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Limit Distribution of the Banach Random Walk

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Abstract

We consider various probability distributions $\{G_n, n \geq 1\}$ concentrated on the interval $[-1, 1] \subset \mathbb{R}$ and investigate basic properties of the limit distribution Γ of the Banach random walk in a Banach space $\mathbb{B}$ generated by $\{G_n, n \geq 1\}$. In particular, we describe assumptions ensuring that the support of Γ is equal to the unit sphere in $\mathbb{B}$ and, on the other hand, we find conditions under which every ball of radius smaller than 1 has a positive measure Γ.

Keywords Banach random walk · Limit distribution · Support of the measure · Quasi-orthogonal Schauder basis

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1 Banach Random Walk in a Banach space

Construction of the Banach Random Walk in a Banach space was given in [3], so we present here only a brief description of this process.

Let $(\mathbb{B}, \|\cdot\|)$ be an infinite-dimensional Banach space with a Schauder basis $\{b_n, n \geq 1\}$ and let $\{\pi_n, n \geq 0\}$ be a sequence of projections $\pi_n : \mathbb{B} \to \mathbb{B}$, given by $\pi_0(x) \equiv 0 \in \mathbb{B}$ and $\pi_n(x) = \sum_{k=1}^{n} x_k b_k$ for $x = \sum_{k=1}^{\infty} x_k b_k \in \mathbb{B}$, $n \geq 1$. Denote

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$$B = \{x \in \mathbb{B} : \|x\| \le 1\}, \quad B_n(0, r) = \{\pi_n(x) \in \mathbb{B} : \|\pi_n(x)\| \le r\}, \quad n, r \ge 0,$$

and for $\pi_{n-1}(x) \in B_{n-1} = B_{n-1}(0, 1)$, where $n \ge 1$, put

$$\alpha_n = \inf\{t \in \mathbb{R} : \|\pi_{n-1}(x) + tb_n\| \le 1\} = \alpha_n(\pi_{n-1}(x)),$$
$$\beta_n = \sup\{t \in \mathbb{R} : \|\pi_{n-1}(x) + tb_n\| \le 1\} = \beta_n(\pi_{n-1}(x)).$$

Without loss of generality we assume that $\|b_1\| = 1$, but we do not require that $\|b_n\| = 1$ for all $n \ge 2$. Obviously $\beta_1 = -\alpha_1$, and in addition $\alpha_1 = -1$ and $\beta_1 = 1$ whenever $\|b_1\| = 1$, but in general $\beta_n \ne -\alpha_n$ for $n \ge 2$. Therefore we introduce the following notion: the Schauder basis $\{b_n, n \ge 1\}$ is called quasi-orthogonal, if $\alpha_{n+1} = -\beta_{n+1}$ for all $n \ge 1$. Under the above assumption $[\alpha_n, \beta_n]$, $n \ge 1$, are bounded intervals in $\mathbb{R}$ with center zero, but in some situations they are reduced to the single point $[0, 0] = \{0\}$.

Let $\{G_n, n \ge 1\}$ be arbitrary probability distributions satisfying condition $G_n([-1, 1]) = 1$ for all $n \ge 1$. Define inductively on a probability space $(\Omega, \mathcal{F}, P)$ a sequence of dependent real-valued r.v.'s $\{X_n, n \ge 1\}$ and, associated with $\{X_n, n \ge 1\}$, $\mathbb{B}$-valued random elements (r.e.'s) $\{Z_n, n \ge 1\}$ as follows: let X_1 be a r.v. with distribution G_1 and let $Z_1 = X_1 b_1$; then $X_1(\omega) \in [\alpha_1, \beta_1] = [-1, 1]$, i.e., $Z_1(\omega) \in B_1$ a.s., and thus we evaluate $\beta_2(Z_1(\omega))$, define X_2 as a r.v. distributed according to the scaled probability measure

$$G_2(\cdot/\beta_2(X_1(\omega)b_1)) = G_2(\cdot/\beta_2(Z_1(\omega))),$$

whenever $\beta_2(Z_1(\omega)) > 0$, and put $Z_2 = X_1 b_1 + X_2 b_2$. More generally, if r.v.'s $X_1, \ldots, X_{n-1}$ and $Z_1, \ldots, Z_{n-1}$ are already defined in such a manner that $Z_{n-1}(\omega) \in B_{n-1}$ a.s., then X_n is a r.v. with distribution

$$G_n(\cdot/\beta_n(X_1(\omega)b_1 + \cdots + X_{n-1}(\omega)b_{n-1})) = G_n(\cdot/\beta_n(Z_{n-1}(\omega))),$$

provided $\beta_n(Z_{n-1}(\omega)) > 0$, and $Z_n = X_1 b_1 + X_2 b_2 + \cdots + X_n b_n$. As was already mentioned, it may happen that for some $n \ge 1$ and $Z_n(\omega) \in B_n$ the interval $\left[\alpha_{n+1}, \beta_{n+1}\right] = \left[\alpha_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))\right]$ reduces to the one-point set $\{0\}$; in such a case we assume that the measure G_{n+1} is transformed so that it assigns the unit mass to the single point 0. Then $Z_{n+1}(\omega) = Z_n(\omega)$, but the next random interval $[\alpha_{n+2}, \beta_{n+2}] = [\alpha_{n+2}(Z_{n+1}(\omega)), \beta_{n+2}(Z_{n+1}(\omega))]$, defined by means of the successive basic vector b_{n+2}, need not be equal to $\{0\}$, and thus the process is still continued.

According to the definition introduced in [3] the sequence of $\mathbb{B}$-valued r.e.'s $\{Z_n, n \ge 1\}$ obtained in this way is called *Banach Random Walk* (BRW) in the Banach space $\mathbb{B}$.

Construction of the Banach Random Walk in an infinite-dimensional separable Hilbert space $\mathbb{H}$ was motivated by Banach's paper [1], where the so-called $\mathfrak{L}$-integral (i.e., integral of Lebesgue type) in abstract spaces was described. Namely, Banek [2] observed that a purely deterministic Banach's [1] construction of the $\mathfrak{L}$-integral in $\mathbb{H}$ is closely related to the asymptotic properties of the Banach Random Walk in $\mathbb{H}$, and in fact the mentioned integral is equal to the limit of expectations of certain functionals acting on the Banach Random Walk. The main idea of Banach's [1] approach which led to the definition of his $\mathfrak{L}$-integral was the symmetry of mappings as well as the symmetry of considered measures in $\mathbb{R}^n$, $n \geq 1$, and such a concept together with the Hahn–Banach theorem enabled him to prove the existence of the $\mathfrak{L}$-integral functional. Thus it is natural to demand that probability distributions G_n, $n \geq 1$, are symmetric in the sense that $G(-A) = G(A)$ for all $A \in \mathcal{B}(\mathbb{R})$.

It was shown in [3] that under this assumption concerning distributions $\{G_n, n \geq 1\}$, the Banach Random Walk in a Banach space $\mathbb{B}$ is a martingale with respect to the natural filtration $\{\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n), n \geq 1\}$ (and in fact it is also a time-inhomogeneous Markov chain). Moreover, if the Banach space $\mathbb{B}$ in question possesses the Radon–Nikodym Property (RNP), cf. [4,9], or [10] for the definition of this notion, then the process $\{Z_n, n \geq 1\}$ converges strongly a.s. in $\mathbb{B}$ and in $L^p(\mathbb{B})$ for all $1 \leq p < \infty$ to a r.e. ξ. The details of these considerations can be found in [3], thus we omit them here.

The aim of this paper is to describe the main properties of the limit distribution $\Gamma = P \circ \xi^{-1}$ of the BRW $\{Z_n, n \geq 1\}$ in a Banach space $\mathbb{B}$; in particular, we are interested in the description of the support supp Γ. It should be pointed out that for a class of bounded, Borel measurable functions Φ on the unit ball $B \subset \mathbb{B}$, the Banach–Lebesgue $\mathfrak{L}$-integral can be expressed as the expected value $E\Phi(\xi)$, see [3], thus the support of ξ is of the significant importance, for it informs what the minimal domain of the integrand Φ should be.

2 Properties of Limit Distribution of the Banach Random Walk in a Banach Space

Throughout this section we assume that $\mathbb{B}$ is a Banach space which has the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$, and $\{Z_n, n \geq 1\}$ is the BRW in $\mathbb{B}$ generated by a sequence of symmetric probability distributions $\{G_n, n \geq 1\}$ concentrated on the interval $[-1, 1] \subset \mathbb{R}$. Moreover, let ξ denote the a.s. limit of the BRW $\{Z_n, n \geq 1\}$ in $\mathbb{B}$, and let $\Gamma = P \circ \xi^{-1}$ be the measure on the ball $B = \{x \in \mathbb{B} : \|x\| \leq 1\}$ induced by ξ.

Analyzing the construction of the process $\{Z_n, n \geq 1\}$ in a Banach space one may expect that the limit distribution $\Gamma = P \circ \xi^{-1}$ of the BRW is concentrated on the surface $S(0, 1) = \{x \in \mathbb{B} : \|x\| = 1\}$ of the closed unit ball $B = \{x \in \mathbb{B} : \|x\| \leq 1\}$. Obviously such a statement is heavily dependent on distributions $\{G_n, n \geq 1\}$, which exert an influence on r.v.'s $\{X_n, n \geq 1\}$, and in general need not be true. However, in the most interesting situation when $\{X_n, n \geq 1\}$ is a sequence of r.v.'s generated by identical distributions with support equal to the interval $[-1, 1] \subset \mathbb{R}$, this indeed is the case. To examine this problem we consider the BRW in a Banach space $\mathbb{B}$ satisfying all the above requirements. First we prove an auxiliary result.

Lemma 1 *For every $x \in \mathbb{B}$ such that $\|\pi_{n-1}(x)\| \leq r_0 \leq 1$, the mapping*

$$[r_0, \infty) \ni r \mapsto \beta_n(\pi_{n-1}(x)/r), \quad r_0 > 0,$$

is a nondecreasing concave function. In consequence, it is continuous in the open interval (r_0, ∞), and a.e. right-hand side and left-hand side differentiable.

Proof Recall that $\beta_n(\pi_{n-1}(x))$ is defined for $\|\pi_{n-1}(x)\| \leq 1$ in such a way that $\|\pi_{n-1}(x) + \beta_n(\pi_{n-1}(x))\, b_n\| = 1$. Thus, if $\|\pi_{n-1}(x)\| = r_0 \leq 1$, then $\|\pi_{n-1}(x)/r + \beta_n(\pi_{n-1}(x)/r)\, b_n\| = 1$ for each $r_0 \leq r < \infty$. Since the unit ball is convex, for all $r_0 \leq r_1 \neq r_2 < \infty$ and $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$, we have

$$\left\| \lambda_1 \frac{\pi_{n-1}(x)}{r_1} + \lambda_2 \frac{\pi_{n-1}(x)}{r_2} + \left[\lambda_1 \beta_n\left(\frac{\pi_{n-1}(x)}{r_1}\right) + \lambda_2 \beta_n\left(\frac{\pi_{n-1}(x)}{r_2}\right)\right] b_n \right\| \leq 1.$$

Hence and from the definition of $\beta_n(\cdot)$ it follows that

$$\lambda_1 \beta_n\left(\frac{\pi_{n-1}(x)}{r_1}\right) + \lambda_2 \beta_n\left(\frac{\pi_{n-1}(x)}{r_2}\right) \leq \beta_n\left(\lambda_1 \frac{\pi_{n-1}(x)}{r_1} + \lambda_2 \frac{\pi_{n-1}(x)}{r_2}\right),$$

i.e., $[r_0, \infty) \ni r \mapsto \beta_n(\pi_{n-1}(x)/r)$ is a concave function. Consequently, it is continuous in the open interval (r_0, ∞), and a.e. right-hand side and left-hand side differentiable, cf. [5], Ch. V, Sect. 8, Th. 2.

Obviously, $\pi_{n-1}(x)/r \to 0$, $r \to \infty$, therefore $\beta_n(\pi_{n-1}(x)/r) \to 1/\|b_n\|$ as $r \to \infty$. Moreover, $0 \leq \beta_n(\pi_{n-1}(x)/r) \leq 1/\|b_n\|$ for all $r \in [r_0, \infty)$; otherwise, in case when $\beta_n(\pi_{n-1}(x)/r) > 1/\|b_n\|$ for some $r \geq r_0$, we would have

$$\begin{aligned} &\left\| \frac{\pi_{n-1}(x)}{r} + \beta_n\left(\frac{\pi_{n-1}(x)}{r}\right) b_n - \frac{\pi_{n-1}(x)}{r} - \alpha_n\left(\frac{\pi_{n-1}(x)}{r}\right) b_n \right\| \\ &= 2\beta_n\left(\frac{\pi_{n-1}(x)}{r}\right) \cdot \|b_n\| > 2 \cdot \frac{1}{\|b_n\|} \cdot \|b_n\| = 2, \end{aligned}$$

which leads to a contradiction with the conditions

$$\left\| \frac{\pi_{n-1}(x)}{r} + \beta_n\left(\frac{\pi_{n-1}(x)}{r}\right) b_n \right\| \leq 1, \quad \left\| \frac{\pi_{n-1}(x)}{r} + \alpha_n\left(\frac{\pi_{n-1}(x)}{r}\right) b_n \right\| \leq 1.$$

Hence it follows that $\beta_n(\pi_{n-1}(x)/r)$ is nondecreasing as $r_0 \leq r \nearrow \infty$. □

To formulate the next result, some explanations are needed. The Schauder basis $\{b_n, n \geq 1\}$ in a Banach space is called monotone, if for every choice of scalars $\{x_n, n \geq 1\}$ the sequence of real numbers $\left\{ \left\| \sum_{k=1}^n x_k b_k \right\|, n \geq 1 \right\}$ is nondecreasing. It is fairly well known that for each Banach space with a Schauder basis there exists a norm equivalent to the original one, such that a given basis $\{b_n, n \geq 1\}$ in this space equipped with the new norm is monotone, see [6], Part I, Ch. I, p. 2. Thus, to avoid additional complications with a new norm concerning notation, in what follows we assume that the basis $\{b_n, n \geq 1\}$ in $(\mathbb{B}, \|\cdot\|)$ is just monotone.

It is worth mentioning that many typical Schauder bases, such as the sequence of unit vectors in c_0 and ℓ^p for $1 \le p < \infty$, or the system of Haar functions in $L^p[0,1]$ for $1 \le p < \infty$ are monotone; furthermore, to obtain this effect the usual norms of these spaces need not be changed, see, e.g., [6], Part I, Ch. I, p. 3.

Theorem 1 *Suppose that*

$$\lim_{n\to\infty} \prod_{k=1}^{n} G_k\left([-r,r]\right) = 0 \tag{1}$$

for some $0 < r < 1$. *Then for the closed ball* $B(0,r) = \{x \in \mathbb{B} : \|x\| \le r\}$, *where* $0 < r < 1$ *is a fixed number, we have*

$$\Gamma\left(B\left(0,r\right)\right) = 0.$$

In consequence, if condition (1) is satisfied for all $0 < r < 1$, *then the whole mass of the measure* $\Gamma = P \circ \xi^{-1}$ *is concentrated on the unit sphere* $S(0,1) = \{x \in \mathbb{B} : \|x\| = 1\}$, *so that* $\operatorname{supp} \Gamma \subseteq S(0,1)$.

Proof Recall that to define the first n steps of the BRW in a Banach space $\mathbb{B}$ with a quasi-orthogonal Schauder basis $\{b_n, n \ge 1\}$ we have to use the following transformation $\Theta_n : K_n^0(0,1) \to (-1,1)^n \subset \mathbb{R}^n$,

$$\begin{aligned}
y_1 &= x_1, \\
y_2 &= \frac{x_2}{\beta_2\left(x_1 b_1\right)}, \\
y_3 &= \frac{x_3}{\beta_3\left(x_1 b_1 + x_2 b_2\right)}, \\
&\vdots \\
y_n &= \frac{x_n}{\beta_n\left(x_1 b_1 + \cdots + x_{n-1} b_{n-1}\right)},
\end{aligned} \tag{2}$$

where $K_n(0,r) = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : \|x_1 b_1 + \cdots + x_n b_n\| \le r\}$, and $K_n^0(0,r) = \operatorname{Int} K_n(0,r) = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : \|x_1 b_1 + \cdots + x_n b_n\| < r\}$, $0 < r < \infty$, $n \ge 1$. Notice that if $(x_1,\ldots,x_{k-1},0,\ldots,0) \in K_n^0(0,1)$ for some $1 < k \le n$, then there exists an open ball with center at this point contained in $K_n^0(0,1)$, thus $\beta_k\left(x_1 b_1 + \cdots + x_{k-1} b_{k-1}\right) > 0$ and so Θ_n is well defined.

To find the inverse transformation $T_n = \Theta_n^{-1}$ to (2) we introduce recursively a sequence of mappings: $A_1 \equiv 1$, $A_2(y_1) = \beta_2(y_1 A_1 b_1) = \beta_2(y_1 b_1)$, $A_3(y_1,y_2) = \beta_3(y_1 A_1 b_1 + y_2 A_2(y_1) b_2) = \beta_3(y_1 b_1 + y_2 \beta_2(y_1 b_1) b_2), \ldots$

$$\begin{aligned}
A_n\left(y_1,y_2,\ldots,y_{n-1}\right) = \beta_n\big(&y_1 A_1 b_1 + y_2 A_2\left(y_1\right) b_2 + y_3 A_3\left(y_1,y_2\right) b_3 \\
&+ \cdots + y_{n-1} A_{n-1}\left(y_1,y_2,\ldots,y_{n-2}\right) b_{n-1}\big).
\end{aligned} \tag{3}$$

Then the transformation $T_n : (-1,1)^n \to K_n^0(0,1)$ is given by

$$
\begin{aligned}
x_1 &= y_1 \cdot A_1 = y_1, \\
x_2 &= y_2 \cdot A_2(y_1) = y_2 \cdot \beta_2(y_1 b_1), \\
x_3 &= y_3 \cdot A_3(y_1, y_2) = y_3 \cdot \beta_3(y_1 b_1 + y_2 \beta_2(y_1 b_1) b_2), \\
&\vdots \\
x_n &= y_n \cdot A_n(y_1, y_2, \ldots, y_{n-1}). \qquad (4)
\end{aligned}
$$

Equations (4) can be verified by induction on the basis of (2). As can be seen, $\Theta_n\left(K_n^0(0,1)\right) = (-1,1)^n$ along with $T_n\left((-1,1)^n\right) = K_n^0(0,1)$, and both these mappings restricted to the domains considered here are one-to-one. The map T_n is also well defined in the whole closed cube $[-1,1]^n$, but then in general it is not injective, in particular—on the boundary $[-1,1]^n \setminus (-1,1)^n$. Thus, although Θ_n is in fact the inverse mapping to $T_n|_{(-1,1)^n}$, instead of the inverse transformation to T_n acting on $[-1,1]^n$ which need not exist, we must investigate inverse images $T_n^{-1}(B)$ of Borel sets $B \in \mathcal{B}(K_n(0,1))$.

Let $(Y_1, \ldots, Y_n)$ be a random vector with values in $[-1,1]^n$ and distribution $\prod_{k=1}^n G_k$. Taking into account the construction of BRW, we conclude that $(X_1, \ldots, X_n) = T_n(Y_1, \ldots, Y_n)$. Observe that each map $\beta_k(x_1 b_1 + \cdots + x_{k-1} b_{k-1})$ is a continuous function of $(x_1, \ldots, x_{k-1}) \in K_{k-1}(0,1)$; to see this, consider sets of the form $p_{k-1}\left(S_+ \cap (\mathbb{R}^{k-1} \times F)\right) = \left(\beta_k'\right)^{-1}(F)$, where S_+ is the graph of $\beta_k'(x_1, \ldots, x_{k-1}) = \beta_k(x_1 b_1 + \cdots + x_{k-1} b_{k-1})$, $p_{k-1}(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-1})$ is the usual projection of $\mathbb{R}^k$ onto $\mathbb{R}^{k-1}$, and F is a closed subset of $\mathbb{R}$. Since T_n is a composition of continuous functions with β_k, we conclude that T_n is continuous as well and in consequence $(X_1, \ldots, X_n)$ is a random vector. The distribution of $(X_1, \ldots, X_n)$ is equal

$$
P \circ (X_1, \ldots, X_n)^{-1} = P \circ (Y_1, \ldots, Y_n)^{-1} \circ T_n^{-1} = \left(\prod_{k=1}^n G_k\right) \circ T_n^{-1}.
$$

From (4) we infer that for a fixed $0 < r < 1$,

$$
\begin{aligned}
&\|x_1 b_1 + x_2 b_2 + \cdots + x_n b_n\| \le r \\
&\quad \Leftrightarrow \|y_1 b_1 + y_2 A_2(y_1) b_2 + \cdots + y_n A_n(y_1, \ldots, y_{n-1}) b_n\| \le r. \qquad (5)
\end{aligned}
$$

Define

$$
\begin{aligned}
D_n(r) = T_n^{-1}(K_n(0,r)) = \big\{&(y_1, \ldots, y_n) \in [-1,1]^n : \|y_1 b_1 + y_2 A_2(y_1) b_2 \\
&+ \cdots + y_n A_n(y_1, \ldots, y_{n-1}) b_n\| \le r\big\},
\end{aligned}
$$

cf. (5). Since $K_n(0,r)$ is a closed subset of $K_n(0,1)$, the set $D_n(r)$ is a Borel subset of $[-1,1]^n$.

Divide both sides of (5) by r and observe that if $(y_1, \ldots, y_n) \in D_n(r)$, then by definition of $\beta_n(\pi_{n-1}(x))$ we obtain

$$\left|\frac{y_n \cdot A_n(y_1, \ldots, y_{n-1})}{r}\right| \leq \beta_n\left(\frac{y_1 b_1 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}}{r}\right),$$

where $A_n(y_1, y_2, \ldots, y_{n-1})$ is given by (3), i.e.,

$$|y_n| \leq \frac{r \cdot \beta_n\left(\frac{y_1 b_1 + y_2 A_2(y_1)\, b_2 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}}{r}\right)}{\beta_n\left(y_1 A_1 b_1 + y_2 A_2(y_1)\, b_2 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}\right)}. \tag{6}$$

Applying Lemma 1 we have

$$\begin{aligned}&\beta_n\left(\frac{y_1 b_1 + y_2 A_2(y_1)\, b_2 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}}{r}\right)\\ &\quad\leq \beta_n\left(y_1 A_1 b_1 + y_2 A_2(y_1)\, b_2 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}\right),\end{aligned}$$

for $r \leq 1$. Taking into account the above estimate and (6) we conclude that $|y_n| \leq r$. In consequence,

$$D_n(r) \subseteq \left\{(y_1, \ldots, y_n) \in [-1, 1]^n : |y_n| \leq r\right\}.$$

Moreover, since the basis $\{b_n, n \geq 1\}$ is monotone, condition (5) implies that

$$\|y_1 b_1 + y_2 A_2(y_1)\, b_2 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}\| \leq r.$$

In other words,

$$\begin{aligned}D_n(r) \subseteq \Big\{&(y_1, \ldots, y_n) \in [-1, 1]^n : \|y_1 b_1 + \cdots + y_{n-1} A_{n-1}(y_1, \ldots, y_{n-2})\, b_{n-1}\|\\ &\leq r, |y_n| \leq r\Big\}.\end{aligned}$$

Arguing in a similar way as above we infer that $|y_{n-1}| \leq r$, next $|y_{n-2}| \leq r$, etc., and finally, from $\|y_1 b_1\| \leq r$ and $\|b_1\| = 1$, it follows that $|y_1| \leq r$. Thus we conclude that

$$D_n(r) \subseteq \left\{(y_1, \ldots, y_n) \in [-1, 1]^n : |y_1| \leq r, \ldots, |y_n| \leq r\right\} = [-r, r]^n,$$

i.e., $T_n^{-1}(K_n(0,r)) = D_n(r) \subseteq [-r,r]^n$. Hence it follows that

$$\begin{aligned}
\Gamma\left(\pi_n^{-1}(B_n(0,r))\right) &= P \circ \xi^{-1}\left(\pi_n^{-1}(B_n(0,r))\right) \\
&= P\left[\pi_n(\xi) \in B_n(0,r)\right] \\
&= P\left[Z_n \in B_n(0,r)\right] = P\left[(X_1,\ldots,X_n) \in K_n(0,r)\right] \\
&= P\left[T_n(Y_1,\ldots,Y_n) \in K_n(0,r)\right] \\
&= P\left[(Y_1,\ldots,Y_n) \in T_n^{-1}(K_n(0,r))\right] \\
&= \left(\prod_{k=1}^n G_k\right)\left(T_n^{-1}(K_n(0,r))\right) = \left(\prod_{k=1}^n G_k\right)(D_n(r)) \\
&\le \prod_{k=1}^n G_k([-r,r]) . \qquad (7)
\end{aligned}$$

In fact we have

$$\Gamma\left(\pi_n^{-1}(B_n(0,r))\right) = \Gamma\left(\pi_n^{-1}(B_n(0,r)) \cap B\right), \qquad (8)$$

as we already know that supp $\Gamma \subseteq B$. Furthermore,

$$\pi_1^{-1}(B_1(0,r)) \cap B \supseteq \pi_2^{-1}(B_2(0,r)) \cap B \supseteq \cdots \supseteq \pi_n^{-1}(B_n(0,r)) \cap B \supseteq \cdots \qquad (9)$$

and

$$\bigcap_{n=1}^{\infty} \pi_n^{-1}(B_n(0,r)) \cap B = B(0,r) . \qquad (10)$$

Consequently,

$$\begin{aligned}
\Gamma(B(0,r)) &= \Gamma\left(\bigcap_{n=1}^{\infty} \pi_n^{-1}(B_n(0,r)) \cap B\right) \\
&= \lim_{n\to\infty} \Gamma\left(\pi_n^{-1}(B_n(0,r)) \cap B\right) \le \lim_{n\to\infty} \prod_{k=1}^n G_k([-r,r]) = 0.
\end{aligned}$$

If $0 < r < 1$ in (1) can be arbitrary, the final conclusion $\Gamma(S(0,1)) = 1$ of the theorem, which can be rewritten also in the form supp $\Gamma \subseteq S(0,1)$, is evident. □

Corollary 1 *If* $\{G_n, n \ge 1\}$ *is a sequence of identical distributions* $G_n = G$, $n \ge 1$, *such that* $G([-r,r]) < 1$ *for each* $0 < r < 1$, *then the assertion of Theorem 1 remains valid. In particular, if* $G_n = U$, $n \ge 1$, *are identical uniform distributions on* $[-1,1]$, *then Theorem* 1 *holds true.*

We are able to prove as well a result going in the opposite direction. To formulate the next theorem, given any $0 < r \le 1$, we choose a sequence of positive real numbers

$\{q_n, n \geq 1\}$ satisfying condition

$$0 < q_n < \left(\sqrt{r^2 + 4r} - r\right)/2 \leq \left(\sqrt{5} - 1\right)/2, \quad n \geq 1,$$

(i.e., $q_n^2 + q_n^3 + q_n^4 + \cdots = q_n^2/\left(1 - q_n\right) < r$) and put

$$s_{k,n} = q_n^k + q_n^{k+1} + \cdots + q_n^n \quad \text{for} \quad 2 \leq k \leq n, \quad s_{n+1,n} = 0, \quad n \geq 1.$$

Theorem 2 *Assume that for a given* $0 < r \leq 1$, *there exists a sequence of numbers* $\{q_n, n \geq 1\} \subset \mathbb{R}$ *satisfying the above requirements, such that*

$$\begin{aligned}&\limsup_{n\to\infty} G_1\left(\left[-\frac{\left(r - s_{2,n}\right)}{\left(1 - s_{2,n}\right)}, \frac{\left(r - s_{2,n}\right)}{\left(1 - s_{2,n}\right)}\right]\right)\\ &\quad\cdot \prod_{k=2}^{n} G_k\left(\left[-\frac{q_n{}^k}{\left(1 - s_{k+1,n}\right)}, \frac{q_n{}^k}{\left(1 - s_{k+1,n}\right)}\right]\right) = c_r > 0.\end{aligned} \tag{11}$$

Then we have

$$\Gamma\left(B\left(0, r\right)\right) \geq c_r > 0,$$

therefore if $0 < r < 1$, *then the whole mass of the measure* Γ *cannot be concentrated on the unit sphere* $S\left(0, 1\right) = \{x \in \mathbb{B} : \|x\| = 1\}$.

Proof Let Θ_n and T_n be the transformations given by (2) and (4) resp. Notice that then

$$\begin{aligned}&\beta_k\left(y_1 A_1 b_1 + y_2 A_2\left(y_1\right) b_2 + \cdots + y_{k-1} A_{k-1}\left(y_1, y_2, \ldots, y_{k-2}\right) b_{k-1}\right)\\ &\quad= A_k\left(y_1, y_2, \ldots, y_{k-1}\right), \quad 2 \leq k \leq n,\end{aligned}$$

cf. (3). Since for every fixed $x, y \in \mathbb{B}$ the mapping $t \mapsto \|x + ty\|$ is a continuous function of the parameter $t \in \mathbb{R}$, we have the following system of equivalent conditions:

$$\begin{aligned}&\left\|\left(1 - s_{2,n}\right) y_1 b_1\right\| \leq r - s_{2,n} \Leftrightarrow |y_1| \leq \frac{r - s_{2,n}}{1 - s_{2,n}},\\ &\left\|q_n^2 y_1 b_1 + \left(1 - s_{3,n}\right) y_2 A_2 b_2\right\| \leq q_n^2 \Leftrightarrow |y_2| \leq \frac{q_n^2}{1 - s_{3,n}},\\ &\left\|q_n^3\left(y_1 b_1 + y_2 A_2 b_2\right) + \left(1 - s_{4,n}\right) y_3 A_3 b_3\right\| \leq q_n^3 \Leftrightarrow |y_3| \leq \frac{q_n^3}{1 - s_{4,n}},\\ &\qquad\vdots\\ &\left\|q_n^n\left(y_1 b_1 + y_2 A_2 b_2 + \cdots + y_{n-1} A_{n-1} b_{n-1}\right) + y_n A_n b_n\right\| \leq q_n^n \Leftrightarrow |y_n| \leq q_n^n\end{aligned} \tag{12}$$

(to simplify the writing, we put here $A_k = A_k(y_1, y_2, \ldots, y_{k-1})$, $2 \le k \le n$). Summing all the inequalities on the left-hand side of (12) we conclude that

$$\begin{aligned}
&\|y_1 b_1 + y_2 A_2(y_1) b_2 + \cdots + y_n A_n(y_1, \ldots, y_{n-1}) b_n\| \le \left\|\left(1 - s_{2,n}\right) y_1 b_1\right\| \\
&\quad + \left\|q_n^2 y_1 b_1 + \left(1 - s_{3,n}\right) y_2 A_2 b_2\right\| + \left\|q_n^3 (y_1 b_1 + y_2 A_2 b_2) + \left(1 - s_{4,n}\right) y_3 A_3 b_3\right\| \\
&\quad + \cdots + \left\|q_n^n (y_1 b_1 + y_2 A_2 b_2 + \cdots + y_{n-1} A_{n-1} b_{n-1}) + y_n A_n b_n\right\| \\
&\le r - s_{2,n} + q_n^2 + q_n^3 + \cdots + q_n^n = r,
\end{aligned}$$

thus

$$\begin{aligned}
&\Delta_n(r, q_n) \\
&:= \left\{(y_1, \ldots, y_n) \in [-1,1]^n : |y_1| \le \frac{r - s_{2,n}}{1 - s_{2,n}}, |y_2| \le \frac{q_n^2}{1 - s_{3,n}}, \ldots, |y_n| \le q_n^n\right\} \\
&\subseteq \left\{(y_1, \ldots, y_n) \in [-1,1]^n : \|y_1 b_1 + \cdots + y_n A_n(y_1, \ldots, y_{n-1}) b_n\| \le r\right\} = D_n(r).
\end{aligned}$$

Hence, by analogy to (7)–(8), it follows that

$$\begin{aligned}
\Gamma\left(\pi_n^{-1}(B_n(0,r)) \cap B\right) &= \left(\prod_{k=1}^n G_k\right)(D_n(r)) \ge \left(\prod_{k=1}^n G_k\right)(\Delta_n(r, q_n)) \\
&= G_1\left(\left[-\frac{r - s_{2,n}}{1 - s_{2,n}}, \frac{r - s_{2,n}}{1 - s_{2,n}}\right]\right) \prod_{k=2}^n G_k\left(\left[-\frac{q_n^k}{1 - s_{k+1,n}}, \frac{q_n^k}{1 - s_{k+1,n}}\right]\right).
\end{aligned}$$

Passing to the limit as $n \to \infty$, on account of (9)–(10) and the assumption (11) we finally conclude that $\Gamma(B(0,r)) \ge c_r > 0$. □

Combining Theorems 1 and 2 we obtain the following result.

Corollary 2 *Let* $\{G_n, n \ge 1\}$ *be a sequence of probability distributions concentrated on the interval* $[-1,1] \subset \mathbb{R}$ *such that condition* (1) *is satisfied for all* r, $0 < r < r_1 < 1$, *and there exists a sequence of positive numbers* $\{q_n, n \ge 1\} \subset \mathbb{R}$ *such that* $q_n^2 + q_n^3 + q_n^4 + \cdots = q_n^2/(1 - q_n) < r_1$, $n \ge 1$, *along with condition* (11) *satisfied for* $r = r_1$. *Then*

$$\Gamma(B(0,r)) = 0, \quad 0 < r < r_1, \quad \textit{and} \quad \Gamma(B(0,r_1)) \ge c_{r_1} > 0.$$

Thus $\operatorname{supp} \Gamma \subseteq B \setminus B^0(0, r_1)$, *where* $B^0(0, r_1) = \{x \in \mathbb{B} : \|x\| < r_1\}$.

Remark 1 It is obvious that if $c_r = 1$ for some $0 < r < 1$ in condition (11), then $\Gamma(B(0,r)) = 1$, thus in such a case $\operatorname{supp} \Gamma \subseteq B(0,r)$.

3 Limit Distribution of the Banach Random Walk in ℓ^p

The assertion of Theorem 1 is quite clear and undoubtedly the assumptions of this result can be satisfied, but it is not so evident that there can be found a sequence of numbers $\{q_n, n \geq 1\}$ satisfying conditions specified in Theorem 2 or Corollary 2. Therefore to solve the problem, we consider in more detail the space $\mathbb{B} = \ell^p$, i.e., the separable Banach space of all infinite sequences $x = (x_1, x_2, \ldots) \subset \mathbb{R}$ with norm $|x|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty$, $1 \leq p < \infty$. As will be seen later, in such a case not merely a fixed ball $B(0, r) \subset \ell^p$ has a positive measure Γ for suitably chosen distributions $\{G_n, n \geq 1\}$, but even for all $0 < r < 1$ we may have $\Gamma(B(0, r)) > 0$.

Proposition 1 *Let $\{Z_n, n \geq 1\}$ be the BRW in ℓ^p, $1 \leq p < \infty$, generated by a sequence $\{G_n, n \geq 1\}$ of symmetric probability distributions on the interval $[-1, 1]$, let ξ be the a.s. limit of the BRW $\{Z_n, n \geq 1\}$ in ℓ^p, and let $\Gamma = P \circ \xi^{-1}$ denote the measure on $B = \left\{x \in \ell^p : |x|_p \leq 1\right\}$ induced by ξ. Consider a triangular array $\left\{c_{k,n}, 1 \leq k \leq n, n \geq 1\right\}$ of real numbers satisfying the following conditions:*

$$0 < c_{k,n} < 1 \quad \textit{for all} \quad k, n, \quad \textit{and} \quad \sum_{k=1}^{n} c_{k,n} = 1, \quad n = 1, 2, \ldots$$

Assume that the distributions G_n, $n \geq 1$, are chosen in such a way that

$$\limsup_{n \to \infty} \prod_{k=1}^{n} G_k \left(\left[-\left[1 - \left(1 - r^p\right)^{c_{k,n}}\right]^{1/p}, \left[1 - \left(1 - r^p\right)^{c_{k,n}}\right]^{1/p} \right] \right) = c_r > 0$$

for a fixed $0 < r < 1$. Then for the closed ball $B(0, r) = \left\{x \in \ell^p : |x|_p \leq r\right\}$, where $0 < r < 1$, we have

$$\Gamma(B(0, r)) \geq c_r > 0.$$

Consequently, in such a case the whole mass of measure Γ is not concentrated on the unit sphere $S(0, 1) = \left\{x \in \ell^p : |x|_p = 1\right\}$.

Proof As in the proof of Theorem 1, we now consider two transformations: $\Theta_n : K_n^0(0, 1) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p < 1\right\} \to (-1, 1)^n$ and $T_n : (-1, 1)^n \to K_n^0(0, 1)$, given by

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= \frac{x_2}{\left(1 - |x_1|^p\right)^{1/p}}, \\ y_3 &= \frac{x_3}{\left[1 - \left(|x_1|^p + |x_2|^p\right)\right]^{1/p}}, \\ &\vdots \\ y_n &= \frac{x_n}{\left[1 - \left(|x_1|^p + \cdots + |x_{n-1}|^p\right)\right]^{1/p}}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} x_1 &= y_1, \\ x_2 &= y_2 \cdot \left(1 - |y_1|^p\right)^{1/p}, \\ x_3 &= y_3 \cdot \left[\left(1 - |y_1|^p\right) \cdot \left(1 - |y_2|^p\right)\right]^{1/p}, \\ &\vdots \\ x_n &= y_n \cdot \left[\left(1 - |y_1|^p\right) \cdot \ldots \cdot \left(1 - |y_{n-1}|^p\right)\right]^{1/p}, \end{aligned} \tag{14}$$

resp. To derive (14), proceed by induction. We may also extend T_n to the whole closed cube $[-1, 1]^n$ by (14). Then $P \circ (X_1, \ldots, X_n)^{-1} = \left(\prod_{k=1}^n G_k\right) \circ T_n^{-1}$, as well as $\Theta_n^{-1} = T_n|_{(-1,1)^n}$ is the inverse map to Θ_n. Notice next that

$$|x_1|^p + |x_2|^p + \cdots + |x_n|^p = 1 - \left(1 - |y_1|^p\right) \cdot \left(1 - |y_2|^p\right) \cdot \ldots \cdot \left(1 - |y_n|^p\right),$$

thus for a fixed $0 < r < 1$ we have

$$\begin{aligned} &|x_1|^p + |x_2|^p + \cdots + |x_n|^p \leq r^p \\ &\quad \Leftrightarrow \left(1 - |y_1|^p\right) \cdot \left(1 - |y_2|^p\right) \cdot \ldots \cdot \left(1 - |y_n|^p\right) \geq 1 - r^p. \end{aligned} \tag{15}$$

Arguing similarly as above we observe that

$$\begin{aligned} &\left(\bigwedge_{1 \leq k \leq n} \left(1 - |y_k|^p\right) \geq \left(1 - r^p\right)^{c_{k,n}}\right) \\ &\Rightarrow \left(1 - |y_n|^p\right)\left(1 - |y_n|^p\right) \cdot \ldots \cdot \left(1 - |y_n|^p\right) \geq \left(1 - r^p\right)^{\sum_{k=1}^n c_{k,n}} = \left(1 - r^p\right). \end{aligned}$$

Moreover, for each fixed k,

$$\left(1 - |y_k|^p\right) \geq \left(1 - r^p\right)^{c_{k,n}} \quad \Leftrightarrow \quad |y_k| \leq \left[1 - \left(1 - r^p\right)^{c_{k,n}}\right]^{1/p}.$$

Hence

$$\begin{aligned} &\left[-\left[1 - \left(1 - r^p\right)^{c_{1,n}}\right]^{1/p}, \left[1 - \left(1 - r^p\right)^{c_{1,n}}\right]^{1/p}\right] \\ &\quad \times \cdots \times \left[-\left[1 - \left(1 - r^p\right)^{c_{n,n}}\right]^{1/p}, \left[1 - \left(1 - r^p\right)^{c_{n,n}}\right]^{1/p}\right] \subset D_n(r), \end{aligned}$$

where

$$\begin{aligned} D_n(r) = T_n^{-1}\left(K_n(0, r)\right) = \Big\{(y_1, \ldots, y_n) \in [-1, 1]^n : &\left(1 - |y_1|^p\right)\left(1 - |y_2|^p\right) \\ &\cdot \ldots \cdot \left(1 - |y_n|^p\right) \geq 1 - r^p\Big\}. \end{aligned}$$

Therefore, for each $n \geq 1$ we have

$$\prod_{k=1}^{n} G_k \left(\left[-\left[1 - \left(1 - r^p \right)^{c_{k,n}} \right]^{1/p}, \left[1 - \left(1 - r^p \right)^{c_{k,n}} \right]^{1/p} \right] \right)$$
$$\leq \left(G_1 \times G_2 \times \cdots \times G_n \right) \left(D_n \left(r \right) \right) = \Gamma \left(\pi_n^{-1} \left(B_n \left(0, r \right) \right) \cap B \right),$$

cf. (7)–(8). Referring to (9)–(10) we obtain

$$\Gamma \left(B \left(0, r \right) \right) = \Gamma \left(\bigcap_{n=1}^{\infty} \pi_n^{-1} \left(B_n \left(0, r \right) \right) \cap B \right) = \lim_{n \to \infty} \Gamma \left(\pi_n^{-1} \left(B_n \left(0, r \right) \right) \cap B \right)$$
$$\geq \limsup_{n \to \infty} \prod_{k=1}^{n} G_k \left(\left[-\left[1 - \left(1 - r^p \right)^{c_{k,n}} \right]^{1/p}, \left[1 - \left(1 - r^p \right)^{c_{k,n}} \right]^{1/p} \right] \right)$$
$$= c_r > 0,$$

which concludes the proof. □

The example presented below shows that the distribution of the limit random element ξ of the BRW in the Banach space $\mathbb{B} = \ell^p$ may in some sense be split uniformly on balls centered at 0.

Example 1 Let $G_k,\ k \geq 1$, be symmetric probability distributions on $[-1, 1]$ such that

$$G_k \left([-z, z] \right) = \left\{ 1 - \left(1 - z^p \right)^{2^k} \right\}^{1/p2^k} \quad \text{for} \quad 0 \leq z \leq 1,\ k \geq 1. \tag{16}$$

Notice that

$$G_k \left([-z, z] \right) \to 0 \text{ as } z \to 0, \quad G_k \left([-z, z] \right) \to 1 \text{ as } z \to 1,$$

and since

$$\left\{ G_k \left([-z, z] \right)^{p2^k} \right\}' = -2^k \left(1 - z^p \right)^{2^k - 1} \left(-p z^{p-1} \right) = 2^k p z^{p-1} \left(1 - z^p \right)^{2^k - 1} > 0$$

for $0 < z < 1$, it follows that the maps $G_k \left([-z, z] \right)$ are increasing in the interval $0 < z < 1$. Therefore $G_k,\ k \geq 1$, are well defined. Consider the triangular array $\left\{ c_{k,n},\ 1 \leq k \leq n, n \geq 1 \right\}$ of real numbers given by

$$c_{k,n} = 1/2^k \quad \text{for} \quad 1 \leq k \leq n-1, \quad \text{and} \quad c_{n,n} = 1/2^{n-1}.$$

Clearly, we have

$$\sum_{k=1}^{n} c_{k,n} = \sum_{k=1}^{n-1} \frac{1}{2^k} + \frac{1}{2^{n-1}} = \frac{1}{2} \cdot \frac{1 - 1/2^{n-1}}{1 - 1/2} + \frac{1}{2^{n-1}} = 1.$$

Substituting $z = \left[1-(1-r^p)^{c_{k,n}}\right]^{1/p}$ in the definition of $G_k([-z,z])$ we obtain

$$\left\{1-\left(1-z^p\right)^{2^k}\right\}^{1/p2^k} = \left\{1-\left(1-\left[1-\left(1-r^p\right)^{c_{k,n}}\right]^{p\cdot 1/p}\right)^{2^k}\right\}^{1/p2^k}$$
$$= \left\{1-\left(1-r^p\right)^{c_{k,n}\cdot 2^k}\right\}^{1/p2^k} = \left\{r^p\right\}^{1/p2^k} = r^{1/2^k}$$

for $1 \le k \le n-1$, and

$$\left\{1-\left(1-z^p\right)^{2^n}\right\}^{1/p2^n} = \left\{1-\left(1-\left[1-\left(1-r^p\right)^{c_{n,n}}\right]^{p\cdot 1/p}\right)^{2^n}\right\}^{1/p2^n}$$
$$= \left\{1-\left(1-r^p\right)^{c_{n,n}\cdot 2^n}\right\}^{1/p2^n} = \left\{1-\left(1-r^p\right)^2\right\}^{1/p2^n}$$
$$= r^{1/2^n}\cdot\left(2-r^p\right)^{1/p2^n}$$

for $k = n$. Hence

$$\prod_{k=1}^{n} G_k\left(\left[-\left[1-\left(1-r^p\right)^{c_{k,n}}\right]^{1/p}, \left[1-\left(1-r^p\right)^{c_{k,n}}\right]^{1/p}\right]\right)$$
$$= \left(\prod_{k=1}^{n-1} r^{1/2^k}\right)\cdot r^{1/2^n}\cdot\left(2-r^p\right)^{1/p2^n} = r^{\sum_{k=1}^{n-1}(1/2^k)}\cdot r^{1/2^n}\cdot\left(2-r^p\right)^{1/p2^n}$$
$$= r^{1-1/2^{n-1}+1/2^n}\cdot\left(2-r^p\right)^{1/p2^n} = r^{1-1/2^n}\cdot\left(2-r^p\right)^{1/p2^n} \to r,$$

so that

$$\limsup_{n\to\infty}\prod_{k=1}^{n} G_k\left(\left[-\left[1-\left(1-r^p\right)^{c_{k,n}}\right]^{1/p}, \left[1-\left(1-r^p\right)^{c_{k,n}}\right]^{1/p}\right]\right) = r > 0.$$

Applying Proposition 1 we conclude that $\Gamma(B(0,r)) \ge r$ for all $0 < r < 1$. From the last estimate it follows in addition that $\Gamma(S(0,1)) = 0$.

Corollary 3 *For every $1 \le p < \infty$, in the Banach space $\mathbb{B} = \ell^p$ there exists a Borel probability measure Γ with $\operatorname{supp}\Gamma = B(0,1)$, such that $\Gamma(S(0,1)) = 0$ and $\Gamma(B(0,r)) \ge r$ for all $0 < r < 1$.*

A small modification of distributions considered above leads to another interesting situation.

Example 2 Let G_k, $k \ge 1$, be symmetric probability distributions on $[-1,1]$ satisfying condition (16) for all $z \in [r_1, 1]$, and condition (1) for all $r \in (0, r_1)$, where $0 < r_1 < 1$ is a fixed number. In other words, we may assume that apart from (16) valid for $r_1 \le z \le 1$, two equal masses

$$G_k(\{-r_1\}) = \frac{1}{2} \cdot \left\{1 - \left(1 - r_1^p\right)^{2^k}\right\}^{1/p2^k} = G_k(\{r_1\})$$

are assigned to points $\{-r_1\}$, $\{r_1\}$ by distributions G_k, while $G_k([-z, z]) = 0$ whenever $0 < z < r_1$. Then

$$\Gamma(B(0, r)) = 0, \quad 0 < r < r_1, \quad \text{and} \quad \Gamma(B(0, r)) \geq r > 0, \quad r_1 \leq r < 1.$$

In consequence, supp $\Gamma \subseteq B(0, 1) \setminus B^0(0, r_1)$, where $B^0(0, r)$ denotes the open ball $\{x \in \ell^p : |x|_p < r\}$.

We leave to the reader further modifications of distributions G_k, $k \geq 1$, leading to a measure $\Gamma = P \circ \xi^{-1}$ such that supp $\Gamma \subseteq B(0, r_2) \setminus B^0(0, r_1)$, where $0 < r_1 < r_2 < 1$ (cf. remark preceding Sect. 3).

4 BRW in Banach Spaces of Martingale Cotype *q*

The main results given in Sect. 3 for spaces ℓ^p can be extended to Banach spaces of the same martingale cotype as ℓ^p. To this end, the first doubt that arises is the question of convergence of the Banach Random Walk (BRW) $\{Z_n, n \geq 1\}$ in such Banach spaces. We discuss briefly this problem.

Let $\mathbb{B}$ be a Banach space of martingale cotype q for some $2 \leq q < \infty$, i.e., there exists a constant C such that for all $\mathbb{B}$ -valued martingales $\{M_n, n \geq 1\}$ in $L^q(\mathbb{B})$,

$$\sum\nolimits_{n\geq 1} E\,\|dM_n\|^q \leq C \sup\nolimits_{n\geq 1} E\,\|M_n\|^q ,$$

where $dM_n = M_n - M_{n-1}$ for $n > 1$ and $dM_1 = M_1$, see, e.g., [7], Ch. 6, p. 221, and [9], Def. 10.41. By Corollary 4.7, [8], or Corollary 10.7 of [9], there exists a norm $|\cdot|$ equivalent to $\|\cdot\|$ in $\mathbb{B}$ such that for a fixed number $\Delta > 0$,

$$\bigwedge_{x,y\in\mathbb{B}} \quad \left|\frac{x+y}{2}\right|^q + \Delta\left|\frac{x-y}{2}\right|^q \leq \frac{|x|^q}{2} + \frac{|y|^q}{2},$$

which can be rewritten in the form

$$\bigwedge_{x,y\in\mathbb{B}} \quad 1 - \left|\frac{x+y}{2}\right| \geq 1 - \left(\frac{|x|^q}{2} + \frac{|y|^q}{2} - \Delta\left|\frac{x-y}{2}\right|^q\right)^{1/q}.$$

Therefore

$$\delta(\varepsilon) = \inf\left\{1 - \left|\frac{x+y}{2}\right| : |x| \leq 1, |y| \leq 1, |x-y| \geq \varepsilon\right\} \geq 1 - \left(1 - \Delta\left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}.$$

Hence it follows that the space $(\mathbb{B}, |\cdot|)$ is uniformly convex, cf. [7], Th. 6.2, or [9], Th. 10.1 and Prop. 10.31. Since each uniformly convex Banach space is reflexive, cf. Theorem 4.3 of [8], and Theorem 10.3 of [9], taking into account a result of Phillips we conclude that the space $(\mathbb{B}, |\cdot|)$ possesses the RNP, see [4], Ch. III, Sect. 2, Corollary 13, p. 76. Consequently, $(\mathbb{B}, \|\cdot\|)$ also enjoys the RNP.

Assume that $\{Z_n, n \geq 1\}$ is a $\mathbb{B}$-valued BRW constructed by means of a quasi-orthogonal basis $\{b_n, n \geq 1\}$ with respect to $\|\cdot\|$. Applying Lemma 5 of [3] we infer that the BRW $\{Z_n, n \geq 1\}$ converges strongly a.s. in $(\mathbb{B}, \|\cdot\|)$ and in $L^p(\mathbb{B}, \|\cdot\|)$ for each fixed $1 \leq p < \infty$. Now it is evident that all the results given in Sect. 2 are still valid for the Banach space $(\mathbb{B}, \|\cdot\|)$, and to generalize the results of Sect. 3 only a small effort is needed.

Having in mind the additional assumption: $(\mathbb{B}, \|\cdot\|)$ is of martingale cotype q, $2 \leq q < \infty$, we are able to describe convergence of the BRW $\{Z_n, n \geq 1\}$ more precisely. Introduce a function $\|\cdot\|_{(q)} : \mathbb{B} \to [0, \infty]$ given by the formula

$$\|x\|_{(q)} = \left(\sum\nolimits_{k\geq 1} \|x_k b_k\|^q\right)^{1/q} \quad \text{for} \quad x = \sum\nolimits_{k\geq 1} x_k b_k \in \mathbb{B},$$

and define $\mathbb{B}_q = \left\{x \in \mathbb{B} : \|x\|_{(q)} < \infty\right\}$. It can be easily verified that $\mathbb{B}_q$ is a linear space and $\|\cdot\|_{(q)}$ is a norm in $\mathbb{B}_q$. (The triangle condition follows from Minkowski's inequality.) Obviously, $\{b_n, n \geq 1\}$ is a quasi-orthogonal, monotone basis in $(\mathbb{B}_q, \|\cdot\|_{(q)})$.

Let $\widetilde{\mathbb{B}}_q$ denote the completion of $\mathbb{B}_q$ under $\|\cdot\|_{(q)}$. As was already noted, the assumptions imposed in [3] ensure that the BRW $\{Z_n, n \geq 1\}$ converges a.s. in $(\mathbb{B}, \|\cdot\|)$ and in $L^p(\mathbb{B}, \|\cdot\|)$, $1 \leq p < \infty$. Hence it follows that for each $\varepsilon > 0$ (and every fixed $1 \leq p < \infty$) there can be found n_ε such that for all $m > n \geq n_\varepsilon$, we have $\|\, \|Z_m - Z_n\|\, \|_p < \varepsilon$, where $\|\cdot\|_p$ denotes the usual L^p norm. But for a fixed $n \geq n_\varepsilon$, $\{Z_m - Z_n, m \geq n\}$ is a martingale, thus in view of Theorem 4.51 [8], or Theorem 10.59 of [9], and the generalized Doob's inequality, see Corollary 1.13 [8], or Corollary 1.29 [9], we obtain

$$\begin{aligned} \left\| \left(\sum_{n<k\leq m} \|dZ_k\|^q \right)^{1/q} \right\|_p &\leq C \left\| \sup_{n<k\leq m} \|Z_k - Z_n\| \right\|_p \\ &\leq C(p) \sup_{m>n} \|\|Z_m - Z_n\|\|_p \leq C(p)\,\varepsilon \end{aligned}$$

whenever $1 < p < \infty$. Consequently, the BRW $\{Z_n, n \geq 1\}$ converges also in $L^p\left(\widetilde{\mathbb{B}}_q, \|\cdot\|_{(q)}\right)$ for all $1 < p < \infty$. By Theorem 1.14 [8], see also Theorem 2.9 of [9], we conclude in addition that the process $\{Z_n, n \geq 1\}$ converges a.s. in $\left(\widetilde{\mathbb{B}}_q, \|\cdot\|_{(q)}\right)$. Therefore the BRW $\{Z_n, n \geq 1\}$ converges a.s. in the space $\mathbb{B} \cap \widetilde{\mathbb{B}}_q$ equipped with norm $\|\cdot\|_{\max} = \max\{\|\cdot\|, \|\cdot\|_{(q)}\}$.

Suppose next that a quasi-orthogonal basis $\{b_n, n \geq 1\}$ in a Banach space $(\mathbb{B}, \|\cdot\|)$ is normalized so that $\|b_n\| = 1$ for all $n \geq 1$. Notice that then

$$\|x\|_{(q)} = \left(\sum\nolimits_{k\geq 1} |x_k|^q\right)^{1/q}, \quad x = \sum\nolimits_{k\geq 1} x_k b_k \in \mathbb{B}.$$

In such a case the spaces $\left(\widetilde{\mathbb{B}}_q, \|\cdot\|_{(q)}\right)$ and ℓ^q are isometrically isomorphic, and thus we may identify $\widetilde{\mathbb{B}}_q$ with ℓ^q. Therefore the main results of Sect. 3, in particular Proposition 1 and Corollary 3, remain valid provided the space ℓ^q is replaced by $\left(\widetilde{\mathbb{B}}_q, \|\cdot\|_{(q)}\right)$. In this way we obtain the following result.

Theorem 3 *Let* $(\mathbb{B}, \|\cdot\|)$ *be a Banach space of martingale cotype* q *for some* $2 \leq q < \infty$, *with a quasi-orthogonal Schauder basis* $\{b_n, n \geq 1\}$ *normalized so that* $\|b_n\| = 1$, $n \geq 1$. *Moreover, let* $\{G_n, n \geq 1\}$ *be a sequence of symmetric probability distributions on the interval* $[-1, 1]$ *satisfying conditions of Proposition 1 with* p *replaced by* q. *Then for a fixed* $0 < r < 1$, *we have*

$$\Gamma\left(B_q(0, r)\right) \geq c_r > 0,$$

where $B_q(0, r) = \left\{x \in \mathbb{B} : \|x\|_{(q)} \leq r\right\}$, $0 < r < \infty$, $\Gamma = P \circ \xi^{-1}$, *and* ξ *is the a.s. limit of the BRW* $\{Z_n, n \geq 1\}$ *in* $\mathbb{B} \cap \widetilde{\mathbb{B}}_q$ *generated by* $\{G_n, n \geq 1\}$. *Hence it follows that the whole mass of the measure* Γ *is not concentrated on the set* $S_q(0, 1) = \left\{x \in \mathbb{B} : \|x\|_{(q)} = 1\right\}$.

As a consequence of this approach and Corollary 3 we get

Corollary 4 *For every Banach space* $(\mathbb{B}, \|\cdot\|)$ *of martingale cotype* $2 \leq q < \infty$, *with a quasi-orthogonal normalized Schauder basis* $\{b_n, n \geq 1\}$, *there exists a Borel probability measure* Γ *with* $\operatorname{supp}\Gamma = B_q(0, 1)$, *such that* $\Gamma\left(S_q(0, 1)\right) = 0$ *and* $\Gamma\left(B_q(0, r)\right) \geq r$ *for all* $0 < r < 1$.

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