

An Erdős–Révész Type Law of the Iterated Logarithm for Order Statistics of a Stationary Gaussian Process

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Abstract Let $\{X(t) : t \in \mathbb{R}_+\}$ be a stationary Gaussian process with almost surely (a.s.) continuous sample paths, $\mathbb{E}X(t) = 0$, $\mathbb{E}X^2(t) = 1$ and correlation function satisfying (i) $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ as $t \rightarrow 0$ for some $0 \leq \alpha \leq 2$ and $C > 0$; (ii) $\sup_{t \geq s} |r(t)| < 1$ for each $s > 0$ and (iii) $r(t) = O(t^{-\lambda})$ as $t \rightarrow \infty$ for some $\lambda > 0$. For any $n \geq 1$, consider n mutually independent copies of X and denote by $\{X_{r:n}(t) : t \geq 0\}$ the r th smallest order statistics process, $1 \leq r \leq n$. We provide a tractable criterion for assessing whether, for any positive, non-decreasing function f , $\mathbb{P}(\mathcal{E}_f) = \mathbb{P}(X_{r:n}(t) > f(t) \text{ i.o.})$ equals 0 or 1. Using this criterion we find, for a family of functions $f_p(t)$ such that $z_p(t) = \mathbb{P}(\sup_{s \in [0,1]} X_{r:n}(s) > f_p(t)) = O((t \log^{1-p} t)^{-1})$, that $\mathbb{P}(\mathcal{E}_{f_p}) = 1_{\{p \geq 0\}}$. Consequently, with $\xi_p(t) = \sup\{s : 0 \leq s \leq t, X_{r:n}(s) \geq f_p(s)\}$, for $p \geq 0$ we have $\lim_{t \rightarrow \infty} \xi_p(t) = \infty$ and $\limsup_{t \rightarrow \infty} (\xi_p(t) - t) = 0$ a.s. Complementarily, we prove an Erdős–Révész type law of the iterated logarithm lower bound on $\xi_p(t)$, namely, that $\liminf_{t \rightarrow \infty} (\xi_p(t) - t)/h_p(t) = -1$ a.s. for $p > 1$ and $\liminf_{t \rightarrow \infty} \log(\xi_p(t)/t)/(h_p(t)/t) = -1$ a.s. for $p \in (0, 1]$, where $h_p(t) = (1/z_p(t))p \log \log t$.

Keywords Extremes of Gaussian processes · Order statistics process · Law of the iterated logarithm

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1 Introduction and Main Results

Let $X = \{X(t) : t \in \mathbb{R}_+\}$ be a stationary Gaussian process with almost surely (a.s.) continuous sample paths, $\mathbb{E}X(t) = 0$ and $\mathbb{E}X^2(t) = 1$. Suppose that the correlation function of X , $r(t) = \mathbb{E}X(t)X(0)$, satisfies the following regularity assumptions:

$$\begin{aligned} r(t) &= 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as } t \rightarrow 0 \text{ for some } 0 \leq \alpha \leq 2, \quad C > 0, \\ r^*(s) &= \sup_{t \geq s} |r(t)| < 1 \quad \text{for each } s > 0, \end{aligned} \quad (1)$$

$$r(t) = O(t^{-2\lambda}) \quad \text{as } t \rightarrow \infty \text{ for some } \lambda > 0. \quad (2)$$

The analysis of extremes of Gaussian stochastic processes has a long history. The celebrated *double sum* method, primarily developed by Pickands, e.g., [8], and extended by seminal works of Piterbarg, e.g., [10] or monograph [9], plays central role in the extreme value theory of Gaussian processes. The technique developed there appeared to be an universal method, which may deliver answers also to classes of non-Gaussian processes, see for example, recent contributions of [5, 6].

Laws of the iterated logarithm take important place in this theory, providing properties of extremal behavior of stochastic processes on large-time scale. One of important contributions in this domain is a result on the process $\xi = \{\xi(t) : t \geq 0\}$, defined via $\xi(t) = \sup\{s : 0 \leq s \leq t, X(s) \geq (2 \log s)^{1/2}\}$. In particular, the law of the iterated logarithm implies that, see [11, 12],

$$\limsup_{t \rightarrow \infty} (\xi(t) - t) = 0 \quad \text{a.s.}$$

Interestingly, under the above regularity assumptions, [12] gave the lower bound of $\xi(t)$ and obtained an Erdős–Révész type law of the iterated logarithm, that is,

$$\liminf_{t \rightarrow \infty} \frac{\xi(t) - t}{t(\log t)^{(\alpha-2)/(2\alpha)} \cdot \log_2 t} = -\frac{(2+\alpha)\sqrt{\pi}}{\alpha \mathcal{H}_\alpha (2C)^{1/\alpha}} \quad \text{a.s. if } 0 < \alpha < 2, \quad (3)$$

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi(t)/t)}{\log_2 t} = -\frac{2\sqrt{\pi}}{\mathcal{H}_2 \sqrt{2C}} \quad \text{a.s. if } \alpha = 2, \quad (4)$$

where \mathcal{H}_α is the *Pickands' constant* defined by $\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} e^{\sup_{t \in [0, T]} (\sqrt{2} B_{\alpha/2}(t) - t^\alpha)}$, with $B_{\alpha/2} = \{B_{\alpha/2}(t) : t \geq 0\}$ denoting fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, i.e., a continuous, centered Gaussian process with covariance function

$$\mathbb{E} B_{\alpha/2}(s) B_{\alpha/2}(t) = \frac{1}{2} (|s|^\alpha + |t|^\alpha - |t-s|^\alpha).$$

Equation (3) shows that for any t big enough there exists an s in $[t - t(\log t)^{(\alpha-2)/(2\alpha)} \cdot \log_2 t, t]$ such that, almost surely, $X(s) \geq (2 \log s)^{1/2}$ and that the length of the interval $t(\log t)^{(\alpha-2)/(2\alpha)} \cdot \log_2 t$ is smallest possible. Moreover, the bigger the parameter α is, the wider the interval will be.

In this paper, we derive a counterpart of Shao's result for the order statistics process $X_{r:n}$. Namely, for any $n \geq 0$, we consider X_1, \dots, X_n , n mutually independent copies of X and denote by $X_{r:n} = \{X_{r:n}(t) : t \geq 0\}$ the r th smallest order statistics process, that is, for each $t \geq 0$, $1 \leq r \leq n$,

$$X_{1:n}(t) = \min_{1 \leq j \leq n} X_j(t) \leq X_{2:n}(t) \leq \dots \leq X_{n-1:n}(t) \leq \max_{1 \leq j \leq n} X_j(t) = X_{n:n}(t).$$

Our first contribution is the theorem that extends classical findings of Qualls and Watanabe [11].

Theorem 1 *For all functions f that are positive and non-decreasing on some interval $[T, \infty)$, $T > 0$, it follows that*

$$\mathbb{P}(\mathcal{E}_f) := \mathbb{P}(X_{r:n}(t) > f(t) \text{ i.o.}) = 0 \text{ or } 1,$$

as the integral

$$\mathcal{I}_f := \int_T^\infty \mathbb{P}\left(\sup_{t \in [0, 1]} X_{r:n}(t) > f(u)\right) du \text{ is finite or infinite.}$$

[1, Theorem 2.2], see also [3], gave the expression for the asymptotic behavior of the probability in \mathcal{I}_f , namely

$$\mathbb{P}\left(\sup_{t \in [0, 1]} X_{r:n}(t) > u\right) = C^{\frac{1}{\alpha}} \binom{n}{\hat{r}} \mathcal{H}_{\alpha, \hat{r}} u^{\frac{2}{\alpha}} (\Psi(u))^{\hat{r}} (1 + o(1)), \quad \text{as } u \rightarrow \infty, \quad (5)$$

where $\hat{r} = n - r + 1$, $\Psi(u) = 1 - \Phi(u)$ and $\Phi(u)$ is the distribution function of unit normal law,

$$\mathcal{H}_{\alpha, k} = \lim_{T \rightarrow \infty} T^{-1} \mathcal{H}_{\alpha, k}(T) \in (0, \infty),$$

$$\mathcal{H}_{\alpha, k}(T) = \int_{\mathbb{R}^k} e^{\sum_{i=1}^k w_i} \mathbb{P}\left(\sup_{t \in [0, T]} \min_{1 \leq i \leq k} \left(\sqrt{2} B_{\alpha/2}^{(i)}(t) - t^\alpha - w_i\right) > 0\right) dw_1 \dots dw_k$$

and $B_{\alpha/2}^{(i)}$, $1 \leq i \leq n$, are mutually independent fractional Brownian motions. $\mathcal{H}_{\alpha, k}$ is the *generalized Pickands' constant* introduced in [2]; see also [1]. Therefore, Theorem 1 provides a tractable criterion for settling the dichotomy of $\mathbb{P}(\mathcal{E}_f)$.

For instance, let

$$f_p(s) = \left(\frac{2}{\hat{r}} \left(\log s + \left(\frac{2 - \hat{r}\alpha}{2\alpha} + 1 - p\right) \log_2 s\right)\right)^{\frac{1}{2}}, \quad p \in \mathbb{R}.$$

One easily checks that, as $u \rightarrow \infty$,

$$\mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u)\right) = C^{\frac{1}{\alpha}}(\hat{r}) \frac{\mathcal{H}_{\alpha, \hat{r}}}{(2\pi)^{\frac{\hat{r}}{2}}} \left(\frac{2}{\hat{r}}\right)^{\frac{2-\hat{r}\alpha}{2\alpha}} \left(u \log^{1-p} u\right)^{-1} (1 + o(1)). \quad (6)$$

Hence, for any $p \in \mathbb{R}$,

$$\mathbb{P}(X_{r:n}(t) > f_p(t) \text{ i.o.}) = \begin{cases} 1 & \text{if } p \geq 0 \\ 0 & \text{if } p < 0 \end{cases}.$$

Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{X_{r:n}(t)}{\sqrt{\log t}} = \sqrt{\frac{2}{\hat{r}}} \text{ a.s.}$$

Next, consider the process $\xi_p = \{\xi_p(t) : t \geq 0\}$ defined as

$$\xi_p(t) = \sup\{s : 0 \leq s \leq t, X_{r:n}(s) \geq f_p(s)\}.$$

Since $\mathcal{J}_{f_p} = \infty$ for $p \geq 0$, Theorem 1 implies that

$$\lim_{t \rightarrow \infty} \xi_p(t) = \infty \text{ a.s. and } \limsup_{t \rightarrow \infty} (\xi_p(t) - t) = 0 \text{ a.s.}$$

Let, cf. (6),

$$h_p(t) = p \left(\mathbb{P}\left(\sup_{s \in [0,1]} X_{r:n}(s) > f_p(t)\right) \right)^{-1} \log_2 t.$$

The second contribution of this paper is an Erdős–Révész type of law of the iterated logarithm for the process ξ_p .

Theorem 2 *If $p > 1$, then*

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} = -1 \text{ a.s.}$$

If $p \in (0, 1]$, then

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} = -1 \text{ a.s.}$$

Now, let us complementary put $\eta_p = \{\eta_p(t) : t \geq 0\}$, where

$$\eta_p(t) = \inf\{s \geq t : X_{r:n}(s) \geq f_p(s)\}.$$

Since

$$\mathbb{P}(\xi_p(t) - t \leq -x) = \mathbb{P}\left(\sup_{s \in (t-x, t]} \frac{X_{r:n}(s)}{f_p(s)} < 1\right)$$

and

$$\mathbb{P}(z - \eta_p(z) \leq -x) = \mathbb{P}\left(\sup_{s \in [z, z+x]} \frac{X_{r:n}(s)}{f_p(s)} < 1\right),$$

then it follows that

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} = \liminf_{z \rightarrow \infty} \frac{z - \eta_p(z)}{h_p(z)}. \quad (7)$$

Theorem 2 shows that for t big enough, there exists an s in $[t - h_p(t), t]$ (as well as in $[t, t + h_p(t)]$ by (7)) such that $X_{r:n}(s) \geq f_p(s)$ and that the length of the interval $h_p(t)$ is smallest possible. One can retrieve (3)–(4) by setting $n = 1$, and $p = \frac{2-\hat{r}\alpha}{2\alpha} + 1 = \frac{2+\alpha}{2\alpha}$. Theorem 2 not only generalizes [12, Theorem 1.1], it also unveils the lacking so far structure of the lower bound of $\xi_p(t)$ by relating it, via $h_p(t)$, to the asymptotics of the tail distribution of the supremum of the underlying process evaluated at $f_p(t)$; in (3) $t(\log t)^{(\alpha-2)/(2\alpha)}$ is of the same asymptotic order as the reciprocal of $\mathbb{P}(\sup_{s \in [0,1]} X(s) > (2 \log t)^{1/2})$. This shines new light on this type of results, which appear to be intrinsically connected with Gumbel limit theorems; see, e.g., [7], where the function $h_p(t)$ plays crucial role. We shall pursue this elsewhere.

The paper is organized as follows. In Sect. 2, we provide a collection of basic results on order statistics of stationary Gaussian processes, used throughout the paper, and prove auxiliary lemmas, which constitute building blocks of the proofs of the main results. These are given in the final part of the paper, Sect. 3.

2 Auxiliary Lemmas

We begin with some auxiliary lemmas that are later needed in the proofs.

The following lemma is the general form of the Borel–Cantelli lemma; cf. [13].

Lemma 1 *Consider a sequence of events $\{E_k : k \geq 0\}$. If*

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k) < \infty,$$

then $\mathbb{P}(E_n \text{ i.o.}) = 0$. Whereas, if

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k) = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\sum_{1 \leq k \neq t \leq n} \mathbb{P}(E_k E_t)}{(\sum_{k=1}^n \mathbb{P}(E_k))^2} \leq 1,$$

then $\mathbb{P}(E_n \text{ i.o.}) = 1$.

The following two lemmas constitute useful tools for approximating the supremum of $X_{r:n}$ on a fixed interval by its maximum on a grid with a sufficiently dense mesh.

Lemma 2 *There exist positive constants K , c and u_0 such that*

$$\begin{aligned} \mathbb{P}\left(\max_{0 \leq j \leq u^{\frac{2}{\alpha}}/\theta} X_{r:n}(j\theta u^{-\frac{2}{\alpha}}) \leq u - \frac{\theta^{\frac{\alpha}{4}}}{u}, \sup_{t \in [0,1]} X_{r:n}(t) > u\right) \\ \leq K u^{\frac{2\hat{r}}{\alpha}} (\Psi(u))^{\hat{r}} \theta^{\frac{\alpha}{2}-1} \Psi(c\theta^{-\frac{\alpha}{4}}), \end{aligned}$$

for each $\theta > 0$ and $u \geq u_0$.

Proof Note that, by stationarity, there exists a constant K , that may vary from line to line, such that, for sufficiently large u ,

$$\begin{aligned} \mathbb{P}\left(\max_{0 \leq j \leq u^{\frac{2}{\alpha}}/\theta} X_{r:n}(j\theta u^{-\frac{2}{\alpha}}) \leq u - \frac{\theta^{\frac{\alpha}{4}}}{u}, \sup_{t \in [0,1]} X_{r:n}(t) > u\right) \\ \leq \frac{u^{\frac{2}{\alpha}}}{\theta} \mathbb{P}\left(X_{r:n}(0) \leq u - \frac{\theta^{\frac{\alpha}{4}}}{u}, \sup_{t \in [0,1]} X_{r:n}(t) > u\right) \\ \leq \frac{u^{\frac{2}{\alpha}}}{\theta} \binom{n}{r} \binom{n}{n-r+1} \mathbb{P}\left(\forall i=1, \dots, r, X_i(0) \leq u - \frac{\theta^{\frac{\alpha}{4}}}{u}, \forall j=r, \dots, n, \sup_{t \in [0,1]} X_j(t) > u\right) \\ \leq K \frac{u^{\frac{2}{\alpha}}}{\theta} \mathbb{P}\left(X_r(0) \leq u - \frac{\theta^{\frac{\alpha}{4}}}{u}, \sup_{t \in [0,1]} X_r(t) > u\right) \left(\mathbb{P}\left(\sup_{t \in [0,1]} X(t) > u\right)\right)^{n-r} \\ \leq K u^{\frac{2\hat{r}}{\alpha}} (\Psi(u))^{\hat{r}} \theta^{\frac{\alpha}{2}-1} \Psi(c\theta^{-\frac{\alpha}{4}}). \end{aligned}$$

The last inequality follows from (5) and the classical result of [7, Lemma 12.2.5], where the constant $c > 0$ is given therein. \square

The proof of the following lemma follows line-by-line the same reasoning as the proof of [1, Theorem 2.2], and thus we omit it.

Lemma 3 *For any $\theta > 0$, as $u \rightarrow \infty$,*

$$\mathbb{P}\left(\max_{0 \leq j \leq u^{\frac{2}{\alpha}}/\theta} X_{r:n}(j\theta u^{-\frac{2}{\alpha}}) > u\right) = C^{\frac{1}{\alpha}} \binom{n}{\hat{r}} \frac{\mathcal{H}_{\alpha, \hat{r}}(\theta)}{\theta} (\Psi(u))^{\hat{r}} (1 + o(1)).$$

The next lemma follows directly from [4, Theorem 2.4] and is a generalization of the classical Berman's inequality to order statistics.

Lemma 4 *For some $n, d \geq 1$, and any $1 \leq l \leq n$ let $\{\xi_l^{(0)}(i) : 1 \leq i \leq d\}$ and $\{\xi_l^{(1)}(i) : 1 \leq i \leq d\}$ be a sequence of $\mathcal{N}(0, 1)$ variables and set $\sigma_{il,jk}^{(\kappa)} = E\xi_l^{(\kappa)}(i)\xi_k^{(\kappa)}(j)$, $\kappa = 0, 1$. For any $1 \leq r \leq n$ and $1 \leq i \leq d$, let $\xi_{r:n}^{(\kappa)}(i)$ be the r th*

order statistic of $\xi_1^{(\kappa)}(i), \dots, \xi_n^{(\kappa)}(i)$. Suppose that, for any $1 \leq i, j \leq d, 1 \leq l, k \leq n, \kappa = 0, 1$,

$$\sigma_{il,jk}^{(\kappa)} = \sigma_{ij}^{(\kappa)} 1_{\{l=k\}}$$

for some $\sigma_{ij}^{(\kappa)}$. Now define

$$\rho_{ij} = \max \left(\left| \sigma_{ij}^{(0)} \right|, \left| \sigma_{ij}^{(1)} \right| \right), \quad A_{ij}^{(r)} = \int_{\sigma_{ij}^{(0)}}^{\sigma_{ij}^{(1)}} \frac{(1 + |h|)^{(n-r)/2}}{(1 - h^2)^{\hat{r}/2}} dh.$$

Then, for any $u_1, \dots, u_d > 0$, for some positive constant $C_{n,r}$ depending only on n and r ,

$$\begin{aligned} & \mathbb{P} \left(\bigcap_{i=1}^d \left\{ \xi_{r:n}^{(0)}(i) \leq u_i \right\} \right) - \mathbb{P} \left(\bigcap_{i=1}^d \left\{ \xi_{r:n}^{(1)}(i) \leq u_i \right\} \right) \\ & \leq C_{n,r} \sum_{1 \leq i < j \leq d} (u_i + u_j)^{-(n-r)} \left(A_{ij}^{(r)} \right)^+ \exp \left(-\frac{\hat{r} (u_i^2 + u_j^2)}{2(1 + \rho_{ij})} \right). \end{aligned}$$

Lemma 5 Under the conditions of Theorem 2, for any $\varepsilon \in (0, 1)$, there exist positive constants K and ρ depending only on ε, α and λ such that

$$\mathbb{P} \left(\sup_{S \leq t \leq T} \frac{X_{r:n}(t)}{f_p(t)} \leq 1 \right) \leq \exp \left(-\frac{(1 - \varepsilon)}{(1 + \varepsilon)} \int_{S+1}^T \mathbb{P} \left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u) \right) du \right) + K S^{-\rho},$$

for any $T - 1 \geq S \geq K$.

Proof Let, for any $i \geq 0$ and $\varepsilon \in (0, 1)$,

$$s_i = S + i(1 + \varepsilon), \quad t_i = s_i + 1, \quad x_i = f_p(t_i), \quad I_i = (s_i, t_i].$$

For some $\theta > 0$, define grid points in the interval I_i , as follows

$$s_{i,u} = s_i + uq_i, \quad 0 \leq u \leq L_i, \quad L_i = \lceil 1/q_i \rceil, \quad q_i = \theta x_i^{-\frac{2}{\alpha}}. \quad (8)$$

Since f_p is an increasing function, it easily follows that, with $T(S, \varepsilon) = \lceil (T - S - 1)/(1 + \varepsilon) \rceil$,

$$\begin{aligned} \mathbb{P} \left(\sup_{S \leq t \leq T} \frac{X_{r:n}(t)}{f_p(t)} \leq 1 \right) & \leq \mathbb{P} \left(\bigcap_{i=0}^{T(S,\varepsilon)} \left\{ \sup_{t \in I_i} X_{r:n}(t) \leq x_i \right\} \right) \\ & \leq \mathbb{P} \left(\bigcap_{i=0}^{T(S,\varepsilon)} \left\{ \max_{0 \leq u \leq L_i} X_{r:n}(s_{i,u}) \leq x_i \right\} \right). \end{aligned}$$

For any $1 \leq l \leq n$ and $i \geq 0$, let $X_{l,i}$ be an independent copy of the process X_l . Define a sequence of processes $Y_l = \{Y_l(t) : t \in \cup_i I_i\}$ as $Y_l(t) = X_{l,i}(t)$, if $t \in I_i$. Let $Y_{r:n} = \{Y_{r:n}(t) : t \geq 0\}$ be the r th order statistic of Y_1, \dots, Y_n . Put

$$\begin{aligned}\sigma_{il,jk}^{(0)} &:= \mathbb{E}X_l(i)X_k(j) = r(|j-i|)1_{\{l=k\}} =: \sigma_{ij}^{(0)}1_{\{l=k\}}, \\ \sigma_{il,jk}^{(1)} &:= \mathbb{E}Y_l(i)Y_k(j) = r(|j-i|)1_{\{l=k\}}1_{\{\exists m:i,j \in I_m\}} =: \sigma_{ij}^{(1)}1_{\{l=k\}},\end{aligned}$$

and note that

$$\begin{aligned}\rho_{ij} &= \max\left(\left|\sigma_{ij}^{(0)}\right|, \left|\sigma_{ij}^{(1)}\right|\right) = |r(|j-i|)|, \\ A_{ij}^{(r)} &= \int_{\sigma_{ij}^{(0)}}^{\sigma_{ij}^{(1)}} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{\hat{r}/2}} dh = 1_{\{\forall m:i,j \notin I_m\}} \int_0^{r(j-i)} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{\hat{r}/2}} dh \\ &=: 1_{\{\forall m:i,j \notin I_m\}} |\tilde{A}_{ij}^{(r)}|. \quad (9)\end{aligned}$$

Now using Lemma 4 we find that

$$\begin{aligned}& \mathbb{P}\left(\bigcap_{i=0}^{T(S,\varepsilon)} \left\{ \max_{0 \leq u \leq L_i} X_{r:n}(s_{i,u}) \leq x_i \right\}\right) \\ & \leq \prod_{i=0}^{T(S,\varepsilon)} \mathbb{P}\left(\max_{0 \leq u \leq L_i} X_{r:n}(s_{i,u}) \leq x_i\right) \\ & \quad + C_{n,r} \sum_{0 \leq i < j \leq T(S,\varepsilon)} \sum_{\substack{0 \leq u \leq L_i \\ 0 \leq v \leq L_j}} (x_i x_j)^{-(n-r)} \left| \tilde{A}_{s_{i,u} s_{j,v}}^{(r)} \right| \exp\left(-\frac{\hat{r}(x_i^2 + x_j^2)}{2(1+|r(s_{j,v} - s_{i,u})|)}\right) \\ & =: P_1 + P_2.\end{aligned}$$

Estimate of P_1 .

Since $X_{r:n}$ is a stationary process, from Eq. 5 combined with Lemma 3, for any $\varepsilon \in (0, 1)$, sufficiently large θ and S ,

$$\begin{aligned}P_1 & \leq \exp\left(-\sum_{i=0}^{T(S,\varepsilon)} \mathbb{P}\left(\max_{0 \leq u \leq L_i} X_{r:n}(s_{i,u}) > x_i\right)\right) \\ & \leq \exp\left(-(1-\varepsilon) \sum_{i=0}^{T(S,\varepsilon)} \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(t_i)\right)\right) \\ & \leq \exp\left(-\frac{1-\varepsilon}{1+\varepsilon} \int_{S+1}^T \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u)\right) du\right).\end{aligned}$$

Estimate of P_2 .

Noting that, for any $0 \leq i < j$, $0 \leq u \leq L_i$, $0 \leq v \leq L_j$;

$$s_{j,v} - s_{i,u} = s_j + vq_j - s_i - uq_i = (j - i)(1 + \varepsilon) + vq_j - uq_i \geq (j - i)\varepsilon,$$

we have

$$\sup_{\substack{0 \leq u \leq L_i, \\ 0 \leq v \leq L_j}} |r(s_{j,v} - s_{i,u})| \leq \sup_{|s-s'| \geq (j-i)\varepsilon} |r(s - s')| = r^*((j - i)\varepsilon) \leq r^*(\varepsilon) < 1.$$

Without loss of generality assume that $\lambda < 2$. From (2) it follows that there is s_0 such that for every $s > s_0$,

$$r^*(s) \leq s^{-\lambda} \leq \min(1, \lambda)/4.$$

Finally, since the integrand in the definition of $\tilde{A}_{s_{i,u}s_{j,v}}^{(r)}$ is continuous and bounded on $[0, r^*(\varepsilon)]$, there exists a generic constant K not depending on S and T , which may differ from line to line, such that

$$\left| \tilde{A}_{s_{i,u}s_{j,v}}^{(r)} \right| \leq K |r(s_{j,v} - s_{i,u})| \leq Kr^*((j - i)\varepsilon).$$

Therefore, for sufficiently large S ,

$$\begin{aligned} P_2 &\leq K \sum_{0 \leq i < j \leq T(S, \varepsilon)} L_i L_j r^*((j - i)\varepsilon) \exp\left(-\frac{\hat{r}(x_i^2 + x_j^2)}{2(1 + r^*((j - i)\varepsilon))}\right) \\ &\leq K \left(\sum_{\substack{0 < j-i \leq 2s_0 \\ 0 \leq i < j \leq T(S, \varepsilon)}} + \sum_{\substack{j-i > 2s_0 \\ 0 \leq i < j \leq T(S, \varepsilon)}} \right) (\cdot) \\ &\leq K \left(\sum_{i=0}^{\infty} x_i^{\frac{4}{\alpha}} \exp\left(-\frac{\hat{r}x_i^2}{1 + r^*(\varepsilon)}\right) \right. \\ &\quad \left. + \sum_{\substack{j-i > 2s_0 \\ 0 \leq i < j \leq T(S, \varepsilon)}} x_i^{\frac{2}{\alpha}} x_j^{\frac{2}{\alpha}} (j - i)^{-\lambda} \exp\left(-\frac{\hat{r}(x_i^2 + x_j^2)}{2(1 + \frac{\lambda}{4})}\right) \right) \\ &\leq K \left(\sum_{i=0}^{\infty} t_i^{-\frac{2}{1 + \sqrt{r^*(\varepsilon)}}} + \sum_{\substack{j-i > 2s_0 \\ 0 \leq i < j \leq T(S, \varepsilon)}} t_i^{-\frac{1}{1 + \frac{\lambda}{2}}} t_j^{-\frac{1}{1 + \frac{\lambda}{2}}} (j - i)^{-\lambda} \right). \end{aligned}$$

We can bound the first sum from the above by

$$K \sum_{i=0}^{\infty} (S + i)^{-\frac{2}{1 + \sqrt{r^*(\varepsilon)}}} \leq KS^{-\frac{1 - \sqrt{r^*(\varepsilon)}}{4}}.$$

The second sum is bounded from above by

$$\begin{aligned}
 \sum_{S \leq i < j < \infty} i^{-\frac{1}{1+\frac{\lambda}{2}}} j^{-\frac{1}{1+\frac{\lambda}{2}}} (j-i)^{-\lambda} &= \sum_{j=S}^{\infty} j^{-\frac{1}{1+\frac{\lambda}{2}}} \sum_{i=S}^{j-1} i^{-\frac{1}{1+\frac{\lambda}{2}}} (j-i)^{-\lambda} \\
 &\leq \sum_{j=S}^{\infty} j^{-\frac{1}{1+\frac{\lambda}{2}}} \left((j/2)^{-\lambda} \sum_{i=S}^{[j/2]} i^{-\frac{1}{1+\frac{\lambda}{2}}} + (j/2)^{-\frac{1}{1+\frac{\lambda}{2}}} \sum_{i=[j/2]}^{j-1} (j-i)^{-\lambda} \right) \\
 &\leq K \sum_{j=S}^{\infty} j^{-\frac{1}{1+\frac{\lambda}{2}}} \left(j^{-\lambda+1-\frac{1}{1+\frac{\lambda}{2}}} + j^{-\frac{1}{1+\frac{\lambda}{2}}} (\log j \cdot 1_{\{\lambda \in [1,2)\}} + j^{-\lambda+1} 1_{\{\lambda \in (0,1)\}}) \right) \\
 &\leq K \left(\sum_{j=S}^{\infty} j^{-\frac{2}{1+\frac{\lambda}{2}}} \log j \cdot 1_{\{\lambda \in [1,2)\}} + \sum_{j=S}^{\infty} j^{1-\lambda-\frac{2}{1+\frac{\lambda}{2}}} \cdot 1_{\{\lambda \in (0,1)\}} \right) \\
 &\leq K \left(S^{1-\frac{2}{1+\frac{\lambda}{2}}} \log S \cdot 1_{\{\lambda \in [1,2)\}} + S^{2-\lambda-\frac{2}{1+\frac{\lambda}{2}}} \cdot 1_{\{\lambda \in (0,1)\}} \right).
 \end{aligned}$$

Hence, for some positive constant ρ , depending only on ε , α and λ ,

$$P_2 \leq K S^{-\rho},$$

which finishes the proof. \square

Lemma 6 *Under the conditions of Theorem 2, for any $\varepsilon \in (0, 1)$, there exist positive constants K and ρ depending only on ε , α and λ such that*

$$\begin{aligned}
 &\mathbb{P} \left(\bigcap_{i=0}^{[T-S]} \left\{ \max_{0 \leq u \leq [y_i^{\frac{2}{\alpha}}/\theta_i]} X_{r:n}(S+i+u\theta_i y_i^{-\frac{2}{\alpha}}) \leq y_i - \frac{\theta_i^{\alpha/4}}{y_i} \right\} \right) \\
 &\geq \frac{1}{4} \exp \left(-(1+\varepsilon) \int_S^T \mathbb{P} \left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u) \right) du \right) - K S^{-\rho},
 \end{aligned}$$

for any $T-1 \geq S \geq K$, where $y_i = f_p(S+i)$ and $\theta_i = y_i^{-\frac{8}{\alpha}}$.

Proof Let, for any $i \geq 0$, $a_i = S+i$ so that $y_i = f_p(a_i)$. Define grid points in the interval $(a_i, a_{i+1}]$ as follows

$$a_{i,u} = a_i + uq_i, \quad 0 \leq u \leq L_i, \quad L_i = [1/q_i], \quad q_i = \theta_i y_i^{-\frac{2}{\alpha}}. \quad (10)$$

Finally, put $\hat{y}_i = y_i - \theta_i^{\frac{\alpha}{4}}/y_i$. Similarly as in the proof of Lemma 5, using Lemma 4 we have

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{i=0}^{[T-S]} \left\{ \max_{0 \leq u \leq L_i} X_{r:n}(a_{i,u}) \leq \hat{y}_i \right\}\right) \\ & \geq \prod_{i=0}^{[T-S]} \mathbb{P}\left(\max_{0 \leq u \leq L_i} X_{r:n}(a_{i,u}) \leq \hat{y}_i\right) \\ & \quad - C_{n,r} \sum_{0 \leq i < j \leq [T-S]} \sum_{\substack{0 \leq u \leq L_i \\ 0 \leq v \leq L_j}} (\hat{y}_i \hat{y}_j)^{-(n-r)} \left(-\tilde{A}_{a_{i,u}a_{j,v}}^{(r)}\right)^+ \\ & \quad \exp\left(-\frac{\hat{r}(\hat{y}_i^2 + \hat{y}_j^2)}{2(1 + |r(a_{j,v} - a_{i,u})|)}\right) \\ & =: P'_1 - P'_2, \end{aligned}$$

where $\tilde{A}_{a_{i,u}a_{j,v}}^{(r)}$ is as in (9).

Estimate of P'_1 .

Note that, by Lemma 3 combined with Eq. 5,

$$\begin{aligned} P'_1 & \geq \frac{1}{4} \exp\left(-\sum_{i=0}^{[T-S]} \mathbb{P}\left(\max_{0 \leq u \leq L_i} X_{r:n}(a_{i,u}) > \hat{y}_i\right)\right) \\ & \geq \frac{1}{4} \exp\left(-\sum_{i=0}^{[T-S]} \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > \hat{y}_i\right)\right) \\ & \geq \frac{1}{4} \exp\left(-(1+\varepsilon) \sum_{i=0}^{[T-S]} \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > y_i\right)\right) \\ & \geq \frac{1}{4} \exp\left(-(1+\varepsilon) \int_S^T \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u)\right) du\right), \end{aligned}$$

provided that S is sufficiently large.

Estimate of P'_2 .

Noting that, for $j \geq i + 2$, and any $0 \leq u \leq L_i$, $0 \leq v \leq L_j$;

$$a_{j,v} - a_{i,u} = a_j + vq_j - a_i - uq_i \geq j - i - 1,$$

we have

$$\sup_{\substack{0 \leq u \leq L_i \\ 0 \leq v \leq L_j}} |r(a_{j,v} - a_{i,u})| \leq \sup_{|s-s'| \geq j-i-1} |r(s-s')| = r^*(j-i-1) \leq r^*(1) < 1. \quad (11)$$

Since the integrand in definition of $\tilde{A}_{a_{i,u}a_{j,v}}^{(r)}$ is continuous and bounded on $[0, r^*(1)]$, there exists a constant K such that

$$\left| \tilde{A}_{a_{i,u}a_{j,v}}^{(r)} \right| \leq Kr(a_{j,v} - a_{i,u}) \leq Kr^*(j - i - 1) < K.$$

On the other hand, by (1), there exist positive constants $s_0 < 1$, such that, for every $0 \leq s \leq s_0$,

$$\tilde{A}_{0s}^{(r)} \geq r(s) \geq 1 - 2|s|^\alpha > 0.$$

Hence,

$$(-\tilde{A}_{a_{i,u}a_{j,v}}^{(r)})^+ = 0, \quad \text{if } j = i + 1, \quad 1 + vq_j - uq_i \leq s_0, \quad (12)$$

$$|r(a_{j,v} - a_{i,u})| \leq r^*(s_0) < 1, \quad \text{if } j = i + 1, \quad 1 + vq_j - uq_i > s_0 \quad (13)$$

Therefore, by (11)–(13) we obtain

$$\begin{aligned} P'_2 \leq & \sum_{\substack{0 \leq i \leq [T-S]-1 \\ j=i+1}} \sum_{\substack{0 \leq u \leq L_i \\ 0 \leq v \leq L_j}} \frac{1}{\sqrt{1 - r^*(s_0)}} \exp\left(-\frac{\hat{r}(\hat{y}_i^2 + \hat{y}_j^2)}{2(1 + r^*(s_0))}\right) \\ & + \sum_{\substack{0 \leq i \leq [T-S]-2 \\ i+2 \leq j \leq [T-S]}} \sum_{\substack{0 \leq u \leq L_i \\ 0 \leq v \leq L_j}} \frac{r^*(j - i - 1)}{\sqrt{1 - r^*(1)}} \exp\left(-\frac{\hat{r}(\hat{y}_i^2 + \hat{y}_j^2)}{2(1 + r^*(j - i - 1))}\right). \end{aligned}$$

Completely similar to the estimation of P_2 in the proof of Lemma 5, we can arrive that there exist positive constants K and ρ , independent of S and T , such that, for sufficiently large S ,

$$P'_2 \leq KS^{-\rho}.$$

□

The following lemma is a straightforward modification of Lemmas 3.1 and 4.1 of [14] and [11, Lemma 1.4].

Lemma 7 *If Theorem 1 is true under the additional condition that for large t ,*

$$\frac{2}{\hat{r}} \log t \leq f^2(t) \leq \frac{3}{\hat{r}} \log t, \quad (14)$$

it is true without the additional condition.

3 Proofs of the Main Results

Proof of Theorem 1 Note that the case $\mathcal{J}_f < \infty$ is straightforward and does not need any additional knowledge on process $X_{r:n}$ apart from the assumption of stationarity. Indeed, for sufficiently large T ,

$$\sum_{i=[T]+1}^{\infty} \mathbb{P} \left(\sup_{t \in [i, i+1]} X_{r:n}(t) > f(i) \right) = \sum_{i=[T]}^{\infty} \mathbb{P} \left(\sup_{t \in [0, 1]} X_{r:n}(t) > f(i+1) \right) \leq \mathcal{J}_f < \infty,$$

and the Borel–Cantelli lemma completes this part of the proof since f is an increasing function.

Now let f be any increasing function such that $\mathcal{J}_f \equiv \infty$. With the same notation as in Lemma 5 with f instead of f_p , we find that, for any $S > 0$,

$$\begin{aligned} \mathbb{P}(X_{r:n}(s) > f(s) \text{ i.o.}) &\geq \mathbb{P} \left(\left\{ \sup_{t \in I_i} X_{r:n}(t) > x_i \right\} \text{ i.o.} \right) \\ &\geq \mathbb{P} \left(\left\{ \max_{1 \leq u \leq L_i} X_{r:n}(s_{i,u}) > x_i \right\} \text{ i.o.} \right), \end{aligned}$$

where, recall, $s_{i,u} = S + i(1 + \varepsilon) + u\theta x_i^{-2/\alpha}$, $L_i = \lceil 1/(\theta x_i^{-2/\alpha}) \rceil$, $\theta, \varepsilon > 0$. Furthermore, for sufficiently large S and θ , cf. estimation of P_1 ,

$$\sum_{i=0}^{\infty} \mathbb{P} \left(\max_{1 \leq u \leq L_i} X_{r:n}(s_{i,u}) > x_i \right) \geq \frac{1 - \varepsilon}{1 + \varepsilon} \int_S^{\infty} \mathbb{P} \left(\sup_{t \in [0, 1]} X_{r:n}(t) > f(u) \right) du = \infty. \quad (15)$$

Let $E_i = \{\max_{1 \leq u \leq L_i} X_{r:n}(s_{i,u}) \leq x_i\}$, and note that

$$1 - \mathbb{P}(E_i^c \text{ i.o.}) = \lim_{m \rightarrow \infty} \prod_{k=m}^{\infty} \mathbb{P}(E_k) + \lim_{m \rightarrow \infty} \left(\mathbb{P} \left(\bigcap_{k=m}^{\infty} E_k \right) - \prod_{k=m}^{\infty} \mathbb{P}(E_k) \right).$$

The first limit is zero as a consequence of (15), and the second limit will be zero because of the asymptotic independence of the events E_k . Indeed, there exist positive constants K and ρ , such that for any $n > m$,

$$A_{m,n} = \left| \mathbb{P} \left(\bigcap_{k=m}^n E_k \right) - \prod_{k=m}^n \mathbb{P}(E_k) \right| \leq K(S + m)^{-\rho},$$

by the same calculations as in the estimate of P_2 in Lemma 5 after realizing that, by Lemma 7, we might restrict ourselves to the case when (14) holds. Therefore, $\mathbb{P}(E_i^c \text{ i.o.}) = 1$, which finishes the proof. \square

Proof of Theorem 2 Step 1. Let $p > 1$, then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.}$$

□

Proof Let $\{T_k : k \geq 1\}$ be a sequence such that $T_k \rightarrow \infty$, as $k \rightarrow \infty$. Put $S_k = T_k - (1 + 2\varepsilon)^2 h_p(T_k)$. Then by Lemma 5,

$$\begin{aligned} \mathbb{P} \left(\frac{\xi_p(T_k) - T_k}{h_p(T_k)} \leq -(1 + 2\varepsilon)^2 \right) &= \mathbb{P}(\xi_p(T_k) \leq S_k) = \mathbb{P} \left(\sup_{S_k < t \leq T_k} \frac{X_{r:n}(t)}{f_p(t)} < 1 \right) \\ &\leq \exp \left(-\frac{(1 - \varepsilon)}{(1 + \varepsilon)} \int_{S_{k+1}}^{T_k} \mathbb{P} \left(\sup_{t \in [0, 1]} X_{r:n}(t) > f_p(u) \right) du \right) + 2KT_k^{-\rho}, \end{aligned}$$

where the last inequality follows by the fact that $h_p(t) = o(t)$, so that $S_k \sim T_k$. Note that as $k \rightarrow \infty$

$$\begin{aligned} \int_{S_{k+1}}^{T_k} \mathbb{P} \left(\sup_{t \in [0, 1]} X_{r:n}(t) > f_p(u) \right) du &\sim (1 + 2\varepsilon)^2 h_p(T_k) \mathbb{P} \left(\sup_{t \in [0, 1]} X_{r:n}(t) > f_p(T_k) \right) \\ &= (1 + 2\varepsilon)^2 p \log_2 T_k. \end{aligned} \quad (16)$$

Now take $T_k = \exp(k^{1/p})$. Then

$$\sum_{k=0}^{\infty} \mathbb{P}(\xi_p(T_k) \leq S_k) \leq 2K \sum_{k=0}^{\infty} k^{-(1+\varepsilon/2)} < \infty.$$

Hence, by the Borel–Cantelli lemma,

$$\liminf_{k \rightarrow \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.} \quad (17)$$

Since $\xi(t)$ is a non-decreasing random function of t , for every $T_k \leq t \leq T_{k+1}$, we have

$$\frac{\xi_p(t) - t}{h_p(t)} \geq \frac{h_p(T_k)}{h_p(T_{k+1})} \frac{\xi_p(T_k) - T_{k+1}}{h_p(T_k)} = \frac{h_p(T_k)}{h_p(T_{k+1})} \left(\frac{\xi_p(T_k) - T_k}{h_p(T_k)} - \frac{T_{k+1} - T_k}{h_p(T_k)} \right).$$

For $p > 1$ elementary calculus implies

$$\lim_{k \rightarrow \infty} \frac{T_{k+1} - T_k}{h_p(T_k)} = 0, \quad \lim_{k \rightarrow \infty} \frac{h_p(T_k)}{h_p(T_{k+1})} = 1,$$

so that

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq \liminf_{k \rightarrow \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \quad \text{a.s.},$$

which finishes the proof of this step. \square

Step 2. Let $p > 1$, then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t \rightarrow \infty} \frac{\xi_p(t) - t}{h_p(t)} \leq -(1 - \varepsilon) \quad \text{a.s.}$$

Proof As in the proof of the lower bound, put

$$T_k = \exp(k^{(1+\varepsilon^2)/p}), \quad S_k = T_k - (1 - \varepsilon)h_p(T_k), \quad k \geq 1.$$

Let

$$B_k = \{\xi_p(T_k) \leq S_k\} = \left\{ \sup_{S_k < s \leq T_k} \frac{X_{r:n}(s)}{f_p(s)} < 1 \right\}.$$

It suffices to show $\mathbb{P}(B_n \text{ i.o.}) = 1$, that is

$$\lim_{m \rightarrow \infty} \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) = 1. \quad (18)$$

Let $a_i^k = S_k + i$ and define grid points in the interval $[a_i^k, a_{i+1}^k]$ as follows

$$\begin{aligned} a_{i,u}^k &= a_i^k + uq_i^k, \quad 0 \leq u \leq L_i^k, \quad L_i^k = \lfloor 1/q_i^k \rfloor, \quad q_i^k = \theta_i^k (y_i^k)^{-\frac{2}{\alpha}}, \quad \theta_i^k = (y_i^k)^{-\frac{8}{\alpha}}, \\ y_i^k &= f_p(a_i^k). \end{aligned}$$

Put

$$A_k = \bigcap_{i=0}^{\lfloor T_k - S_k \rfloor} \left\{ \max_{0 \leq u \leq L_i^k} X_{r:n}(a_{i,u}^k) \leq y_i^k - \theta_i^k / y_i^k \right\}.$$

Clearly, for $m \geq 1$,

$$\mathbb{P}\left(\bigcup_{k=m}^{\infty} A_k\right) \leq \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) + \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c).$$

Put $\hat{y}_i^k = y_i^k - \theta_i^k / y_i^k$. Then, by Lemma 2, for some constants K independent of S and T , which may vary between (and among) lines,

$$\begin{aligned} \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c) &\leq \sum_{k=m}^{\infty} \sum_{i=0}^{[T_k - S_k]} \mathbb{P}\left(\max_{0 \leq u \leq L_i^k} X_{r:n}(a_{i,u}^k) \leq \hat{y}_i^k, \sup_{s \in [0,1]} X_{r:n}(s) \geq y_i^k\right) \\ &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} (y_i^k)^{\frac{2\hat{r}}{\alpha}} \Psi(y_i^k) (\theta_i^k)^{\frac{\alpha}{2}-1} \Psi\left(K(\theta_i^k)^{-\frac{\alpha}{4}}\right) \\ &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} (a_i^k \log^{1-p} a_i^k)^{-1} (\log a_i^k)^{\frac{4}{\alpha}-3\alpha} \exp\left(-\frac{\log^2 a_i^k}{K}\right) \\ &\leq K \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} (S_k + i)^{-3} (\log(S_k + i))^{\frac{4}{\alpha}-3\alpha+p-1} \\ &\leq K \sum_{k=m}^{\infty} S_k^{-1} \leq K m^{-4}, \end{aligned}$$

provided m is large enough. Therefore,

$$\lim_{m \rightarrow \infty} \sum_{k=m}^{\infty} \mathbb{P}(A_k \cap B_k^c) = 0$$

and

$$\lim_{m \rightarrow \infty} \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) \geq \lim_{m \rightarrow \infty} \mathbb{P}\left(\bigcup_{k=m}^{\infty} A_k\right).$$

To finish the proof of (18), we only need to show that

$$\mathbb{P}(A_n \text{ i.o.}) = 1. \quad (19)$$

Similarly to (16), we have

$$\int_{S_k}^{T_k} \mathbb{P}\left(\sup_{t \in [0,1]} X_{r:n}(t) > f_p(u)\right) du \sim (1 - \varepsilon) p \log_2 T_k.$$

Now from Lemma 6 it follows that

$$\mathbb{P}(A_k) \geq \frac{1}{4} \exp\left(-(1 - \varepsilon^2) p \log_2 T_k\right) - K S_k^{-\rho} \geq \frac{1}{8} k^{-(1-\varepsilon^4)},$$

for every k sufficiently large. Hence,

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty. \quad (20)$$

Applying Lemma 4, we get for $0 \leq t < k$

$$\mathbb{P}(A_k A_t) \leq \mathbb{P}(A_k) \mathbb{P}(A_t) + M_{k,t}, \quad (21)$$

where, similarly to the proof of Lemma 5,

$$M_{k,t} = C_{n,r} \sum_{\substack{0 \leq i \leq [T_k - S_k] \\ 0 \leq j \leq [T_t - S_t]}} \sum_{\substack{0 \leq u \leq L_i^k \\ 0 \leq v \leq L_j^t}} \left(\hat{y}_i^k \hat{y}_j^t \right)^{-(n-r)} \left| \tilde{A}_{s_{i,u}^k, s_{j,v}^t}^{(r)} \right| \exp \left(- \frac{\hat{r} \left((\hat{y}_i^k)^2 + (\hat{y}_j^t)^2 \right)}{2(1 + |r(s_{i,u}^k - s_{j,v}^t)|)} \right),$$

where

$$\left| \tilde{A}_{s_{i,u}^k, s_{j,v}^t}^{(r)} \right| \leq K \left| r(s_{i,u}^k - s_{j,v}^t) \right|.$$

It is easy to see that,

$$\frac{S_{k+1} - T_k}{T_{k+1} - T_k} \sim 1, \text{ as } k \rightarrow \infty,$$

so that, for $0 \leq t < k$ and k large enough, and assuming without loss of generality that $\lambda < 2$,

$$\begin{aligned} \left| r(s_{i,u}^k - s_{j,v}^t) \right| &\leq r^*(S_k - T_t) \leq r^*(S_k - T_{k-1}) \leq K r^* \left(\frac{1}{2} (T_k - T_{k-1}) \right) \\ &\leq 2K (T_k - T_{k-1})^{-\lambda} \\ &\leq \min(1, \lambda)/16. \end{aligned}$$

Therefore,

$$\begin{aligned} M_{k,t} &\leq K (T_k - T_{k-1})^{-\lambda} \sum_{\substack{0 \leq i \leq [T_k - S_k] \\ 0 \leq j \leq [T_t - S_t]}} L_i^k L_j^t \exp \left(- \frac{\hat{r} \left((\hat{y}_i^k)^2 + (\hat{y}_j^t)^2 \right)}{2(1 + \frac{\lambda}{8})} \right) \\ &\leq K (T_k - T_{k-1})^{-\lambda} L_{[T_k - S_k]}^k L_{[T_t - S_t]}^t \sum_{\substack{0 \leq i \leq [T_k - S_k] \\ 0 \leq j \leq [T_t - S_t]}} (a_i^k)^{-\frac{1}{1+\frac{\lambda}{4}}} (a_j^t)^{-\frac{1}{1+\frac{\lambda}{4}}} \end{aligned}$$

$$\begin{aligned} &\leq K(T_k - T_{k-1})^{-\lambda} \log^{\frac{5}{\alpha}} T_k \log^{\frac{5}{\alpha}} T_t \cdot T_k^{\frac{\lambda}{4}} T_t^{\frac{\lambda}{4}} \\ &\leq K T_k^{-\frac{\lambda}{4}} \leq K \exp(-\lambda k^{(1+\varepsilon^2)/p}/4). \end{aligned}$$

Hence we have,

$$\sum_{0 \leq t < k < \infty} M_{k,t} < \infty. \quad (22)$$

Now (19) follows from (21), (22) and (20) and the general form of the Borel–Cantelli lemma. \square

Step 3. If $p \in (0, 1]$, then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.} \quad (23)$$

and

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \leq -(1 - \varepsilon) \quad \text{a.s.} \quad (24)$$

Proof Put

$$T_k = \exp(k^{1/p}), \quad S_k = T_k \exp\left(-(1 + 2\varepsilon)^2 h_p(T_k)\right).$$

Proceeding the same as in the proof of (17), one can obtain that

$$\liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.}$$

On the other hand it is clear that

$$\liminf_{t \rightarrow \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \geq \liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \quad \text{a.s.}$$

since

$$\lim_{k \rightarrow \infty} \frac{\log(T_k/T_{k+1})}{h_p(T_k)/T_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{h_p(T_k)}{T_k} \frac{T_{k+1}}{h_p(T_{k+1})} = 1.$$

This proves (23).

Let

$$T_k = \exp\left(k^{(1+\varepsilon^2)/p}\right), \quad S_k = T_k \exp\left(-(1 - \varepsilon)h_p(T_k)\right).$$

Noting that

$$\frac{S_{k+1} - T_k}{S_{k+1}} \sim 1 \quad \text{as } k \rightarrow \infty,$$

along the same lines as in the proof of (18), we also have

$$\liminf_{k \rightarrow \infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \leq -(1 - \varepsilon) \quad \text{a.s.},$$

which proves (24). \square

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