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Random Walk in Balls and an Extension of the Banach Integral in Abstract Spaces

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Abstract

We describe the construction of a random walk in a Banach space $\mathbb{B}$ with a quasi-orthogonal Schauder basis and show that it is a martingale. Next we prove that under certain additional assumptions the described random walk converges a.s. and in $L^p(\mathbb{B})$, $1 \leq p < \infty$, to a random element ξ, which generates a probability measure with support contained in the unit ball $B \subset \mathbb{B}$. Moreover, we define the Banach integral with respect to the distribution of ξ for a class of bounded, Borel measurable real-valued functions on B. Next some examples of nonstandard Banach spaces with quasi-orthogonal Schauder bases are presented; furthermore, examples which demonstrate the possibility of applications of all the obtained results in spaces ℓ^p, $1 \leq p < \infty$ and $L^p[0, 1]$, $1 < p < \infty$ are given.

Keywords Banach random walk · Martingale · Radon–Nikodym property · Quasi-orthogonal Schauder basis

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1 Introduction

Let $\{e_i, i \geq 1\}$ be a complete orthonormal system (CONS) in a separable Hilbert space $(\mathbb{H}, |\cdot|)$ and let $\pi_n(x) = \sum_{i=1}^{n} x_i e_i$ for $x = \sum_{i=1}^{\infty} x_i e_i \in \mathbb{H}$. Denote by K_n and B the unit balls in $\mathbb{R}^n$ and $\mathbb{H}$, respectively.

In a paper published as addendum to the Saks monograph *Theory of the Integral*, Banach [1] described the most general form of a nonnegative linear functional F (satisfying additional conditions analogous to some properties of the Lebesgue integral, thus called by Banach $\mathfrak{L}$-integral), defined on the linear set $\mathcal{L}$ of bounded, Borel measurable functions $\Phi\colon B \to \mathbb{R}$, namely

$$F(\Phi) = \lim_{n\to\infty} F_n(\Phi),$$

where

$$F_n(\Phi) = \int_{K_n} \Phi(\pi_n(x))\, g_n(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

$$g_n(x_1, \ldots, x_n) = \mathbb{1}_{K_n}(x_1, \ldots, x_n) \frac{g(x_1)\, g\left(x_2/\sqrt{1-x_1^2}\right) \ldots\, g\left(x_n/\sqrt{1-\left(x_1^2+\cdots+x_{n-1}^2\right)}\right)}{\sqrt{\left(1-x_1^2\right)\cdot\left[1-\left(x_1^2+x_2^2\right)\right]\cdot\ldots\cdot\left[1-\left(x_1^2+\cdots+x_{n-1}^2\right)\right]}},$$

$g\colon [-1, 1] \to [0, \infty)$ is Borel measurable and integrable with $\int_{-1}^{1} g(t)\,\mathrm{d}t = 1$, and $\mathbb{1}_A$ denotes the indicator of the set A.

In fact, Banach [1] considered only the case when g is the density of the uniform distribution on $[-1, 1]$, and a more general case was treated by Banek [2]. Furthermore, Banek [2] observed that

$$F_n(\Phi) = E\Phi(Z_n),$$

where $\{Z_n\}$ is the so-called Banach random walk (BRW) in $B \subset \mathbb{H}$ given by the random linear combination $Z_n = \sum_{i=1}^{n} X_i e_i,\ n \geq 1$, of elements of CONS $\{e_i\}$ in $\mathbb{H}$, with coefficients X_i that are dependent r.v.'s defined recursively as follows: X_1 is a r.v. having density g concentrated on the interval $[-1, 1]$, and if the r.v.'s $X_1, \ldots, X_{n-1}$ are already defined, then X_n is defined as a r.v. with probability density $g\left(x_n/\sqrt{1-(X_1^2+\cdots+X_{n-1}^2)}\right)$ in the random interval

$$\left[-\sqrt{1-\left(X_1^2+\cdots+X_{n-1}^2\right)}, \sqrt{1-\left(X_1^2+\cdots+X_{n-1}^2\right)}\right].$$

The last observation forms a probabilistic background to the purely deterministic Banach construction of the $\mathfrak{L}$-integral for a class of bounded, Borel measurable functions defined in $B \subset \mathbb{H}$.

It is worth mentioning that Banach [1] considered two various special cases: (1) the mapping Φ is defined on a compact metric space and (2) Φ is defined in the unit ball of a separable Hilbert space.

In this paper, we describe a generalized BRW $\{Z_n, n \geq 1\}$ with values in the unit ball of a Banach space. Moreover, we give a criterion for the existence of the Banach–Lebesgue integral

$$F(\Phi) = \lim_{n\to\infty} F_n(\Phi) \tag{1}$$

in terms of the constructed BRW $\{Z_n, n \geq 1\}$, where Φ is a bounded, Borel measurable real-valued function defined in the unit ball of a Banach space.

2 Banach Random Walk in a Banach space

Let $(\mathbb{B}, \|\cdot\|)$ be an infinite-dimensional Banach space with a Schauder basis $\{b_n, n \geq 1\}$. Then each vector $x \in \mathbb{B}$ possesses a unique series expansion $x = \sum_{k=1}^{\infty} x_k b_k$, and thus, for all $n \geq 1$, the projections $\pi_n\colon \mathbb{B} \to \mathbb{B}$, given by $\pi_n(x) = \sum_{k=1}^{n} x_k b_k$, are well defined. Denote

$$B_n = \{\pi_n(x) \in \mathbb{B}\colon \|\pi_n(x)\| \leq 1\}, \quad B = \{x \in \mathbb{B}\colon \|x\| \leq 1\},$$

and put

$$\alpha_1 = \inf\{t \in \mathbb{R}\colon \|tb_1\| \leq 1\}, \quad \beta_1 = \sup\{t \in \mathbb{R}\colon \|tb_1\| \leq 1\} \, (= -\alpha_1).$$

Furthermore, given any point $\pi_{n-1}(x) \in B_{n-1}$, $n \geq 2$, define inductively

$$\begin{aligned} \alpha_n &= \alpha_n(\pi_{n-1}(x)) = \inf\{t \in \mathbb{R}\colon \|\pi_{n-1}(x) + tb_n\| \leq 1\}, \\ \beta_n &= \beta_n(\pi_{n-1}(x)) = \sup\{t \in \mathbb{R}\colon \|\pi_{n-1}(x) + tb_n\| \leq 1\}. \end{aligned}$$

It is clear that $\alpha_n \leq 0 \leq \beta_n$, and $[\alpha_n, \beta_n]$, $n \geq 1$, are bounded intervals in $\mathbb{R}$, for

$$\|\pi_{n-1}(x) + tb_n\| \leq 1 \;\Rightarrow\; |t| \cdot \|b_n\| \leq \|\pi_{n-1}(x)\| + 1, \; n \geq 1,$$

where $\pi_{n-1}(x) = \pi_0(x) = 0$ for $n = 1$. Obviously, α_n and β_n depend on $\pi_{n-1}(x) \in B_{n-1}$ and b_n, and in general the intervals $[\alpha_n, \beta_n]$, $n \geq 2$, need not be symmetric about 0. In addition, it may happen that for some $n \geq 2$ the interval $[\alpha_n, \beta_n]$ reduces to the single point $[0, 0] = \{0\}$. To fix a standard length of the first interval, without loss of generality we may and do assume that $\|b_1\| = 1$ (but we do not require that $\|b_n\| = 1$ for all $n \geq 2$). In such a situation, $\alpha_1 = -1$ and $\beta_1 = 1$.

Let G_n, $n \geq 1$, be arbitrary probability distributions concentrated on $[-1, 1] \subset \mathbb{R}$, i.e., $G_n([-1, 1]) = 1$ for all $n \geq 1$. Define inductively on a probability space $(\Omega, \mathcal{F}, P)$ a sequence of (dependent) real r.v.'s $\{X_n, n \geq 1\}$ and, associated with it, a sequence of $\mathbb{B}$-valued random elements (r.e.) $\{Z_n, n \geq 1\}$. Namely, let X_1 be a r.v. with distribution G_1 and let $Z_1 = X_1 b_1$. Then $X_1(\omega) \in [\alpha_1, \beta_1] = [-1, 1]$ a.s., and thus, we may define X_2 as a r.v. distributed according to G_2, scaled linearly in such a way that it is concentrated on $[\alpha_2, \beta_2] = [\alpha_2 (Z_1 (\omega)), \beta_2 (Z_1 (\omega))]$. In other words, X_2 is a r.v. with distribution function

$$G_2 \left(\frac{2t - [\beta_2 (Z_1 (\omega)) + \alpha_2 (Z_1 (\omega))]}{\beta_2 (Z_1 (\omega)) - \alpha_2 (Z_1 (\omega))} \right), \ t \in \mathbb{R},$$

whenever $\beta_2 - \alpha_2 > 0$, and then we put $Z_2 = X_1 b_1 + X_2 b_2$. Next, given any value $X_2 (\omega)$, and *a fortiori* $Z_2 (\omega) \in B_2$ a.s., we define X_3 as a r.v. with distribution G_3 scaled linearly in such a way that it is concentrated on the interval $[\alpha_3, \beta_3] = [\alpha_3 (Z_2 (\omega)), \beta_3 (Z_2 (\omega))]$, and then put $Z_3 = X_1 b_1 + X_2 b_2 + X_3 b_3$, etc. More generally, if $X_1, \ldots, X_{n-1}$ and $Z_1, \ldots, Z_{n-1}$ are already defined in such a manner that $Z_{n-1} (\omega) \in B_{n-1}$ a.s., then X_n is a r.v. with distribution function

$$G_n \left(\frac{2t - \left[\beta_n (Z_{n-1} (\omega)) + \alpha_n (Z_{n-1} (\omega))\right]}{\beta_n (Z_{n-1} (\omega)) - \alpha_n (Z_{n-1} (\omega))} \right), \ t \in \mathbb{R},$$

provided $\beta_n - \alpha_n > 0$ and $Z_n = X_1 b_1 + X_2 b_2 + \cdots + X_n b_n = Z_{n-1} + X_n b_n$.

As was already mentioned, it may happen that for some $n \geq 1$ and $Z_n (\omega) \in B_n$ the interval $\left[\alpha_{n+1}, \beta_{n+1}\right] = \left[\alpha_{n+1} (Z_n (\omega)), \beta_{n+1} (Z_n (\omega))\right]$ is reduced to the single point $\{0\}$; in this situation, we assume that the distribution G_{n+1} is transformed in such a way that it assigns the unit mass to the one point set $\{0\}$. Although in such a case $Z_{n+1}(\omega) = Z_n(\omega)$, the next random interval $[\alpha_{n+2}, \beta_{n+2}] = [\alpha_{n+2} (Z_{n+1}(\omega)), \beta_{n+2} (Z_{n+1}(\omega))]$, defined by means of the successive basic vector b_{n+2}, need not be equal to $\{0\}$; thus, the process is still continued.

Definition 1 The sequence of $\mathbb{B}$-valued r.e.'s $\{Z_n, n \geq 1\}$ obtained in the way described above is called *Banach random walk* (BRW) in a Banach space $\mathbb{B}$.

It seems that the main idea of Banach's [1] construction of $\mathfrak{L}$-integral was the symmetry of mappings corresponding to the symmetry of Lebesgue measures in $\mathbb{R}^n$, $n \geq 1$, which led to convergence of the integral functional in (1). Therefore, we introduce in addition the following notion:

Definition 2 The Schauder basis $\{b_n, n \geq 1\}$ is called *quasi-orthogonal*, if

$$\beta_{n+1}(\pi_n (x)) = -\alpha_{n+1} (\pi_n (x)) \tag{2}$$

for all $n \geq 1$ and $x \in \mathbb{B}$ such that $\pi_n (x) \in B_n$.

Recall that the basis $\{b_n, n \geq 1\}$ in a Banach space $(\mathbb{B}, \| \cdot \|)$ is said to be unconditional, if for all $n \geq 1$, $x_k \in \mathbb{R}$ and $\epsilon_k = \pm 1$, $1 \leq k \leq n$, we have

$$\left\| \sum_{k=1}^{n} \epsilon_k x_k b_k \right\| = \left\| \sum_{k=1}^{n} x_k b_k \right\|. \tag{3}$$

The definition of a quasi-orthogonal Schauder basis in a Banach space seems to be similar to the condition defining an unconditional basis, but in spite of this, these two notions are not equivalent. If the basis $\{b_n, n \geq 1\}$ is unconditional, i.e., (3) holds, then it is obviously quasi-orthogonal, but the converse need not be true. To explain the notion of quasi-orthogonality, below we present the construction of relevant examples of Banach spaces with quasi-orthogonal Schauder bases which are not unconditional. It should be pointed out that many familiar Banach spaces possess unconditional Schauder bases (thus in fact quasi-orthogonal) consisting of unit vectors, but the class of Banach spaces with quasi-orthogonal bases is substantially larger than the class of spaces with unconditional bases.

3 Banach Spaces with Quasi-Orthogonal Bases

The quasi-orthogonal basis is constructed sequentially, step by step: given any basic vectors $b_1, b_2, \ldots, b_n$, the element b_{n+1} of the basis is chosen in such a way that for arbitrary $x_1, x_2, \ldots, x_n \in \mathbb{R}$ satisfying condition $x_1 b_1 + \cdots + x_n b_n = \pi_n(x) \in B_n$, Eq. (2) is satisfied. As will be seen condition (3) for this property is not necessary.

1. *Spaces of bounded sums and conditionally convergent series*

Let $S = \mathbb{R}^{\mathbb{N}} = \{x = (x_1, x_2, \ldots) : x_k \in \mathbb{R} \text{ for all } k \geq 1\}$ be the set of all infinite sequences of real numbers. Define a function $\|\cdot\| : S \to [0, \infty]$ by

$$\begin{aligned} \|x\| &= \sup\{|x_1 + x_2|, |x_1 - x_2|, |x_1 + x_2 + x_3|, |x_1 + x_2 - x_3|, \\ &\qquad \ldots, |x_1 + \cdots + x_{n-1} + x_n|, |x_1 + \cdots + x_{n-1} - x_n|, \ldots\} \\ &= \sup\{|x_1| + |x_2|, |x_1 + x_2| + |x_3|, \ldots, |x_1 + \cdots + x_n| + |x_{n+1}|, \ldots\}, \end{aligned}$$

and next put

$$S_b = \{x = (x_1, x_2, \ldots) \in S\colon \|x\| < \infty\}.$$

Then $\|\cdot\|$ is a norm in S_b, and $(S_b, \|\cdot\|)$ is a Banach space. The space S_b consists of all bounded sequences $(x_1, x_2, \ldots) \in S$ of real numbers with bounded partial sums $s_n = x_1 + \cdots + x_n$; namely, if $x \in S_b$ and $\|x\| = M < \infty$, then $|x_n| \leq M$ and $|s_n| \leq M$ for all $n \geq 1$. Conversely, if there exists a constant $0 \leq M < \infty$ such that $|x_n| \leq M$ and $|s_n| \leq M$ for all $n \geq 1$, then $|s_n \pm x_{n+1}| \leq |s_n| + |x_{n+1}| \leq 2M$, and thus, $\|x\| \leq 2M$, i.e., $x = (x_1, x_2, \ldots) \in S_b$. Therefore, $(S_b, \|\cdot\|)$ may be called the *space of bounded sums*.

However, the space S_b of sequences of real numbers is not separable. To show this, consider the family $2^{\mathbb{N}}$ of all the subsets of the set $\mathbb{N} = \{1, 2, \ldots\}$, and for $\emptyset \neq A = \{k_1, k_2, \ldots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, define x_A as the sequence with terms $x_{k_{2j-1}} = 1$, $x_{k_{2j}} = -1$, $j \geq 1$, and $x_i = 0$ otherwise, along with $x_\emptyset = (0, 0, \ldots)$. Then $\|x_A\| \leq 2$, while $\|x_A - x_B\| \geq 1$ whenever $A, B \subseteq \mathbb{N}$, $A \neq B$. Since the set $2^{\mathbb{N}}$ is uncountable, the space S_b is nonseparable. Hence in the context of our requirements, the space S_b is inadequate.

Consider the set

$$S_c = \left\{ x = (x_1, x_2, \ldots) \in S_b\colon \text{ the series } \sum_k x_k \text{ converges} \right\}.$$

In other words, S_c is the set of all sequences $x = (x_1, x_2, \ldots) \in S_b$ for which a finite limit $\lim_n s_n = s \in \mathbb{R}$ exists. It can be easily verified that $(S_c, \|\cdot\|)$ is also a Banach space. Moreover, the space S_c is separable. Indeed, the set of elements

$$\{e_n = (\delta_{kn}, k \geq 1)\,,\ n = 1, 2, \ldots\}\,,$$

where $\delta_{kn} = 0$ for $k \neq n$ and $\delta_{nn} = 1$, is a basis of the space S_c, and finite linear combinations of vectors e_n with rational coefficients from a countable dense subset in S_c. Moreover, from the definition of the norm $\|\cdot\|$ it follows that the basis $\{e_n,\ n \geq 1\}$ is quasi-orthogonal, but it is not unconditional, because sums of the form $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n,\ x = (x_1, x_2, \ldots) \in S_c$, need not be convergent for all combinations of signs $\epsilon_k = \pm 1$. Taking into account the above properties, $(S_c, \|\cdot\|)$ may be called the Banach *space of conditionally convergent series*.

Since the existence of $\lim_n s_n = s \in \mathbb{R}$ implies that $\lim_n x_n = 0$, we conclude that $x \in S_c \Rightarrow x \in c_0$. The Banach space c_0 of sequences of real numbers convergent to 0 is usually considered with the supremum norm $|x|_\infty = \sup\{|x_1|, |x_2|, \ldots\}$, but the two norms $\|\cdot\|$ and $|\cdot|_\infty$ restricted to S_c are not equivalent. To see this, consider the sequence of points $\left\{x^{(n)}, n \geq 1\right\}$,

$$\begin{aligned}
x^{(1)} &= (1, -1/2, 1/3, -1/4, 1/5, -1/6, 1/7, -1/8, \ldots)\,,\\
x^{(2)} &= (1, 1/2, 1/3, -1/4, 1/5, -1/6, 1/7, -1/8, \ldots)\,,\\
x^{(3)} &= (1, 1/2, 1/3, 1/4, 1/5, -1/6, 1/7, -1/8, \ldots)\,,\\
&\ \ \vdots
\end{aligned}$$

and put $x^{(\infty)} = (1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots, 1/n, 1/(n+1), \ldots)$.

Then $\left\|x^{(n)}\right\| < \infty$ for all $n \geq 1$, so that $\left\{x^{(n)}, n \geq 1\right\} \subset S_c$. Moreover, $\left|x^{(n)} - x^{(\infty)}\right|_\infty = 1/n \to 0$, while $\left\|x^{(n)} - x^{(\infty)}\right\| = \infty$, $n = 1, 2, \ldots$ which is a consequence of the fact that $\sum_n 1/n = \infty$. Therefore, the inclusion $S_c \subset c_0$ is valid only for sets, but it is not true for Banach spaces, $(S_c, \|\cdot\|) \subsetneq (c_0, |\cdot|_\infty)$.

A similar effect as in the case of the space S_c can be obtained for every fixed system of signs $\epsilon = (\epsilon_1, \epsilon_2, \ldots) \in \{-1, 1\}^{\mathbb{N}}$ and the norm given by

$$\|x\|_\epsilon = \|(\epsilon_1 x_1, \epsilon_2 x_2, \epsilon_3 x_3, \ldots)\| \text{ for } x = (x_1, x_2, \ldots).$$

In this way, we obtain a Banach space $\left(S_{c,\epsilon}, \|\cdot\|_\epsilon\right)$, where

$$S_{c,\epsilon} = \left\{ x = (x_1, x_2, \ldots) \in S\colon \|x\|_\epsilon < \infty \text{ and the series } \sum_k \epsilon_k x_k \text{ converges} \right\}.$$

The basis $\{e_n,\ n \geq 1\}$ in $S_{c,\epsilon}$ is quasi-orthogonal, but it is not unconditional. Note now that $\ell^1 \subseteq S_{c,\epsilon}$ for each $\epsilon \in \{-1,1\}^{\mathbb{N}}$, therefore $\ell^1 \subseteq \bigcap_{\epsilon\in\{-1,1\}^{\mathbb{N}}} S_{c,\epsilon}$. On the other hand, assuming that $x = (x_1, x_2, \ldots) \in \bigcap_{\epsilon\in\{-1,1\}^{\mathbb{N}}} S_{c,\epsilon}$ and taking $\epsilon^{(x)} = (\operatorname{sign} x_1, \operatorname{sign} x_2, \ldots, \operatorname{sign} x_n, \ldots)$ we have

$$\epsilon_1^{(x)} x_1 + \epsilon_2^{(x)} x_2 + \cdots + \epsilon_n^{(x)} x_n = |x_1| + |x_2| + \cdots + |x_n|\,,\ n \geq 1.$$

Hence, we infer that

$$\sum_{n=1}^{\infty} |x_n| = \sup_n \left\{ \left| \epsilon_1^{(x)} x_1 + \epsilon_2^{(x)} x_2 + \cdots + \epsilon_n^{(x)} x_n \right| \right\} \leq \|x\|_{\epsilon^{(x)}}\,,$$

where $\epsilon^{(x)} \in \{-1,1\}^{\mathbb{N}}$, thus $x \in \ell^1$. Consequently, $\ell^1 = \bigcap_{\epsilon\in\{-1,1\}^{\mathbb{N}}} S_{c,\epsilon}$.

Define next a function $\|\cdot\|_{\cap} : \bigcap_{\epsilon\in\{-1,1\}^{\mathbb{N}}} S_{c,\epsilon} \to [0,\infty]$ by the formula: $\|x\|_{\cap} = \sup\left\{\|x\|_{\epsilon}\,,\ \epsilon \in \{-1,1\}^{\mathbb{N}}\right\}$. Since

$$|\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_{n-1} x_{n-1} \pm \epsilon_n x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for each $\epsilon \in \{-1,1\}^{\mathbb{N}}$, where the inequality $\leq$ may be replaced by the equality whenever $\epsilon_k = \operatorname{sign} x_k$, $1 \leq k \leq n-1$ and $\pm\epsilon_n = \operatorname{sign} x_n$, the function $\|\cdot\|_{\cap}$ assumes only finite values and in fact $\|x\|_{\cap} = \sum_n |x_n|$, i.e., $\|\cdot\|_{\cap}$ is the norm equal precisely to the norm $|x|_1 = \sum_n |x_n|$ of the space ℓ^1. Therefore, one can write

$$\left(\ell^1, |\cdot|_1\right) = \bigcap_{\epsilon\in\{-1,1\}^{\mathbb{N}}} \left(S_{c,\epsilon}, \|\cdot\|_{\epsilon}\right).$$

In this sense, the Banach space $\left(\ell^1, |\cdot|_1\right)$ in comparison with the space $(S_c, \|\cdot\|)$ is "*relatively small*". It is also worth mentioning that the basis $\{e_n, n \geq 1\}$ in S_c (as well as in $S_{c,\epsilon}$) is monotone, i.e., for every choice of scalars $\{x_n, n \geq 1\} \subseteq \mathbb{R}$, the sequence of numbers $\left\{\|\sum_{k=1}^n x_n e_n\|, n \geq 1\right\}$ is nondecreasing.

Recall now that a basis $\{b_n, n \geq 1\}$ of a Banach space $(\mathbb{B}, \|\cdot\|)$ is called boundedly complete if, for every sequence of scalars $\{x_n, n \geq 1\} \subseteq \mathbb{R}$ such that $\sup_{n\geq 1} \|\sum_{k=1}^n x_k b_k\| < \infty$, the series $\sum_{n=1}^{\infty} x_n b_n$ converges in norm of $\mathbb{B}$. Unfortunately, the basis $\{e_n, n \geq 1\}$ in S_c or $S_{c,\epsilon}$ is not boundedly complete.

2. *Spaces of conditionally convergent series with rates of convergence*

The space S_c described here may be the prototype for a wide class of various spaces with quasi-orthogonal Schauder bases that are not unconditional. For instance, consider the spaces S_c^p of (conditionally) summable sequences of real numbers spanned on vectors of the basis $\{e_n, n \geq 1\}$, with norms likewise in ℓ^p, $1 \leq p < \infty$, (that describe rates of convergence)

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \|R_n(x)\|^p\right)^{1/p},$$

where $R_n(x) = \sum_{k=n}^{\infty} x_k e_k$ for $x = (x_1, x_2, \ldots) \in S$; spaces $S_{c,w}$ with norms determined by some positive weights $w = (w_1, w_2, \ldots)$, $w_i > 0$,

$$\|x\|_{1,w} = \sum_{n=1}^{\infty} w_n \|R_n(x)\|,$$

say geometrical weights $w = (w_1, w_2, \ldots) = \left(q^1, q^2, \ldots\right)$, $q > 0$, or more generally, spaces $S_{c,w}^p$ equipped with norms

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n \|R_n(x)\|^p\right)^{1/p}, \ 1 \le p < \infty, \text{ etc.}$$

Since convergence of the series on the right-hand side of the definition of $\|x\|_p$ implies that partial sums of the series $\sum_n x_n$ satisfy Cauchy's criterion, the basis $\{e_n, n \ge 1\}$ in $\left(S_c^p, \|\cdot\|_p\right)$ is evidently monotone and boundedly complete. Thus in view of Dunford's theorem [4, Ch. III, §1, Th. 6, p. 64], these spaces possess the Radon–Nikodym property (RNP)—see [4, Ch. III, §1, p. 61] for the definition of this notion. Moreover, the equality

$$\left\|\sum_{j=k}^{n} x_j e_j\right\| = \left\|\sum_{j=k}^{n-1} x_j e_j - x_n e_n\right\|$$

valid for all $(x_1, x_2, \ldots) \in S$ and $1 \le k < n < \infty$ implies that the basis $\{e_n, n \ge 1\}$ in $\left(S_c^p, \|\cdot\|_p\right)$ is again quasi-orthogonal, but it is not unconditional. A similar conclusion can be also derived for spaces equipped with norms $\|\cdot\|_{p,w}$, provided that w is a suitable sequence of weights.

3. *Spaces of bounded sums of functions and conditionally convergent function series*

Let $\{q_1 = 0, q_2 = 1, q_3, q_4, \ldots\} \subset [0, 1] \subset \mathbb{R}$ be a countable set of numbers dense in $[0, 1]$ (arranged in any order), and let $\{e_n(t), n \ge 1\}$ be the system of Schauder hat functions in $[0, 1]$ defined as follows: $e_1(t) = 1 - t$ and $e_2(t) = t$, $0 \le t \le 1$; for $n > 2$ the points $q_1, \ldots, q_{n-1}$ divide the interval $[0, 1]$ into $n - 2$ subintervals, and if $\left[q_i, q_j\right]$ is the subinterval which contains the point q_n, then $e_n(t) = 0$ for $t \in [0, q_i] \cup [q_j, 1]$, $e_n(q_n) = 1$, and e_n is a linear function in the interval $[q_i, q_n]$ as well as in $\left[q_n, q_j\right]$. It is known that the described system of Schauder hat functions forms a basis for the space $C[0, 1]$ of real-valued continuous functions in $[0, 1]$ with the supremum norm

$$\|f\|_\infty = \sup_{0 \le t \le 1} |f(t)|,$$

see, e.g., [10, Prop. 2.3.5, p. 29].

Now with every sequence $x = (x_1, x_2, \ldots) \subset \mathbb{R}^{\mathbb{N}}$, we associate an element g having coefficients $(x_1, x_2, \ldots)$ in the basis $\{e_n(t), n \ge 1\}$, formally written as $g := (x_1 e_1 + x_2 e_2 + \cdots)$, and define

$$\|g\| = \|\sup\{|x_1e_1 + x_2e_2|, |x_1e_1 - x_2e_2|, \ldots, |x_1e_1 + \cdots + x_{n-1}e_{n-1} + x_ne_n|, \\ |x_1e_1 + \cdots + x_{n-1}e_{n-1} - x_ne_n|, \ldots\}\|_\infty \\ = \left\| \sup_{n\geq 1}\{|x_1e_1 + \cdots + x_ne_n| + |x_{n+1}e_{n+1}|\}\right\|_\infty,$$

along with

$$S_b[0,1] = \{g = (x_1e_1 + x_2e_2 + \cdots) : \|g\| < \infty\}.$$

Then $(S_b[0,1], \|\cdot\|)$ is a (nonseparable) Banach space. The space $S_b[0,1]$ consists of elements $g = (x_1e_1 + x_2e_2 + \cdots)$ with finite sums of functions $x_1e_1 + x_2e_2 + \cdots + x_ne_n$ bounded uniformly in $0 \leq t \leq 1$ and $n \geq 1$, but not necessarily convergent function series $x_1e_1 + x_2e_2 + \cdots$

Furthermore, denote

$$S_c[0,1] = \left\{g = (x_1e_1 + x_2e_2 + \cdots) \in S_b[0,1] : \text{the series } \sum_n x_ne_n \right. \\ \left. \text{converges in norm } \|\cdot\| \right\}.$$

It can be shown that $S_c[0,1]$ as a set of functions is identically equal to $C[0,1]$, and on account of the well-known open mapping theorem, the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$; therefore, $(S_c[0,1], \|\cdot\|)$ treated as a function Banach space with its norm topology is the same as $(C[0,1], \|\cdot\|_\infty)$. Since finite linear combinations $\sum_{k=1}^n w_ke_k$ with rational coefficients form a countable dense set in $C[0,1]$, the space $S_c[0,1]$ is separable. The Schauder basis $\{e_n(t), n \geq 1\}$ in $(S_c[0,1], \|\cdot\|)$ is monotone, because finite linear combinations of hat functions $e_n(t),\ n \geq 1$, are piecewise linear with an increasing number of nodes. However, the basis $\{e_n(t), n \geq 1\}$ is not boundedly complete, for bounded finite linear combinations of basic functions need not define a conditionally uniformly convergent series of functions. Moreover, the basis $\{e_n(t), n \geq 1\}$ in $S_c[0,1]$ is quasi-orthogonal with respect to $\|\cdot\|$, but it is not quasi-orthogonal in $(C[0,1], \|\cdot\|_\infty)$, and it is not unconditional. More precisely, properties of a given Schauder basis in $C[0,1]$ depend on the shape of the unit sphere, and from our considerations it follows that for each Schauder basis $\{e_n(t), n \geq 1\}$ there can be defined a norm $\|\cdot\|$ equivalent to the original supremum norm $\|\cdot\|_\infty$ in $C[0,1]$, such that the basis $\{e_n(t), n \geq 1\}$ becomes quasi-orthogonal with respect to $\|\cdot\|$, although the same basis need not be quasi-orthogonal with respect to $\|\cdot\|_\infty$.

By analogy to $S_{c,\epsilon}$ one can now define the spaces $S_{c,\epsilon}[0,1]$ consisting of function series $\sum_n \epsilon_nx_ne_n$ convergent (conditionally) uniformly in $0 \leq t \leq 1$ with norms

$$\|(x_1e_1 + x_2e_2 + \cdots)\|_\epsilon = \|(\epsilon_1x_1e_1 + \epsilon_2x_2e_2 + \cdots)\|,$$

for all sequences of signs $\epsilon = (\epsilon_1, \epsilon_2, \ldots) \in \{-1, 1\}^{\mathbb{N}}$. The intersection

$$\bigcap_{\epsilon \in \{-1,1\}^{\mathbb{N}}} S_{c,\epsilon}[0,1] := \ell^1[0,1]$$

is then the Banach space of function series convergent absolutely uniformly in $0 \leq t \leq 1$, with the norm

$$\sup\left\{\|(x_1 e_1 + x_2 e_2 + \cdots)\|_\epsilon : \epsilon \in \{-1,1\}^{\mathbb{N}}\right\} = \sup_{0 \leq t \leq 1} \sum_{n=1}^{\infty} |x_n e_n(t)| .$$

The last formula follows from the fact that Schauder hat functions $e_n(t)$, $n \geq 1$, are nonnegative, and in a more general case this is a consequence of a theorem by Sierpiński, cf. [10, Prop. 1.5.7, p. 19]. Clearly, the basis $\{e_n(t), n \geq 1\}$ in $\ell^1[0,1]$ is quasi-orthogonal, monotone, unconditional and boundedly complete; thus, $\ell^1[0,1]$ possesses the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64 and Ch. III, §3, Corollary 8, p. 83].

To estimate the rate of (uniform) convergence of function series of the form $g(t) = \sum_{n=1}^{\infty} x_n e_n(t)$, $t \in [0,1]$, we may introduce various norms similar as in ℓ^p, $1 \leq p < \infty$; namely, let $(R_n g)(t) = \sum_{k \geq n} x_k e_k(t)$, and let

$$S_c^p[0,1] = \left\{g = \sum_{n=1}^{\infty} x_n e_n(t) \in S_c[0,1]\colon \|g\|_p = \left(\sum_{n=1}^{\infty} \|(R_n g)\|^p\right)^{1/p} < \infty\right\}.$$

Arguing as in example 2, we conclude that the basis $\{e_n(t), n \geq 1\}$ in the space $\left(S_c^p[0,1], \|\cdot\|_p\right)$ is quasi-orthogonal, but it is not unconditional. Moreover, the basis $\{e_n(t), n \geq 1\}$ in $\left(S_c^p[0,1], \|\cdot\|_p\right)$ is monotone and boundedly complete; thus, by Dunford's theorem the spaces $\left(S_c^p[0,1], \|\cdot\|_p\right)$ possess the RNP, see [4, Ch. III, §1, Th. 6, p. 64].

Remark 1 The same idea as above leads to other examples of Banach spaces of a similar kind. For example, let $I = \{i_n, n \geq 1\}$ be a sequence of positive integers such that $1 \leq i_n \leq n$, $i_n \nearrow \infty$ and $n - i_n \nearrow \infty$ as $n \to \infty$, and for $x = \{x_n\} \in \mathbb{R}^\infty$, let

$$\|x\|_{I,p} = \left\{\sum_{n=1}^{\infty} \left(\left|x_{i_n} + \cdots + x_n\right| + |x_{n+1}|\right)^p\right\}^{1/p},$$

where $1 \leq p < \infty$, and

$$\|x\|_{I,\infty} = \sup_{n \geq 1} \left\{\left|x_{i_n} + \cdots + x_n\right| + |x_{n+1}|\right\}.$$

Define

$$\mathbb{B}_{I,p} = \left\{x = \{x_n\} \in \mathbb{R}^\infty\colon \text{ the series } \sum_n x_n e_n \text{ converges in norm } \|\cdot\|_{I,p}\right\},$$

$1 \leq p \leq \infty$. Then $\mathbb{B}_{I,p}$, $1 \leq p \leq \infty$, are Banach spaces such that $\{e_n, n \geq 1\}$ is a quasi-orthogonal basis with respect to $\|\cdot\|_{I,p}$, but in general not unconditional. Moreover, for $1 \leq p < \infty$ the basis $\{e_n, n \geq 1\}$ in $\mathbb{B}_{I,p}$ is boundedly complete, and thus, these spaces possess the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64]. Next, if $I^{(1)} = \left\{i_n^{(1)}, n \geq 1\right\}$, $I^{(2)} = \left\{i_n^{(2)}, n \geq 1\right\}, \ldots, I^{(k)} = \left\{i_n^{(k)}, n \geq 1\right\}$ are some sequences of positive integers satisfying conditions: $1 \leq i_n^{(1)} < \cdots < i_n^{(k)} \leq n$, $i_n^{(j)} \nearrow \infty$, $1 \leq j \leq k$, $i_n^{(j+1)} - i_n^{(j)} \nearrow \infty$, $1 \leq j < k$, and $n - i_n^{(k)} \nearrow \infty$ as $n \to \infty$, then for $n - i_n \geq k \geq 2$ (replacing i_n by $i_n^{(1)}$), the sums $\left|x_{i_n} + \cdots + x_n\right| + |x_{n+1}|$ can be divided into blocks of the form

$$\left|x_{i_n^{(1)}} + \cdots + x_{i_n^{(2)}-1}\right| + \left|x_{i_n^{(2)}} + \cdots + x_{i_n^{(3)}-1}\right| + \cdots + \left|x_{i_n^{(k)}} + \cdots + x_n\right| + |x_{n+1}|;$$

in addition, for $i \geq 1$, instead of $x_i e_i$, the terms $\epsilon_i x_i e_i$ can be used, where $\{\epsilon_i, i \geq 1\}$ are various sequences of signs ± 1. By means of these expressions, in an analogous way as above, new norms and new Banach spaces with quasi-orthogonal bases can be defined.

4 Convergence of the Banach Random Walk

According to the standard terminology, the measure G on the Borel σ-field $\mathcal{B}(\mathbb{R})$ is called here symmetric, if $G(-A) = G(A)$ for all $A \in \mathcal{B}(\mathbb{R})$.

Lemma 1 *Let $\{b_n, n \geq 1\}$ be a quasi-orthogonal Schauder basis in a real Banach space $(\mathbb{B}, \|\cdot\|)$, and let $\{G_n, n \geq 1\}$ be a sequence of symmetric probability distributions concentrated on the interval $[-1, 1] \subset \mathbb{R}$. Then the BRW $\{Z_n, n \geq 1\}$ is a $\mathbb{B}$-valued martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n) = \sigma(Z_1, Z_2, \ldots, Z_n)$, $n \geq 1$.*

Proof From the construction of BRW in a Banach space, it follows that $Z_n \in B$ for all $n \geq 1$, i.e., the r.e.'s Z_n are bounded and therefore Bochner integrable. Moreover, each Z_n is $\mathcal{F}_n$-measurable; thus,

$$E\left[Z_{n+1}|\mathcal{F}_n\right] = E\left[Z_n + X_{n+1}b_{n+1}|\mathcal{F}_n\right] = Z_n + E\left[X_{n+1}|\mathcal{F}_n\right]b_{n+1} \text{ a.s.}$$

Hence it suffices to show the equality $E\left[X_{n+1}|\mathcal{F}_n\right] = 0$ a.s. But the last statement is obvious, since it is known that X_{n+1} possesses a symmetric distribution in the a.s. symmetric interval $\left[\alpha_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))\right] = \left[-\beta_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))\right]$. □

Lemma 2 *Let $\{Z_n, n \geq 1\}$ be the BRW in a Banach space $\mathbb{B}$ with a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$ and let $\{\mathcal{F}_n, n \geq 1\}$ be the defined above filtration associated with $\{Z_n, n \geq 1\}$. Denote by $\mathcal{F}_\infty$ the σ-field generated by the field $\bigcup_{n=1}^{\infty} \mathcal{F}_n$, i.e., $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$. Then the following statements are true:*

(a) *There exists a vector measure* $\upsilon: \mathcal{F}_\infty \to \mathbb{B}$ *with bounded variation, absolutely continuous with respect to* P, *such that*

$$\left\langle Z_n, x^* \right\rangle \to \frac{\mathrm{d}\left\langle \upsilon, x^* \right\rangle}{\mathrm{d}P} \ \text{a.s. for all } x^* \in \mathbb{B}^*,$$

(b) *If there is a r.e.* $\xi \in L^1(\mathcal{F}_\infty; \mathbb{B})$ *such that*

$$\left\langle Z_n, x^* \right\rangle \to \left\langle \xi, x^* \right\rangle \ \text{a.s.}$$

for each $x^* \in \mathbb{B}^*$, *then*

$$\| Z_n - \xi \| \to 0 \ \text{a.s.}$$

Proof Since the BRW $\{Z_n, n \geq 1\}$ satisfies the condition $Z_n \in B$, $n \geq 1$, we have $\sup_{n \geq 1} E \|Z_n\| \leq 1 < \infty$. Therefore, our result is a direct consequence of a theorem given by Stegall, which can be found in [11, Ch. II, §4.3, Prop. 4.3, p. 132]. □

Lemma 3 *For each set* $A \in \mathcal{F}_\infty$ *there exists*

$$\lim_{n \to \infty} \int_A Z_n \mathrm{d}P = V(A)$$

in the strong topology of $\mathbb{B}$, *and the mapping* $V: \mathcal{F}_\infty \to \mathbb{B}$ *is a countably additive vector measure.*

Proof Observe first that in view of Jensen's inequality for conditional expectations in a Banach space, the sequence $\{\|Z_n\|, \mathcal{F}_n, n \geq 1\}$ is a real-valued submartingale, cf. [11, Ch. II, §4.1, (g), p. 127], or [12]. Furthermore,

$$\sup_{n \geq 1} E \|Z_n\|^p \leq 1 < \infty \ \text{ for each } 1 \leq p < \infty,$$

so that r.v.'s $\{\|Z_n\|, n \geq 1\}$ are uniformly integrable, which implies a.s. convergence $\|Z_n\| \to Z_\infty$ (and in L^1), where $Z_\infty \in L^p = L^p(\mathbb{R})$ for every fixed $1 \leq p < \infty$, see, e.g., [8, Ch. IV, Th. IV-1-2, p. 62, and Prop. IV-5-24, p. 91]. In particular,

$$\int_A \|Z_n\| \mathrm{d}P \to \int_A Z_\infty \mathrm{d}P$$

for each measurable set $A \in \mathcal{F}$. Next, if $B \in \bigcup_n \mathcal{F}_n$, then by the martingale property of $\{Z_n, n \geq 1\}$,

$$\int_B Z_n \mathrm{d}P \to V(B)$$

strongly in $\mathbb{B}$. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be chosen in such a way that $\int_C Z_\infty \mathrm{d}P < \varepsilon/3$ whenever $C \in \mathcal{F}_\infty$ and $P[C] < \delta$. Given any set $A \in \mathcal{F}_\infty$, select $B \in \bigcup_n \mathcal{F}_n$ satisfying condition $P[A \div B] < \delta$. Obviously,

$$\left\| \int_A Z_n \mathrm{d}P - \int_A Z_m \mathrm{d}P \right\| \le \left\| \int_A Z_n \mathrm{d}P - \int_B Z_n \mathrm{d}P \right\| + \left\| \int_B Z_n \mathrm{d}P - \int_B Z_m \mathrm{d}P \right\| + \left\| \int_B Z_m \mathrm{d}P - \int_A Z_m \mathrm{d}P \right\|.$$

Moreover,

$$\left\| \int_A Z_n \mathrm{d}P - \int_B Z_n \mathrm{d}P \right\| \le \int_{A \setminus B} \|Z_n\| \,\mathrm{d}P + \int_{B \setminus A} \|Z_n\| \,\mathrm{d}P = \int_{A \div B} \|Z_n\| \,\mathrm{d}P.$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_{A \div B} \|Z_n\| \,\mathrm{d}P = \int_{A \div B} Z_\infty \mathrm{d}P < \varepsilon/3,$$

thus

$$\left\| \int_A Z_n \mathrm{d}P - \int_B Z_n \mathrm{d}P \right\| < \varepsilon/3$$

for sufficiently large $n \ge n_0$. Since the sequence $\left\{\int_B Z_n \mathrm{d}P, n \ge 1\right\}$ is Cauchy in $\mathbb{B}$, we also conclude that

$$\left\| \int_B Z_n \mathrm{d}P - \int_B Z_m \mathrm{d}P \right\| < \varepsilon/3$$

for all large enough $m > n \ge n_1$. Consequently,

$$\left\| \int_A Z_n \mathrm{d}P - \int_A Z_m \mathrm{d}P \right\| < \varepsilon,$$

whenever $m > n \ge \max\{n_0, n_1\}$. In other words, the sequence of integrals $\left\{\int_A Z_n \mathrm{d}P, n \ge 1\right\}$ is Cauchy in $(\mathbb{B}, \|\cdot\|)$, and therefore, there exists

$$\lim_{n\to\infty} \int_A Z_n \mathrm{d}P = V(A)$$

in the strong topology of $\mathbb{B}$ for each set $A \in \mathcal{F}_\infty$. It can be easily seen that $V\colon \mathcal{F}_\infty \to \mathbb{B}$ is finitely additive. Let $A_1, A_2, \ldots \in \mathcal{F}_\infty$ be an arbitrary sequence of pairwise disjoint sets. Notice that

$$\left\| \lim_{n\to\infty} \int_{\bigcup_{k=1}^\infty A_k} Z_n \mathrm{d}P - \lim_{n\to\infty} \int_{\bigcup_{k=1}^m A_k} Z_n \mathrm{d}P \right\| = \lim_{n\to\infty} \left\| \int_{\bigcup_{k=m+1}^\infty A_k} Z_n \mathrm{d}P \right\|,$$

and thus, to prove countable additivity of V it is enough to show that

$$\lim_{n\to\infty}\left\|\int_{\bigcup_{k=m+1}^{\infty}A_k} Z_n \mathrm{d}P\right\| \to 0$$

as $m \to \infty$. Taking m_0 so large that $P\left[\bigcup_{k=m+1}^{\infty}A_k\right] < \delta$ for $m \geq m_0$, we obtain

$$\lim_{n\to\infty}\left\|\int_{\bigcup_{k=m+1}^{\infty}A_k} Z_n \mathrm{d}P\right\| \leq \lim_{n\to\infty}\int_{\bigcup_{k=m+1}^{\infty}A_k}\|Z_n\|\,\mathrm{d}P = \int_{\bigcup_{k=m+1}^{\infty}A_k} Z_\infty \mathrm{d}P < \varepsilon/3$$

provided $m \geq m_0$, which terminates the proof. □

The above Lemma 3 enables us to apply the Lebesgue decomposition theorem for the vector measure V, see [4, Ch. I, §5, Th. 9, p. 31].

Lemma 4 *Let*

$$V(A) = \lim_{n\to\infty}\int_A Z_n \mathrm{d}P, \quad A \in \mathcal{F}_\infty,$$

and let $V = H + J$, $|H| \ll P$, $|J| \perp P$, *be the Lebesgue decomposition of* V *with respect to* P, *where* $|H|$, $|J|$ *are variations of* H *and* J, *respectively. Then* $\lim_{n\to\infty} Z_n$ *exists a.s. if and only if* H *has a Radon–Nikodym derivative* $h \in L^1(\mathcal{F}; \mathbb{B})$. *Moreover, in this case* $\lim_{n\to\infty} Z_n = E(h|\mathcal{F}_\infty)$ *a.s.*

Proof Arguing as above, we easily note that $\{Z_n, \mathcal{F}_n, n \geq 1\}$ is an $L^1(\mathbb{B})$-bounded martingale (here, and in the sequel $L^p(\mathbb{B}) = L^p(\Omega, \mathcal{F}, P; \mathbb{B})$, $1 \leq p < \infty$); thus, the conclusion follows from the martingale pointwise convergence theorem given in [4, Ch. V, §2, Th. 9, p. 130]. □

Lemma 5 *Let* $\mathbb{B}$ *be a Banach space with the RNP and a quasi-orthogonal Schauder basis* $\{b_n, n \geq 1\}$. *Moreover, let* $\{G_n, n \geq 1\}$ *be a sequence of symmetric probability distributions concentrated on* $[-1, 1] \subset \mathbb{R}$. *Then the BRW martingale* $\{Z_n, n \geq 1\}$ *converges strongly a.s. and in* $L^p(\mathbb{B})$ *for each fixed* $1 \leq p < \infty$.

Proof The limit $\lim_{n\to\infty} Z_n$ of the martingale $\{Z_n, n \geq 1\}$ exists in $L^p(\mathbb{B})$-norm, if and only if $\sup_{n\geq 1}\|Z_n\|_p^p = \sup_{n\geq 1} E\|Z_n\|^p < \infty$, where $1 < p < \infty$, which is evident as $Z_n \in B$ for $n \geq 1$. The last observation implies uniform integrability of random elements $\{Z_n, n \geq 1\}$, and we have obviously $\sup_{n\geq 1}\|Z_n\|_1 = \sup_{n\geq 1} E\|Z_n\| \leq 1 < \infty$. Thus $\lim_{n\to\infty} Z_n$ exists as well in $L^1(\mathbb{B})$-norm in view of the martingale mean convergence theorem, cf. [4, Ch. V, §2, Corollary 4, p. 126]. It is also well known that an $L^1(\mathbb{B})$ convergent martingale converges a.s. to its $L^1(\mathbb{B})$-limit, see [4, Ch. V, §2, Th. 8, p. 129], or [11, Ch. II, §4.3, Th. 4.2, p. 131 and Th. 4.3, p. 136]. □

Corollary 1 *If* $\mathbb{B}$ *is a Banach space with a quasi-orthogonal boundedly complete Schauder basis* $\{b_n, n \geq 1\}$, *then the BRW* $\{Z_n, n \geq 1\}$ *in* $\mathbb{B}$ *converges strongly a.s. and in* $L^p(\mathbb{B})$ *for each fixed* $1 \leq p < \infty$.

Proof By a theorem of Dunford, if a Banach space $\mathbb{B}$ possesses a boundedly complete Schauder basis, then $\mathbb{B}$ has the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64]. Hence and from Lemma 5, the assertion of Corollary 1 follows. □

Corollary 2 *Let $\mathbb{B}$ be a reflexive Banach space which has a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$. Then the $\mathbb{B}$-valued BRW $\{Z_n, n \geq 1\}$ converges strongly a.s. and in $L^p(\mathbb{B})$ for each fixed $1 \leq p < \infty$. In particular, if $\mathbb{B} = \mathbb{H}$ is a Hilbert space with a basis $\{b_n, n \geq 1\}$ which forms a CONS in $\mathbb{H}$, then the last statement remains valid.*

Proof It is fairly well known from a theorem of Phillips that reflexive Banach spaces have the RNP, see [4, Ch. III, §3, Corollary 4, p. 82]. Since each Hilbert space is reflexive, we conclude that $\mathbb{B} = \mathbb{H}$ has the RNP. Thus, an application of Lemma 5 concludes the proof. □

Theorem 1 *Let $\phi: \mathbb{B} \to \mathbb{R}$ be a bounded and continuous mapping in a Banach space $\mathbb{B}$ which has the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$. If $\{Z_n, n \geq 1\}$ is the BRW in $\mathbb{B}$, then*

$$\phi(Z_n) \to \phi(\xi) \ \text{ a.s. and in } L^p = L^p(\mathbb{R}), \quad 1 \leq p < \infty,$$

where $\xi = \lim_{n\to\infty} Z_n$ a.s. and in $L^p(\mathbb{B})$-norm for all $1 \leq p < \infty$. In particular, there exists

$$\lim_{n\to\infty} E\phi(Z_n) = E\phi(\xi).$$

Proof Since ϕ is continuous and the assumptions of Lemma 5 are fulfilled, we conclude that $\phi(Z_n) \to \phi(\xi)$ a.s. But in addition ϕ is assumed to be bounded, thus using the Lebesgue-dominated convergence theorem we obtain also convergence $\phi(Z_n) \to \phi(\xi)$ in L^p, $1 \leq p < \infty$. The last statement of the theorem follows from the estimate

$$|E\phi(Z_n) - E\phi(\xi)| \leq E\,|\phi(Z_n) - \phi(\xi)| \to 0.$$

□

Corollary 3 *The assertion of Theorem 1 remains valid for a Banach space $\mathbb{B}$ with a boundedly complete quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$, as well as for a Hilbert space $\mathbb{B} = \mathbb{H}$ with the Schauder basis $\{b_n, n \geq 1\}$ that forms a CONS in $\mathbb{H}$.*

5 The Banach Functional Integral in a Class of Banach Spaces

Let $\mathcal{C}_b = \{\phi: B \to \mathbb{R}; \phi\text{-bounded and continuous}\}$. Observe that $\mathcal{C}_b$ has the following properties:
(i_1) the set $\mathcal{C}_b$ is a real linear space,
(i_2) if $\phi \in \mathcal{C}_b$, then $|\phi| \in \mathcal{C}_b$.

Define a functional $f : \mathcal{C}_b \to \mathbb{R}$ by the formula:

$$f(\phi) = \lim_{n\to\infty} E\phi(Z_n) = E\phi(\xi), \tag{4}$$

where $\{Z_n, n \geq 1\}$ is a BRW in the Banach space $\mathbb{B}$, and $\xi = \lim_{n\to\infty} Z_n$ a.s. and in $L^p(\mathbb{B})$, $1 \leq p < \infty$.

It can be easily seen that the mapping f satisfies the following conditions:
(ii$_1$) $f : \mathcal{C}_b \to \mathbb{R}$ is a linear functional,
(ii$_2$) the functional f is nonnegative, i.e., $f(\phi) \geq 0$ whenever $\phi \in \mathcal{C}_b$ and $\phi \geq 0$,
(ii$_3$) if 1^0 $\{\phi_n\} \subset \mathcal{C}_b$, $\psi \in \mathcal{C}_b$, 2^0 $|\phi_n| \leq \psi$ for $n \geq 1$, and 3^0 $\lim_{n\to\infty} \phi_n(x) = 0$ for all $x \in B$, then $\lim_{n\to\infty} f(\phi_n) = 0$.

Notice that 3^0 implies P-a.s. convergence $\phi_n(\xi) \to 0$; thus, the last condition follows from the classical Lebesgue-dominated convergence theorem applied to integrals $E\phi_n(\xi)$, $n = 1, 2, \ldots$ (Actually, in our approach we can even replace condition 3^0 by a weaker assumption $\phi_n \to 0$ in $P \circ \xi^{-1}$-measure.) Consequently, the functional f satisfies all the conditions given in §2 of the Banach paper [1]. Therefore, for our functional f the Banach Th. 1, §3, p. 322, [1] is valid. In this way, we obtain the following result.

Theorem 2 *Let $\{Z_n, n \geq 1\}$ be a BRW in a Banach space $\mathbb{B}$ with the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$, in particular in a Banach space $\mathbb{B}$ with a boundedly complete quasi-orthogonal Schauder basis. Then, the functional f given by* (4) *has an extension to the additive functional F on the linear set $\mathcal{L} \supset \mathcal{C}_b$ of all bounded, Borel measurable functions $\Phi : B \to \mathbb{R}$. Moreover, the extended functional F on $\mathcal{L}$ possesses all the properties* (A)–(E) *and* (R) *specified in §1 of the Banach paper* [1], *analogous to the Lebesgue integral.*

Remark 2 The approach presented above is a generalization of the method proposed by Banach [1] for the construction of the so-called $\mathfrak{L}$-integral—an analogue of the Lebesgue integral in abstract spaces.

Theorem 3 *Let $\mathbb{B}$ be a Banach space with the RNP and a quasi-orthogonal Schauder basis, in particular—a Banach space with a boundedly complete quasi-orthogonal Schauder basis. Then each sequence of symmetric probability distributions $\{G_n, n \geq 1\}$ concentrated on the interval $[-1, 1] \subset \mathbb{R}$ generates a probability measure Γ on the Borel σ-field $\mathcal{B}$ in $\mathbb{B}$, given by*

$$\Gamma(A) = E\mathbb{1}_A(\xi), \quad A \in \mathcal{B}.$$

The measure Γ is equal to the limit distribution of the described above BRW $\{Z_n, n \geq 1\}$ in $\mathbb{B}$, thus $\operatorname{supp} \Gamma \subseteq B$.

Proof Obviously, Γ is nonnegative and normalized so that $\Gamma(\mathbb{B}) = \Gamma(B) = 1$. It suffices to verify countable additivity of Γ, but it follows immediately from the properties of the integral $E(\cdot)$. The last conclusion can also be easily shown in a direct way. To this end, let $A_1, A_2, \ldots \in \mathcal{B}$ be arbitrary disjoint sets. Since $\operatorname{supp} \Gamma \subseteq B$, we have $\Gamma(A) = \Gamma(A \cap B) = E\mathbb{1}_{A\cap B}(\xi)$, $A \in \mathcal{B}$. Observe next that

$$E\mathbb{1}_{\bigcup_{j=1}^{n}(A_j\cap B)}(\xi) = E\left(\sum_{j=1}^{n}\mathbb{1}_{A_j\cap B}(\xi)\right) = \sum_{j=1}^{n}E\mathbb{1}_{A_j\cap B}(\xi) = \sum_{j=1}^{n}\Gamma\left(A_j\right),$$

and

$$0 \le \sum_{j=1}^{n}\mathbb{1}_{A_j\cap B}(x) = \mathbb{1}_{\bigcup_{j=1}^{n}(A_j\cap B)}(x) \nearrow \mathbb{1}_{\bigcup_{n=1}^{\infty}(A_n\cap B)}(x) \le \mathbb{1}_B(x),\ x \in B.$$

Hence, on account of the Lebesgue monotone convergence theorem,

$$\begin{aligned}\sum_{n=1}^{\infty}\Gamma(A_n) &= \lim_{n\to\infty}\sum_{j=1}^{n}\Gamma\left(A_j\right) = \lim_{n\to\infty}E\mathbb{1}_{\bigcup_{j=1}^{n}(A_j\cap B)}(\xi)\\ &= E\left(\lim_{n\to\infty}\mathbb{1}_{\bigcup_{j=1}^{n}(A_j\cap B)}(\xi)\right) = E\mathbb{1}_{(\bigcup_{n=1}^{\infty}A_n)\cap B}(\xi) = \Gamma\left(\bigcup_{n=1}^{\infty}A_n\right).\end{aligned}$$

□

From the construction of the BRW in a Banach space, it follows immediately that the limit distribution $\Gamma = P\circ\xi^{-1}$ of the BRW is sign-invariant with respect to the Schauder basis $\{b_n, n \ge 1\}$, in the sense that for each set $A \in \mathcal{B}$ and every sequence $\epsilon = \{\epsilon_1, \epsilon_2, \ldots\}$ of signs $\epsilon_k \in \{-1, 1\},\ k \ge 1$, we have

$$\Gamma(A) = \Gamma(\epsilon A),$$

where $\epsilon A = \left\{\sum_{k=1}^{\infty}\epsilon_k x_k b_k \in \mathbb{B}\colon \sum_{k=1}^{\infty}x_k b_k \in A\right\}$. It is also clear that each sign-invariant measure is symmetric, thus

$$\Gamma(A) = \Gamma(-A)\ \text{ for all } A \in \mathcal{B}.$$

By analogy to the notion of the Wiener measure, we propose to call Γ *the Banach measure* in a Banach space. One may expect that the Banach measure will play a similarly important role in Banach spaces as is played by the Gaussian measure constructed by Gross [5], cf. Bogachev [3], or Kuo [6].

6 Examples

1. Let $\mathbb{B} = \ell^p,\ 1 \le p < \infty$, and let $b_n = e_n = (0, \ldots, 0, 1, 0, \ldots),\ n \ge 1$, where 1 is the nth term of the sequence $(0, \ldots, 0, 1, 0, \ldots)$. Then $\pi_n(x) = \sum_{k=1}^{n}x_k b_k = (x_1, \ldots, x_n, 0, \ldots)$ for $x = \sum_{n=1}^{\infty}x_n b_n = (x_1, x_2, \ldots) \in \ell^p$, and thus

$$|\pi_n(x) + tb_{n+1}|_p^p = \sum_{k=1}^{n}|x_k|^p + |t|^p \le 1 \ \Leftrightarrow\ |t| \le \left(1 - \sum_{k=1}^{n}|x_k|^p\right)^{1/p}.$$

Hence it follows that $\{b_n, n \geq 1\}$ is a quasi-orthogonal Schauder basis in ℓ^p. (In fact, the considered basis is unconditional.) Moreover, if

$$\sup_{n\geq 1} |\pi_n(x)|_p = \sup_{n\geq 1} \left| \sum_{k=1}^{n} x_k b_k \right|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \leq M < \infty,$$

then the series $\sum_{n=1}^{\infty} x_n b_n$ converges in ℓ^p. Thus, the basis $\{b_n, n \geq 1\}$ is boundedly complete and in consequence each space $\ell^p,\ 1 \leq p < \infty$, has the RNP. It is also well known that for $p > 1$ the spaces ℓ^p are reflexive, which implies as well that they have the RNP. Therefore, all the above results are valid for Banach spaces $\mathbb{B} = \ell^p,\ 1 \leq p < \infty$.

2. Let $\mathbb{B} = L^p\,[0, 1]$, where $1 \leq p < \infty$. Consider the system of Haar functions: $h_1^0\,(s) = 1,\ s \in [0, 1]$, and

$$h_k^n\,(s) = 2^n \cdot \mathbb{1}_{[(2k-2)/2^{n+1},\,(2k-1)/2^{n+1})}\,(s) - 2^n \cdot \mathbb{1}_{[(2k-1)/2^{n+1},\,(2k)/2^{n+1})}\,(s)$$

for $k = 1, 2, \ldots, 2^n,\ n = 1, 2, \ldots,\ s \in [0, 1]$. It is known that the system of Haar functions forms a Schauder basis in $L^p\,[0, 1]$, see, e.g., [9, Th. 24.17, pp. 290–295], or [7, Part II, Prop. 2.c.1, p. 150]. For convenience of the reader, we sketch here the proof that the Haar basis in $L^p\,[0, 1]$ is quasi-orthogonal.

Proposition 1 *The system of Haar functions is a quasi-orthogonal basis in* $L^p\,[0, 1]$, $1 \leq p < \infty$.

Proof Let the Haar functions be arranged in a sequence that is divided into blocks, each of 2^n members, numbered by upper indices $n = 0, 1, 2, \ldots,$

$$\Lambda = \left\{ \left(h_1^0\right), \left(h_1^1, h_2^1\right), \left(h_1^2, h_2^2, h_3^2, h_4^2\right), \left(h_1^3, h_2^3, h_3^3, h_4^3, h_5^3, h_6^3, h_7^3, h_8^3\right), \ldots \right\}.$$

We make now two crucial observations: 1^0 for a fixed $n \geq 0$ within the same nth block the Haar functions $h_k^n,\ 1 \leq k \leq 2^n$, have nonoverlapping supports, 2^0 the function of the form $c_1^0 h_1^0 + c_1^1 h_1^1 + c_2^1 h_2^1 + \cdots + c_1^{n-1} h_1^{n-1} + \cdots + c_{2^{n-1}}^{n-1} h_{2^{n-1}}^{n-1}$, where $c_j^i \in \mathbb{R}$ are arbitrarily fixed coefficients, is constant in each interval of the form $\left[(k-1)/2^n, k/2^n\right),\ k = 1, 2, \ldots, 2^n$.

Suppose now that $\pi_{k-1}^n\,(x) = x_1^0 h_1^0 + x_1^1 h_1^1 + x_2^1 h_2^1 + \cdots + x_{k-1}^n h_{k-1}^n$ and the next element of Λ is h_k^n. Then for any parameter $t \in \mathbb{R}$,

$$\begin{aligned}\left\|\pi_{k-1}^n\,(x) + t h_k^n\right\|_p^p &= \int_0^1 \left|\pi_{k-1}^n\,(x)\,(s) + t h_k^n\,(s)\right|^p \mathrm{d}s = \int_0^{(k-1)/2^n} \left|\pi_{k-1}^n\,(x)\,(s)\right|^p \mathrm{d}s \\ &+ \int_{(k-1)/2^n}^{k/2^n} \left|\pi_{2^{n-1}}^{n-1}\,(x)\,(s) + t h_k^n\,(s)\right|^p \mathrm{d}s \\ &+ \int_{k/2^n}^{1} \left|\pi_{2^{n-1}}^{n-1}\,(x)\,(s)\right|^p \mathrm{d}s = I_1 + I_2 + I_3,\end{aligned}$$

where $\pi_{2^{n-1}}^{n-1}(x)(s) = \pi_{2^{n-1}}^{n-1}(x)\left((2k-1)/2^{n+1}\right) = c,\ s \in \left[(k-1)/2^n, k/2^n\right)$. The first and third integrals on the right-hand side do not depend on the parameter t, and the middle term is equal to

$$I_2 = \int_{(k-1)/2^n}^{(2k-1)/2^{n+1}} \left|c + t2^n\right|^p \mathrm{d}s + \int_{(2k-1)/2^{n+1}}^{k/2^n} \left|c - t2^n\right|^p \mathrm{d}s$$
$$= \left|c + t2^n\right|^p \cdot \frac{1}{2^{n+1}} + \left|c - t2^n\right|^p \cdot \frac{1}{2^{n+1}} := r(t).$$

Since $r(t) = r(-t)$, and

$$\inf\{t \in \mathbb{R}: r(t) \le 1 - I_1 - I_3\} = -\sup\{-t \in \mathbb{R}: r(-t) \le 1 - I_1 - I_3\}$$
$$= -\sup\left\{t' \in \mathbb{R}: r\left(t'\right) \le 1 - I_1 - I_3\right\},$$

we conclude that $\alpha_k^n = -\beta_k^n$. The same argument remains valid when π_{k-1}^n is replaced by $\pi_{2^{n-1}}^{n-1}$, and π_k^n is replaced by π_1^n, and thus, the system of Haar functions forms a quasi-orthogonal basis in $L^p[0,1]$. □

Remark 3 It is clear that the Haar functions for $n \ge 1$ can be modified as follows:

$$h_k^n(s) = 2^{n/p} \cdot \mathbb{1}_{[(2k-2)/2^{n+1},\,(2k-1)/2^{n+1})}(s) - 2^{n/p} \cdot \mathbb{1}_{[(2k-1)/2^{n+1},\,(2k)/2^{n+1})}(s),$$

$k = 1, 2, \ldots, 2^n,\ s \in [0,1]$. Then the above Proposition 1 for modified Haar functions remains true, and in addition, we have $\|h_k^n\|_p = 1$ for all k, n.

Evidently, all the spaces $L^p[0,1]$, $1 < p < \infty$, are reflexive Banach spaces, and thus, they possess the RNP, which is a straightforward consequence of Phillips' theorem, cf. [4, Ch. III, §3, Corollary 6, p. 82]. Unfortunately, the space $L^1[0,1]$ does not have the RNP, see [4, Ch. VII, p. 219].

Hence, it follows that all the results presented in previous sections are valid for Banach spaces $L^p[0,1]$, $1 < p < \infty$.

Remark 4 Proposition 1 together with observation that the space $L^1[0,1]$ does not possess the RNP implies the following conclusion: the existence of a quasi-orthogonal Schauder basis in a Banach space is not a sufficient condition for the RNP. The same conclusion follows from the fact that the space $(S_c[0,1], \|\cdot\|)$ (isometrically isomorphic to $(C[0,1], \|\cdot\|_\infty)$) does not have the RNP.

3. To illustrate the technique of computations of Banach $\mathfrak{L}$-integrals based on the method described here, we calculate, for instance, two "rarefied" absolute pth moments of $\xi = \sum_{n=1}^{\infty} X_n b_n$ in ℓ^p, $1 \le p < \infty$, determined by lacunary series $\sum_{n=1}^{\infty} X_{2n-1} b_{2n-1}$ and $\sum_{n=1}^{\infty} X_{2n} b_{2n}$, namely $E\left(\sum_{n=1}^{\infty} |X_{2n-1}|^p\right)$ and $E\left(\sum_{n=1}^{\infty} |X_{2n}|^p\right)$, where ξ is the limit random element of the BRW generated by the sequence of identical uniform distributions on $[-1,1] \subset \mathbb{R}$. Since the basis $\{b_n, n \ge 1\}$ in ℓ^p is unconditional, the lacunary series considered here are strongly convergent, see, e.g., [7, Prop. 1.c.1, p. 15, and p. 19].

Observe first that if $\pi_{\text{odd}}(x) = \sum_{n=1}^{\infty} x_{2n-1} b_{2n-1}$ for $x = \sum_{n=1}^{\infty} x_n b_n \in \ell^p$, then $|\pi_{\text{odd}}(x)|_p = \left(\sum_{n=1}^{\infty} |x_{2n-1}|^p\right)^{1/p}$, thus

$$\left| \left(\sum_{n=1}^{\infty} |x_{2n-1}|^p\right)^{1/p} - \left(\sum_{n=1}^{\infty} \left|x'_{2n-1}\right|^p\right)^{1/p} \right| = \left| |\pi_{\text{odd}}(x)|_p - \left|\pi_{\text{odd}}\left(x'\right)\right|_p \right|$$
$$\leq \left|\pi_{\text{odd}}\left(x - x'\right)\right|_p = \left(\sum_{n=1}^{\infty} \left|x_{2n-1} - x'_{2n-1}\right|^p\right)^{1/p} \leq \left|x - x'\right|_p \to 0$$

as $x \to x'$ in ℓ^p-norm $|\cdot|_p$. It follows that $\ell^p \ni x \mapsto \left(\sum_{n=1}^{\infty} |x_{2n-1}|^p\right)^{1/p}$ is a continuous function. Therefore, $\ell^p \ni x \mapsto \sum_{n=1}^{\infty} |x_{2n-1}|^p$, as well as $\ell^p \ni x \mapsto \sum_{n=1}^{\infty} |x_{2n}|^p$ are continuous, and both these maps are bounded in the unit ball $B \subset \ell^p$.

According to the construction of the BRW in ℓ^p, the density of X_1 is equal to $f_1(x_1) = \mathbb{1}_{[-1,1]}(x_1)/2$, and the density of $(X_1, X_2, \ldots, X_n)$ for $n > 1$ is given by

$$f_n(x_1, x_2, \ldots, x_n)$$
$$= \frac{\mathbb{1}_{K_n}(x_1, x_2, \ldots, x_n)}{2^n \cdot \left(\left[1 - |x_1|^p\right] \cdot \left[1 - \left(|x_1|^p + |x_2|^p\right)\right] \cdot \ldots \cdot \left[1 - \left(|x_1|^p + \cdots + |x_{n-1}|^p\right)\right]\right)^{1/p}},$$

where $K_n = K_n(0, 1)$ is the unit ball with center zero and radius 1 in $\mathbb{R}^n$, equipped with the ℓ^p-norm $|(x_1, \ldots, x_n)|_{n,p} = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$. Hence,

$$E|X_1|^p = \int_{-1}^{1} \frac{|x_1|^p}{2} \mathrm{d}x_1 = 2\int_{0}^{1} \frac{x_1^p}{2} \mathrm{d}x_1 = \left.\frac{x_1^{p+1}}{p+1}\right|_0^1 = \frac{1}{p+1}.$$

Moreover, for $n > 1$,

$$\begin{aligned}
E|X_n|^p &= \int_{K_n} |x_n|^p \cdot f_n(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \\
&= \int_{K_{n-1}} \left(2\int_0^{\left[1-\left(|x_1|^p+\cdots+|x_{n-1}|^p\right)\right]^{1/p}} x_n^p \cdot f_n(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_n\right) \\
&\quad \times \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \\
&= \int_{K_{n-1}} \left.\frac{x_n^{p+1}}{p+1}\right|_0^{\left[1-\left(|x_1|^p+\cdots+|x_{n-1}|^p\right)\right]^{1/p}} \\
&\quad \times \frac{f_{n-1}(x_1, x_2, \ldots, x_{n-1})}{\left[1 - \left(|x_1|^p + \cdots + |x_{n-1}|^p\right)\right]^{1/p}} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \\
&= \int_{K_{n-1}} \frac{\left[1 - \left(|x_1|^p + \cdots + |x_{n-1}|^p\right)\right]}{p+1} \cdot f_{n-1}(x_1, x_2, \ldots, x_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \\
&= \frac{1}{p+1} \left\{ 1 - \left(E\,|X_1|^p + E\,|X_2|^p + \cdots + |X_{n-1}|^p \right) \right\}.
\end{aligned}
$$

Thus,

$$
E\,|X_2|^p = \frac{1}{p+1} \left\{ 1 - E\,|X_1|^p \right\} = \frac{1}{p+1} \left(1 - \frac{1}{p+1} \right) = \frac{1}{p+1} \frac{p}{p+1},
$$

and, by induction,

$$
\begin{aligned}
E\,|X_n|^p &= \frac{1}{p+1} \left\{ 1 - \left(\frac{1}{p+1} + \frac{p}{(p+1)^2} + \cdots + \frac{p^{n-2}}{(p+1)^{n-1}} \right) \right\} \\
&= \frac{1}{(p+1)} \left\{ 1 - \frac{1}{p+1} \cdot \frac{1 - p^{n-1}/(p+1)^{n-1}}{1 - p/(p+1)} \right\} = \frac{p^{n-1}}{(p+1)^n}.
\end{aligned}
$$

Therefore,

$$
E \left(\sum_{n=1}^{\infty} |X_{2n-1}|^p \right) = \sum_{n=1}^{\infty} \frac{p^{2n-1-1}}{(p+1)^{2n-1}} = \frac{1}{p+1} \cdot \frac{1}{1 - p^2/(p+1)^2} = \frac{p+1}{2p+1},
$$

and

$$
E \left(\sum_{n=1}^{\infty} |X_{2n}|^p \right) = \sum_{n=1}^{\infty} \frac{p^{2n-1}}{(p+1)^{2n}} = \frac{p}{(p+1)^2} \cdot \frac{1}{1 - p^2/(p+1)^2} = \frac{p}{2p+1}.
$$

In consequence,

$$
E\,|\pi_{\mathrm{odd}}(\xi)|_p^p = \frac{p+1}{2p+1}, \quad E\,|\xi - \pi_{\mathrm{odd}}(\xi)|_p^p = \frac{p}{2p+1},
$$

so that

$$
E\,|\xi|_p^p = E \left(\sum_{n=1}^{\infty} |X_n|^p \right) = \frac{p+1}{2p+1} + \frac{p}{2p+1} = 1.
$$

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References

1. Banach, S.: The Lebesgue integral in abstract spaces. In: Saks, S. (ed.) Theory of the Integral. Dover Publication, New York (1964)
2. Banek, T.: Banach random walk in the unit ball $S \subset \ell^2$ and chaotic decomposition of $\ell^2(S, P)$. J. Theor. Probab. **29**(4), 1728–1735 (2016)
3. Bogachev, V.I.: Gaussian Measures. American Mathematical Society, Providence (1998)
4. Diestel, J., Uhl, J.J.: Vector Measures. American Mathematical Society, Providence (1977)
5. Gross, L.: Abstract Wiener spaces. In: Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability, vol. 2, pp. 31–42 (1965)
6. Kuo, H.H.: Gaussian Measures in Banach Spaces. Lecture Notes in Mathematics, vol. 463. Springer, Berlin (1975)
7. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I, II. Springer, Berlin (1977, 1979)
8. Neveu, J.: Discrete Parameter Martingales. North Holland, Amsterdam (1975)
9. Schilling, R.L.: Measures, Integrals and Martingales. Cambridge University Press, Cambridge (2005)
10. Semadeni, Z.: Schauder Bases in Banach Spaces of Continuous Functions. Lecture Notes in Mathematics, vol. 918. Springer, Berlin (1982)
11. Vakhania, N.N., Tarieladze, V.I., Chobanyan, S.A.: Probability Distributions on Banach Spaces. D. Reidel, Kluwer, Dordrecht (1987)
12. Zapała, A.M.: Jensen's inequality for conditional expectations in Banach spaces. Real Anal. Exch. **26**(2), 541–552 (2000/2001)