3, p. 1122–1126]. The modal aspect of negation is explained in a very straightforward way for the reason that *incompatibility* is a modal relation. We explain the exclusion-expressing aspect of negation by appeal to negation’s role in conversations, that is to express the incompatibility of certain assertions and denials or states of affairs.³ In other words, because the role of negation in a conversation is to express incompatibilities, it expresses exclusion. So, Australian negation finds its philosophical basis in being grounded in incompatibility.⁴

Berto and Restall show that their semantics can accommodate many different negations, along with negation of relevant logics, by imposing different conditions on the frames.⁵ One family of negations accommodated by their semantics is the family of constructive negations in which negation is characterized by implication of something repugnant or undesirable. One account of this family of negations can be found in Curry’s conception of negation as *refutability* [6, p. 255]. According to Curry, just as we define axioms to be the set of sentences that are taken to be primitively true, we can also define, *counteraxioms*, a set of sentences that are taken to be primitively false. In the way that the truths of a system are the formulas that are deducible from the axioms; the falsities of a system are the formulas from which a counteraxiom is deducible. In other words, a sentence A is refutable just in case a counteraxiom is deducible from A .⁶ This entails that the refutations or negations of a system are equivalent to implication of a counteraxiom on this conception. Moreover,

²Their work on the semantics of negation on the Australian Plan is not entirely new. For the earlier developments of negation as a modal operator see [2, 9, 10, 12, 25].

³According to Berto and Restall, we can understand the incompatibility relation in two different ways: pragmatically or metaphysically [3, p. 1122]. The former uses the primitive incompatibility of assertion and denial, whereas the latter takes incompatibility to be a primitive metaphysical relation between worldly things such as situations, worlds, facts etc. to explain the role of negation. (For the former see [22], and for the latter see [2].) I will be silent about my interpretation of the philosophical account of incompatibility because the arguments provided in this paper are consistent with both of the accounts.

⁴There are other semantic treatments of the modal negation of relevant logics such as the ones in [19] and [13], and some of the issues we will discuss throughout the paper may also be discussed in these alternative semantic treatments. However, we will only focus on the Australian Plan as construed by [3], since investigating these alternative semantics is beyond the scope of this paper. This will also help us to concentrate on how weaker modal negations are modeled when other connectives are modeled classically. Thus, I leave the research on the weaker negations in other frameworks for future research.

⁵Most of these correspondence results can be found in the earlier work by Dunn [9, 10, 12] and Restall [23–25].

⁶Since in the system developed in [6] a formula B being deducible from a set of formulas $\Gamma \cup \{A\}$ is equivalent to $A \supset B$ being deducible from Γ , I take the liberty to use deducibility and implication interchangeably in contexts where they are equivalent. For details on Curry’s deducibility relation see [6, p. 185–189].

we accommodate different negations according to the refutability conception, such as minimal negation or intuitionist negation, depending on the conditions imposed on the counteraxioms⁷ and, as mentioned above, many of them are also accommodated by the Australian semantics. However, the weakest member of this family, subminimal negation, is not *fully* accommodated. To see why it is not fully accommodated let us investigate the features of subminimal negation.⁸

Proof theoretically, subminimal negation can be characterized by Local Contraposition [LC]⁹:

$$\frac{A \succ B}{\neg B \succ \neg A} [LC]$$

It is easy to check the validity of [LC] on the refutability conception. Given that we have characterized negation as a counteraxiom being deducible from a formula, $\neg B$ means that a counteraxiom is deducible from B . Holding of the premiss-sequent tells us that B is deducible from A . If a counteraxiom is deducible from B ($\neg B$), we can conclude that A implies the same counteraxiom by transitivity of the deducibility relation. Consequently, $\neg A$ is deducible from $\neg B$. Furthermore, a sentence can be subminimally negated by implying different counteraxioms, and as a consequence, the following DeMorgan law is subminimally invalid:

$$[\wedge \neg] \neg A \wedge \neg B \vdash \neg(A \vee B)$$

We can check its subminimal invalidity by considering the cases when A and B imply two different counteraxioms. In these cases, even though we have $\neg A \wedge \neg B$, we do not have any basis to conclude that a counteraxiom is implied by $A \vee B$, given that we have multiple counteraxioms that are not closed under disjunction.¹⁰ Hence, we cannot infer $\neg(A \vee B)$ from $\neg A \wedge \neg B$.

Subminimal invalidity of $[\wedge \neg]$ reveals that subminimal negation is not fully accommodated by the Australian Plan, because, $[\wedge \neg]$ is valid according to the

⁷For the correspondence between the conditions on the counteraxioms and inference rules see [6, p. 257–261], [17, p. 1259].

⁸Subminimal negation is not studied in depth by Curry, but it has been studied by many others. See [4, 9, 10, 14, 15, 17].

⁹There is a difference between the way some of the constructive negations are characterized in [6] (and in the literature related to Curry’s account such as [14, 15]) and how they are characterized on the Australian Plan because of a difference in their respective deducibility relations. In Curry’s account, deducibility relation holds between a set of formulas and a formula, it is monotonic, transitive and as mentioned in footnote 6 it enjoys an equivalence with the implication connective. However, these properties of Curry’s deducibility relation combined with the refutability conception of negation makes the relevantly unacceptable rule of selective contraposition [SC] the characteristic rule of subminimal negation: $\frac{\Gamma, A \succ B}{\Gamma, \neg B \succ \neg A} [SC]$. (Observe that in the presence of thinning A , $\neg A \succ \neg B$ is provable using [SC].) Hence, accommodating subminimal negation in a relevantly acceptable context requires us to drop one of the properties of Curry’s deducibility relation.

In negation on the Australian Plan literature, i.e., in [3] and [9, 10, 12], structural properties of the deducibility relation are preserved but its connection to the implication connective is dropped out. This results in [LC] (Observe that we omit the context formula set.) becoming the rule that proof theoretically characterizes subminimal negation, and in this paper, I will keep the Australian Plan convention. As a consequence of this difference, $A, \neg A \succ \neg B$ is now subminimally invalid on the Australian Plan and it is associated with stronger negations such as minimal negation. (See Dunn’s kite of negations in [9, 12] and [10].)

¹⁰For a discussion of this property of subminimal negation in particular see [17, p. 1265–1266] and [15].

semantics presented by Berto and Restall as it will be shown in the next section. However, when the language is \wedge and \vee -free, incompatibility frames can accommodate subminimal negation as briefly discussed by [9, 12]. Consequently, in the \wedge and \vee -free fragment of the language subminimal negation can be grounded in incompatibility, and this indicates that it can be accommodated on the Australian Plan. In this paper I will present a multi-relational frame semantics, which I will call multi-incompatibility frames, where we can fully accommodate subminimal negation on the Australian Plan. I will show this by proving the completeness and soundness results for a subminimal logic and by proving the correspondence theorems of stronger principles. I will also argue that the multi-incompatibility semantics preserve the philosophical merits of negation on the Australian Plan by grounding negation in incompatibility. As a consequence, multi-incompatibility frames will also be philosophically motivated and it will accommodate more negations.

2 Australian Plan Semantics

In this section, I will present the frame semantics of negation on the Australian Plan and show that it cannot accommodate subminimal negation when \wedge and \vee are present. I will then provide two reasons, one semantic and one proof theoretic, for wanting to accommodate subminimal negation on the Australian Plan.

Let me start by introducing our language \mathcal{L} . It consists of atomic sentences $p_1, p_2, \dots, p_n, \dots$, (The first three of which I will abbreviate to p, q, r), two binary connectives, \wedge and \vee and one singulary connective \neg . I will use $A_1, A_2, \dots, A_n, \dots$ as metavariables (The first three of which I will abbreviate to A, B, C) for formulas. The well-formed-formulas (wffs) are the atomic sentences and if A_i and A_j are wffs, then, $\neg A_i, A_i \wedge A_j$ and $A_i \vee A_j$ are wffs. I will use the capital Greek letters for sets of formulas. I will use not, and, or, only if, iff, and universal and existential quantifiers with their usual meaning in the metalanguage, and I will also use the notations $\&, \supset, \forall, \exists$ when expressing the frame conditions. Lastly, \mathcal{L}_\neg denotes the pure negation fragment of our language, i.e., the sublanguage of \mathcal{L} that consists of the atomic sentences and wffs generated only by \neg .

An incompatibility-frame is a triple $\mathfrak{F} = \langle U, \sqsubseteq, \perp \rangle$, where U is a non-empty set (of states),¹¹ \sqsubseteq is a partial order on U , and \perp is a binary relation of incompatibility on U .¹² Incompatibility frames satisfy the Forwards condition:

¹¹It is important to note that the elements of U are not to be interpreted as maximally-consistent worlds as they are in the Kripke semantics for normal modal logics. One can interpret them as mere information states, situations, or parts of worlds instead of complete worlds.

¹²We can present the formal semantics of negation on the Australian Plan in two different ways. One way would be using the compatibility relation as the accessibility relation in the semantic clause for negation as Berto and Restall do [2, 3, 23, 25], the other way would be using the incompatibility relation (perp relation) as Dunn does [9, 10]. Since they are interdefinable, the choice among the two does not affect the merits of Australian negation. I will use the incompatibility relation because it illustrates the similarities between the frame semantics and negation as refutability more vividly. Because of this choice, I will use the name “incompatibility frames” to refer to our frames as opposed to other common names such as perp frames or compatibility frames.

Forwards: $(x \perp y \text{ and } x \sqsubseteq x' \text{ and } y \sqsubseteq y')$ implies $x' \perp y'$

An incompatibility frame becomes a model \mathfrak{M} when we add the function v to our frame, $\mathfrak{M} = \langle U, \sqsubseteq, \perp, v \rangle$, where v is a function that associates every atomic sentence with a set of states. We assume that v satisfies the Heredity Constraint (HC) for atoms:

(HC) $x \in v(p)$ and $x \sqsubseteq y$ implies $y \in v(p)$

We define the relation of truth-at-a-state in a model (\Vdash) inductively as follows:

- (S-AT) $x \Vdash p$ iff $x \in v(p)$
- (S \wedge) $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.
- (S \vee) $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.
- (S \neg) $x \Vdash \neg A$ iff for all $y \in U$ ($y \Vdash A$ only if $x \perp y$)

Given our inductive definitions, (HC) holds for every formula.¹³ Moreover, we say that an argument from Γ to A holds in a model \mathfrak{M} ($\Gamma \vDash_{\mathfrak{M}} A$) just in case for every $x \in U$ if $x \Vdash B$ for all $B \in \Gamma$, then $x \Vdash A$. We say that the argument from Γ to A is valid in a frame \mathfrak{F} ($\Gamma \vDash_{\mathfrak{F}} A$) iff it holds for every model \mathfrak{M} built on \mathfrak{F} . An argument from Γ to A is valid with respect to a class of incompatibility frames \mathcal{F} ($\Gamma \vDash_{\mathcal{F}} A$) just in case it is valid in every frame $\mathfrak{F} \in \mathcal{F}$.

Before getting into the issues surrounding subminimal negation, let us look into our semantics to see how it manifests the aforementioned philosophical merits of the Australian Plan, namely the relationship between negation and incompatibility. We interpret the \perp relation in (S \neg) as an incompatibility relation, a relation that holds between two information states x, y where x rules y out. For instance, if “The table is brown.” is true at x and “The table is green.” is true at y , then x and y are incompatible ($x \perp y$). Moreover, we read the semantic clause for this particular example as follows “It is not the case that the table is green.” is true at x just in case all green-table states are incompatible with x . This example illustrates how the incompatibility relation between states is related to negation in our semantics: the implication from the truth of a sentence at a state to its incompatibility with x is necessary and sufficient for the truth of the sentence’s negation at x . Hence, truth value of the negated sentence at a state x is dependent on the truth value of the sentence at a state that is incompatible with x .

As mentioned above, different conditions on the frames, such as the symmetry or irreflexivity of the incompatibility relation, correspond to (are all and only frames that validate) different negation principles, so we can obtain different negations from incompatibility frames. For instance, double negation introduction [DNI] corresponds to frames with symmetric incompatibility relations ($\mathfrak{F}s$), i.e., $(A \vDash_{\mathfrak{F}s} \neg\neg A)$. Given that quasi-minimal negation is characterized by [DNI], the negation determined by the symmetric incompatibility frames is quasi-minimal negation.¹⁴ There are many more negations we can get from our incompatibility frames (including

¹³We prove this claim (also known as the persistence lemma) by induction on the complexity of formulas. The proofs for the cases of \wedge and \vee are straightforward and the proof for \neg appeals to the Forwards condition. Since the proof is very similar to the proof of Theorem 1 in Section 3, I will not prove it here.

¹⁴I will be using the more recent naming convention associated with each negation used by Dunn and Zhou [12] and Horn and Wansing [16].

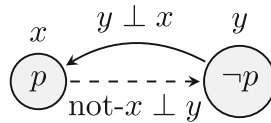


Fig. 1 Countermodel to [DNI]— Normal arrows represent \perp -related states and dashed arrows represent the crucial not- \perp related states. I will use the same convention throughout the paper

negation of relevant logics), but I will not provide a detailed account of the other negations in incompatibility frames because they are well-studied in the literature [3, 9, 10, 12, 23, 25].¹⁵

The negation determined by the class of all incompatibility frames (\mathcal{F}) is quite interesting. First, the rule [LC] preserves the property of holding in a model with respect to \mathcal{F} .¹⁶ Second, the negation determined by \mathcal{F} is weaker than quasi-minimal negation because [DNI] is invalid with respect to \mathcal{F} , i.e., $A \not\models_{\mathcal{F}} \neg\neg A$. The countermodel consists of $\mathfrak{M} = \langle U, \sqsubseteq, \perp, v \rangle$, where $U = \{x, y\}$, $\perp = \{(y, x)\}$, $\sqsubseteq = \{(x, x), (y, y)\}$, and $x \Vdash p$ and $y \Vdash \neg p$. (See Fig. 1.)

This countermodel has a state, x , where $x \Vdash p$ but $x \not\models \neg\neg p$. Having $x \not\models \neg\neg p$ only forces us to have a state (y) where $\neg p$ is verified ($y \Vdash \neg p$) that is not incompatible with x ($\text{not-}x \perp y$) and having $x \Vdash p$ and $y \Vdash \neg p$ only forces y to be incompatible with x ($y \perp x$). Since we do not end up with a contradiction, this is a countermodel to [DNI]. It is easy to see that [DNI] would be valid if the incompatibility relation were to be symmetric, since $y \perp x$ would entail $x \perp y$ which would contradict with $\text{not-}x \perp y$.

Lastly, even though the negation determined by \mathcal{F} is weaker than quasi-minimal negation, it is also not as weak as subminimal negation, because the subminimally invalid DeMorgan Law [$\wedge\neg$] is valid with respect to \mathcal{F} , i.e., $(\neg A \wedge \neg B \models_{\mathcal{F}} \neg(A \vee B))$.

Proof Suppose $x \Vdash \neg A \wedge \neg B$ for an arbitrary $x \in U$ in an arbitrary model \mathfrak{M} built on an arbitrary frame $\mathfrak{F} \in \mathcal{F}$. We need to prove that for each y ($y \Vdash A \vee B$ only if $x \perp y$) in order to obtain $y \Vdash \neg(A \vee B)$. To prove it, we will suppose $y \Vdash A \vee B$ and apply proof by cases to the disjunction we obtain from ($S\vee$). Suppose $y \Vdash A$. Since for every y ($y \Vdash A$ only if $x \perp y$) follows from $x \Vdash \neg A \wedge \neg B$ by ($S\wedge$) and ($S\neg$), we obtain $x \perp y$ by eliminating the quantifier and Modus Ponens. We similarly obtain $x \perp y$ from the other disjunct $y \Vdash B$ as well. So, we can conclude that ($y \Vdash A \vee B$ only if $x \perp y$) by conditional introduction, and consequently, $x \Vdash \neg(A \vee B)$ by universal quantifier introduction on y and ($S\neg$). Therefore, $\neg A \wedge \neg B \models_{\mathcal{F}} \neg(A \vee B)$. \square

¹⁵Conditions in some of the referred works are provided with the compatibility relation. Given that incompatibility and compatibility are interdefinable those conditions can be easily translated to the incompatibility frames.

¹⁶Since its proof is similar to the [LC] case in the proof of Theorem 3 in Section 3 I will not repeat it here.

This negation is known as *preminimal* negation in the literature and proof theoretically characterized by [LC] and $[\wedge\lrcorner]$ (or its rule version). Consequently, subminimal negation is not accommodated by the Australian Plan because the negation determined by the class of all incompatibility frames is preminimal negation, i.e., preminimal negation is the weakest negation of the Australian Plan.

However, there are good reasons for wanting to have subminimal negation on the Australian Plan. One reason is that preminimal negation and subminimal negation coincide in \mathcal{L}_\lrcorner [9, 12]. This is because the problem is caused by $[\wedge\lrcorner]$ which is absent in the \wedge and \vee -free fragment in the language. The negation determined by the class of all incompatibility frames in \mathcal{L}_\lrcorner only validates [LC] and nothing stronger. This implies that the incompatibility frames can accommodate subminimal negation in the \wedge and \vee -free fragment of our language \mathcal{L} because subminimal negation is characterized proof theoretically by validating only [LC]. As a consequence of this, we can claim that subminimal negation is a modal exclusion-expressing operator as well because it can also be grounded in incompatibility, as other negations we discussed so far. Since it is also a modal-exclusion expressing operator, accommodating subminimal negation in an incompatibility frame where we have \wedge and \vee in our language is desirable.

Another reason for wanting to accommodate subminimal negation on the Australian Plan is proof theoretic. If we want to have a proof system that is sound and complete with respect to the validities determined by the class of all incompatibility frames, we will have to use the rule version of $[\wedge\lrcorner]$ as a sequent to sequent rule or an axiom, since the weakest negation determined by it is preminimal negation.¹⁷ But, this means that one of the primitive rules for negation in this proof system will contain \wedge and \vee , and consequently, our proof system will end up having primitive impure rules.¹⁸ Given that having pure rules over impure rules is more desirable—because only by pure rules are we able to proof theoretically provide a meaning to the connective independently from other connectives— a proof system that consists of pure rules for the weakest negation on the Australian Plan is also more desirable.¹⁹ Hence, we might want to have a weaker negation than preminimal negation in order to have a proof system that consists of only pure rules. Therefore, there are both philosophical and proof theoretic motivations for wanting to accommodate subminimal negation on the Australian Plan.

3 Multi-Incompatibility Frames

In this section, I will introduce a multi-relational frame semantics of negation that can accommodate subminimal negation as well as all the other negations that can

¹⁷An axiomatic system that has preminimal negation can be found in [7].

¹⁸For a discussion on the purity of inference rules see [8, p.245–274], [17, 519–521].

¹⁹Note that although [LC] is pure, it violates another desirable proof theoretic property, simplicity, since it mentions negation twice.

be obtained from incompatibility frames. I will then argue that our new semantics is also in accordance with the philosophical motivations behind the Australian Plan and it can be construed as an improvement on the incompatibility semantics. Then I will prove the soundness and completeness results of the logic \mathcal{N} that consists of a proof system that only has [LC] and the usual rules of inference for \wedge and \vee . Lastly, I will prove some key correspondence results of our new semantics in order to show that the negations that can be obtained from incompatibility frames can also be obtained from our multi-incompatibility frames, and I will present a more fine-grained kite of negations.

But before the formal presentation, I want to emphasize two different points that have already been mentioned above to motivate our semantics. First, as Hazen also points out, the subminimal invalidity of $[\wedge\neg]$ is due to having multiple counteraxioms rather than having a single counteraxiom: “ P and Q might both be false, but by implying different counteraxioms, without there being a single counteraxiom implied by both ...” [15, p. 106] But, it is important to note that if the set of our counteraxioms were to be closed under disjunction, there would be a single counteraxiom (the disjunctive counteraxiom that is composed of the counteraxioms implied by P and Q) implied by both and $[\wedge\neg]$ would be valid [17, p. 1266]. So, in order to have subminimal negation on the refutability conception, we need the set of multiple counteraxioms to not be closed under disjunction. Second, $(S\neg)$ and negation on the refutability conception are structurally similar, both of them take negation of a formula to be the implication of something undesirable. We construe negation on the refutability reading as an implication to something undesirable such as a counteraxiom. In $(S\neg)$, a formula is negated at a state x just in case the truth of a sentence at a state y implies something undesirable, i.e., incompatibility of the states x and y . Combining these two points provides us a way to solve our problem. If we have multiple incompatibility relations that are not closed under union, as we have multiple counteraxioms that are not closed under disjunction, we can invalidate $[\wedge\neg]$; because $[\wedge\neg]$ would be invalid when a state verifying A and B were to imply different incompatibilities without there being a single incompatibility implied by both. We can illustrate this idea more rigorously by revising $(S\neg)$ where A 's truth-at- y implies *an* incompatibility relation that holds between states, rather than implying *the* incompatibility relation between states. Since we want to negate a formula when and only when its truth-at-a-state implies an incompatibility, any one of the incompatibility relations is supposed to be necessary and sufficient to negate a formula:²⁰

$$x \Vdash \neg A \text{ iff for all } y (y \Vdash A \text{ only if } x \perp_1 y) \vee \text{ for all } y (y \Vdash A \text{ only if } x \perp_2 y) \\ \vee \dots \vee \text{ for all } y (y \Vdash A \text{ only if } x \perp_i y)$$

When we generalize this clause with an existential quantifier over different incompatibility relations, we get our new semantic clause for negation:

$$(S'\neg) \quad x \Vdash \neg A \text{ is true iff there is a } \perp_n \text{ such that for all } y (y \Vdash A \text{ only if } x \perp_n y)$$

²⁰This disjunctive approach is inspired by Humberstone's discussion on subminimal negation [17, p. 1265-1269]. For a different frame semantics for subminimal negation that is based on a similar idea also see [14, p. 6].

We will use multi-relational frames to implement the multiple incompatibility relations in our semantics.²¹ In multi-relational frames, we have a set of multiple accessibility relations rather than having a unique accessibility relation. A multi-incompatibility frame is a triple $\mathfrak{FM} = \langle U, \sqsubseteq, Q \rangle$ where U is a non-empty set of states, \sqsubseteq is a partial order relation on U , and Q is a non-empty set of \perp_i relations on U .²² Forwards is also a condition on \mathfrak{FM} :

Forwards: $(x \perp_i y \text{ and } x \sqsubseteq x' \text{ and } y \sqsubseteq y')$ only if $x' \perp_i y'$

A multi-incompatibility frame \mathfrak{FM} becomes a model \mathfrak{M} when we add the function v that assigns atomic sentences to sets of states in U . We assume that v satisfies the Heredity Constraint (HC) for atoms:

(HC) $x \in v(p)$ and $x \sqsubseteq y$ implies $y \in v(p)$

We define the relation of truth-at-a-state in a model (\Vdash) inductively as follows:

- (S-AT) $x \Vdash p$ iff $x \in v(p)$
- (S \wedge) $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.
- (S \vee) $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.
- (S' \neg) $x \Vdash \neg A$ iff there is a $\perp_i \in Q$ for all y ($y \Vdash A$ only if $x \perp_i y$)

Given our inductive definitions, (HC) holds for every formula.

Theorem 1 $x \Vdash A$ and $x \sqsubseteq y$ implies $y \Vdash A$.

Proof We will prove this theorem by doing induction on the complexity of formulas. The basis case directly follows from (HC). For the inductive cases, we will only provide the case for \wedge and \neg and leave the case for \vee to the reader.

(Case for \wedge) Suppose $x \Vdash A \wedge B$ and $x \sqsubseteq y$. We have $x \Vdash A$ and $x \Vdash B$ by (S \wedge). We get $y \Vdash A$ and $y \Vdash B$ by the inductive hypothesis and we conclude that $y \Vdash A \wedge B$ by (S \wedge).

(Case for \neg) Suppose $x \Vdash \neg A$ and $x \sqsubseteq y$. We need to prove there is a $\perp_i \in Q$ such that for all z ($z \Vdash A$ only if $y \perp_i z$) to get $y \Vdash \neg A$. Suppose $z \Vdash A$ for conditional introduction for an arbitrary z . By (S' \neg), we get there is a $\perp_i \in Q$ for all y ($y \Vdash A$ only if $x \perp_i y$) from $x \Vdash \neg A$. We eliminate the quantifiers and get $x \perp_i z$ by Modus Ponens with $z \Vdash A$. Since we have $x \sqsubseteq y$ by assumption, $z \sqsubseteq z$ by the reflexivity of \sqsubseteq , and $x \perp_i z$, we get $y \perp_i z$ by Forwards. So, we have ($z \Vdash A$ only if $y \perp_i z$) by conditional introduction. By appropriate quantifier introductions, we get there is a $\perp_i \in Q$ for all z ($z \Vdash A$ only if $y \perp_i z$). □

²¹The use of multi-relational frames is inspired by Schotch and Jennings’s multi-relational approach in deontic logic [28]. For a brief discussion of Schotch and Jennings’s system see [18, p. 245-247].

²²Allowing an empty set of \perp relations would make every negated formula false, which does not have any impact on subminimal negation in our language \mathcal{L} , since [LC] still preserves the property of holding in a model in those frames as well. But, we will have to impose this restriction to our frames for every negation stronger than subminimal negation given that they all validate [DNI]. Since I want to single out the properties of the incompatibility relation when proving the correspondence results between the frame conditions and stronger negation principles, I will restrict Q to be non-empty for all of our frames. However, there will be a discussion below on the impact of allowing an empty Q for subminimal negation when nullary connectives are added to the language.

We say that an argument from Γ to A holds in a model \mathfrak{M} ($\Gamma \models_{\mathfrak{M}} A$) just in case for every $x \in U$ if $x \Vdash B$ for all $B \in \Gamma$, then $x \Vdash A$. We say that the argument from Γ to A is valid in a multi-incompatibility frame $\mathfrak{F}\mathfrak{M}$ ($\Gamma \models_{\mathfrak{F}\mathfrak{M}} A$) iff it holds in every model \mathfrak{M} built on $\mathfrak{F}\mathfrak{M}$. An argument from Γ to A is valid with respect to a class of multi-incompatibility frames $\mathcal{F}\mathcal{M}$, ($\Gamma \models_{\mathcal{F}\mathcal{M}} A$) just in case it is valid in every $\mathfrak{F}\mathfrak{M} \in \mathcal{F}\mathcal{M}$.

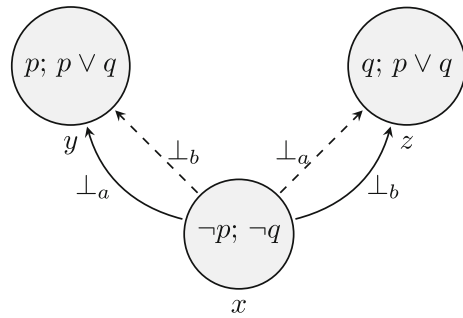
We can interpret $(S'\neg)$ similar to how we interpreted $(S\neg)$ above, but we first need to have a better understanding of having multiple incompatibility relations in order to claim that $(S\neg)$ and $(S'\neg)$ have similar interpretations. One way to interpret them is that each incompatibility relation is a different way to be incompatible. For instance, the incompatibility between the states where “Sherlock Holmes is a fictional character.” is true and where “Sherlock Holmes exists.” is true denotes one way to be incompatible, and the incompatibility between a state where “The table is green.” is true and where “The table is brown.” is true denotes another way to be incompatible. Having different ways to be incompatible can further be explained by the reasons why states are incompatible. Going back to our example, the first two are incompatible because one thing cannot exist and be fictional at the same time, whereas the other two are incompatible because an object cannot instantiate different color properties.²³ In other words, we can have a philosophically plausible reading of multiple incompatibility relations by showing that different ways to be incompatible can model the difference in reasons for being incompatible. Moreover, since $(S'\neg)$ also uses an incompatibility relation to define the truth of a negated sentence and we at least have a plausible reading of the multiple incompatible relations, we can retain Berto and Restall’s argument for the modality and exclusion-expressing aspect of negation as well, and consequently claim that multi-incompatibility frames preserve the philosophical merits of incompatibility frames.

Now, we should check whether this semantics can accommodate subminimal negation, and then check whether multi-incompatibility frames accommodate all the other negations that incompatibility frames accommodate. We can check that $[\wedge\neg]$ is invalid with respect to the class of all multi-incompatibility frames $\mathcal{F}\mathcal{M}$ by the following countermodel $\mathfrak{M} = \langle U, \sqsubseteq, Q, v \rangle$ where $U = \{x, y, z\}$, $Q = \{\perp_a = \{\langle x, y \rangle\}, \perp_b = \{\langle x, z \rangle\}\}$, $\sqsubseteq = \{\langle x, x \rangle, \langle y, y \rangle, \langle z, z \rangle\}$ and $x \Vdash \neg p, x \Vdash \neg q, y \Vdash p, z \Vdash q$ (Fig. 2).

This countermodel shows that there is a state x such that $x \Vdash \neg p \wedge \neg q$ while $x \not\Vdash \neg(p \vee q)$. By having the state y where $y \Vdash p$ and $x \perp_a y$, we satisfy the conditions for $x \Vdash \neg p$ and since also $y \Vdash p \vee q$ and not- $x \perp_b y$ we satisfy the conditions for $x \not\Vdash \neg(p \vee q)$. Similarly, by having the state z where $z \Vdash q$ and $x \perp_b z$, we satisfy the conditions for $x \Vdash \neg q$ and since also $z \Vdash p \vee q$ and not- $x \perp_a z$ we satisfy the conditions for $x \not\Vdash \neg(p \vee q)$. It is important to note that if we did not have different multiple incompatibility relations such as \perp_a and \perp_b , we

²³As I have mentioned, I am not taking a stance on whether incompatibility is a pragmatic relation between assertion and denials, or a metaphysical relation between things. So, I take multiple ways to be incompatible to apply to both of these readings.

Fig. 2 Countermodel to $[\wedge\neg]$



would end up with a contradiction, since we need to have x to be both \perp related and not- \perp related to y and z .

Invalidity of the De Morgan law $[\wedge\neg]$ is the first step towards our goal to fully accommodate subminimal negation. We need to show that this semantics only validates $[LC]$ and does not extend beyond that. We can show this by providing the soundness and completeness results of a logic \mathcal{N} that has the proof system in Fig. 3.²⁴

Definition 1 There is a derivation of $\Gamma \succ A$ just in case there is a tree of sequents where its root is $\Gamma \succ A$ and the leaves of which are the instances of $[R]$ and where each sequent on the tree is a direct consequence of one of the rules applied to the sequents which are immediately above them in the tree.

Definition 2 A is derivable from Γ in \mathcal{N} ($\Gamma \vdash_{\mathcal{N}} A$) just in case there is a derivation of $\Gamma \succ A$.

Theorem 2 (Variable sharing property) Let $Var(A)$ denote the set of atomic sentences occurring in the formula A and $\bigwedge \Gamma$ denote the conjunction of formulas in Γ . If $\Gamma \vdash_{\mathcal{N}} B$, then $Var(\bigwedge \Gamma) \cap Var(B) \neq \emptyset$.

Proof Consider the lattice $\mathbf{L4} = \langle \mathbf{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}, \leq, \wedge, \vee, \neg \rangle$ where \mathbf{V} is the set of objects, \leq is a partial order on \mathbf{V} that goes upwards on the Hasse diagram in Table 1, and \wedge, \vee and \neg are operators on \mathbf{V} that behave according to the matrices in Table 1. We define s to be a function from \mathcal{L} to \mathbf{V} .

We will try to establish two conditionals to prove our theorem. First, we will show that if $\Gamma \vdash_{\mathcal{N}} A$, then $s(\bigwedge \Gamma) \leq s(A)$. Second, we will show that if $s(\bigwedge \Gamma) \leq s(A)$, then $Var(\bigwedge \Gamma) \cap Var(B) \neq \emptyset$.

Establishing our first conditional is quite straightforward and well-known since our logic is a sublogic of first degree entailment (FDE) and FDE is sound with respect to $\mathbf{L4}$. (See [1] and [21].) We only need to show that our rules preserve the \leq ordering, given that \leq is reflexive, \wedge and \vee are meet and join, and \neg behaves according to the matrix above showing it is trivial and left to the reader.

²⁴The comma denotes the set union in the context where uppercase Greek letters stand for sets of formulas and a formula occurring alone is, in this context, standing for its singleton.

$$\begin{array}{c}
 \frac{}{A \succ A} \text{ [R]} \qquad \frac{\Gamma \succ A}{\Gamma, \Gamma' \succ A} \text{ [Th]} \qquad \frac{\Gamma \succ A \quad \Gamma', A \succ B}{\Gamma, \Gamma' \succ B} \text{ [Cut]} \\
 \\
 \frac{\Gamma, A \succ B}{\Gamma, A \wedge C \succ B} \text{ [\wedge L]} \qquad \frac{\Gamma, A \succ B}{\Gamma, C \wedge A \succ B} \text{ [\wedge L]} \qquad \frac{\Gamma \succ A \quad \Gamma \succ B}{\Gamma \succ A \wedge B} \text{ [\wedge R]} \\
 \\
 \frac{\Gamma \succ A}{\Gamma \succ A \vee B} \text{ [\vee R]} \qquad \frac{\Gamma \succ A}{\Gamma \succ B \vee A} \text{ [\vee R]} \qquad \frac{\Gamma, A \succ C \quad \Gamma, B \succ C}{\Gamma, A \vee B \succ C} \text{ [\vee L]} \\
 \\
 \frac{A \succ B}{\neg B \succ \neg A} \text{ [LC]}
 \end{array}$$

Fig. 3 Structural and operational rules of \mathcal{N}

Our next step will be to establish that if $s(\wedge \Gamma) \leq s(A)$, then $Var(\wedge \Gamma) \cap Var(A) \neq \emptyset$. We prove this by contraposition, i.e., we will show how to construct a **L4**-countermodel for cases where $Var(\wedge \Gamma) \cap Var(A) = \emptyset$. Suppose $Var(\wedge \Gamma) \cap Var(p) = \emptyset$, then take a function s where $s(p) = \mathbf{n}$ for every atomic variable $p \in Var(A)$ and $s(q) = \mathbf{b}$ for every atomic variable $q \in Var(\wedge \Gamma)$. By inspection of the matrices we can establish that $s(\wedge \Gamma) = \mathbf{b}$ and $s(A) = \mathbf{n}$, and as a consequence $s(\wedge \Gamma) = \mathbf{b} \not\leq s(A) = \mathbf{n}$ \square

Theorem 3 (Soundness) $\Gamma \vdash_{\mathcal{N}} A$ only if $\Gamma \vDash_{\mathfrak{F}\mathfrak{M}} A$ for all multi-incompatibility frames $\mathfrak{F}\mathfrak{M}$.

Proof [R] is valid and the structural rules [Th] and [CUT] preserve the property of holding in a model given that $\vDash_{\mathfrak{F}\mathfrak{M}}$ is a reflexive, transitive and a monotonic relation. The proofs for the rules that govern \wedge and \vee are well-known, so I will not provide their proofs here. We only need to prove that [LC] preserves the property of holding in a model:

[LC] Suppose $A \vDash_{\mathfrak{M}} B$ and suppose $x \Vdash \neg B$. From $x \Vdash \neg B$, we have there is a $\perp_i \in Q$ such that for each y ($y \Vdash B$ only if $x \perp_i y$) by $(S'\neg)$. From $A \vDash_{\mathfrak{M}} B$,

Table 1 Lattice **L4** and matrices for \wedge, \vee and \neg on **V**

\vee	t	b	n	f	\wedge	t	b	n	f	\neg	
t	t	t	t	t	t	t	b	n	f	t	f
b	t	b	t	b	b	b	b	f	f	b	b
n	t	t	n	n	n	n	f	n	f	n	n
f	t	b	n	f	f	f	f	f	f	f	t

we have for each y (if $y \Vdash A$, then $y \Vdash B$). By transitivity, for all y (if $y \Vdash A$, then $x \perp_i y$) which gives us $x \Vdash \neg A$ by existential generalization on $x \perp_i y$ and $(S'\neg)$.

Since all of our inference rules preserve the property of holding in a model and [R] is valid, every provable sequent in our proof system is valid. \square

Now we will prove the completeness direction. But, we will need to prove a couple of theorems and lemmas before. First, we will prove the pair extension theorem. The proofs of the pair extension theorem (Theorem 4) and Lemmas 1-3 presented here can be found in [23, p. 92–95] with slight differences.

Definition 3 A set of formulas Γ is

- non-trivial iff $\Gamma \neq \emptyset$ and $\Gamma \neq \mathcal{L}$.
- a \vdash -theory iff $A_1, \dots, A_n \in \Gamma$ and $A_1, \dots, A_n \vdash_{\mathcal{N}} B$ only if $B \in \Gamma$.
- prime iff $A \vee B \in \Gamma$ only if $A \in \Gamma$ or $B \in \Gamma$.

Definition 4 An ordered pair $\langle \Gamma, \Delta \rangle$ of sets of formulas is a \vdash -pair iff there are no formulas $A_1, \dots, A_n \in \Gamma$ and $B_1, \dots, B_n \in \Delta$ where $A_1, \wedge \dots \wedge A_n \vdash_{\mathcal{N}} B_1 \vee \dots \vee B_n$

Definition 5 A \vdash -pair $\langle \Gamma', \Delta' \rangle$ extends $\langle \Gamma, \Delta \rangle$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

Definition 6 A \vdash -pair $\langle \Gamma, \Delta \rangle$ is full iff $\Gamma \cup \Delta = \mathcal{L}$

Lemma 1 If Γ is a non-trivial set of formulas such that $\Gamma \not\vdash_{\mathcal{N}} A$, then $\langle \Gamma, \{A\} \rangle$ is a \vdash -pair.

Proof Suppose $\Gamma \not\vdash_{\mathcal{N}} A$ and $\langle \Gamma, \{A\} \rangle$ is not a \vdash -pair. From $\langle \Gamma, \{A\} \rangle$ not being a \vdash -pair we get that there are formulas $B_i \dots B_j \in \Gamma$ where $B_i \wedge \dots \wedge B_j \vdash_{\mathcal{N}} A$ by Definition 4. Since we also have $\Gamma \vdash_{\mathcal{N}} B_i \wedge \dots \wedge B_j$ by [R], [Th] and [$\wedge R$], we get $\Gamma \vdash_{\mathcal{N}} A$ by [CUT] giving us a contradiction. \square

Lemma 2 If $\langle \Gamma, \Delta \rangle$ is a full \vdash -pair, then Γ is a prime \vdash -theory.

Proof Suppose $\langle \Gamma, \Delta \rangle$ is a full \vdash -pair. Suppose $A \in \Gamma$ and $A \vdash_{\mathcal{N}} B$. Given that $B \notin \Delta$ by Definition 4 and $\Gamma \cup \Delta = \mathcal{L}$ by Definition 6, B must be an element of Γ . So, Γ is a \vdash -theory.

Now let us prove its primeness. Suppose $A \vee B \in \Gamma$ but $A \notin \Gamma$ and $B \notin \Gamma$. So, both $A, B \in \Delta$ because $\Gamma \cup \Delta = \mathcal{L}$. But, since $A \vee B \vdash_{\mathcal{N}} A \vee B$ and $\langle \Gamma, \Delta \rangle$ is a \vdash -pair, this leads to contradiction because there is a formula in Γ and formulas in Δ , namely $A \vee B$ and A, B , where $A \vee B \vdash_{\mathcal{N}} A \vee B$ which contradicts with the Definition 4 of a \vdash -pair. So either A or B must be in Γ . And consequently, Γ is also prime. \square

Lemma 3 Any \vdash -pair $\langle \Gamma, \Delta \rangle$ can be extended by some full \vdash -pair $\langle \Gamma', \Delta' \rangle$.

Proof Let $\langle \Gamma, \Delta \rangle$ be a \vdash -pair. We start by showing that at least one of $\langle \Gamma \cup \{C\}, \Delta \rangle$ and $\langle \Gamma, \Delta \cup \{C\} \rangle$ is a \vdash -pair. Suppose $\langle \Gamma \cup \{C\}, \Delta \rangle$ is not a \vdash -pair, then there is a formula A in the conjunction of formulas $A_1 \wedge \dots \wedge A_n$ in Γ and a formula B in the disjunction of formulas $B_1 \vee \dots \vee B_n$ in Δ such that $A \wedge C \vdash_{\mathcal{N}} B$. If $\langle \Gamma, \Delta \cup \{C\} \rangle$ also were not to be a \vdash -pair we would have had $A' \vdash_{\mathcal{N}} B' \vee C$ where A' is in the conjunction and B' is in the disjunction. From $A \wedge C \vdash_{\mathcal{N}} B$ and $A' \vdash_{\mathcal{N}} B' \vee C$ We can derive $A \wedge A' \vdash_{\mathcal{N}} B \vee B'$ by the derivations π_1 and π_2 and π_3 and applications of [CUT], but this contradicts with our assumption that $\langle \Gamma, \Delta \rangle$ is a \vdash -pair. Hence, we prove that at least one of them should be a \vdash -pair.

$$\begin{array}{c}
 \pi_1 \\
 \frac{\frac{\frac{\overline{B' \succ B'} \text{ [R]}}{B' \succ B' \vee (A \wedge C)} \text{ [\vee R]}}{B', A \succ B' \vee (A \wedge C)} \text{ [Th]}}{\frac{\frac{\frac{\overline{A \succ A} \text{ [R]}}{A, C \succ A} \text{ [Th]}}{A, C \succ A \wedge C} \text{ [\wedge R]}}{A, C \succ B' \vee (A \wedge C)} \text{ [\vee R]}}{\frac{B' \vee C, A \succ B' \vee (A \wedge C)}{(B' \vee C) \wedge A, A \succ B' \vee (A \wedge C)} \text{ [\wedge L]}} \text{ [\vee L]}} \\
 \pi_2 \\
 \frac{\frac{\frac{\overline{B' \succ B'} \text{ [R]}}{B' \succ B \vee B'} \text{ [\vee R]}}{B' \vee (A \wedge C) \succ B \vee B'} \text{ [\vee L]}}{B' \vee (A \wedge C) \succ B \vee B'} \\
 \pi_3 \\
 \frac{\frac{\frac{A' \succ B' \vee C}{A' \wedge A \succ B' \vee C} \text{ [\wedge L]}}{A' \wedge A \succ (B' \vee C) \wedge A} \text{ [\wedge R]}}{\frac{\frac{\overline{A \succ A} \text{ [R]}}{A' \wedge A \succ A} \text{ [\wedge L]}}{A' \wedge A \succ (B' \vee C) \wedge A} \text{ [\wedge R]}}
 \end{array}$$

We define the series of \vdash -pairs $\langle \Gamma_n, \Delta_n \rangle$ as follows. Let $\langle \Gamma_0, \Delta_0 \rangle$ be $\langle \Gamma, \Delta \rangle$ and if $\langle \Gamma_n, \Delta_n \rangle$, then

$$\langle \Gamma_{n+1}, \Delta_{n+1} \rangle = \begin{cases} \langle \Gamma_n \cup \{C_n\}, \Delta_n \rangle & \text{If } \langle \Gamma_n \cup \{C_n\}, \Delta_n \rangle \text{ is a } \vdash\text{-pair} \\ \langle \Gamma_n, \Delta_n \cup \{C_n\} \rangle & \text{Otherwise} \end{cases}$$

If $\langle \Gamma_n, \Delta_n \rangle$ is a \vdash -pair, then so is the $\langle \Gamma_{n+1}, \Delta_{n+1} \rangle$ because, as shown in the previous paragraph, at least one of $\langle \Gamma \cup \{C\}, \Delta \rangle$ and $\langle \Gamma, \Delta \cup \{C\} \rangle$ is a \vdash -pair. We reiterate this process until we put every formula in our language \mathcal{L} to either left or right. Given that at the end of the process the union set of the sets in the pair is going to be identical to \mathcal{L} and the pair will be a \vdash -pair, we will have a full \vdash -pair by Definition 6. Hence, there is a full \vdash -pair $\langle \Gamma', \Delta' \rangle$ that extends the \vdash -pair $\langle \Gamma, \Delta \rangle$. \square

Theorem 4 (Pair Extension) *If $\Gamma \not\vdash_{\mathcal{N}} A$, then there is a prime \vdash -theory $\Gamma' \supseteq \Gamma$ such that $A \notin \Gamma'$.*

Proof Suppose $\Gamma \not\vdash_{\mathcal{N}} A$, so by Lemma 1, $\langle \Gamma, \{A\} \rangle$ is a \vdash -pair. Then, by Lemma 3, there is a full \vdash -pair $\langle \Gamma', \Delta \rangle$ s.t., $\Gamma' \supseteq \Gamma$ and $\Delta \supseteq \{A\}$, that extends $\langle \Gamma, \{A\} \rangle$. By Lemma 2, Γ' is a prime \vdash -theory. Now, we only need to show that $A \notin \Gamma'$. Suppose $A \in \Gamma'$, then $\Gamma' \vdash_{\mathcal{N}} A$, but this contradicts with $\langle \Gamma', \Delta \rangle$ being a full \vdash -pair because by Definition 4 there are no formulae $B \in \Gamma'$ and $C \in \Delta$ where $B \vdash_{\mathcal{N}} C$. Hence, $A \notin \Gamma'$ \square

Definition 7 The canonical frame \mathfrak{FM}_c is a triple $\mathfrak{FM}_c = \langle \mathbb{U}, \subseteq, \mathbb{Q} \rangle$. \mathbb{U} is the set of all non-trivial prime \vdash -theories where non-trivial prime \vdash -theories in \mathbb{P} are ordered by the subset relation \subseteq . \mathbb{Q} is the non-empty finite set of canonical \perp_A relations for $A \in \mathcal{L}$ where $x \perp_A y$ iff $\neg A \in x$ and $A \in y$.²⁵ We get the Canonical Model \mathfrak{M}_c by adding the function V to \mathfrak{FM}_c where $V(p) = \{x \in \mathbb{U} \mid p \in x\}$. The valuation function V assigns atomic formulas to upward closed subsets of \mathbb{U} , i.e., $x \in V(p)$ and $x \subseteq y$ implies $y \in V(p)$. We define the canonical truth-at-a-state relation $x \Vdash_c p$ as $x \in V(p)$.

In order to show that our Canonical Model is actually a model we need to show that (1) accessibility relations in the canonical model satisfy the Forwards condition, (2) semantic clauses are satisfied in the canonical model, (3) the relation \Vdash_c can be generalized to all formulas, and (4) (HC) holds for all formulas in the canonical model.

Lemma 4 Every $\perp_i \in \mathbb{Q}$ on the Canonical Model \mathfrak{M}_c satisfies Forwards.

Proof (Forwards)

Suppose $x \perp_A y$ and $x \subseteq x'$ and $y \subseteq y'$ for an arbitrary \perp_A . Given $x \perp_A y$ and Definition 7, $\neg A \in x$ and $A \in y$. Consequently, $\neg A \in x'$ and $A \in y'$ by $x \subseteq x'$ and $y \subseteq y'$. Therefore, $x' \perp_A y'$ by Definition 7. \square

Corollary 1 \mathfrak{FM}_c is a canonical frame.

Lemma 5 For any non-trivial prime \vdash -theory x and a formula A

- $p \in x$ iff $x \in V(p)$
- $A \wedge B \in x$ iff $A \in x$ and $B \in x$
- $A \vee B \in x$ iff $A \in x$ or $B \in x$
- $\neg A \in x$ iff there is a $\perp_i \in \mathbb{Q}$, for each $y (A \in y$ only if $x \perp_i y)$

Proof Cases for \wedge and \vee are satisfied by definition of non-trivial prime \vdash -theories.

(LTR): Suppose $\neg A \in x$ and $A \in y$ for arbitrary x, y . We have $x \perp_A y$ by Definition 7. Then, we get $A \in y$ implies $x \perp_A y$ by conditional introduction. Lastly, by universal generalization on y , and existential generalization on \perp_A , we prove the left to right direction.

²⁵Note that there are many canonical incompatibility relations, given that it is not difficult to satisfy the definition. This fact comes handy when proving Lemma 5.

(RTL) We will prove this direction by contraposition. Suppose $\neg A \notin x$. We will prove $\forall \perp_i \exists y(A \in y \text{ and not-}x \perp_i y)$. For every formula B either $A \not\vdash_{\mathcal{N}} B$ or $A \vdash_{\mathcal{N}} B$. We will proceed by proof by cases. Suppose $A \not\vdash_{\mathcal{N}} B$. Let y be the set $\{A\}$. Then there is a non-trivial prime theory y' where $y' \supseteq y$ and $B \notin y'$ by Theorem 4 and by Definition 3. So, we have $A \in y'$ for a non-trivial prime theory. It is straightforward to show that $\text{not-}x \perp_B y'$ because $B \notin y'$ is sufficient to show that $\text{not-}x \perp_B y'$ by definition of the canonical \perp relation (Definition 7). Hence we get $\exists y(A \in y \ \& \ \text{not-}x \perp_B y)$.

Now suppose the other disjunct ($A \vdash_{\mathcal{N}} B$) and let y be the set $\{A\}$. First, by Theorem 2, there is a propositional variable p such that $A \not\vdash_{\mathcal{N}} p$ where $p \notin \text{Var}(A)$. Consequently, we can show that y can be extended to a non-trivial prime y' by Theorem 4 where $p \notin y'$, hence y' is a non-trivial prime theory. Now we need to show $\text{not-}x \perp_B y$. Given that $A \vdash_{\mathcal{N}} B$ we get that $\neg B \vdash_{\mathcal{N}} \neg A$ by [LC]. From this we can infer that $\neg B \notin x$ because otherwise $\neg A$ would be an element of x which contradicts with our assumption $\neg A \notin x$. Given the definition of the canonical \perp relation (Definition 7), $\neg B \notin x$ is sufficient for us to show that $\text{not-}x \perp_B y'$. So, we also get $\exists y(A \in y \ \& \ \text{not-}x \perp_B y)$. Hence, $\exists y(A \in y \ \& \ \text{not-}x \perp_B y)$ holds for every B given that we get it from both of the disjuncts, and consequently we get for every $\perp_i \in \mathbb{Q} \exists y(A \in y \ \& \ \text{not-}x \perp_i y)$. □

Lemma 6 $x \Vdash_c A$ iff $A \in x$.

Proof The proof is by induction on complexity for each direction. Cases for \wedge and \vee are trivial. For negation, we adapt the proof in [12] to our canonical model. (RTL) Suppose $\neg A \in x$. We have there is a $\perp_i \in \mathbb{Q}$, for each $y(A \in y$ only if $x \perp_i y)$ by Lemma 5. By induction hypothesis we have there is a $\perp_i \in \mathbb{Q}$, for each $y(x \Vdash_c A$ only if $x \perp_i y)$. Hence, we have $x \Vdash_c \neg A$.

(LTR) The other direction is proved with the reverse order of the proof above. □

Lemma 7 (HC) generalizes to all formulas, i.e., $x \Vdash_c A$ and $x \subseteq y$ implies $y \Vdash_c A$

Proof Its proof is trivial from Lemma 6 and the set theoretic properties of \in and \subseteq relations. □

Theorem 5 (Completeness) $\Gamma \vdash_{\mathcal{N}} A$ if $\Gamma \models_{\mathfrak{F}\mathfrak{M}} A$ for all multi-incompatibility frames $\mathfrak{F}\mathfrak{M}$.

Proof Given Lemmas 4,5,6,7, we conclude that our canonical model \mathfrak{M}_c is a model. Suppose $\Gamma \not\vdash_{\mathcal{N}} A$, then we can conclude that there is a non-trivial prime theory x in our canonical countermodel such that $C \in x$ for every $C \in \Gamma$ and $A \notin x$ from Theorem 4. □

Soundness and Completeness Theorems (Theorems 3 and 5) show us that multi-incompatibility frames can accommodate subminimal negation since [LC] is the only rule that governs negation in the proof system of \mathcal{N} . As discussed, we also preserve the philosophical motivations of the Australian Plan in multi-incompatibility

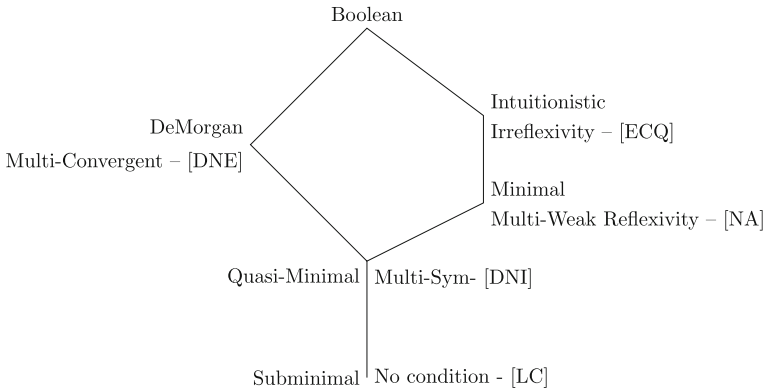


Fig. 4 New Kite of Negations

semantics. So, we have a semantics for subminimal negation on the Australian Plan. Lastly, we need to show that our multi-incompatibility frames can accommodate other negations already encompassed in incompatibility frames in order to claim that the multi-incompatibility frame semantics is in accordance with the Australian Plan. It is quite easy to show this. We can impose a condition like the following to restrict our frames to have a unique relation, i.e., making them identical to incompatibility frames:

$$(U-\perp)\forall x\forall y\forall \perp_m \forall \perp_n (x \perp_m y \text{ iff } x \perp_n y)$$

Once we have the incompatibility frames, we have the preminimal negation and consequently we can get the rest of the negations by imposing the well-known conditions on our frames [9, 10, 12]. However, even though the uniqueness condition validates $[\wedge\neg]$ it does not correspond to it, because it is stronger than needed. This means that the conditions that correspond to principles that characterize stronger negations are different from the conditions in incompatibility frames. If we look back on our discussion of subminimal negation on the refutability conception, we can see that there is a weaker condition that can get us stronger negations. We claimed that we get subminimal negation when we have multiple counteraxioms that are not closed under disjunction. So, if the set of counteraxioms is closed under disjunction, we can get stronger negations without imposing a uniqueness condition. Similarly, if we close our set of incompatibility relations under union, we can get stronger negations without having to impose a stronger condition such as $(U-\perp)$. This idea gives rise to novel frame conditions and a new kite of negations (See Fig. 4) where we associate well-known valid inferences (See Fig. 5) with frame conditions.

Following theorems are the correspondence results of these new conditions. I only provide the completeness directions given that the soundness directions are straightforward:

$$\frac{}{A \succ \neg\neg A} \text{ [DNI]} \quad \frac{}{A \wedge \neg A \succ \neg B} \text{ [NA]} \quad \frac{}{A \wedge \neg A \succ B} \text{ [ECQ]} \quad \frac{}{\neg\neg A \succ A} \text{ [DNE]}$$

Fig. 5 Stronger Negation Inferences

Fig. 6 Rules for **T** and **F**

$$\frac{}{\Gamma \succ \mathbf{T}} [\mathbf{T}] \quad \frac{}{\mathbf{F} \succ A} [\mathbf{F}] \quad \frac{}{\mathbf{T} \succ \neg \mathbf{F}} [\mathbf{NOR}]$$

Theorem 6 [DNI] is valid in a frame \mathfrak{FM} ($A \models_{\mathfrak{FM}} \neg\neg A$) iff \mathfrak{FM} satisfies Multi-Sym:

$$\forall x \exists \perp_i \forall \perp_m \forall y (y \perp_m x \supset x \perp_i y).$$

Proof Consider a frame where Multi-Sym does not hold, i.e. ($y \perp_m x$ & not- $x \perp_i y$) for some x , an arbitrary \perp_i , some \perp_m and some y . Let us define our valuation as $v(p) = \{w \mid x \sqsubseteq w\}$. Now, suppose $z \Vdash p$ for an arbitrary z to show that $y \Vdash \neg p$. From $z \Vdash p$ we get $x \sqsubseteq z$ from $v(p)$. Since we have $y \perp_m x$, $x \sqsubseteq z$ and $y \sqsubseteq y$, we get $y \perp_m z$ by Forwards which gives us $y \Vdash \neg p$. Lastly, since not- $(x \perp_i y)$, we get $x \not\Vdash \neg\neg p$. Since we also have $x \Vdash p$ by $v(p)$, we have $A \not\models \neg\neg A$.²⁶ \square

Theorem 7 [DNE] is valid in a frame \mathfrak{FM} ($\neg\neg A \models_{\mathfrak{FM}} A$) iff it satisfies Multi-Convergence:

$$\forall x \forall \perp_i \exists y (\text{not-}x \perp_i y \ \& \ \exists \perp_j \forall z ((\text{not-} y \perp_j z) \supset z \sqsubseteq x))$$

Proof The proof of this theorem is very similar to the proof in [24, p. 858] where the only difference lies in having multiple incompatibility relations and is left to the reader. \square

The proofs of the correspondence theorems for [ECQ] and [NA] require us to extend our language to \mathcal{L}^+ by adding the nullary connectives **T** and **F**. We add the usual truth conditions to include them in our semantics, where $x \Vdash \mathbf{T}$ and $x \not\Vdash \mathbf{F}$ for every x and the rules in Fig. 6 to include them in our proof theory.

Theorem 8 [NA] is valid in a frame \mathfrak{FM} ($A \wedge \neg A \models_{\mathfrak{FM}} \neg B$) iff \mathfrak{FM} satisfies the condition Multi-Weak Reflexivity:

$$\forall x \forall \perp_m \exists \perp_i \forall y (x \perp_m x \supset x \perp_i y)$$

Proof We have ($x \perp_m x$ & not- $x \perp_i y$) for some \perp_m and for some x, y and an arbitrary \perp_i . Let us set our valuation as $v(p) = \{x \perp_m x\}$. This gives us $x \Vdash p \wedge \neg p$. From this we can prove that $x \Vdash \neg \mathbf{T}$ given that [NA] is valid. However, since $y \Vdash \mathbf{T}$ and not- $x \perp_i y$, we can get for each $\perp_i \exists y (y \Vdash \mathbf{T} \ \& \ \text{not-}x \perp_i y)$ which gives us $x \not\Vdash \neg \mathbf{T}$. \square

Theorem 9 [ECQ] is valid in a frame \mathfrak{FM} ($A \wedge \neg A \models_{\mathfrak{FM}} B$) iff it satisfies Irreflexivity: $\forall x \forall \perp_i \text{not-}x \perp_i x$.

Proof Suppose $x \perp_i x$ for some \perp_i and x . We set our valuation $v(p) = \{x \mid x \perp_i x\}$. This gives us $x \Vdash A \wedge \neg A$, and consequently, $x \Vdash \mathbf{F}$. But this contradicts with our definition of \Vdash in a model of \mathfrak{FM} , where $x \not\Vdash \mathbf{F}$. \square

²⁶The proof of this direction for *incompatibility* frames can be found in [23, p. 263].

There are two different interesting issues with respect to the addition of the nullary connectives. The first one is related to the non-emptiness of the set of incompatibility relations in our frames and the rules mentioned above. If we were to allow an empty set of incompatibility relations, [NOR] rule would be invalidated, since in a model where there is no incompatibility relation all the negated formulas are false in every point. But, all the other rules of \mathcal{N} along with [FE] and [TI] would still be valid. As a consequence, it is reasonable to think that there are two different candidates for a subminimal logic in the language \mathcal{L}^+ , one where [NOR] is a primitive rule and we have non-empty set of incompatibility relations, and the other does not have [NOR] and allows for an empty set of incompatibility relations.²⁷ However, there are two different reasons to favor non-empty set of incompatibility relations. The first one is a technical reason. Note that in our proof of Lemma 5, we make an essential use of the variable sharing property, but when we extend our language to \mathcal{L}^+ we lose that property. In the absence of a different proof we will have to rely on the rule [NOR] to prove the Lemma 5, similar to the way it is implicitly used in [12, p. 240].²⁸ Our second reason to have [NOR] as a subminimally valid rule is conceptual. The question whether [NOR] is a subminimally valid rule amounts to asking whether $\neg\mathbf{F}$ is a theorem of a subminimal logic. Since on the refutability conception every counteraxiom is deducible from itself, every counteraxiom is refutable. In our language, we can express this fact with $\neg\mathbf{F}$. This provides some additional reason to suppose non-emptiness of Q on our frames.

Second, we had to extend our language to \mathcal{L}^+ to prove Theorems 8 and 9. This is quite expected, because the principles [NA] and [ECQ] characterize minimal and intuitionistic negations which on the refutability conception require expressions of theorems and anti-theorems, and in our language we express them with \mathbf{T} and \mathbf{F} . However, this yields an interesting consequence about subminimal negation in the multi-incompatibility frames, namely, we do not have to use \mathcal{L}^+ to have a sound and complete logic with subminimal negation, yet, they are required for intuitionist and minimal negations. Hence, subminimal negation shows features that are different from its cousins on the refutability conception when it is accommodated on the Australian Plan.

²⁷In fact, one can prove that [NOR] corresponds to the non-emptiness restriction. For the completeness direction, consider a frame that allows for an empty set of incompatibility relations. A model built on that frame will have $v(\mathbf{T}) = U$ and for every $x \in U$. But, we will also have $x \not\models \neg\mathbf{F}$, since there is no incompatibility relation. Hence, there will be a point where $x \Vdash \mathbf{T}$ but $x \not\models \neg\mathbf{F}$. Soundness direction is left to the reader.

²⁸Also note that [NOR] is provable for every negation that validates [DNI], [LC], [FE] and [CUT]:

$$\frac{\frac{}{\mathbf{T} \succ \neg\neg\mathbf{T}} [DNI] \quad \frac{\frac{}{\mathbf{F} \succ \neg\mathbf{T}} [\mathbf{FE}]}{\neg\neg\mathbf{T} \succ \neg\mathbf{F}} [\mathbf{LC}]}{\mathbf{T} \succ \neg\mathbf{F}} [\mathbf{CUT}]$$

This shows us that (1) intuitionist and minimal negation should have non-empty set of incompatibility relations given the correspondence result and (2) the lack of variable sharing property in these logics will not affect our completeness theorem since [NOR] is available.

4 Conclusion and Future Work

The problem we tried to tackle in this paper was to fully accommodate subminimal negation on the Australian Plan. We showed that multi-incompatibility frames can accommodate subminimal negation by providing the soundness and completeness results of \mathcal{N} . We also argued that multi-incompatibility frames preserve all the philosophical motivations behind the negation on the Australian Plan and showed that they accommodate the negations of incompatibility frames. Hence, we successfully solved the problem of accommodating subminimal negation on the Australian Plan. However, there is more work to be done with multi-incompatibility frames in future work. For instance, it may be interesting to investigate whether we can use multi-incompatibility frames to account for another closely related type of modal negation—negation as unnecessary [12]— and provide similar results for dual subminimal negation (and all the other dual negations). In other words, we can investigate the limitations of the multi-incompatibility approach when modeling negation. One issue is whether we can preserve the relationship between compatibility and incompatibility relations while holding the philosophical interpretation of multi-incompatibility relations. Even though we provided a plausible reading of having multiple incompatibilities, we have not investigated whether that reading can translate to having multiple compatibility relations. In other words, we justify using multiple incompatibility relations by associating them with having different reasons to be incompatible, but can we use a similar justification for compatibility relations? This would be a highly desirable translation in case one wants to hold the interdefinability of incompatibility and compatibility. Moreover, one can further investigate the pragmatic and metaphysical interpretations of incompatibility with respect to the multiple-incompatibility frames. Throughout the paper, we have not favored one interpretation over the other and assumed that both of the accounts are suitable for multi-incompatibility frames. However, a fully developed account of multi-incompatibility frames for either of the interpretations may lead to fruitful theories in philosophy of language or metaphysics.

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References

1. Belnap, N.D. In J.M. Dunn, & G. Epstein (Eds.) (1977). *A useful 4-valued logic*. Reidel: Dordrecht.
2. Berto, F. (2015). A modality called negation. *Mind*, 124(495), 761–793. <https://doi.org/10.1093/mind/fzv026>.
3. Berto, F., & Restall, G. (2019). Negation on the Australian plan. *Journal of Philosophical Logic*, 48, 1–26. <https://doi.org/10.1007/s10992-019-09510-2>.
4. Colacito, A., de Jongh, D., & Vargas, A L (2017). Subminimal negation. *Soft Computing*, 21(11), 165–174. <https://doi.org/10.1007/s00500-016-2391-8>.
5. Copeland, B.J. (1979). On when a semantics is not a semantics: Some reasons for disliking the Routley-Meyer semantics for relevance logic. *Journal of Philosophical Logic*, 8, 399–413.
6. Curry, H. (1963). *Foundations of mathematical logic*. New York: Dover Publication.
7. Došen, K. (1986). Negation as a modal operator. *Reports on Mathematical Logic*, 20, 15–27.
8. Dummett, M. (1991). *The logical basis of metaphysics*. Cambridge, Massachusetts: Harvard University Press.
9. Dunn, J.M. In J.E. Tomberlin (Ed.) (1994). *Star and perp: Two treatments of negation*. California: Ridgeview Publishing Company.
10. Dunn, J.M. In H. Wansing (Ed.) (1996). *Generalised ortho negation*. Berlin: Walter De Gruyter.
11. Dunn, J.M. (1976). Intuitive semantics for first-degree entailments and ‘coupled trees’. *Philosophical Studies*, 29, 149–168. <https://doi.org/10.1007/BF00373152>.
12. Dunn, J.M., & Zhou, C. (2005). Negation in the context of gaggle theory. *Studia Logica*, 80, 235–264. <https://doi.org/10.1007/s11225-005-8470-y>.
13. Fine, K. (1974). Models for entailment. *Journal of Philosophical Logic*, 3(4), 347–372. <https://doi.org/10.1007/BF00257480>.
14. Hazen, A. (1992). Subminimal negation. Philosophy Department PrDOI 1/92, University of Melbourne, Melbourne.
15. Hazen, A.P. (1995). Is even minimal negation constructive?. *Analysis*, 55(2), 105–107. <https://doi.org/10.2307/3328907>.
16. Horn, L.R., & Wansing, H. In E.N. Zalta (Ed.) (2020). *Negation*, Spring 2020: Metaphysics Research Lab, Stanford University.
17. Humberstone, L. (2011). *The connectives*. Cambridge: MIT Press.
18. Humberstone, L. (2016). *Philosophical applications of modal logic*. London: College Publications.
19. Logan, S.A. (2020). Putting the stars in their places. *Thought: A Journal of Philosophy*, 9(3), 188–197. <https://doi.org/10.1002/tht3.462>.
20. Meyer, R., & Martin, E.P. (1986). Logic on the Australian plan. *The Journal of philosophical logic*, 15(3), 305–332. <https://doi.org/10.1007/BF00248574>.
21. Omori, H., & Wansing, H. (2017). 40 years of FDE: An introductory overview. *Studia Logica*, 105, 1021–1049. <https://doi.org/10.1007/s11225-017-9748-6>.
22. Price, H. (1990). Why ‘not’?. *Mind*, 99(394), 221–238. <https://doi.org/10.1093/mind/XCIX.394.221>.
23. Restall, G. (2000). *An introduction to substructural logics*. London: Routledge.
24. Restall, G. (2000). Defining double negation elimination. *Logic Journal of the IGPL*, 8(6), 853–860. <https://doi.org/10.1093/jigpal/8.6.853>.
25. Restall, G. In D. Gabbay, & H. Wansing (Eds.) (1999). *Negation in Relevant Logics: How I Stopped Worrying and Learned to Love the Routley Star*. Dordrecht: Kluwer Academic Publishers.
26. Routley, R., & Meyer, R.K. (1972). The semantics of entailment–2. *Journal of Philosophical Logic*, 1, 53–73. <https://doi.org/10.1007/BF00649991>.
27. Routley, R., & Routley, V. (1972). Semantic of first degree entailment. *Nous*, 6, 335–359. <https://doi.org/10.2307/2214309>.
28. Schotch, P., & Jennings, R. In R. Hilpinen (Ed.) (1981). *Non-Kripkean deontic logic*. Dordrecht: D. Reidel Publishing Company.