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Rewriting the History of Connexive Logic

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Abstract

The "official" history of connexive logic was written in 2012 by Storrs McCall who argued that connexive logic was founded by ancient logicians like Aristotle, Chrysippus, and Boethius; that it was further developed by medieval logicians like Abelard, Kilwardby, and Paul of Venice; and that it was rediscovered in the 19th and twentieth century by Lewis Carroll, Hugh MacColl, Frank P. Ramsey, and Everett J. Nelson. From 1960 onwards, connexive logic was finally transformed into non-classical calculi which partly concur with systems of relevance logic and paraconsistent logic. In this paper it will be argued that McCall's historical analysis is fundamentally mistaken since it doesn't take into account two versions of connexivism. While "humble" connexivism maintains that connexive properties (like the condition that no proposition implies its own negation) only apply to "normal" (e.g., self-consistent) antecedents, "hardcore" connexivism insists that they also hold for "abnormal" propositions. It is shown that the overwhelming majority of the forerunners of connexive logic were only "humble" connexivists. *Their* ideas concerning ("humbly") connexive implication don't give rise, however, to anything like a non-classical logic.

Keywords History of logic · Connexive logic · Ex contradictorio quodlibet · Relevance logic · Paraconsistent logic

1 Introduction

What is connexive logic? According to [57]:

> Systems of connexive logic are *contra-classical* in the sense that they are neither subsystems nor extensions of classical logic. Connexive logics have a standard logical vocabulary and comprise certain non-theorems of classical logic as theses. Since classical propositional logic is Post-complete, any additional axiom in its

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> language gives rise to the trivial system, so that any non-trivial system of connexive logic will have to leave out some theorems of classical logic. The name 'connexive logic' was introduced by Storrs McCall [...] and suggests that systems of connexive logic are motivated by some ideas about coherence or connection between the premises and the conclusions of valid inferences or between the antecedent and the succedent (consequent) of valid implications. The kind of coherence in question concerns the meaning of implication and negation […] One basic idea is that no formula provably implies or is implied by its own negation.

The *basic idea* of the "connexivity" of an implication operator has been paraphrased by Pizzi & Williamson as follows:

> Consider the following claims about an intuitively conceived relation of implication:
> (1) No proposition implies its own negation.
> (2) No proposition implies each of two contradictory propositions.
> (3) No proposition implies every proposition.
> (4) No proposition is implied by every proposition.
> (1) is often called 'Aristotle's Thesis'. (2) will be called here 'Weak Boethius' Thesis' [...] (1)–(4) have often been found plausible in the history of logic, although the historical details cannot be discussed here. ([47], 569)

In contrast to Pizzi & Williamson's objective, the main aim of the present paper just is to take a closer look at the historical details and check whether, and to which extent, Aristotle, Boethius and other ancient logicians may have defended the above theses. For this purpose, it will be of utmost importance to distinguish two variants: *restricted*, or "humble" connexivism and *unrestricted*, or "hardcore" connexivism.[1] Before explaining this distinction in more detail, let me answer an objection raised by a referee who thinks that modern systems of connexive logic only deal with conditions (1) and (2) of Pizzi & Williamson's list but not with (3) and (4).

The earliest occurrence of the *notion* of connexive implication appears to be McCall's paper [38] where (1) and (2) are defined as characteristic axioms of connexive logic. McCall refers to (1) (or, more exactly, to its formal counterpart $\neg(\neg p \rightarrow p)$) as *Aristotle's thesis* and to (2) (or, more exactly, to its formal counterpart $(q \rightarrow r) \rightarrow \neg(q \rightarrow \neg r)$) as *Boethius' thesis*.[2] It is true, though, that McCall nowhere considered Pizzi & Williamson's principles (3) and (4) as characterizing the connexivity of implication. Yet there are two compelling reasons why contemporary investigations of connexive logic should take also (3) and (4) into account.

[1] The terms 'humble connexivity' and 'hardcore connexivity' have been coined in [16, 25], respectively. For a recent defence of "humble" connexivism cf. [13]; similarly, in [8] it is argued that any "Evidential Conditional" should only be "humbly" connexive.

[2] As a matter of fact, McCall makes use of the Polish notation and formalizes the theses as '*NCNpp*' and '*CCqrNCqNr*', respectively. Note that the usual "translation" of Aristotle's thesis into ordinary language ("No proposition is entailed by its own negation") contains the "passive" verb form 'is entailed' and might thus be distinguished from its "active" variant "No proposition entails its own negation" (i.e., Pizzi & Williamson's (1)). Similarly, besides the "active" version of Boethius thesis (as Pizzi & Williamson's (2)) one can consider the "passive" variant 'No proposition is implied by each of two contradictory propositions'. The latter principle had already been defended by Aristotle; cf. section 2 below.

First, (3) is an *immediate consequence* of (1) because, if (3) doesn't hold, i.e., if there exists some proposition q which is implied by *every* proposition p, then q will be implied *both* by p *and* by $\neg p$, in violation of (1). Similarly, (4) is an immediate consequence of the "passive" variant of (1), i.e., of 'No proposition is implied by its own negation'. Hence it remains theoretically possible to construct non-classical logics which satisfy (3)and/or(4) without satisfying (1).

Second, a good deal of the medieval discussions of Aristotle's theses centres around the validity of the "anti-connexive" principles "Necessarium ex quodlibet" and "Ex impossibili quodlibet":

> NEQ For every proposition q: If q if necessarily true, then q is implied, or entailed, by any proposition p.

> EIQ For every proposition p: If p if necessarily false (or impossible), then p implies, or entails, any arbitrary proposition q.

Clearly, if either of these principles holds, then there exist certain *exceptions* to the validity of Aristotle's thesis. No proposition p *entails* its own negation *–unless p itself is impossible*; and no proposition q *is entailed by* its own negation, *unless it is necessary*! This observation (made already in the thirteenth century by Robert Kilwardby, in the fourteenth century by Walter Burleigh, and in the seventeenth century by Gottfried Wilhelm Leibniz) gave rise to the subsequent distinction between "hardcore" connexivism and "humble" connexivism.

While "hardcore" connexivism insists that *no proposition at all* violates the crucial conditions (1)–(4), "humble" connexivism is ready to admit that there may be some *exceptions*. E.g., one may plausibly think of *tautological* propositions as exceptions to condition (4). After all, a tautological proposition q *can't be false*; hence it can't ever happen that p is true and yet q be false. Thus, q is implied, or entailed, by *every* proposition p, provided that one subscribes to the basic idea of *strict implication*:

> STRICT p strictly implies q, formally $p \rightarrow_{str} q$, if and only if ('iff', for short) it is impossible that p is true and yet q is false.

This semantic definition entails the validity of NEQ. A "humble" connexivist will therefore suggest modifying Pizzi & Williamson's condition (4) so that only:

> (4*) No *non-tautological* proposition is implied by every proposition.

Similarly, since a self-contradictory proposition p can't ever be true, it can never happen that (p is true and q is false). Hence definition STRICT also warrants the validity of EIQ. So, again, a "humble" connexivist will suggest modifying Pizzi & Williamson's condition (3) so that only:

> (3*) No *self-consistent* proposition implies every proposition.

As a matter of course, corresponding qualifications apply to the remaining principles (1) and (2) as well. Their "humble" versions say:

(1*) No *self-consistent* proposition implies its own negation.
(2*) No *self-consistent* proposition implies each of two contradictory propositions.

None of these *restricted* theses gives rise to anything like a contra-classical logic. As a matter of fact, (1*) – (4*) are *theorems* of almost all systems of strict implication! E.g., in order to prove (1*), one only has to assume that proposition p implies its own negation, $p \rightarrow_{str} \neg p$; if the implication operator is *reflexive*, one further has $p \rightarrow_{str} p$; thus, if $\rightarrow_{str}$ also satisfies the *rule of conjunction.*

Conj 1 $(p \rightarrow q), (p \rightarrow r) \Rightarrow (p \rightarrow (q \wedge r))$,

one obtains $(p \rightarrow_{str} (p \wedge \neg p))$, i.e., p strictly implies a contradiction, so that p itself is self-*inconsistent*.

As the formalization of principle Conj 1 indicates, we are using here, besides '¬', '∧', '∨' as symbols for the truth-functional operators of *negation*, *conjunction*, and *disjunction*:

- '→' as a symbol for arbitrary *implication* operators (e.g., material implication, strict implication, counterfactual conditionals, etc.);
- '◊' and '□' as symbols for the modal operators 'it is possible that' and 'it is necessary that';
- '⇒' as a symbol for logical *inferences* (from premises p_1, ..., p_n to a conclusion q).

In his contribution to the *Handbook of the History of Logic*, [39], Storrs McCall presented the "official" version of the "History of Connexivity". It is argued there:

- that connexive implication was *discovered* more than 2300 years ago by the ancient logicians Chrysippus and/or Aristotle;
- that it was *defended* by many medieval logicians like Boethius, Abelard, Kilwardby, and Paul of Venice;
- that it was *rediscovered* by Lewis Carroll, Hugh McColl, Frank P. Ramsey, and Everett J. Nelson.

In the following sections, it will be shown that, with the only exception of the Stoic Chrysippus (and his modern follower Nelson), the vast majority of these logicians had only "humble" versions of the connexive principles in mind.

2 Aristotle (ca. 384–322 BC)

In an oft-quoted passage from *Prior Analytics* (57b3–14) Aristotle showed (according to McCall's interpretation in [39], 415):

> [...] that two implications of the form 'If p then q' and 'If not-p then q' cannot both be true. The first yields, by contraposition, 'If not-q then not-p', and this together with the second gives 'If not-q then q' by transitivity. But, Aristotle says, this is impossible: a proposition cannot be implied by its own negation.

This interpretation is largely undisputed. Aristotle evidently believed that “no” proposition p is implied, or entailed, by its own negation:

ARIST 1 $\neg(\neg p \rightarrow p)$.

He therefore believed that “no” proposition q can be implied, or entailed, by each of two contradictory propositions p and $\neg p$:

ARIST 2 $\neg[(p \rightarrow q) \wedge (\neg p \rightarrow q)]$.

For, by means of the principles of *contraposition* and *transitivity*:

CONTRA If $(p \rightarrow q)$, then $(\neg q \rightarrow \neg p)$
TRANS If $(p \rightarrow q)$ and $(q \rightarrow r)$, then $(p \rightarrow r)$,

the Stagirite proved that “Aristotle’s Second Thesis”, ARIST 2, follows from “Aristotle’s First Thesis”, ARIST 1.

The decisive question, however, is whether Aristotle meant these principles in the sense of a “humble” connexivism or in the sense of a “hardcore” connexivism.[3] In other words, would he have been willing to grant that *tautological* propositions represent possible counter-examples to ARIST 1, 2; or would he have insisted that even a tautology is not entailed by its own negation, i.e., by a self-contradictory proposition? Unfortunately, Aristotle’s logical writings don’t provide enough evidence to decide this issue in a definite way. His logic is primarily not a *propositional* logic, but a *term* logic which basically deals with the four *categorical forms*:

- Universal affirmative proposition (UA) Every S is P
- Universal negative proposition (UN) No S is P
- Particular affirmative proposition (PA) Some S is P
- Particular negative proposition (PN) Some S isn’t P.

When developing his theory of the syllogism, Aristotle carries out *propositional* inferences (such as TRANS and CONTRA) only *implicitly*. In particular, he nowhere *explicitly* stated the basic laws of *conjunction*:

CONJ 2 $(p \wedge q) \rightarrow p$
CONJ 3 $(p \wedge q) \rightarrow q$.

[3] Routley & Montgomery ([51], 83) seem to have been the first contemporary logicians who mentioned that Aristotle might have understood the connexive principles in the weak or “humble” sense confined to *contingent* propositions.

Yet, as has been argued elsewhere,[4] it seems very plausible to assume that he would have accepted the following *refutation* of the “hardcore” version of Arist 1:

1. $(p \wedge \neg p) \rightarrow p$ Conj 2
2. $(p \wedge \neg p) \rightarrow \neg p$ Conj 3
3. $\neg p \rightarrow \neg(p \wedge \neg p)$ Contra, (1)
4. $\neg\neg p \rightarrow \neg(p \wedge \neg p)$ Contra, (2).

As lines (3) and (4) show, one and the same proposition, $\neg(p \wedge \neg p)$, is implied by each of the contradictory premises $\neg p$ and $\neg\neg p$! Therefore, it seems safe to conclude that Aristotle was only a “humble” connexivist.[5] This diagnosis was made, incidentally, already in the fourteenth century by Walter Burleigh. In *De Puritate Artis Logicae* (written around 1325) he explained:

> And if it is said that contradictory propositions cannot follow from the same antecedent [...] I answer that the same consequent doesn’t follow from the affirmation and the negation of the same antecedent, *unless the negation of that consequent contains a contradiction. And in this way Aristotle has to be understood.*[6]

3 Chrysippus (ca. 279–206 BC)

According to [39], 415, connexive implication was “first defined in the 4th Century B.C.” by a logician who defended the following conception:

> (3) And those who introduce the notion of connection say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.[7]

This conception is contrasted with (1) *material* implication (as defended by Philo) and with (2) *strict* implication (as defended by Diodorus) and with yet another conception (probably defended by the Peripatetics). The Kneales point out that “the names of the

[4] Cf. section 3 of [26]; in particular it is shown there that Patzig’s critique of Aristotle, [45], is mistaken, and that the critique both of Weidemann, [58], and of Strobach & Malink [55] rest on the untenable assumption that Aristotle was thinking of *material* implication instead of strict implication.

[5] Interestingly, Priest takes the simultaneous derivability of $\neg(p \wedge \neg p)$ from p and from $\neg p$ not as an indication that Aristotle intended his connexive thesis only in a “humble” sense. *Presupposing* that Aristotle was a “hardcore” connexivist, Priest instead concludes that “as must have been obvious to Aristotle, a contradiction cannot entail both conjuncts, and so, presumably, either conjunct” [48: 142]. Thus, according to Priest, Aristotle *rejected* the laws of conjunction (at least for the special case where one conjunct is the negation of the other). I leave it to the reader to judge which of these rivalling interpretations is historically more plausible.

[6] Cf. [5], 74/10–16: “Et si dicatur, quod ad idem antecedens non sequuntur contradictoria [...] Dico quod ad idem antecedens affirmatum et negatum non sequitur idem consequens, nisi oppositum illius consequentis includat contradictoria. Et sic debet intelligi dictum Aristotelis.”

[7] McCall here quotes [17], 129, who in turn translated a passage from Sextus Empiricus’ *Outlines of Scepticism*.

authors of the third and fourth views have been lost" ([17], 129), but they guess "that the third view may be that of Chrysippus". Although McCall refrains from referring to the proponent of connexive implication by the name 'Chrysippus', we will here adopt this identification, no matter whether it is historically correct or not.

Now McCall hastens to add that "it follows from this definition [i.e. from (3)] that no conditional of the form 'If p then not-p' can be true, since the contradictory of not-p, i.e. p, is never incompatible with p" ([39], 415). This claim, however, is *not* (entirely) *correct*! The relation of *compatibility* is "normally" interpreted as *compossibility* so that 'p is compatible with q' becomes equivalent to $\Diamond(p \wedge q)$. Given this understanding, the Chrysippian account (3) would simply coincide with the Diodorean conception, because – according to the standard definition Strict – p strictly implies q iff $\neg \Diamond(p \wedge \neg q)$, or in other words iff "the contradictory of its consequent is [*incompossible*] with its antecedent". But – as was stressed in section 2 above – strict implication necessarily gives rise to the "paradoxes of strict implication" so that $\rightarrow_{str}$ *violates* the "hardcore version" of connexive implication. In particular, the self-inconsistent proposition $(p \wedge \neg p)$ strictly implies its own negation!

McCall, in contrast, maintains that for Chrysippus $(p \wedge \neg p)$ does *not* imply $\neg(p \wedge \neg p)$ because no proposition at all, not even a self-contradictory one, is "incompatible with itself".[8] Hence Chrysippus's understanding of the compatibility relation can't be "normal" but rather requires:

Chrys 1 (Absolutely) Each proposition is compatible with itself.

Unfortunately, McCall didn't provide any textual evidence *why* Chrysippus made this assumption. An explanation may, however, be obtained when one considers the following strange conditional which Diodorus, but not Chrysippus, accepted as sound:

Atoms "If atomic elements of things do not exist, then atomic elements of things do exist." ([17], 129)

This proposition has the structure $(\neg p \rightarrow p)$, where 'p' abbreviates the dogma of the Stoic's conception of nature 'Each material thing is composed of indivisible parts'. The *existence* of "atomic elements of things" thus is thought to be (nomologically) *necessary*. Hence the antecedent of Atoms is "impossible" while its consequent is "necessary". Diodorus, a defender of *strict* implication, accordingly subscribed to the truth of Atoms because – in accordance with Eiq and Neq – the "impossible" antecedent $\neg p$ implies the "necessary" consequent p.

Chrysippus, in contrast, *denied* the soundness of Atoms. Prima facie, this refusal might be due to Chrysippus's reluctance to consider the dogma of the existence of atoms as *necessary*. After all, from a modern perspective, it may be doubted whether the assumption of the existence of atoms is *true* at all.[9] However, O'Toole & Jennings ([43], 481–2) argued "that the axiōma 'Atomic elements of existents are without parts' is conceptually or analytically true, and hence necessary [...] *according to the versions*

[8] Cf. [39], 415: "Thus even 'p & ~ p' is not incompatible with itself, and 'If p & ~ p, then not-(p & ~ p)' is connexively false."

[9] The hypothesis that Chrysippus's rejection of Atoms might be due to his doubts concerning the "necessity" of the existence of atoms was put forward in [26] but revised in [28].

of necessity both of Diodorus and Chrysippus." Therefore, one would better look for an alternative explanation why Chrysippus rejected Atoms.

In their edition of Sextus Empiricus's *Outlines of Scepticism*, Annas & Barnes paraphrase Chrysippus's definition by the requirement that "the opposite of its consequent *conflicts* with its antecedent" ([2], 12). O'Toole & Jennings wittily explain that the primary meaning of the verb 'μαχεται':

> [...] is *to fight* or *to battle* or *to war*. Now, one would hardly want to translate the term 'μαχεται' in a logical context as 'fights' or 'battles' or 'wars'. [...] Probably 'conflicts' is just the right compromise. It is bloodless enough for a logic book, yet it remains faithful to the etymological origins of the Greek term. ([43], 490)

Thus, for Chrysippus, the incompatibility of two propositions *p*, *q* has to arise from a "conflict" between *p and q*: One proposition, say *p*, somehow has to *contradict* the *other* proposition, *q*; otherwise, there can be no "war", no "battle" and no "conflict" between them. In particular, as captured by Chrys 1, no proposition can be incompatible with itself![10] Or, as Sanford put it, proposition Atoms fails to satisfy Chrysippus's definition of a sound conditional simply because "the denial of the main clause is *not incompatible* with the if-clause. Indeed, it *is* the if-clause" ([52], 24).

Given this interpretation of the notion 'μαχεται', Chrysippus's idea of a sound conditional may be formalized in terms of modern modal logic as follows:

Chrys 2 $p \rightarrow_{\mathrm{Chr}} q =_{\mathrm{df}} \neg \Diamond (p \wedge \neg q) \wedge \Diamond p \wedge \Diamond \neg q$.[11]

On the one hand, the antecedent has to be incompatible with the negation of the consequent. On the other hand, this incompatibility must not be caused by one of the propositions alone, i.e., *p* must be possible, and $\neg q$ must be possible, too; otherwise, there is no real "conflict" *between* them. Chrys 2 thus reconciles the two faces of Janus' head in an elegant way. Because of condition $\neg \Diamond(p \wedge \neg q)$, Chrysippian implication remains sort of a *strict* implication, while condition $\Diamond p \wedge \Diamond\neg q$ also makes it *connexive*.[12]

4 Manlius Severinus Boethius (Ca. 480–525)

McCall ([39]: 416–417) spends comparatively much space on the role which Boethius played for the development of connexive logic. He starts with quoting the famous passage from *De hypotheticis syllogismis*:

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[10] Of course, from a *contemporary* point of view, every impossible proposition is incompatible (i.e., *incompossible*) with any proposition, hence also with itself!

[11] This definition is closely related not only to Nasti De Vincenti's proposal in [41] but also to Priest's suggestion to "[…] define a connexivist conditional, $\alpha \rightarrow \beta$, as $\Diamond\alpha \wedge (\alpha \Rightarrow \beta) \wedge \Diamond\neg\beta$ [… where] $\Rightarrow$ can be any strict conditional […]" ([48]: 146).

[12] The logical properties of the "hybrid" operator $\rightarrow_{\mathrm{Chr}}$ are investigated in [28].

'Si est *A*, cum sit *B*, est *C*; [...] atqui cum sit *B*, non est *C*; non est igitur *A*.'[13]

Next McCall "transliterates" the underlying idea into the inference:

If *p*, then if *q* then *r*,
if *q* then not-*r*,
therefore, not-*p*

and he justifies this transformation as follows:

> The reasoning that led Boethius to assert the validity of this schema was presumably this. Since the two implications 'If *q* then *r*' and 'If *q* then not-*r*' are incompatible, the second premiss contradicts the consequent of the first premiss. Hence, by *modus tollens*, we get the negation of the antecedent of the first premise, namely 'not-*p*'. [...] The corresponding conditional, 'If $q \to r$ then $\neg(q \to \neg r)$' will be denoted Boethius' thesis, and serves with the thesis $\neg(p \to \neg p)$ as the distinguishing mark of connexive logic.

"Boethius' Thesis" may be regarded as an "active" counterpart of the "passive" principle Arist 1 insofar as Aristotle's expression 'is implied by' is simply replaced by 'implies'. This "active" principle shall here be reformulated as:

Boeth If $(p \to q)$, then $\neg(p \to \neg q)$.

McCall rightly stresses that in Boethius's opinion the two implications $(p \to q)$ and $(p \to \neg q)$ "*contradict*" each other and that they are thus "*incompatible*". Other logicians, however, believe that they are *contradictories*, which means that besides Boeth also the "Converse Boethius Thesis" holds:

$\text{Boeth}_{\text{conv}}$ If $\neg(p \to q)$, then $(p \to \neg q)$.

Thus, e.g., the Kneales maintained that Boethius "said that the negative [!] of *Si est A, est B* was *Si est A, non est B*" ([17], 191), and they supported this claim by referring to a passage from *De hypotheticis syllogismis* where Boethius explained:

> Some hypothetical propositions, however, are affirmative and others are negative […] Affirmatives are when we say 'If it is *A*, it is *B*', or 'If it is not *A*, it is *B*'; negative 'If it is *A*, it is not *B*', 'If it is not *A*, it is not *B*'. For it depends on the consequent whether the proposition is judged to be affirmative or negative.[14]

[13] Cf. [40], 851B-C; in Migne's edition the text bears the title "De Syllogismo Hypothetico Libri Duo"; McCall and others usually refer to it as "De hypotheticis syllogismis".

[14] Cf. [40], 843 D: "Sunt autem hypotheticae propositiones, aliae quidem affirmativae, aliae negativae […] affirmativa quidem, ut cum dicimus, si est *a*, est *b* si non est *a*, [est] *b* negativa, si est *a* non est *b* si non est *a*, non est *b*. Ad consequentem enim propositionem respiciendum est., ut an affirmativa an negativa sit propositio judicetur". In Migne's edition the second example 'si non est *a* est *b*' is erroneously formulated as 'si non est *a* non est *b*'; this mistake was already pointed out in [53], 18, fn. 27.

However, as the last sentence of this quotation makes clear, Boethius is far from maintaining that 'Si est *A*, non est *B*' would be "*the negative*", i.e., the negation of 'Si est *A*, est *B*'. Boethius only *classifies* the conditionals into "affirmative" and "negative" ones, depending on whether the *consequent* is affirmative (like 'est *B*') or negative (like 'non est *B*').

A referee of this paper drew my attention to Priest's paper [48] where it was similarly maintained (p. 144) that "Boethius [...] states, for example, that 'α entails $\neg\beta$' is the negation of 'α entails β' [...] He went even as far as to endorse the converse of [BOETH]". Priest tried to back up this claim by referring (in fn. 11) to "Kneale and Kneale (1982), p. 191, who call these [principles] 'mistakes'." As the foregoing explanations should have made clear, however, the only mistake is to attribute to Boethius the view that the falsity of, e.g., '$\alpha \rightarrow \beta$' entails the truth of '$\alpha \rightarrow \neg\beta$'.[15]

McCall goes on to discuss the question whether the passage "Si est *A*, cum sit *B*, est *C* ..." may be interpreted as a principle of *propositional logic* at all. Philosophers like John Marenbon and Christopher Martin had objected that Boethius' logic is a *term logic* and hence "cannot treat sentential connectives as propositional [...] operators".[16] Now it can hardly be denied that Boethius's logic is a term logic in the sense that the variables *A*, *B*, ... stand for terms or concepts like 'animal', 'man', etc. Thus, immediately after the quoted passage, Boethius illustrates his thesis by the example "Si est *A* homo, cum sit *B* animatum, est *C* animal" ([40], 852B). Moreover, as has been argued by Whity ([59]), the Boethian formula 'Si est *A*' should always be understood in the sense of 'If (something, *x*) is *A*', so that the conditional 'Si est *A* est *B*' expresses the universal affirmative proposition 'Every *A* is *B*'. Accordingly for Boethius (as well as for Abelard) the syllogistic mood BARBARA takes the shape "*Si est A, est B; si est B, est C; ergo si est A, est C*".[17]

But, trivially, *term logic* also contains *propositions*, and propositions can be negated. Therefore, negation remains an indispensable ingredient of term logic, and it seems safe to conclude that McCall's *propositional* "transliteration" ('If *p*, then if *q* then *r*; if *q* then not-*r*; therefore not-*p*') correctly reflects Boethius' *term-logical* principle 'Si est *A*, cum sit *B*, est *C*; atqui cum sit *B*, non est *C*, non est igitur *A*'. In this sense McCall's conclusion ([39], 417) "that, for Boethius, $\neg(p \rightarrow \neg q)$ follows from $p \rightarrow q$" is entirely correct, and Boethius may rightly be considered as an advocate of principle BOETH. Somewhat more exactly, the *term-logical* interpretation of "Boethius' Thesis" says: If the conditional 'If (*x*) is *B*, then (*x*) is *C*' is true, then the "contrary" conditional 'If (*x*) is *B*, then (*x*) is not-*C*' can't be true as well, i.e.

$\text{BOETH}_{\text{term}}$ If 'Every *B* is *C*', then not also 'Every *B* is not-*C*'.

This principle just expresses the traditional doctrine of the *contrariety* of *universal affirmative* and *universal negative* propositions (which is turn is basically equivalent to

[15] In this connection it should also be noticed that in his contribution to the *Stanford Encyclopedia of Philosophy*, Wansing originally adopted Kneale's interpretation by explaining "Moreover, Boethius, for instance, holds that 'if *A* then ~*B*' is the negation of 'if *A* then *B*'." In the meantime, however, Wansing corrected this as follows: "Boethius here draws a distinction between affirmative and negative conditionals and explains that negative conditionals have the form 'if *a*, then not *b*' and 'if not *a*, then not *b*.' This statement is quite different from the reading offered by Kneale and Kneale". Cf. the (2016) vs. (2020) version of [57].

[16] Cf. [36], 288, and the discussion in [39], 416–417.

[17] Cf. [17], 191 and the discussion of the proper interpretation of Boethius's formula 'Si est *A*' in chapter 22 of [29].

the law of *subalternation* saying that the UA entails the PA). Hence it is obvious that Boethius would never have accepted the "Converse Boethius Thesis" since it amounts to the assumption that from the *falsity* of the UN, i.e., from the truth of the PA, one might infer the truth of the UA!

To conclude this section, let us briefly discuss whether Boethius was a "hardcore" or only a "humble" connexivist. As has been argued elsewhere, the strong parallelism between the connexive principles and their term-logical counterparts supports the following argument.[18] The inference of *subalternation* holds in the vast majority of cases where the subject term, *S*, is "*normal*", i.e., not "empty". Thus, one may reasonably assume that the intuitions, which guided traditional logicians to consider the laws of subalternation as valid, were based on the tacit assumption that the terms of the categorical forms are "normal". This hypothesis is at least much more likely than to assume that they would have been willing to defend these laws also for "empty" terms. In a similar way, the connexivist inference from $(p \rightarrow q)$ to $\neg(p \rightarrow \neg q)$ holds whenever the antecedent is "*normal*", i.e., self-consistent. Therefore, one may reasonably assume that the intuitions, which guided Aristotle and Boethius in putting forward their laws of connexive implication, were based on the tacit assumption that the antecedents are "normal". This hypothesis, again, appears much more likely than to assume that they would have been willing to defend their laws also for "abnormal", self-inconsistent propositions.

5 Peter Abelard (1079–1142)

McCall ([39]: 417) points out that in Abelard's *Dialectica*, "we find [...] four connexive principles that Abelard makes 'the centrepieces of his theory of conditionals'", namely Arist 1, Arist 2, Boeth, and a "[v]ariant of Aristotle's thesis, $\neg(p \rightarrow \neg p)$."[19] For textual evidence, McCall quotes a passage from Bocheński's [3], which is not, however, very clear. Much better formulations can be found in *Dialectica* where Abelard explains that one and the same consequent cannot follow "from the affirmation and from the negation of the same" proposition, and where furthermore "the truth of one of two contradictories doesn't require the truth of the other; instead it expels and destroys it".[20] E.g., the two propositions 'If someone is a man, he is an animal' and 'If someone is not a man, he is an animal' cannot both be true because otherwise one might derive the inconveniency:

> If someone is not an animal, he is an animal
> *Si non est animal, est animal* ([10], 290/16-24).

Similarly, the conditionals 'If someone is a man, he is an animal' and 'If someone is a man, he is not an animal' cannot both be true because otherwise one might derive the inconveniency:

[18] Cf. [26]. The parallel between the law of subalternation and principle Arist 1 had already been pointed out in [46]. We will return to this issue in sections 9 and 10 below.
[19] McCall adopted this quote from Martin [35, 37], and he referred to Arist 1 as "Aristotle's *second* thesis".
[20] Cf. [10], 290/14–15: "[…] ad affirmationem et negationem eiusdem non sequitur idem consequens" and 290/25–27: "[…] cum alterius dividentium veritas non solum veritatem alterius non exigat, immo omnino eam expellat et extinguat".

If someone is a man, he is not a man
'si *est homo, non est homo*' ([10], 292/2).

In sum, then, Abelard evidently was convinced of the validity of principles Arist 1, 2 and of their counterparts

Abel 1 $\neg[(p \to q) \wedge (p \to \neg q)]$
Abel 2 $\neg(p \to \neg p)$.

However, Abelard was soon confronted with *counterexamples*. Thus, Martin reconstructed the following "embarrassing argument" which contemporary logicians had put forward against Abel 2:

1. If Socrates is a man and a stone, Socrates is a man.
2. If Socrates is a man, Socrates is not a stone
So 3. If Socrates is a man and a stone, Socrates is not a stone.
But 4. If Socrates is not a stone, Socrates is not a man and a stone
So 5. If Socrates is a man and a stone, Socrates is not a man and a stone.[21]

Abelard wasn't too much worried by *this* argument, however, because he considered step (2) as *not valid*. According to the logical theory established by Boethius, (2) may be justified by the "locus ab oppositis". This principle says that if one of two opposed terms (like 'man' and 'stone') is predicated of a subject (like 'Socrates'), then the other term has to be denied of the same subject.[22] Now Abelard knew quite well that the implication 'If Socrates is a man, Socrates is not a stone', or, equivalently, 'If Socrates is a man, Socrates is a not-stone' becomes *true* when interpreted as a *strict* implication. But he argued in favour of the "stricter" requirement that the consequent must be "contained" in the antecedent. Such "containment" obtains, in particular, when the antecedent refers to the concept of a *species* and the consequent refers to the concept of the corresponding *kind*, e.g.: 'If Socrates is a man, Socrates is an animal'. But, in Abelard's opinion, the *negative* concept 'not-stone' cannot be regarded as a *kind* from which the species 'man' might be singled out by a "differentia specifica".

The rejection of the "locus ab oppositis" enabled Abelard to cope with two further counterexamples to Abel 2. The first is a generalization of the above argument. If one assumes that Socrates is every bodily substance ("omne corpus"), i.e., an ass, a horse, a tree, a flower, etc., then Socrates is *not* every bodily substance." For if Socrates is *every* substance, he is in particular a *man*; hence, according to the "locus ab oppositis", he is *not* a *stone* and *a fortiori* he is *not every* kind of substance.[23]

Abelard also mentions a geographic counterpart of this "paradox":

If Socrates is everywhere, he is not everywhere

[21] Cf. [34], 569–570. In [10], 395/9–17, Abelard presented an abridged version of this argument: "Sequitur autem ex ista '*si Socrates est homo, non est lapis*', ista: '*si est utrumque,* idest homo et lapis*, non est utrumque*'. Ex his quoque duabus consequentiis: '*si est homo, <non > est lapis*' '*si est lapis, non est homo*' ista infertur: '*si est homo et lapis, non est homo et lapis*'."
[22] Cf. [10], 393/24–25: "si aliquid oppositorum predicatur de aliquo, oppositum ipsius removetur ab eodem".
[23] Cf. [10], 396/14: "[...] unde et si est omne corpus, non est omne corpus."

'*Si* [*Socrates*] *est ubique, non est ubique*'. ([10], 396/19)

The truth of this conditional is substantiated thus: If Socrates is everywhere, he is in Rome; but if he is in Rome, he is *not* in Paris; hence he is not everywhere!

As was mentioned before, Abelard considered these arguments not as a refutation of the connexive principles but rather as a refutation of the "locus ab oppositis". But other contemporary logicians provided further counterexamples which couldn't be rejected in this way. The most important examples shall be discussed in the subsequent section.

6 Some Twelfth Century Debates

The 12th and thirteenth century is perhaps the most important period for the development of connexive logic. In particular De Rijk's *Logica Modernorum* [9] contains many intriguing discussions of the validity of the connexive principles. E.g., Alberic of Paris raised the following argument:

1. If Socrates is a man and is not an animal, Socrates is not an animal
2. If Socrates is not an animal, Socrates is not a man
3. If Socrates is not a man, it is not the case that Socrates is a man and [not][24] an animal

C* ∴ If Socrates is a man and not an animal, it is not the case that Socrates is a man and not an animal.[25]

Since the conclusion of this argument has the structure $(p \wedge \neg q) \rightarrow \neg(p \wedge \neg q)$, it represents a counterexample to Abel 2. Furthermore, the argument does *not* rely on the "locus ab oppositis". Line 2 is obtained by applying the principle of *contraposition* to the unproblematic conditional 'If Socrates is a man, Socrates is an animal'. The remaining proof makes only use of laws which Abelard regarded as indispensable, namely the rules of *conjunction* and the *transitivity* of implication. Therefore, "[...] confronted with this argument Master Peter [i.e., Abelard] essentially threw up his hands and granted its necessity" ([35], 395).

According to [60], 142, the "most notable logical text to emerge from any of the schools of Abelard's rivals is the *Ars Meliduna*, an immense work which provides a rich and varied conspectus of the views of the *Melidunenses*". The work was first published 1967 in vol. II/1 of *Logica Modernorum*.[26] It contains many interesting counterexamples to the connexive principles. E.g., the following consideration is apt to refute at one stroke both Abel 1 and Abel 2:

[24] This negation is missing in Martin's text but it is clearly required for the conclusiveness of the argument.

[25] Cf. [35], 394–5; the original argument is to be found in "Introductiones Montane Minores", ed. in [9], 65–66.

[26] Cf. [9], II/1, 264–390; some important details missing in De Rijk's edition have been edited in [14]; for a closer discussion cf. [30].

> Furthermore, if both ‘Every man is a man’ and ‘Not every man is a man’, not every man is a man. [...] Therefore the negation of the consequent entails the negation of the antecedent. Hence if every man is a man, then not both ‘Every man is a man’ and ‘Not every man is a man’. But if both ‘Every man is a man’ and ‘Not every man is a man’, then every man is a man. Therefore, if ‘Every man is a man’ and ‘Not every man is a man’, then not both ‘Every man is a man’ and ‘Not every man is a man’.[27]

If one abbreviates the (tautological) proposition ‘Every man is a man’ by ‘*p*’, the proof can be formalized as follows[28]:

1. $(p \wedge \neg p) \rightarrow \neg p$ CONJ 3
2. $[\neg\neg]p \rightarrow \neg(p \wedge \neg p)$ CONTRA (1)[29]
3. $(p \wedge \neg p) \rightarrow p$ CONJ 2
4. $(p \wedge \neg p) \rightarrow \neg(p \wedge \neg p)$ TRANS (3, 2).

While line 4 represents a counterexample to ABEL 2, lines 1 and 3 taken together refute ABEL 1. This shows that “hardcore” connexivism is incompatible with such elementary laws as CONJ 2, 3![30]

Another elegant counterexample arises in the field of *epistemic logic*[31]:

> If Socrates knows that he is a stone, Socrates is a stone. If he is a stone, he knows nothing. Therefore, if he knows that he is a stone, he knows nothing. But since he knows himself to be stone, he knows something.[32]

With ‘$K(x,p)$’ symbolizing ‘*x* knows, that *p*’, ‘$\exists p$’ symbolizing the existential quantifier ‘there is a proposition *p*’, and ‘*s*’ and ‘*S*’ abbreviating the name ‘*s*ocrates’ and the predicate ‘*S*tone’, respectively, the argument can be formalized as follows

- E1 $K(s,S(s)) \rightarrow S(s)$
- E2 $S(s) \rightarrow \neg\exists p(K(s, p))$
- E3 $K(s, S(s)) \rightarrow \neg\exists p(K(s, p))$
- E4 $K(s, S(s)) \rightarrow \exists p(K(s, p))$.

[27] Cf. [14], 143: “Item, si et omnis homo est homo et non omnis homo est homo, non omnis homo est homo. [...] Ergo ad contradictoriam consequentis sequitur contradictoria antecedentis. Ergo si omnis homo est homo, non et omnis homo est homo et non omnis homo est homo. Sed si + et + omnis homo est homo et non omnis homo est homo, omnis homo est homo. Ergo si omnis homo est homo et non omnis homo est homo, non et omnis homo est homo et non omnis homo est homo”.

[28] The conclusiveness of the proof doesn’t depend on the particular choice of *p* as a *tautology*. Any other choice, e.g. ‘Every man is an animal’, would have done as well.

[29] This inference presupposes the law of double negation: $p \rightarrow \neg\neg p$.

[30] Interestingly, this had already been noted by Martin: “[...] giving up contraposition is not enough; for, unlike Aristotle’s connexive principle, Abelard’s permit a very quick argument against him. Thus, ‘if *p* and not *p*, then *p*’ by simplification, but also ‘if *p* and not *p*, then not *p*’” ([34], 570). Martin hesitated, however, “to give up the connexive principles or else say that they do not apply in the paradoxical cases” ([34], 571).

[31] Cf. [14], 143. Variants of this paradox re-occur in many later medieval logic texts; cf., e.g., [5], 70–71.

[32] Cf. [14], 143: “Si Socrates scit se esse lapidem, Socrates est lapis. Si est lapis, nihil scit. Ergo si scit se esse lapidem, nihil scit. Sed etsi scit se esse lapidem, aliquid scit.”

E1 is an instance of the so-called "truth-axiom" for knowledge, saying that if anything is *known* (to be true), then it must be *true*. E2 is an application of the "law of nature", that stones don't know (or believe, or feel, ...) *anything*: Hence if Socrates is (or were) a stone, he wouldn't know anything. E3 follows from E1 and E2 by transitivity. E4 is an instance of the (second order) principle of existential generalization: If someone, x, knows that q (where q is any proposition whatsoever), then there exists a proposition p such that x knows that p. Hence two contradictory conclusions {'Socrates knows nothing', 'Socrates knows something'} follow from one and the same assumption 'Socrates knows that he is a stone'.

Another interesting counterexample falls into the field of *alethic modal logic*:

> If a certain proposition necessarily is a true contingent, it is necessarily true. And if it is necessarily true, it is necessary. If it is necessary, it is not contingent. Thus if, necessarily, it is a true contingent, it is not contingent. Furthermore, if it necessarily is a true contingent, it is necessarily contingent. If it is necessarily contingent, it is contingent. Thus if, necessarily, it is a true contingent, it is contingent. So, if necessarily it is a true contingent, it is both contingent and not contingent.[33]

During the history of logic, different conceptions of contingency have been suggested. Proposition p may be regarded as contingent iff p is *not necessary*; other logicians consider p as contingent iff p is *neither necessary nor impossible*. Here we only presuppose the following principle where 'C(p)' symbolizes 'p is contingent':

CONT $C(p) \rightarrow \neg\Box(p)$.

For a full formalization of the above argument, it seems advisable to introduce another "modal" operator 'T' expressing the simple *truth* of p. Hence:

TRUE $T(p) \leftrightarrow p$.

This operator enables us to make a superficial difference between the statement that, necessarily, proposition p is true, $\Box(T(p))$, and the statement that p is necessary, $\Box p$. The aim of the above argument is to show that from the assumption that, necessarily, a certain proposition p is contingently true, one can derive the contradictory conclusions that *p is contingent* and that *p is not contingent*. The single steps of the proof proceed as follows:

- M1 $\Box(T(p) \wedge C(p)) \rightarrow \Box(T(p))$
- M2 $\Box(T(p)) \rightarrow \Box(p)$
- M3 $\Box(p) \rightarrow \neg C(p)$
- M4 $\Box(T(p) \wedge C(p)) \rightarrow \neg C(p)$

[33] Cf. [14], 143: "Si hoc enuntiabile necessario est verum contingens, necessario est verum. Et si necessario est verum, est necessarium. Ergo si est necessarium, non est contingens. Ergo si necessario est verum contingens, non est contingens. Item si necessario est verum contingens, necessario est contingens. Si necessario est contingens, est contingens. Ergo si necessario est verum contingens, est contingens. Quare si necessario est verum contingens, et est contingens et non est contingens."

- M5 $\Box(T(p) \wedge C(p)) \to \Box(C(p))$
- M6 $\Box(C(p)) \to C(p)$
- M7 $\Box(T(p) \wedge C(p)) \to C(p) \wedge \neg C(p)$.

M1 constitutes an instance of the *modal* conjunction law

Conj 4 $\Box(p \wedge q) \to \Box p \wedge \Box q$,

which might be derived from the "normal" principles Conj 2, 3 in conjunction with what is nowadays called the rule of necessitation:

Necess If $(p \to q)$, then $(\Box p \to \Box q)$.

M2 is similarly obtained from principles True plus Necess. M3 follows from Cont by means of *contraposition*. M4 follows from M1-M3 by *transitivity*. M5 is another instance of Conj 4. M6 is an immediate consequence of the "truth-axiom" saying that any *necessary* proposition *a fortiori* has to be *true*. Finally, M7 is obtained from M4, M5, and M6 by means of transitivity (M5, M6) and the "normal" conjunction principle Conj 1.

Altogether, then, the above proof conclusively shows that, in contradiction to Abel 1, the *apparently consistent* premise 'proposition [*p*] necessarily is a true contingent' entails both the conclusion '[*p*] is contingent' and its negation '[*p*] is not contingent'. What makes this counterexample so interesting is the fact that – so to speak – the "inner content" of the premise *is consistent*: Clearly, *p* might be a proposition which is contingently true; but *the proposition that p is contingently true*, even if true, cannot be *necessarily true*!

The (anonymous) author of the *Ars Meliduna* did not, however, use these counterexamples primarily to refute the connexive principles (as defended, e.g., by Abelard). He rather considered all these "inconveniencies" as arguments for the thesis that *nothing* follows from a "false" proposition ("nihil ex falso accidere").[34] The context of the inquiry might suggest that this thesis shall be understood in the weaker sense 'Ex *impossibili* nihil sequitur'

Eins Nothing follows from a proposition which is *necessarily* false, i.e., impossible.

However, the *Meludinenses* appear to have subscribed to the stronger, literal version 'Ex *falso* nihil sequitur':

Efns Nothing follows from a *false* proposition.[35]

[34] Cf. [9], 387: "We learn from this passage that in the author's days there were these four theses [...] about the matter involved: (*a*) *nil ex falso accidere* (*b*) *ex nulla affirmativa sequi negativam* [...] (*d*) *ex quolibet* per se *impossibili quidlibet sequitur*."

[35] Cf. [9], vol. II/1, 283, where among the many doctrines of the "secta Meludina" the following very strange claims are listed: "(11) <N > ichil sequitur ex falso; (12) <N > ullum falsum sit." For a closer discussion cf. [30].

They evidently believed that their counterexamples constituted paradoxes in the sense that these inferences lead from "false" premises to outright *inconsistencies*. Upon closer analysis, however, the counterexamples only lead to conflicting *conditionals* of the form '*If p* then *q*' and '*If p* then ¬*q*'. As the author of *Ars Meliduna* rightly emphasizes, no proposition *q* "can simultaneously be [true] and not be [true]". Therefore, it is also correct to maintain that *q* and ¬*q* cannot be *entailed* by one and the same *true* proposition. But this does *not* mean that *q* and ¬*q* can't simultaneously be entailed by *any proposition at all*. The italicized part of the claim:

> [...] for just as nothing can both be [true] and not be [true], *so also no two [opposite consequents] can [be true, i.e., follow] from one and the same [antecedent].*
> [...] nam quemadmodum nihil potest simul esse et non esse, *ita nec ad eandem duae* ([14], 143)

is unwarranted. According to Conj 2, 3, both *q* and ¬*q* do follow from $(q \wedge \neg q)$! The simultaneous truth of $((q \wedge \neg q) \rightarrow q)$ and $((q \wedge \neg q) \rightarrow \neg q)$ is *not* itself a *contradiction*!

7 Robert Kilwardby (1222–1277)

The next prominent logician considered by McCall ([39], 417–418) is Robert Kilwardby who, in his *Notule libri Priorum*, provided:

> [...] a new criticism of Aristotle's [first] thesis which asserts the incompatibility of 'If *p* then *q*' and 'If not-*p* then *q*'. Kilwardby gives two examples of pairs of such propositions which are *not* incompatible:
>
> (i) If you are seated, God exists
> If you are not seated, God exists
>
> (ii) If you are seated, then either you are seated or you are not seated
> If you are not seated, then either you are seated or you are not seated.
>
> The first pair is true because 'God exists', being a necessary proposition, follows from anything – *quia necessarium sequitur ad quodlibet*: an early formulation of the positive paradox of strict implication. But here we must distinguish, Kilwardby says, two kinds of implication: *consequentia essentialis* or *naturalis*, and *consequentia accidentalis*. In the former case the consequent must be 'understood' (*intelligitur*) in the antecedent, and such is not the case with 'If you are seated, God exists'. The latter is a *consequentia accidentalis*, '*et de tali non intelligendum sermo Aristotelis'*.

It should be emphasized (a bit more than McCall does) that Kilwardby's main intention was not to *criticize* Aristotle, but rather to defend the Stagirite against criticisms which other logicians had raised. These logicians were well acquainted with the "anti-connexive" principles Neq and Eiq which they applied to "necessary" consequents such as ‘God exists’ or to “impossible” antecedents such as ‘You are an ass’.[36] Thus Kilwardby explains that principle Arist 2:

> [...] does not seem to be unacceptable, since one opposite may well follow from another as in ‘If you are an ass, you are not an ass’ because anything follows from the impossible and the necessary follows from anything. ([56], 1143)

Now, if one accepts that propositions like ‘You are an ass’, ‘God does not exists’, and ‘Atoms do not exist’ are in some sense “impossible” (so that their negations are in some sense “necessary”), then, according to Eiq and Neq, the “anti-connexive” conditionals ‘If you are an ass, you are not an ass’, ‘If God doesn’t exist, God does exist’, and ‘If atoms don’t exist, atoms do exist’ result as true. However, as Kilwardby objected, these consequences are not “*natural*”.

Let it be mentioned in passing that this objection somehow anticipates the reservations of proponents of modern relevance logic who – according to [33] – reject implications when “the antecedent seems irrelevant to the consequent”, i.e. when “the antecedents and consequents (or premises and conclusions) are on completely different topics”. In contrast to today’s relevance logicians, however, Kilwardby accepts all conditionals as “natural” which are based on the laws of *disjunction*:

Disj 1 $p \rightarrow (p \vee q)$
Disj 2 $q \rightarrow (p \vee q)$.[37]

Therefore, he had to admit that the second counterexample (quoted above by McCall) consists of “natural consequences”. Nevertheless, he tried to save Aristotle’s Thesis by introducing another requirement:

> To the second objection it should be said that the same thing can follow in two ways, viz. either by virtue of the same thing in it [...] or by virtue of different things in it [...]. So Aristotle understands that something does not follow of necessity from the same thing’s being so and not being so, and by virtue of the same thing. ([56], 1141-1143).

In a recent paper ([15]) S. Johnston accepted this ad hoc condition as a reasonable requirement for connexive implication and set out to develop a formal semantics that would fit this idea. This project, however, is rather misguided because the remaining considerations of the *Notule libri Priorum*– which were entirely ignored by Johnston – show that Kilwardby eventually recognized that the *unrestricted* versions of Aristotle’s

[36] For readers who are not so familiar with medieval logic it may be helpful to point out that ‘You are an ass’ was never meant as an *affront*. Rather, it’s a standard example of an “impossible” proposition because the addressee of any assertion is a human being. But ‘No human being is an ass’ is an analytic truth. Hence, for any person P, the proposition ‘P is an ass’ is (analytically) “impossible”.

[37] Cf. [56], 1141: “Further, a disjunctive follows from either of its parts, and in a natural inference”. The laws of disjunction are rejected in some modern systems of relevance logic such as Parry’s “Analytic implication”. Cf. [33], section 6: “[...] the principle of disjunction introduction needs to be restricted. So, instead of having $A \rightarrow A \vee B$ as a theorem for all formulas A and B, this schema is valid only when all the propositional variables in B are also in A”.

Theses are bound to fail. Towards the end of "Lesson 55" Kilwardby presented the following "Solution":

> So it should be granted that from the impossible its opposite follows, and that the necessary follows from its opposite.[38]

Hence, for Kilwardby, the following principles are incontrovertible counterexamples to Arist 2 (or Abel 2):

Kilw 1 $(p \wedge \neg p) \rightarrow \neg(p \wedge \neg p)$
Kilw 1 $\neg(p \vee \neg p) \rightarrow (p \vee \neg p)$.

Interestingly, McCall himself seems to have recognized that Kilwardby's considerations seriously threaten the very enterprise of "hardcore" connexivism. On the one hand, he rejected Kilwardby's attempt to save Arist 1, 2 by recourse to the ad hoc requirement 'by virtue of the same thing'.[39] On the other hand he frankly confessed:

> It appears we must accept the fact that the type of implication for which Aristotle's thesis holds cannot consistently admit of conditionals of the form 'if *p*, then either *p* or *q*'.

Of course, it remains possible for a "hardcore" connexivist like McCall (or, perhaps, also Priest)[40] to *give up* principles Disj 1, 2. But one can no longer claim the ancient and medieval logicians as proponents of "hardcore" connexivism because, *for them*, the validity of the fundamental laws of disjunction and conjunction seem to have been beyond any reasonable doubt.

8 Proving (and Justifying) "Ex contradictorio quodlibet"

The notion of *impossibility* – as it occurs in the definition of strict implication – admits of various interpretations. Thus, the Stoics considered the non-existence of atoms as *nomologically* impossible, while medieval logicians considered the non-existence of God as *theologically* impossible. According to the proponents of relevance logic, inferences based on such extra-logical impossibilities are problematic. Anyway, it appears reasonable to strengthen former principle Strict in such a way that not just any weird kind of "impossibility" is admitted. Only the *logical* impossibility of $(p \wedge \neg q)$ suffices to warrant the truth of '$p \rightarrow_{str} q$':

[38] Cf. [56], 1145, and the more detailed discussion in [27].
[39] Cf. [39], 418: "But it is doubtful that Aristotle intended any such thing, and Kilwardby seems to be leaning over backwards here."
[40] Priest [48], 144 remarks that Kilwardby defended Aristotle's theses but "Kilwardby also endorsed extensional disjunction principles [i.e., Disj 1, 2] and so was in some trouble." The expression 'trouble' is somewhat inadequate, however. If Kilwardby would have been trying to defend "hardcore" connexivism, one might speak of "trouble"; but Kilwardby eventually recognized that Aristotle's theses only hold in the restricted "humble" form, i.e., restricted to self-consistent antecedents.

STRICT$_{log}$ p strictly implies q iff it is *logically impossible* that p is true and yet q is false.

This revised definition transforms the former principle "Ex *impossibili* quodlibet" into the more specific "Ex *contradictorio* quodlibet":

ECQ If p if *logically* impossible, or self-contradictory, then p implies any arbitrary proposition q.[41]

Sometime in the twelfth century, clever logicians discovered that ECQ can be *proven*, i.e., derived from other fundamental laws of logic. The Kneales pointed out that

> [...] a certain William of Soissons [...] 'produced an engine for capturing, as his friends say, the citadel of the old logic, building up unexpected links of arguments and demolishing the opinions of the ancients' [...] How exactly he set to work we do not know; but apparently he proved [...] that from one impossible all impossibles follow ([17], 201).

Martin suspected "that most probably [William's] machine was a version of what we now know as C. I. Lewis's proof that anything follows from a contradiction" ([34], 565). Furthermore, Martin pointed out that a certain Adam of the Little Bridge (alias Adam Parvipontanus) "and his followers were famous in the twelfth century for their adoption of the Parvipontanian principles that anything follows from an impossibility and a necessity follows from anything". A clear *proof* of ECQ was at any rate presented by Alexander Neckham in his *De Naturis Rerum* composed around 1180:

> I wonder that certain men oppose the thesis that from a *per se* impossibility anything whatsoever follows. [...] For doesn't it follow that if Socrates is a man and not a man, then Socrates is a man, but if Socrates is a man, then Socrates is a man or a stone. Therefore, if Socrates is a man and not a man, then Socrates is a man or a stone. But if Socrates is a man and Socrates is not a man, then Socrates is not a man. Therefore, if Socrates is a man and Socrates is not a man, then Socrates is a stone.
> By means of a similar deduction it is proved that if Socrates is a man and Socrates is not a man, then Socrates is a crab, and so on for other things, for example a rose, a lily and the rest. Why, then, can't they see how from an impossibility [...] anything you like follows?[42]

This "standard" proof relies on the principles of disjunction introduction, DISJ 1, 2, and, in a decisive way, also on what later came to be called "disjunctive syllogism":

DISJ 3 $(p \vee q), \neg p \Rightarrow q$
DISJ 4 $(p \vee q), \neg q \Rightarrow p$.

[41] Similarly, instead of the all too liberal principle "*Necessarium* ex quodlibet", one obtains: If q is *logically* necessary, or tautological, then q is implied by any arbitrary proposition p.
[42] Cf. [61], 288–289; our translation has been adopted with slight variations from [34], 571, and [35], 400.

The "Avranches Text" edited in [14] contains an interesting variant of the "standard" proof. On the one hand, the Anonymous Author (AA) apparently succeeds in showing that "everything follows" not only from a *pair* of contradictory propositions $\{p, \neg p\}$, but even from a single "impossible" proposition such as 'Socrates is an ass'. On the other hand, AA shows that if an impossible proposition entails every other proposition, then conversely a necessary proposition is entailed by every other proposition.[43] The former proof may be divided into two segments; the first leads from the conditionals.

(A) If Socrates is an ass, Socrates is [i.e., he exists].
(B) If Socrates is, Socrates is Socrates.
(C) If Socrates is Socrates, Socrates is a man.
(D) If Socrates is a man, Socrates is not an ass.

to the "anti-connexive" conclusion:

(E) If Socrates is an ass, Socrates is not an ass.[44]

The second "half" makes use of the laws of disjunction to infer (on the basis of (E)):

(F) If Socrates is an ass, Socrates is an ass or whatsoever is true.
(G) If Socrates is an ass, Socrates is not an ass and (Socrates is an ass or whatsoever is true)
(H) If Socrates is an ass and (Socrates is not an ass or whatsoever is true), whatsoever is true.
(I) If Socrates is an ass, whatsoever is true.

The decisive step, in (H), is the application of disjunctive syllogism which was generalized by AA as the principle that "from a disjunction and the destruction of one of its parts the position of the other part follows".[45]

In modern times, paraconsistent logicians raised objections to Ecq. Thus, in a contribution to the *Stanford Encyclopedia of Philosophy*, Priest, Tanaka & Weber explain:

> Contemporary logical orthodoxy has it that, from contradictory premises, anything follows. A logical consequence relation is *explosive* if according to it any arbitrary conclusion B is entailed by any arbitrary contradiction A, $\neg A$ (*ex contradictione quodlibet* (ECQ)). Classical logic, and most standard 'non-classical' logics too [...], are explosive. Inconsistency, according to received wisdom, cannot be coherently reasoned about.
> Paraconsistent logic challenges this orthodoxy. A logical consequence relation is said to be *paraconsistent* if it is not explosive. Thus, if a consequence relation is paraconsistent, then even in circumstances where

[43] Cf. [14], 136: "Eodem modo ostenditur de quolibet alio inpossibili, scilicet ex eo sequitur quidlibet. Ex hoc facile est habere quod necessarium sequitur ad quidlibet".
[44] Cf. [14], 135–136. Some minor problems associated with this "proof" are analysed in [30].
[45] Cf. [14], 136: "[...] ex disiuncta enim et destructa parte illius sequitur positio reliquae partis".

> the available information is inconsistent, the consequence relation does not explode into *triviality*. ([49])

The use of the word 'explosive' indicates that paraconsistent logicians are somehow *afraid* of a situation where the whole *world* (or at least the world of propositions) *collapses* because *everything* becomes *provable*. But this fear is unjustified. From the earliest beginnings of logic, *reductio ad absurdum* has always been acknowledged as an important method of proof. It says:

> REDUCTIO If from the assumption that all propositions of a certain set $\{p_1, ..., p_n\}$ are true, it can be concluded that, e.g., q and $\neg q$ would both be true, it follows that at least one of the p_i must be false.

After an application of this method of indirect proof, no sane logician would ever want to use ECQ and argue that, since both q and $\neg q$ *have been shown to be true*, it follows that Socrates is a stone, and that Socrates is a lily, etc. Of course, in a certain way one may formulate ECQ with the help of the words "that from contradictory premises *anything follows*". But it appears much more adequate to paraphrase ECQ in the *subjunctive* (or *counterfactual*) mood: 'If contradictory premises *would* be true, then anything else *would* be true as well'. This important point has been nicely emphasized by the seventeenth century logician Juan Caramuel y Lobkowitz:

> But which bad things would occur in the world if, *per impossibile*, two contradictory propositions would be true together? Or, what if, *per impossibile*, one and the same proposition would be both true and false? I answer: Then in the whole world not a single truth would remain.[46]

To be sure, "bad things" *would* happen, *if* – *per impossibile*! – two contradictory propositions like q and $\neg q$ *would* both be true. But don't worry: it is *impossible* that both q and $\neg q$ *are* true! Therefore, nobody has to be afraid that the set of true propositions "explodes" into the universal set.

9 Lewis Carroll, Hugh MacColl and Frank P. Ramsey

After his discussion of Kilwardby's (and of Paul of Venice's)[47] defence of connexive implication, McCall ([39]: 418) jumps ahead to the end of the nineteenth century in

[46] Cf. [6], 215: "Sed quid in Mundo mali accideret, si per impossibile essent duae Contradictoriae simul verae? aut *Quid si per impossibile, una et eadem Propositio, vera & falsa esset simul*? Respondeo *Nullam* in toto *Mundo Veritatem mansuram*. Vel si dubites, da mihi has Propositiones, *Petrus currit*, & *Petrus non currit* (aut, si has nolis, alias quascunque Contradictorias) esse simul veras: & Ego tibi quidquid volueris demonstrabo." Caramuel's ensuing proof, again, rests on the "disjunctive syllogism".

[47] McCall considers also Paul of Venice as a forerunner of connexive logic because "[...] in listing no fewer than 10 interpretations of the meaning of 'if ... then', [... Paul] reiterates Sextus Empiricus' connexive category: 'Tenthly people say that for the truth of a conditional it is required that the opposite of the consequent be incompatible with the antecedent." However, the latter condition may as well be interpreted as defining *strict* implication; and even if Paul of Venice should have favoured *connexive* implication, then probably only the "humble" and not the "hardcore" version.

order to call Lewis Carroll, Hugh MacColl, and Frank P. Ramsey as additional approvers of connexivism. Carroll's "Barbershop Paradox" at best shows, however, that the author of 'Alice's Adventures in Wonderland' favoured a "humbly" *connexive* implication over a merely *material* implication. Perhaps Carroll fancied a "*causal* implication" as it was later developed, e.g., by Burks & Copi in [4]. Such a conception is "humbly" connexive in so far as it satisfies "the inference from $p \rightarrow q$ to $\neg(p \rightarrow \neg q)$".[48] But the antecedent of the conditionals in the "Barbershop Paradox" ('If Allen is out') as well as the antecedent of the conditionals in Burks & Copi's example ('If the Conservatives win the election in 1950') are all "normal", i.e. self-consistent, and there is no evidence for the assumption that Carroll, or Burks & Copi, might have meant BOETH to be extended to self-*inconsistent* antecedents!

A similar comment applies to Ramsey who suggested in [50] "that in a sense 'If *p*, *q*' and 'If *p*, not-*q*' are contradictories". To be sure, unlike *material* implication, most versions of a *more demanding* conception of implication do satisfy the condition that if one of the conditionals $(p \rightarrow q)$, $(p \rightarrow \neg q)$ is true, the other can't be true as well. But this only means that they are "*humbly*" connexive.[49] Ramsey's investigation was typically concerned with situations where two people "are arguing 'If *p* will *q*' and are both in doubt as to *p*". They therefore add "*p* to their stock of knowledge and argu[e] on that basis about *q*".[50] But, clearly, adding *p* hypothetically to one's stock of knowledge requires in particular that *p* is *possible*, i.e., self-consistent. There is not the slightest evidence that Ramsey would have wanted to extend his principle to *impossible* antecedents!

This verdict also applies to MacColl who – according to McCall – suggested to "base Aristotelian syllogistic, at that time the standard school-book 'logic', upon the logic of propositions" ([39], 419). As was explained already in section 4 above, it was quite usual for Boethius, Abelard, and other medieval logicians to formulate the UA 'Every *A* is *B*' in the form of an implication: 'Si est *A*, est *B*'. In [32], Hugh MacColl picked up this idea and formally represented the UA by '$a \rightarrow b$'. As Storrs McCall explains:

> The syllogistic mood Barbara ('If all *A* is *B*, and all *B* is *C*, then all *A* is *C*') then becomes: $[(a \rightarrow b) \& (b \rightarrow c)] \rightarrow (a \rightarrow c)$, and MacColl is on his way to reducing syllogistic to propositional logic. ([39], 420)

In particular, the traditional law of *subalternation*, i.e., the inference from 'All *A* is *B*' to 'Some *A* is *B*' is transformed:

> [...] in MacColl's system, [into] Boethius thesis $(a \rightarrow b) \rightarrow \neg(a \rightarrow \neg b)$. Using connexive implication as his basis, MacColl is able to show that all nineteen moods of the traditional syllogistic are valid [...].

[48] Cf. [39], 419; McCall's negation symbol ~ has been replaced by '¬'.

[49] This appears to hold true also of the theories of conditionals developed by Goodman, [12], Stalnaker, [54], and Lewis, [31]. For reasons of space, a discussion of this issue cannot be given here.

[50] We adopt the quoted passage of [50] from [39], 420.

It should be observed, however, that this transformation had been discovered long before MacColl by *Leibniz*![51]

10 Gottfried Wilhelm Leibniz (1646–1716)

One of the many brilliant ideas of the polymath Leibniz consisted in simplifying the UA 'Every *A* is *B*' into the semi-formal expression '*A contains B*' or '*A is B*'. Since, according to the traditional doctrine of *obversion*, 'No *A* is *B*' is equivalent to 'Every *A* is not-*B*', the UN can similarly be simplified to '*A* contains *not-B'* or '*A* is *not-B*'. Furthermore 'Some *A* is *B*' and 'Some *A* isn't *B*' (as negations of UN and UA) re-appear as '*A* doesn't contain not-*B*' and '*A* doesn't contain *B*', respectively. Thus, for Leibniz the inference of subalternation takes the form 'If *A* is *B* then *A* isn't not-*B*',[52] and he sets out to *prove* this law as follows:

> (91) *A* is *B*, therefore *A* is not not-*B*. For let it be true that *A* is not-*B*, assuming that this is possible. Now *A* is *B* (by hypothesis), therefore *A* is *B* not-*B*, which is absurd. (Add no. 100 below]. [...]
> (100) If *A* is *B*, it follows that *A* isn't not-*B*, i.e., it is false that every *A* is not-*B*. For if *A* is *B*, then no A is not-*B*, i.e., it is false that some *A* is not-*B* (by 87). Therefore […] it is much more false that every *A* is not-*B*.

Leibniz goes one step further and states that "whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows".[53] Thus, each law of Leibniz's "intensional" algebra of *terms* (which, by the way, is provably equivalent to Boole's extensional algebra of *sets*) may be transformed into an algebra of *propositions*.[54] In particular, the term-logical laws

> LEIB 1_{term} If *A* contains *B*, then *A* doesn't contain not-*B*
> LEIB 2_{term} *A* doesn't contain not-*A*[55]

are transformed into the following laws of propositional logic:

[51] A summary of Leibniz's most important logical innovations is given in [24]. Fuller expositions may be found in [22, 23].

[52] Cf. [18], § 91. Leibniz, however, had notorious difficulties distinguishing '*A* non est *B*' from '*A* est non *B*'. E.g., in § 82 he erroneously maintained: "It is also possible to say that '*A* isn't *B*' [*A* non est *B*] is the same as '*A* is not-*B*' [*A* est non-*B*]". A few paragraphs he corrected himself: "(92) The inference 'If *A* isn't not-*B*, then *A* is *B*' is invalid. That is, it is indeed false that every animal is a not-man, but it does not follow that every animal is a man." The translation has been adopted with minor changes from [44], 68–69. For a detailed discussion of this issue cf. [20].

[53] Cf. [44], 85, and 87: "[...] that a proposition follows from a proposition is simply that a consequent is contained in an antecedent, as a term [is contained] in a term. By this method we reduce inferences to propositions, and propositions to terms."

[54] The equivalence between Leibniz's algebra of concepts and Boole's algebra of sets was proved in [19]. Leibniz's "idée capitale" of deriving an algebra of propositions from the algebra of concepts was first praised by Couturat ([7], 354), and it was formally elaborated in [21].

[55] This law is formulated in § 43 of [18] as follows: "It is false that *B* contains not-*B*; or, *B* does not contain not-*B*" ([43], 49).

LEIB 1_{prop} If p entails q, then p doesn't entail not-q
LEIB 2_{prop} p doesn't entail not-p.

As Leibniz himself noticed, his proofs contain a minor flaw, however. The assumption that concept A contains both B and not-B is not *absolutely* "absurd"; it is only absurd when A itself is possible, i.e., self-consistent. Similarly, the assumption that concept B contains its own negation is absurd only if B itself is possible. As a matter of fact, Leibniz *defines* a concept A to be possible iff A does not simultaneously contain contradictory concepts like B and not-B. Therefore, the term-logical laws have to be restricted to *possible* or *self-consistent* concepts A, and the corresponding propositional laws only hold for *self-consistent* antecedents. Leibniz formulated the latter restriction quite explicitly by saying: "That it is false that B contains not-B is also to be understood with respect to a proposition B which doesn't contain a contradiction".[56] The *amended* versions of Leibniz's laws of consistency:

LEIB 1^*_{prop} If $\Diamond p$ and if $(p \rightarrow q)$, then not also $(p \rightarrow \neg q)$
LEIB 2^*_{prop} If $\Diamond p$, then not $(p \rightarrow \neg p)$

express just the conditions of "humble" connexivity, i.e., principles (1*) and (2*) from section 1 above.

11 Conclusion

According to McCall ([39]: 415), we can look back on a period of "Two thousand three hundred years of connexive implication". Our investigation of the first 2200 years (from Aristotle to Ramsey) has shown, however, that the vast majority of the logicians either understood their claims only in the sense of "humble" connexivism; or, if they originally believed in "hardcore" connexivism (as, e.g., Abelard), they were eventually convinced by other logicians to give up this belief since it is incompatible with the validity of other, better entrenched laws of logic. There is only one likely exception: *Chrysippus*. This Stoic logician not only rejected the *example* 'If atomic elements of things do not exist, then atomic elements of things do exist', but, apparently, each conditional of the form $(p \rightarrow \neg p)$ because, for him, an implication is sound iff the negation of the consequent "conflicts" with the antecedent. In the case of $(p \rightarrow \neg p)$, this means that p would have to "conflict" with p, but in Chrysippus' opinion:

CHRYS 1′ (Absolutely) No proposition "conflicts" with itself.

The very same idea was picked up, some 2150 years after Chrysippus, by Everett Nelson who maintained:

[56] Cf. [18], end of § 43: "Falsum esse B continere non-B, intelligendum est et de propositione B, quae non continet contradictionem." Unfortunately, in Parkinson's translation ([44], 59), this important explanation is missing!

> For example, in regard to the proposition 'All men are mortal and some men are not mortal', the two component propositions are inconsistent with each other, but the whole compound is not inconsistent with itself [...]. ([42], 447).

Since Nelson adopts Chrysippus's idea of entailment according to which "'*p* entails *q*' means that *p* is inconsistent with the [...] contradictory of *q*" ([42], 445), he avoids the "paradoxes" of strict implication and obtains a connexive logic in which Abel 1 and Abel 2 become *theorems*. This result, however, has a price. In particular, the usual laws of *conjunction* and *disjunction* have to be given up. Nelson tried to justify these failures as follows:

> Though '*p* and *q* entail *p*' cannot be asserted on logical grounds, I do not deny that from '*p* is true and *q* is true' we can pass to '*p* is true'. All I deny is that such a passage is in virtue of an entailment relation holding between '*p* is true and *q* is true' and '*p* is true'.
> Furthermore, '*p* entails *p* or *q*' cannot be asserted on logical grounds [...]. Of course, if *p* has truth, then '*p* or *q*' has truth, but here [...] we are dealing neither with truth-values nor with material implication, but with propositional functions in their essence and with entailment. ([42], 448)

This is not the place to discuss the *plausibility* of Nelson's defence of an "intensional implication",[57] since the focus of this paper lay on the *history* of connexive logic. Yet it seems appropriate to close with the following remarks. "*Humble*" connexivity as characterized by the conditions

- that a *self-consistent* antecedent *p* cannot imply *its own negation*,
- that it cannot imply *both of two contradictory propositions*,
- that it cannot therefore imply *every proposition*,

is an absolute plausible property that may be imposed on any reasonable conception of implication. This property has been approved by practically every ancient and medieval logician considered in this paper. "*Hardcore*" connexivity, however, transcends this requirement by maintaining that also

- a *self-inconsistent* proposition doesn't imply every (other) proposition,
- in particular, a *self-inconsistent conjunction* like $(p \wedge \neg p)$ doesn't imply the single conjuncts p and $\neg p$; and furthermore
- $(p \wedge \neg p)$ does not even imply its own (tautological) negation $\neg(p \wedge \neg p)$!

[57] A defence of "hardcore" connexivism based on Nelson's idea has recently been attempted in [11]. McCall attributes to Nelson "the credit for being the first logician to give formal equivalents of Sextus' implication operator" but he criticizes him because "he made no attempt to incorporate his insights into a full-fledged logical system" [39], 421.

Whatever the motives may be which guide contemporary logicians in building calculi which satisfy such "hardcore" conditions,[58] they should at least stop claiming that *their* logics are elaborations of ideas that can be traced back to Aristotle, Boethius, Abelard, Kilwardby, etc.

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[58] According to [39], 421, the first attempt to construct a "full-fledged logical system" of connexive logic was made by Angell in 1962. Interestingly, McCall considers Angell's logic as "an intuitively plausible axiomatic system", while Angell himself admitted that his system "is not a completely satisfactory formalization of the logic of propositions" ([1], 342). One reason for the inadequacy of Angell's system is the validity of the "Converse Boethius Thesis", which appears to be a rather unwelcome by-product of Angell's 4-valued semantics.

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