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Functional Convergence of Linear Processes with Heavy-Tailed Innovations

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Abstract We study convergence in law of partial sums of linear processes with heavy-tailed innovations. In the case of summable coefficients, necessary and sufficient conditions for the finite dimensional convergence to an α-stable Lévy Motion are given. The conditions lead to new, tractable sufficient conditions in the case $\alpha \leq 1$. In the functional setting, we complement the existing results on M_1-convergence, obtained for linear processes with nonnegative coefficients by Avram and Taqqu (Ann Probab 20:483–503, 1992) and improved by Louhichi and Rio (Electr J Probab 16(89), 2011), by proving that in the general setting partial sums of linear processes are convergent on the Skorokhod space equipped with the S topology, introduced by Jakubowski (Electr J Probab 2(4), 1997).

Keywords Limit theorems · Functional convergence · Stable processes · Linear processes

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1 Introduction and Announcement of Results

Let $\{Y_j\}_{j\in\mathbb{Z}}$ be a sequence of independent and identically distributed random variables. By a *linear process built on innovations* $\{Y_j\}$, we mean a stochastic process

$$X_i = \sum_{j\in\mathbb{Z}} c_j Y_{i-j}, \quad i \in \mathbb{Z}, \tag{1}$$

where the constants $\{c_j\}_{j\in\mathbb{Z}}$ are such that the above series is $\mathbb{P}$-a.s. convergent. Clearly, in non-trivial cases, such a process is dependent and stationary, and due to the simple linear structure, many of its distributional characteristics can be easily computed (provided they exist). This refers not only to the expectation or the covariances, but also to more involved quantities, like constants for regularly varying tails (see e.g., [21] for discussion) or mixing coefficients (see e.g., [10] for discussion).

There exists a huge literature devoted to applications of linear processes in statistical analysis and modeling of time series. We refer to the popular textbook [6] as an excellent introduction to the topic.

Here, we would like to stress only two particular features of linear processes.

First, linear processes provide a natural illustration for phenomena of *local* (or *weak*) dependence and *long-range* dependence. The most striking results go back to Davydov [9], who obtained a rescaled fractional Brownian motion as a functional weak limit for suitable normalized partial sums of $\{X_i\}$'s.

Another important property of linear processes is the propagation of big values. Suppose that *some* random variable Y_{j_0} takes a big value, then this big value is propagated along the sequence X_i (everywhere, where Y_{j_0} is taken with a big coefficient c_{i-j_0}). Thus, linear processes form the simplest model for phenomena of clustering of big values, what is important in models of insurance (see e.g., [21]).

In the present paper, we shall deal with heavy-tailed innovations. More precisely, we shall assume that the law of Y_i belongs to the domain of strict attraction of a non-degenerate strictly α-stable law μ_α, i.e.,

$$Z_n = \frac{1}{a_n}\sum_{i=1}^{n} Y_i \underset{\mathcal{D}}{\longrightarrow} Z, \tag{2}$$

where $Z \sim \mu_\alpha$.

Let us observe that by the Skorokhod theorem [25], we also have

$$Z_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} Y_i \underset{\mathcal{D}}{\longrightarrow} Z(t), \tag{3}$$

where $\{Z(t)\}$ is the stable Lévy process with $Z(1) \sim \mu_\alpha$, and the convergence holds on the Skorokhod space $\mathbb{D}([0, 1])$, equipped with the Skorokhod J_1 topology.

Recall, that if the variance of Z is *infinite*, then (2) implies the existence of $\alpha \in (0, 2)$ such that

$$\mathbb{P}(|Y_j| > x) = x^{-\alpha} h(x), \quad x > 0, \tag{4}$$

where h is a function that varies slowly at $x = +\infty$, and also

$$\lim_{x\to\infty} \frac{\mathbb{P}(Y_j > x)}{\mathbb{P}(|Y_j| > x)} = p \quad \text{and} \quad \lim_{x\to\infty} \frac{\mathbb{P}(Y_j < -x)}{\mathbb{P}(|Y_j| > x)} = q, \quad p + q = 1. \tag{5}$$

The norming constants a_n in (3) must satisfy

$$n\mathbb{P}(|Y_j| > a_n) = \frac{nh(a_n)}{a_n^{\alpha}} \to C > 0, \tag{6}$$

hence are necessarily of the form $a_n = n^{1/\alpha} g(n^{1/\alpha})$, where the slowly varying function $g(x)$ is the de Bruijn conjugate of $(C/h(x))^{1/\alpha}$ (see [5]). Moreover, if $\alpha > 1$, then $\mathbb{E}Y_j = 0$, and if $\alpha = 1$, then $p = q$ in (5).

Conversely, conditions (4), (5) and

$$\mathbb{E}[Y_j] = 0, \text{ if } \alpha > 1, \tag{7}$$

$$\{Y_j\} \text{ are symmetric, if } \alpha = 1, \tag{8}$$

imply (3).

If a_n is chosen to satisfy (6) with $C = 1$, then μ_α is given by the characteristic function

$$\hat{\mu}(\theta) = \begin{cases} \exp\left(\int_{\mathbb{R}^1} (e^{i\theta x} - 1) f_{\alpha,p,q}(x)\,\mathrm{d}x\right) & \text{if } 0 < \alpha < 1, \\ \exp\left(\int_{\mathbb{R}^1} (e^{i\theta x} - 1) f_{1,1/2,1/2}(x)\,\mathrm{d}x\right) & \text{if } \alpha = 1, \\ \exp\left(\int_{\mathbb{R}^1} (e^{i\theta x} - 1 - i\theta x) f_{\alpha,p,q}(x)\,\mathrm{d}x\right) & \text{if } 1 < \alpha < 2, \end{cases} \tag{9}$$

where

$$f_{\alpha,p,q}(x) = (p\,\mathbb{I}(x > 0) + q\,\mathbb{I}(x < 0))\,\alpha|x|^{-(1+\alpha)}.$$

We refer to [12] or any of contemporary monographs on limit theorems for the above basic information.

Suppose that the tails of $|Y_j|$ are regularly varying, i.e., (4) holds for some $\alpha \in (0, 2)$, and the (usual) regularity conditions (7) and (8) are satisfied. It is an observation due to Astrauskas [1] (in fact: a direct consequence of the Kolmogorov Three Series Theorem—see Proposition 5.4 below) that the series (1) defining the linear process X_i is $\mathbb{P}$-a.s. convergent if, and only if,

$$\sum_{j\in\mathbb{Z}} |c_j|^{\alpha} h(|c_j|^{-1}) < +\infty. \tag{10}$$

Given the above series is convergent we can define

$$S_n(t) = \frac{1}{b_n} \sum_{i=1}^{[nt]} X_i, \quad t \geq 0, \tag{11}$$

and it is natural to ask for convergence of S_n's, when b_n is suitably chosen. Astrauskas [1] and Kasahara & Maejima [16] showed that *fractional stable Lévy Motions* can appear in the limit of $S_n(t)$'s, and that some of the limiting processes can have *regular* or even *continuous* trajectories, while trajectories of other can be *unbounded on every interval*.

In the present paper, we consider the important case of summable coefficients:

$$\sum_{j \in \mathbb{Z}} |c_j| < +\infty. \tag{12}$$

In Sect. 2, we give necessary and sufficient conditions for the finite dimensional convergence

$$S_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i \underset{f.d.d.}{\longrightarrow} A \cdot Z(t), \tag{13}$$

where the constants a_n are the same as in (2), $A = \sum_{j \in \mathbb{Z}} c_j$ and $\{Z(t)\}$ is an α-stable Lévy Motion such that $Z(1) \sim Z$. The obtained conditions lead to tractable sufficient conditions, which in case $\alpha < 1$ are new and essentially weaker than condition

$$\sum_{j \in \mathbb{Z}} |c_j|^\beta < +\infty, \quad \text{for some } 0 < \beta < \alpha,$$

considered in [1], [8] and [16]. See Sect. 4 for details. Notice that in the case $A = 0$, another normalization b_n is possible with a non-degenerate limit. We refer to [22] for comprehensive analysis of dependence structure of infinite variance processes.

Section 3 contains strengthening of (13) to a functional convergence in some suitable topology on the Skorokhod space $\mathbb{D}([0, 1])$. Since the paper [2], it is known that in non-trivial cases (when at least two coefficients are nonzero) the convergence in the Skorokhod J_1 topology cannot hold. In fact, none of Skorokhod's J_1, J_2, M_1 and M_2 topologies are applicable. This can be seen by analysis of the following simple example ([2], p. 488). Set $c_0 = 1$, $c_1 = -1$ and $c_i = 0$ if $j \neq 0, 1$. Then $X_i = Y_i - Y_{i-1}$ and (13) holds with $A = \sum_j c_j = 0$, i.e.,

$$S_n(t) \underset{\mathcal{P}}{\longrightarrow} 0, \quad t \geq 0.$$

But we see that

$$\sup_{t \in [0,1]} S_n(t) = \max_{k \leq n} (Y_k - Y_0) / a_n$$

converges in law to a Fréchet distribution. This means that *supremum* is not a continuous (or almost surely continuous) functional, what excludes convergence in Skorokhod's topologies in the *general* case.

For linear processes with *nonnegative* coefficients c_i, partial results were obtained by Avram and Taqqu [2], where convergence in the M_1 topology was considered. Recently, these results have been improved and developed in various directions in [20] and [3]. We use the linear structure of processes and the established convergence in the M_1 topology to show that in the general case, the finite dimensional convergence (13) can be strengthen to convergence in the so-called S topology, introduced in [13]. This is a sequential and non-metric, but fully operational topology, for which addition is *sequentially* continuous.

Section 5 is devoted to some consequences of results obtained in previous sections. We provide examples of functionals continuous in the S topology. In particular, we show that for every $\gamma > 0$

$$\frac{1}{na_n^\gamma}\sum_{k=1}^n\left(\sum_{i=1}^k\left(\sum_j c_{i-j}Y_j\right) - AY_i\right)^\gamma \xrightarrow[\mathcal{P}]{} 0.$$

We also discuss possible extensions of the theory to linear sequences built on *dependent* summands.

The "Appendix" contains technical results of independent interest.

Conventions and notations. Throughout the paper, in order to avoid permanent repetition of standard assumptions and conditions, we adopt the following conventions. We will say that $\{Y_j\}$'s satisfy *the usual conditions* if they are *independent identically distributed* and (4), (5), (7) and (8) hold. When we write X_i, it is always the linear process given by (1) and is well-defined, i.e., satisfies (10). Similarly, the norming constants $\{a_n\}$ are defined by (6) and the normalized partial sums $S_n(t)$ and $Z_n(t)$ are given by (11) with $b_n = a_n$ and (3), respectively, where Z is the limit in (2) and $Z(t)$ is the stable Lévy Motion such that $Z(1) \sim Z$.

2 Convergence of Finite Dimensional Distributions for Summable Coefficients

We begin with stating the main result of this section followed by its important consequence.

Theorem 2.1 *Let $\{Y_j\}$ be an i.i.d. sequence satisfying the usual conditions. Suppose that*

$$\sum_j |c_j| < +\infty.$$

Then

$$S_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i \xrightarrow[f.d.d.]{} A\cdot Z(t), \quad \textit{where } A = \sum_j c_j,$$

if, and only if,

$$\sum_{j=-\infty}^{0} \frac{\left|d_{n,j}\right|^{\alpha}}{a_n^{\alpha}} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right) \to 0, \quad \textit{as } n \to \infty,$$
$$\sum_{j=n+1}^{\infty} \frac{\left|d_{n,j}\right|^{\alpha}}{a_n^{\alpha}} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right) \to 0, \quad \textit{as } n \to \infty, \tag{14}$$

where

$$d_{n,j} = \sum_{k=1-j}^{n-j} c_k, \quad n \in \mathbb{N}, j \in \mathbb{Z}.$$

Corollary 2.2 *Under the assumptions of Theorem 2.1, define*

$$U_i = \sum_j |c_{i-j}| Y_j, \quad X_i^+ = \sum_j c_{i-j}^+ Y_j, \quad X_i^- = \sum_j c_{i-j}^- Y_j, \tag{15}$$

where $c^+ = c \vee 0,\ c^- = (-c) \vee 0,\ c \in \mathbb{R}^1$, *and set*

$$T_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} U_i, \quad T_n^+(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i^+, \quad T_n^-(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i^-. \tag{16}$$

Then

$$T_n(t) \underset{f.d.d.}{\longrightarrow} A_{|\cdot|} \cdot Z(t), \quad \textit{where } A_{|\cdot|} = \sum_j |c_j|,$$

implies

$$T_n^+(t) \underset{f.d.d.}{\longrightarrow} A_+ \cdot Z(t), \quad \textit{where } A_+ = \sum_j c_j^+,$$
$$T_n^-(t) \underset{f.d.d.}{\longrightarrow} A_- \cdot Z(t), \quad \textit{where } A_- = \sum_j c_j^-,$$
$$S_n(t) = T_n^+(t) - T_n^-(t) \underset{f.d.d.}{\longrightarrow} A \cdot Z(t), \quad \textit{where } A = \sum_j c_j.$$

Proof of Corollary 2.2 In view of Theorem 2.1, it is enough to notice that

$$\frac{\left|d_{n,j}\right|^{\alpha}}{a_n^{\alpha}} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right) = \mathbb{P}\left(\left|\sum_{k=1-j}^{n-j} c_k\right| \cdot |Y_j| > a_n\right) \leq \mathbb{P}\left(\left(\sum_{k=1-j}^{n-j} |c_k|\right) \cdot |Y_j| > a_n\right).$$

Proof of Theorem 2.1 Using Fubini's theorem, we obtain that

$$S_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} \sum_{j \in \mathbb{Z}} c_{i-j} Y_j = \sum_{j \in \mathbb{Z}} \frac{1}{a_n} \left(\sum_{k=1-j}^{[nt]-j} c_k\right) Y_j = \sum_{j \in \mathbb{Z}} \frac{1}{a_n} d_{[nt],j} Y_j. \tag{17}$$

Further, we may decompose

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \frac{1}{a_n} d_{[nt],j} Y_j = & \sum_{j=-\infty}^{0} \frac{1}{a_n} d_{[nt],j} Y_j \\
& + \sum_{j=1}^{[nt]} \frac{1}{a_n} d_{[nt],j} Y_j \\
& + \sum_{j=[nt]+1}^{\infty} \frac{1}{a_n} d_{[nt],j} Y_j \\
= & S_n^-(t) + S_n^0(t) + S_n^+(t). \qquad (18)
\end{aligned}
$$

Let us consider the partial sum process:

$$
Z_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} Y_i, \quad t \geq 0.
$$

First we will show

Lemma 2.3 *Under the assumptions of Theorem 2.1 we have for each $t > 0$*

$$
S_n^0(t) - A \cdot Z_n(t) \underset{\mathcal{P}}{\longrightarrow} 0. \qquad (19)
$$

In particular,

$$
S_n^0(t) \underset{\mathcal{D}}{\longrightarrow} A \cdot Z(t). \qquad (20)
$$

Proof of Lemma 2.3 Define

$$
V_n^0 = \sum_{j=1}^{[nt]} \frac{\left(A - d_{[nt],j}\right)}{a_n} Y_j = A \cdot Z_n(t) - S_n^0(t). \qquad (21)
$$

To prove that $V_n^0 \longrightarrow_{\mathcal{P}} 0$, we apply Proposition 5.5. We have to show that

$$
\begin{aligned}
& \sum_{j=1}^{[nt]} \frac{\left|A - d_{[nt],j}\right|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{\left|A - d_{[nt],j}\right|} \right) \\
& = \sum_{j=1}^{[nt]} \mathbb{P}\left(\left|A - d_{[nt],j}\right| \cdot |Y_j| > a_n \right) \to 0, \quad \text{as } n \to \infty.
\end{aligned} \qquad (22)
$$

Since $a_n \to \infty$ and $\left|A - d_{[nt],j}\right| \leq \sum_{k\in\mathbb{Z}} |c_k|$, we have

$$
\max_{1\leq j\leq [nt]} \mathbb{P}\left(\left|A - d_{[nt],j}\right| \cdot |Y_j| > a_n \right) \to 0. \qquad (23)
$$

We need a simple lemma.

Lemma 2.4 *Let* $\{a_{n,j}\,;\ 1 \le j \le n,\ n \in \mathbb{N}\}$ *be an array of numbers such that*

$$\max_{1\le j\le n} |a_{n,j}| \to 0, \quad \textit{as } n \to \infty.$$

Then there exists a sequence $j_n \to \infty$, $j_n = o(n)$, *such that*

$$\sum_{j=1}^{j_n} |a_{n,j}| \to 0.$$

Proof of Lemma 2.4 For each $m \in \mathbb{N}$ there exists $N_m > \max\{N_{m-1}, m^2\}$ such that for $n \ge N_m$

$$\sum_{j=1}^{m} |a_{n,j}| < \frac{1}{m}.$$

Set $j_n = m$, if $N_m \le n < N_{m+1}$. By the very definition, if $N_m \le n < N_{m+1}$ then

$$\sum_{j=1}^{j_n} |a_{n,j}| < \frac{1}{m} \qquad \text{and} \qquad \frac{j_n}{n} \le \frac{j_n}{N_m} = \frac{m}{N_m} \le \frac{m}{m^2} = \frac{1}{m}.$$

By the above lemma and (23), we can find a sequence $j_n \to \infty$, $j_n = o(n)$, increasing so slowly that still

$$\sum_{j=1}^{j_n} \mathbb{P}\left(\left|A - d_{[nt],j}\right| \cdot |Y_j| > a_n\right) + \sum_{j=[nt]-j_n+1}^{[nt]} \mathbb{P}\left(\left|A - d_{[nt],j}\right| \cdot |Y_j| > a_n\right) \to 0.$$

For the remaining part we have

$$\max_{j_n<j\le[nt]-j_n} \left|A - d_{[nt],j}\right| = \max_{j_n<j\le[nt]-j_n} \left|A - \sum_{k=1-j}^{[nt]-j} c_k\right| = \delta_n \to 0,$$

hence for $\delta \ge \delta_n$

$$\begin{aligned}\sum_{j=j_n+1}^{[nt]-j_n} \mathbb{P}\left(\left|A - d_{[nt],j}\right| \cdot |Y_j| > a_n\right) &\le \sum_{j=j_n+1}^{[nt]-j_n} \mathbb{P}\left(|\delta_n||Y_j| > a_n\right)\\ &\le \sum_{j=1}^{[nt]} \mathbb{P}\left(|\delta_n||Y_j| > a_n\right)\end{aligned}$$

$$\begin{aligned} &\leq [nt]\frac{\delta^\alpha}{a_n^\alpha} h(a_n/\delta) \\ &= [nt]\delta^\alpha \frac{h(a_n)}{a_n^\alpha}\frac{h(a_n/\delta)}{h(a_n)}. \end{aligned}$$

Since $na_n^{-\alpha}h(a_n) = n\mathbb{P}(|Y| > a_n) \to 1$ and h varies slowly we have

$$[nt]\delta^\alpha \frac{h(a_n)}{a_n^\alpha}\frac{h(a_n/\delta)}{h(a_n)} \sim [nt]\delta^\alpha \frac{1}{n} \to t\delta^\alpha, \text{ as } n \to \infty.$$

But $\delta > 0$ is arbitrary, hence we have proved (22) and

$$V_n^0 = A \cdot Z_n(t) - S_n^0(t) \underset{\mathcal{P}}{\longrightarrow} 0.$$

Since

$$A \cdot Z_n(t) \underset{\mathcal{D}}{\longrightarrow} A \cdot Z(t),$$

Lemma 2.3 follows.

In the next step, we shall prove

Lemma 2.5 *Under the assumptions of Theorem 2.1, the following items (i)–(iii) are equivalent.*

(i)

$$S_n(1) \underset{\mathcal{D}}{\longrightarrow} A \cdot Z(1), \tag{24}$$

(ii)

$$S_n^-(1) + S_n^+(1) \underset{\mathcal{P}}{\longrightarrow} 0. \tag{25}$$

(iii) For every $t \in [0, 1]$

$$S_n(t) - A \cdot Z_n(t) \underset{\mathcal{P}}{\longrightarrow} 0. \tag{26}$$

Proof of Lemma 2.5 By Lemma 2.3 we know that $S_n^0(1) - A \cdot Z_n(1) \longrightarrow_{\mathcal{P}} 0$ and $S_n^0(1) \longrightarrow_{\mathcal{D}} A \cdot Z(1)$. Since $S_n(1) = S_n^-(1) + S_n^0(1) + S_n^+(1)$, (26) implies (25) and the latter implies (24).

So let us assume (24). By regular variation of a_n, we have for each $t \in (0, 1]$

$$S_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i = \frac{a_{[nt]}}{a_n}\frac{1}{a_{[nt]}}\sum_{i=1}^{[nt]} X_i \underset{\mathcal{D}}{\longrightarrow} t^{1/\alpha} A \cdot Z(1) \sim A \cdot Z(t).$$

It follows that

$$\mathbb{E}\big[e^{i\theta S_n(t)}\big] = \mathbb{E}\left[e^{i\theta S_n^0(t)}\right]\mathbb{E}\left[e^{i\theta\left(S_n^-(t)+S_n^+(t)\right)}\right] \to \mathbb{E}\left[e^{i\theta A\cdot Z(t)}\right], \quad \theta \in \mathbb{R}^1.$$

Since also

$$\mathbb{E}\left[e^{i\theta S_n^0(t)}\right] \to \mathbb{E}\left[e^{i\theta A\cdot Z(t)}\right], \quad \theta \in \mathbb{R}^1,$$

and $\mathbb{E}\left[e^{i\theta A\cdot Z(t)}\right] \neq 0,\ \theta \in \mathbb{R}^1$ (for $Z(t)$ has infinitely divisible law), we conclude that

$$\mathbb{E}\left[e^{i\theta(S_n^-(t)+S_n^+(t))}\right] \to 1, \quad \theta \in \mathbb{R}^1.$$

Thus $S_n^-(t) + S_n^+(t) \longrightarrow_{\mathcal{P}} 0$ and by Lemma 2.3 also $S_n^0(t) - A\cdot Z(t) \longrightarrow_{\mathcal{P}} 0$. Hence (26) follows.

Let us observe that by Proposition 5.5 (25) holds if, and only if,

$$\sum_{j=-\infty}^{0} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) + \sum_{j=n+1}^{\infty} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty, \tag{27}$$

i.e., relation (14) holds. Therefore the Proof of Theorem 2.1 will be complete, if we can show that convergence of *one-dimensional* distributions implies the finite dimensional convergence. But this is obvious in view of (26):

$$(S_n(t_1), S_n(t_2), \ldots, S_n(t_m)) - A\cdot(Z_n(t_1), Z_n(t_2), \ldots, Z_n(t_m)) \underset{\mathcal{P}}{\longrightarrow} 0,$$

and the finite dimensional distributions of stochastic processes $A\cdot Z_n(t)$ are convergent to those of $A\cdot Z(t)$.

Remark 2.6 Observe that for one-sided moving averages, the two conditions in (14) reduce to one (the expression in the other equals 0). This is the reason we use in Theorem 2.1 two conditions replacing the single statement (27).

Remark 2.7 In the Proof of Proposition 5.5, we used the Three Series Theorem with the level of truncation 1. It is well-known that any $r \in (0, +\infty)$ can be chosen as the truncation level. Hence, conditions (14) admit an equivalent reformulation in the "r-form"

$$\sum_{j=-\infty}^{0} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{r\cdot a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty.$$

$$\sum_{j=n+1}^{\infty} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{r\cdot a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty.$$

3 Functional Convergence

3.1 Convergence in the M_1 Topology

As outlined in Introduction (see also Sect. 5.2 below), the convergence of finite dimensional distributions of linear processes built on heavy-tailed innovations cannot be, in general, strengthened to functional convergence in any of Skorokhod's topologies J_1, J_2, M_1, M_2.

The general linear process $\{X_i\}$ can be, however, represented as a difference of linear processes with nonnegative coefficients. Let us recall the notation introduced in Corollary 2.2:

$$X_i^+ = \sum_j c_{i-j}^+ Y_j, \qquad T_n^+(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i^+,$$
$$X_i^- = \sum_j c_{i-j}^- Y_j, \qquad T_n^-(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i^-.$$

Notice, that in general $X_i^{\pm}(\omega)$ *is not* equal to $(X_i(\omega))^{\pm}$ and that we have

$$S_n(t) = T_n^+(t) - T_n^-(t). \tag{28}$$

The point is that both $T_n^+(t)$ and $T_n^-(t)$ are partial sums of *associated* sequences in the sense of [11] (see e.g., [7] for the contemporary theory) and thus exhibit much more regularity.

Theorem 1 of Louhichi and Rio [20] can be specified to the case of linear processes considered in our paper in the following way.

Proposition 3.1 *Let the innovation sequence* $\{Y_j\}$ *satisfies the usual conditions. Let*

$$c_j \geq 0,\ j \in \mathbb{Z},\ \textit{and} \sum_j c_j < +\infty. \tag{29}$$

If the linear process $\{X_i\}$ *is well-defined and*

$$S_n(t) \underset{f.d.d.}{\longrightarrow} A \cdot Z(t),$$

then also functionally

$$S_n \underset{\mathcal{D}}{\longrightarrow} A \cdot Z$$

on the Skorokhod space $\mathbb{D}([0, 1])$ *equipped with the* M_1 *topology.*

Remark 3.2 The first result of this type was obtained by Avram and Taqqu [2]. They required however more regularity on coefficients (e.g., monotonicity of $\{c_j\}_{j\geq 1}$ and $\{c_{-j}\}_{j\geq 1}$).

3.2 M_1-Convergence Implies S-Convergence

Let us turn to linear processes with coefficients of arbitrary sign. Given decomposition (28) and Proposition 3.1, the strategy is now clear: Choose any *linear* topology τ on $\mathbb{D}([0,1])$ which is *coarser* than M_1, then

$$S_n(t) \underset{f.d.d.}{\longrightarrow} A \cdot Z(t),$$

should imply

$$S_n \underset{\mathcal{D}}{\longrightarrow} A \cdot Z$$

on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the topology τ. Since convergence of càdlàg functions in the M_1 topology is bounded and implies pointwise convergence outside of a countable set, there are plenty of such topologies. For instance, any space of the form $L^p([0,1],\mu)$, where $p \in [0,\infty)$ and μ is an *atomless* finite measure on $[0,1]$, is suitable. The point is to choose the *finest* among linear topologies with required properties, for we want to have the maximal family of continuous functionals on $\mathbb{D}([0,1])$.

Although we are not able to identify such an "ideal" topology, we believe that this distinguished position belongs to the S topology, introduced in [13]. This is a non-metric sequential topology, with sequentially continuous addition, which is stronger than any of mentioned above $L^p(\mu)$ spaces and is functional in the sense it has the following classic property (see Theorem 3.5 of [13]).

Proposition 3.3 *Let $\mathbb{Q} \subset [0,1]$ be dense, $1 \in \mathbb{Q}$. Suppose that for each finite subset $\mathbb{Q}_0 = \{q_1 < q_2 < \cdots < q_m\} \subset \mathbb{Q}$ we have as $n \to \infty$*

$$(X_n(q_1), X_n(q_2), \ldots, X_n(q_m)) \underset{\mathcal{D}}{\longrightarrow} (X_0(q_1), X_0(q_2), \ldots, X_0(q_m)),$$

where X_0 is a stochastic process with trajectories in $\mathbb{D}[0,1])$. If $\{X_n\}$ is uniformly S-tight, then

$$X_n \underset{\mathcal{D}}{\longrightarrow} X_0,$$

on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the S topology.

For readers familiar with the limit theory for stochastic processes, the above property may seem obvious. But it is trivial only for processes with continuous trajectories. It is not trivial even in the case of the Skorokhod J_1 topology, since the point evaluations

$$\pi_t : \mathbb{D}([0,1]) \to \mathbb{R}^1, \quad \pi_t(x) = x(t),$$

can be J_1-discontinuous at some $x \in \mathbb{D}([0,1])$ (see [26] for the result corresponding to Proposition 3.3). In the S topology, the point evaluations are *nowhere* continuous (see [13], p. 11). Nevertheless, Proposition 3.3 holds for the S topology, while it *does not hold* for the linear metric spaces $L^p(\mu)$ considered above. It follows that the S topology is suitable for the needs of limit theory for stochastic processes. It admits even such efficient tools like the a.s Skorokhod representation for subsequences [14]. On the other hand, since $\mathbb{D}([0,1])$ equipped with S is non-metric and sequential, many of apparently standard reasonings require special tools and careful analysis. This will be seen below.

Before we define the S topology, we need some notation. Let $\mathbb{V}([0,1]) \subset \mathbb{D}([0,1])$ be the space of (regularized) functions of finite variation on $[0,1]$, equipped with the norm of total variation $\|v\| = \|v\|(1)$, where

$$\|v\|(t) = \sup\left\{|v(0)| + \sum_{i=1}^{m} |v(t_i) - v(t_{i-1})|\right\},$$

and the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_m = t$. Since $\mathbb{V}([0,1])$ can be identified with a dual of $(\mathbb{C}([0,1]), \|\cdot\|_\infty)$, we have on it the weak-$*$ topology. We shall write $v_n \Rightarrow v_0$ if for every $f \in \mathbb{C}([0,1])$

$$\int_{[0,1]} f(t)\mathrm{d}v_n(t) \to \int_{[0,1]} f(t)\mathrm{d}v_0(t).$$

Definition 3.4 (*S-convergence and the S topology*) We shall say that x_n S-converges to x_0 (in short $x_n \to_S x_0$) if for every $\varepsilon > 0$ one can find elements $v_{n,\varepsilon} \in \mathbb{V}([0,1])$, $n = 0, 1, 2, \ldots$ which are ε-uniformly close to x_n's and weakly-$*$ convergent:

$$\|x_n - v_{n,\varepsilon}\|_\infty \le \varepsilon, \quad n = 0, 1, 2, \ldots, \tag{30}$$

$$v_{n,\varepsilon} \Rightarrow v_{0,\varepsilon}, \quad \text{as } n \to \infty. \tag{31}$$

The S topology is the sequential topology determined by the S-convergence.

Remark 3.5 This definition was given in [13], and we refer to this paper for detailed derivation of basic properties of S-convergence and construction of the S topology, as well as for instruction how to effectively operate with S. Here, we shall stress only that the S topology emerges naturally in the context of the following *criteria of compactness*, which will be used in the sequel.

Proposition 3.6 (2.7 in [13]) *For $\eta > 0$, let $N_\eta(x)$ be the* number of η-oscillations *of the function $x \in \mathbb{D}([0,1])$, i.e., the largest integer $N \ge 1$, for which there exist some points*

$$0 \le t_1 < t_2 \le t_3 < t_4 \le \cdots \le t_{2N-1} < t_{2N} \le 1,$$

such that

$$|x(t_{2k}) - x(t_{2k-1})| > \eta \quad \textit{for all } k = 1, \ldots, N.$$

Let $\mathcal{K} \subset \mathbb{D}$. Assume that

$$\sup_{x\in\mathcal{K}} \|x\|_\infty < +\infty, \tag{32}$$

$$\sup_{x\in\mathcal{K}} N_\eta(x) < +\infty, \text{ for each } \eta > 0. \tag{33}$$

Then from any sequence $\{x_n\} \subset \mathcal{K}$ one can extract a subsequence $\{x_{n_k}\}$ and find $x_0 \in \mathbb{D}([0,1])$ such that $x_{n_k} \longrightarrow_S x_0$.

Conversely, if $\mathcal{K} \subset \mathbb{D}([0,1])$ is relatively compact with respect to $\longrightarrow_S$, then it satisfies both (32) and (33).

Corollary 3.7 (2.14 in [13]) *Let $\mathbb{Q} \subset [0,1]$, $1 \in \mathbb{Q}$, be dense. Suppose that $\{x_n\} \subset \mathbb{D}([0,1])$ is relatively S-compact and as $n \to \infty$*

$$x_n(q) \to x_0(q), \quad q \in \mathbb{Q}.$$

Then $x_n \to x_0$ in S.

Remark 3.8 The S topology is *sequential*, i.e., it is generated by the convergence $\longrightarrow_S$. By the Kantorovich–Kisyński recipe [17] $x_n \to x_0$ in S topology if, and only if, in each subsequence $\{x_{n_k}\}$ one can find a further subsequence $x_{n_{k_l}} \longrightarrow_S x_0$. This is the same story as with a.s. convergence and convergence in probability of random variables.

According to our strategy, we are going to prove that Skorokhod's M_1-topology is stronger than the S topology or, equivalently, that $x_n \longrightarrow_{M_1} x_0$ implies $x_n \longrightarrow_S x_0$. We refer the reader to the original Skorohod's article [24] for the definition of the M_1 topology, as well as to Chapter 12 of [28] for a comprehensive account of properties of this topology.

The M_1 convergence can be described using a suitable modulus of continuity. We define for $x \in \mathbb{D}([0,1])$ and $\delta > 0$

$$w^{M_1}(x,\delta) := \sup_{0\vee(t_2-\delta)\le t_1<t_2<t_3\le 1\wedge(t_2+\delta)} H\left(x(t_1), x(t_2), x(t_3)\right), \tag{34}$$

where $H(a,b,c)$ is the distance between b and the interval with endpoints a and c:

$$H(a,b,c) = (a\wedge c - a\wedge c\wedge b) \vee (a\vee c\vee b - a\vee c).$$

Proposition 3.9 (2.4.1 of [24]) *Let $(x_n)_{n\ge1}$ and x_0 be arbitrary elements in $\mathbb{D}([0,1])$. Then*

$$x_n \underset{M_1}{\longrightarrow} x_0$$

if, and only if, for some dense subset $\mathbb{Q} \subset [0,1]$ containing 0 and 1,

$$x_n(t) \to x(t), \quad t \in \mathbb{Q}, \tag{35}$$

and

$$\lim_{\delta\to 0}\limsup_{n\to\infty} w^{M_1}(x_n,\delta) = 0. \tag{36}$$

In particular, if $x_n \longrightarrow_{M_1} x_0$*, then*

$$x_n(t) \to x_0(t)$$

for $t = 1$ *and at every point of continuity of* x_0*.*

Lemma 3.10 *For any* $a, b, c, d \in \mathbb{R}^1$

$$|a-b| \le |c-d| + H(c,a,d) + H(c,b,d).$$

Proof If $c \le a \le b \le d$, then $b-a \le d-c = d-c+H(c,a,d)+H(c,b,d)$. If $a \le c \le b \le d$ then $b-a = b-c+c-a \le d-c+H(c,a,d) = d-c+H(c,a,d)+H(c,b,d)$. If $a \le c \le d \le b$ then $b-a = b-d+d-c+c-a = H(c,b,d)+d-c+H(c,a,b)$. If $a \le b \le c \le d$, then $b-a \le |b-c|+|c-a| = H(c,b,d)+H(c,a,d) \le H(c,b,d)+H(c,a,d)+d-c$. The other cases can be reduced to the considered above.

Corollary 3.11 *Let* $x \in \mathbb{D}([0,1])$*. For any* $0 \le s \le u < v \le t \le 1$*,*

$$|x(u)-x(v)| \le |x(s)-x(t)| + H(x(s),x(u),x(t)) + H(x(s),x(v),x(t)).$$

Lemma 3.12 *Let* $x \in \mathbb{D}([0,1])$*. For* $0 \le s < t \le 1$*, define*

$$\beta = \sup_{s\le u<v<w\le t} H(x(u),x(v),x(w)).$$

If $\eta > 2\beta$ *then*

$$N_\eta(x;[s,t]) \le \frac{|x(t)-x(s)|+\beta}{\eta-\beta},$$

where $N_\eta(x;[s,t])$ *denotes the number of* η*-oscillations of* x *in the interval* $[s,t]$*.*

Proof Let $s \le t_1 < t_2 \le t_3 < t_4 \le \cdots \le t_{2N-1} < t_{2N} \le t$ be such that

$$|x(t_{2k}) - x(t_{2k-1})| > \eta \quad \text{for all } \ k = 1,\ldots,N.$$

Assume first that $x(t_2) - x(t_1) > \eta$. We claim that

$$x(t_3) \ge x(t_2) - \beta \quad \text{and} \qquad x(t_4) - x(t_3) > \eta.$$

To see this, suppose that $x(t_3) < x(t_2) - \beta$. Then the distance between $x(t_2)$ and the interval with endpoints $x(t_1)$ and $x(t_3)$ is greater than β, which is a contradiction.

Hence $x(t_3) \geq x(t_2) - \beta$. On the other hand, if we assume that $x(t_4) - x(t_3) < -\eta$, we obtain that

$$x(t_1) = x(t_1) - x(t_2) + x(t_2) - x(t_3) + x(t_3) < -\eta + \beta + x(t_3) < x(t_3) - \beta,$$

which means that the distance between $x(t_3)$ and the interval with endpoints $x(t_1)$ and $x(t_4)$ is greater than β, again a contradiction.

Repeating this argument, we infer that:

$$x(t_{2k}) - x(t_{2k-1}) > \eta, \quad \text{for all } k = 1, \ldots, N$$

and

$$x(t_{2k+1}) - x(t_{2k}) > -\beta \quad \text{for all } k = 1, \ldots, N-1.$$

Taking the sum of these inequalities, we conclude that:

$$x(t_{2N}) - x(t_1) > N\eta - (N-1)\beta = N(\eta - \beta) + \beta. \tag{37}$$

On the other hand, by Corollary 3.11, we have:

$$|x(t_{2N}) - x(t_1)| \leq |x(t) - x(s)| + 2\beta. \tag{38}$$

Combining (37) and (38), we obtain that

$$N \leq \frac{|x(t) - x(s)| + \beta}{\eta - \beta},$$

which is the desired upper bound.

Assuming that $x(t_2) - x(t_1) < -\eta$, we come in a similar way to the inequality

$$x(t_{2N}) - x(t_1) < -N\eta + (N-1)\beta = -N(\eta - \beta) - \beta$$

or

$$|x(t_{2N}) - x(t_1)| \; N(\eta - \beta) + \beta.$$

This again allows us to use Corollary 3.11 and gives the desired bound for N

The following result was stated without proof in [13]. A short proof can be given using Skorohod's criterion 2.2.11 (page 267 of [24]) for the M_1-convergence, expressed in terms of the number of upcrossings. This proof has a clear disadvantage: It refers to an equivalent definition of the M_1-convergence, but the equivalence of both definitions was not proved in Skorokhod's paper. In the present article, we give a complete proof.

Theorem 3.13 *The S topology is weaker than the M_1 topology (and hence, weaker than the J_1 topology). Consequently, a set $A \subset \mathbb{D}([0,1])$ which is relatively M_1-compact is also relatively S-compact.*

Proof Let $x_n \longrightarrow_{M_1} x_0$. By Proposition 3.9

$$x_n(t) \to x_0(t),$$

on the dense set of points of continuity of x_0 and for $t = 1$. Suppose, we know that $\mathcal{K} = \{x_n\}$ satisfies conditions (32) and (33). Then by Proposition 3.6 $\{x_n\}$ is relatively S-compact and by Corollary 3.7 $x_n \to x_0$ in S. Thus, it remains to check conditions

$$K_{\sup} = \sup_n \|x_n\|_\infty < +\infty, \tag{39}$$

$$K_\eta = \sup_n N_\eta(x_n) < \infty, \quad \eta > 0. \tag{40}$$

First suppose that $x_0(1-) = x_0(1)$. Then, $\mathbb{D}([0,1]) \ni x \mapsto \|x\|_\infty$ is M_1-continuous at x_0. Consequently, $x_n \longrightarrow_{M_1} x_0$ implies $\|x_n\|_\infty \to \|x_0\|_\infty$ and (39) follows.

If $x_0(1-) \neq x_0(1)$, we have to proceed a bit more carefully. Consider (36) and take $\delta > 0$ and n_0 such that $w(x_n, \delta) \leq 1$, $n \geq n_0$. Find $t_0 \in (1-\delta, 1)$ which is a point of continuity of x_0. Then,

$$\sup_{t\in[0,t_0]} |x_n(t)| \to \sup_{t\in[0,t_0]} |x_0(t)|,$$

hence $\sup_n \sup_{t\in[0,t_0]} |x_n(t)| < +\infty$. We also know that $x_n(t_0) \to x_0(t_0)$ and $x_n(1) \to x_0(1)$. Choose $n \in \mathbb{N}$ and $u \in (t_0, 1)$. By the very definition of the modulus H

$$\begin{aligned}|x_n(u)| &\leq |x_n(t_0)| + |x_n(1)| + H\left(x_n(t_0), x_n(u), x_n(1)\right)\\ &\leq \sup_n |x_n(t_0)| + \sup_n |x_n(1)| + 1, \quad n \geq n_0.\end{aligned}$$

It follows that also

$$\sup_n \sup_{t\in(t_0,1]} |x_n(t)| < +\infty,$$

and so (39) holds.

In order to prove (40), choose $\eta > 0$ and $0 < \varepsilon < \eta/2$. By Proposition 3.9, there exist some $\delta > 0$ and an integer $n_0 \geq 1$ such that $w^{M_1}(x_n, \delta) < \varepsilon$, $n \geq n_0$. Next, we find a partition $0 = t_0 < t_1 < \cdots < t_M = 1$ consisting of points of continuity of x_0 and such that

$$t_{j+1} - t_j < \delta, \quad j = 0, 1, \ldots, M-1.$$

Again by Proposition 3.9, there exists an integer $n_1 \geq n_0$ such that for any $n \geq n_1$

$$|x_n(t_j) - x(t_j)| < \varepsilon, \quad j = 0, 1, \ldots, M. \tag{41}$$

Fix an integer $n \geq n_1$. Suppose that $N_\eta(x_n) \geq N$, i.e., there exist some points

$$0 \leq s_1 < s_2 \leq s_3 < s_4 \leq \cdots \leq s_{2N-1} < s_{2N} \leq 1, \tag{42}$$

such that

$$|x_n(s_{2k}) - x_n(s_{2k-1})| > \eta, \quad \text{for all} \quad k = 1, 2, \ldots, N. \tag{43}$$

The Proof of (40) will be complete once we estimate the number N by a constant independent of n.

The η-oscillations of x_n determined by (42) can be divided into two (disjoint) groups. The first group (Group 1) contains the oscillations for which the corresponding interval $[s_{2k-1}, s_{2k})$ contains at least one point $t_{j'}$. Since the number of points t_j is M,

$$\text{the number of oscillations in Group 1 is at most } M. \tag{44}$$

In the second group (Group 2), we have those oscillations for which the corresponding interval $[s_{2k-1}, s_{2k})$ contains no point t_j, i.e.,

$$t_j \leq s_{2k-1} < s_{2k} \leq t_{j+1} \quad \text{for some } j = 0, 1, \ldots, M-1. \tag{45}$$

We now use Lemma 3.12 in each of intervals $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, m$. Note that

$$\beta_{n,j} := \sup_{t_j \leq u < v < w \leq t_{j+1}} H\left(x_n(u), x_n(v), x_n(w)\right) \leq w^{M_1}(x_n, \delta) < \varepsilon,$$

hence,

$$N_\eta(x_n, [t_j, t_{j+1}]) \leq \frac{|x_n(t_{j+1}) - x_n(t_j)| + \beta_{n,j}}{\eta - \beta_{n,j}} < \frac{2K_{\sup} + \varepsilon}{\eta - \varepsilon}.$$

Since there are M intervals of the form $[t_j, t_{j+1}]$, we conclude that

$$\text{the number of oscillations in Group 2 is at most } M \cdot \frac{2K_{\sup} + \varepsilon}{\eta - \varepsilon} \tag{46}$$

Summing (44) and (46), we obtain that

$$N \leq M\left(1 + \frac{2K_{\sup} + \varepsilon}{\eta - \varepsilon}\right) = M\frac{2K_{\sup} + \eta}{\eta - \varepsilon},$$

which does not depend on n. Theorem 3.13 follows.

For the sake of completeness, we provide also a typical example of a sequence $(x_n)_{n \geq 1}$ in $\mathbb{D}[0, 1]$ which is S-convergent, but does not converge in the M_1 topology.

Example 3.14 Let $x = 0$ and

$$x_n(t) = 1_{[1/2-1/n,1]}(t) - 1_{[1/2+1/n,1]}(t) = \begin{cases} 1 & \text{if } \frac{1}{2} - \frac{1}{n} \le t < \frac{1}{2} + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then $x_n \longrightarrow_S x$. To see this, we take $v_{n,\varepsilon} = x_n$. Then $v_{n,\varepsilon} \Rightarrow v_\varepsilon = 0$ since for any $f \in C[0,1]$,

$$\int_0^1 f(t)\mathrm{d}v_n(t) = f\left(\frac{1}{2} - \frac{1}{n}\right) - f\left(\frac{1}{2} + \frac{1}{n}\right) \to 0.$$

The fact that $(x_n)_{n\ge 1}$ cannot converge in M_1 follows by Proposition 3.9 since if $t_1 < \frac{1}{2} - \frac{1}{n} < t_2 < \frac{1}{2} + \frac{1}{n} < t_3$, then $H(x_n(t_1), x_n(t_2), x_n(t_3)) = 1$.

3.3 Convergence in Distribution in the S Topology

Now, we are ready to specify results on functional convergence of stochastic processes in the S topology, which are suitable for needs of linear processes. They follow directly from Proposition 3.6 and Proposition 3.3.

Proposition 3.15 (3.1 in [13]) *A family $\{X_\gamma\}_{\gamma\in\Gamma}$ of stochastic processes with trajectories in $\mathbb{D}([0,1])$ is uniformly S-tight if, and only if, the families of random variables $\{\|X_\gamma\|_\infty\}_{\gamma\in\Gamma}$ and $\{N_\eta(X_\gamma)\}_{\gamma\in\Gamma}$, $\eta > 0$, are uniformly tight.*

Proposition 3.16 *Let $\{X_n\}_{n\ge 0}$ and $\{Y_n\}_{n\ge 0}$ be two sequences of stochastic processes with trajectories in $\mathbb{D}([0,1])$ such that as $n \to \infty$*

$$\begin{aligned}&(X_n(q_1) + Y_n(q_1), X_n(q_2) + Y_n(q_2), \ldots, X_n(q_k) + Y_n(q_k)) \\ &\quad \underset{\mathcal{D}}{\longrightarrow} (X_0(q_1) + Y_0(q_1), X_0(q_2) + Y_0(q_2) \ldots, X_0(q_k) + Y_0(q_k)),\end{aligned}$$

for each subset $\mathbb{Q}_0 = \{0 \le q_1 < q_2 < \cdots < q_k\}$ of a dense set $\mathbb{Q} \subset [0,1]$, $1 \in \mathbb{Q}$.

If $\{X_n\}$ and $\{Y_n\}$ are uniformly S-tight, then

$$X_n + Y_n \underset{\mathcal{D}}{\longrightarrow} X_0 + Y_0$$

on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the S topology.

Proof of Proposition 3.16 According to Proposition 3.3, it is enough to establish the uniform S-tightness of $X_n + Y_n$. This follows immediately from Proposition 3.15 and from the inequalities $\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty$ and

$$N_\eta(x + y) \le N_{\eta/2}(x) + N_{\eta/2}(y),$$

valid for arbitrary functions $x, y \in \mathbb{D}[0,1]$ and $\eta > 0$.

Remark 3.17 In linear topological spaces, the algebraic sum $\mathcal{K}_1+\mathcal{K}_2 = \{x_1+x_2\,;\ x_1 \in \mathcal{K}_1, x_2 \in \mathcal{K}_2\}$ of *compact* sets $\mathcal{K}_1$ and $\mathcal{K}_2$ *is compact*. It follows directly from the continuity of the operation of addition and trivializes the proof of uniform tightness of sum of uniformly tight random elements. In $\mathbb{D}([0,1])$ equipped with S, we are, however, able to prove that the addition is only sequentially continuous, i.e., if $x_n \longrightarrow_S x_0$ and $y_n \longrightarrow_S y_0$, then $x_n + y_n \longrightarrow_S x_0 + y_0$. In general, it does not imply continuity (see [13], p. 18, for detailed discussion). Sequential continuity gives a weaker property: the sum $\mathcal{K}_1 + \mathcal{K}_2$ of relatively S-compact $\mathcal{K}_1$ and $\mathcal{K}_2$ is relatively S-compact. For the uniform tightness purposes, we also need that the S-closure of $\mathcal{K}_1 + \mathcal{K}_2$ is again relatively S-compact. This is guaranteed by the lower-semicontinuity in S of $\|\cdot\|_\infty$ and N_η (see [13], Corollary 2.10).

3.4 The Main Result

Theorem 3.18 *Let $\{Y_j\}$ be an i.i.d. sequence satisfying the usual conditions and $\sum_j |c_j| < +\infty$. Let $S_n(t)$ be defined by (11) and $T_n(t)$ by (16). Then*

$$T_n(t) \underset{f.d.d.}{\longrightarrow} A_{|\cdot|} \cdot Z(t), \quad \textit{where } A_{|\cdot|} = \sum_j |c_j|,$$

implies

$$S_n \underset{\mathcal{D}}{\longrightarrow} A \cdot Z, \quad \textit{where } A = \sum_j c_j,$$

on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the S topology.

Proof By Corollary 2.2

$$T_n^+(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i^+ \underset{f.d.d.}{\longrightarrow} A_+ \cdot Z(t), \quad T_n^-(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i^- \underset{f.d.d.}{\longrightarrow} A_- \cdot Z(t),$$

where $A_+ = \sum_{i\in\mathbb{Z}} c_i^+$ and $A_- = \sum_{i\in\mathbb{Z}} c_i^-$. It follows from Proposition 3.1 that $T_n^+ \longrightarrow_{\mathcal{D}} A_+ \cdot Z$ on $\mathbb{D}([0,1])$ equipped with the M_1 topology. A similar result holds for T_n^-. Since the law of every càdlàg process is M_1-tight, Le Cam's theorem [19] (see also Theorem 8 in Appendix III of [4]) guarantees that both sequences $\{T_n^+\}$ and $\{T_n^-\}$ are uniformly M_1-tight. By Theorem 3.13 we obtain the uniform S-tightness of both $\{T_n^+\}$ and $\{T_n^-\}$. Again by Corollary 2.2

$$S_n(t) = T_n^+(t) - T_n^-(t) \underset{f.d.d.}{\longrightarrow} A \cdot Z(t).$$

Now a direct application of Proposition 3.16 completes the proof of the theorem.

4 Discussion of Sufficient Conditions

Conditions (14) do not look tractable. In what follows, we shall provide three types of checkable sufficient conditions. In both cases, the following slight simplification (47) of (14) will be useful. As in Proof of Lemma 2.3, we can find a sequence $j_n \to \infty$, $j_n = o(n)$, such that

$$\sum_{j=-j_n+1}^{0} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty.$$
$$\sum_{j=n+1}^{n+j_n-1} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty.$$

Hence, it is enough to check

$$\begin{aligned} \sum_{j=-\infty}^{-j_n} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) &\to 0, \quad \text{as } n \to \infty. \\ \sum_{j=n+j_n}^{+\infty} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{47}$$

The advantage of this form of the conditions consists in the fact that

$$\begin{aligned} \sup_{j \le -j_n} |d_{n,j}| &\to 0, \quad \text{as } n \to \infty, \\ \sup_{j \ge n+j_n} |d_{n,j}| &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{48}$$

We will write $\longrightarrow_{\mathcal{D}(S)}$ when convergence in distribution with respect to the S topology takes place.

Corollary 4.1 *Under the assumptions of Theorem 2.1, if there exists* $0 < \beta < \alpha$, $\beta \le 1$ *such that*

$$\sum_{j \in \mathbb{Z}} |c_j|^\beta < +\infty, \tag{49}$$

then

$$S_n(t) \underset{\mathcal{D}(S)}{\longrightarrow} A \cdot Z(t).$$

Proof We have to check (47). By simple manipulations and taking into account that due to (6) $K = \sup_n n a_n^{-\alpha} h(a_n) < +\infty$ we obtain

$$\sum_{j=-\infty}^{-j_n} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right)$$

$$
\begin{aligned}
&= \frac{1}{n} \sum_{j=-\infty}^{-j_n} \Big| \sum_{k=1-j}^{n-j} c_k \Big|^{\beta} \frac{n h(a_n)}{a_n^{\alpha}} \big|d_{n,j}\big|^{\alpha-\beta} \frac{1}{h(a_n)} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right) \\
&\leq K \frac{1}{n} \sum_{j=-\infty}^{-j_n} \sum_{k=1-j}^{n-j} |c_k|^{\beta} \Psi_{\alpha-\beta}\left(a_n, \frac{a_n}{\left|d_{n,j}\right|}\right),
\end{aligned}
$$

where

$$
\Psi_{\alpha-\beta}(x, y) = \left(\frac{x}{y}\right)^{\alpha-\beta} \frac{h(y)}{h(x)}.
$$

Let

$$
h(x) = c(x) \exp\left(\int_a^x \frac{\epsilon(u)}{u} \, \mathrm{d}u\right),
$$

where $\lim_{x\to\infty} c(x) = c \in (0, \infty)$ and $\lim_{x\to\infty} \epsilon(x) = 0$, be the Karamata representation of the slowly varying function $h(x)$ (see e.g., Theorem 1.3.1 in [5]). Take $0 < \gamma < \min\{\alpha - \beta, c\}$ and let $L > a$ be such that for $x > L$

$$
\epsilon(x) \leq \gamma \text{ and } c - \gamma < c(x) < c + \gamma.
$$

Then, we have for $x \geq y \geq L$

$$
\frac{h(y)}{h(x)} = \frac{c(y)}{c(x)} \exp\left(\int_y^x \frac{\epsilon(u)}{u} \, \mathrm{d}u\right) \leq \frac{c+\gamma}{c-\gamma} \exp\left(\gamma \log\left(\frac{x}{y}\right)\right) = \frac{c+\gamma}{c-\gamma} \left(\frac{x}{y}\right)^{\gamma},
$$

and so

$$
\Psi_{\alpha-\beta}(x, y) \leq K \left(\frac{y}{x}\right)^{\alpha-\beta-\gamma}, \quad x \geq y \geq L.
$$

It follows from that fact and (48) that

$$
\sup_{j \leq -j_n} \Psi_{\alpha-\beta}\left(a_n, \frac{a_n}{\left|d_{n,j}\right|}\right) \to 0, \text{ as } n \to \infty.
$$

Hence it is sufficient to show that

$$
\sup_n \frac{1}{n} \sum_{j=-\infty}^{-j_n} \sum_{k=1-j}^{n-j} |c_k|^{\beta} < +\infty.
$$

In fact, more is true.

Lemma 4.2 *If* $\sum_{j=0}^{\infty} |b_j| < +\infty$*, then for each* $t > 0$

$$\frac{1}{n} \sum_{j=0}^{\infty} \sum_{k=1+j}^{n+j} b_k \to 0, \text{ as } n \to \infty.$$

Proof of Lemma 4.2 We have

$$\begin{aligned} \left| \frac{1}{n} \sum_{j=0}^{\infty} \sum_{k=1+j}^{n+j} b_k \right| &\leq \frac{1}{n} \sum_{j=0}^{\infty} \sum_{k=1+j}^{n+j} |b_k| \\ &= \frac{1}{n} \sum_{k=1}^{\infty} (k \wedge n) |b_k| \\ &= \left(\frac{1}{n} \sum_{k=1}^{n} k|b_k| + \sum_{k=n+1}^{\infty} |b_k| \right). \end{aligned}$$

The first sum in the last line converges to 0 by Kronecker's lemma. The second is the rest of a convergent series.

Returning to the Proof of Corollary 4.1, let us notice that convergence

$$\sum_{j=n+j_n}^{+\infty} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|} \right) \to 0, \quad \text{as } n \to \infty,$$

can be checked the same way.

Corollary 4.3 *Under the usual conditions, if* $\alpha \in (1, 2)$ *and* $\sum_{j \in \mathbb{Z}} |c_j| < +\infty$*, then*

$$S_n(t) \underset{\mathcal{D}(S)}{\longrightarrow} A \cdot Z(t).$$

Remark 4.4 Corollaries 4.1 and 4.3 were proved independently by Astrauskas [1] and Davis and Resnick [8]. Our approach follows direct manipulations of Astrauskas, while Davis and Resnick involved point process techniques.

Remark 4.5 For $\alpha \leq 1$ assumption (49) is unsatisfactory, for it excludes the case of strictly α-stable random variables $\{Y_j\}$ with $\sum_j |c_j|^\alpha < +\infty$, but $\sum_j |c_j|^\beta = +\infty$ for every $\beta < \alpha$. With our criterion given in Theorem 2.1 we can easily prove the needed result.

Corollary 4.6 *Suppose that* $\alpha \leq 1$, $\sum_{j \in \mathbb{Z}} |c_j|^\alpha < +\infty$*, the usual conditions hold and* h *is such that*

$$h(\lambda x)/h(x) \leq M, \quad \lambda \geq 1, x \geq x_0, \tag{50}$$

for some constants M, x_0*. If the linear process* $\{X_i\}$ *is well-defined, then*

$$S_n(t) \underset{\mathcal{D}(S)}{\longrightarrow} A \cdot Z(t).$$

Proof of Corollary 4.6 First notice that $\sum_j |c_j| < +\infty$ so that A is defined. Proceeding like in the Proof of Corollary 4.1, we obtain

$$\begin{aligned}
\sum_{j=-\infty}^{-j_n} \frac{\left|d_{n,j}\right|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right)
&= \frac{1}{n} \sum_{j=-\infty}^{-j_n} \left|\sum_{k=1-j}^{n-j} c_k\right|^\alpha \frac{nh(a_n)}{a_n^\alpha} \frac{1}{h(a_n)} h\left(\frac{a_n}{\left|d_{n,j}\right|}\right) \\
&\leq K \cdot M \frac{1}{n} \sum_{j=-\infty}^{-j_n} \sum_{k=1-j}^{n-j} |c_k|^\alpha \to 0,
\end{aligned}$$

where the convergence to 0 holds by Lemma 4.2.

Remark 4.7 As mentioned before, the above corollary covers the important case when $h(x) \to C > 0$, as $x \to \infty$, i.e., when the law of Y_i is in the domain of *strict (or normal)* attraction. Many other examples can be produced using Karamata's representation of slowly varying functions. Assumption (50) is much in the spirit of Lemma A.4 in [21]. Our final result goes in different direction.

Remark 4.8 Notice that if $\alpha < 1$, then $\sum_j |c_j|^\alpha h(|c_j|^{-1}) < +\infty$, with h slowly varying, automatically implies $\sum_j |c_j| < +\infty$.

Corollary 4.9 *Under the usual conditions, if $\alpha < 1$, then*

$$S_n(t) \underset{\mathcal{D}(S)}{\longrightarrow} A \cdot Z(t),$$

if

$$\sum_{j \in \mathbb{Z}} |c_j|^\alpha < +\infty,$$

and the coefficients c_j are regular in a very weak sense: there exists a constant $0 < \gamma < \alpha$ such that

$$\frac{\max_{j+1 \leq k \leq j+n} |c_k|^{\frac{(1-\alpha)(\alpha-\gamma)}{(1-\alpha+\gamma)}}}{\sum_{k=j+1}^{j+n} |c_k|^\alpha} \leq K_+ < +\infty, \quad j \geq 0. \tag{51}$$

$$\frac{\max_{j-n \leq k \leq j-1} |c_k|^{\frac{(1-\alpha)(\alpha-\gamma)}{(1-\alpha+\gamma)}}}{\sum_{k=j-n}^{j-1} |c_k|^\alpha} \leq K_- < +\infty, \quad j \leq 0. \tag{52}$$

(with the convention that $0/0 \equiv 1$.)

Remark 4.10 Notice that we always assume that the linear process is well- defined. This may require more than demanded in Corollary 4.9.

Proof of Corollary 4.9 As before, we have to check (47).

$$\begin{aligned}
&\sum_{j=-\infty}^{-j_n} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) \\
&\quad = \frac{1}{n} \sum_{j=-\infty}^{-j_n} \Big|\sum_{k=1-j}^{n-j} c_k\Big|^{\alpha-\gamma} \frac{nh(a_n)}{a_n^\alpha} |d_{n,j}|^\gamma \frac{1}{h(a_n)} h\left(\frac{a_n}{|d_{n,j}|}\right) \\
&\quad \leq K \frac{1}{n} \sum_{j=-\infty}^{-j_n} \Big|\sum_{k=1-j}^{n-j} c_k\Big|^{\alpha-\gamma} \Psi_\gamma\left(a_n, \frac{a_n}{|d_{n,j}|}\right),
\end{aligned}$$

where $\Psi_\gamma(x, y)$ was defined in the Proof of Corollary 4.1 and

$$\sup_{j \leq -j_n} \Psi_\gamma\left(a_n, \frac{a_n}{|d_{n,j}|}\right) \to 0, \text{ as } n \to \infty.$$

Thus it is enough to prove

$$\sup_n \frac{1}{n} \sum_{j=-\infty}^{-j_n} \Big|\sum_{k=1-j}^{n-j} c_k\Big|^{\alpha-\gamma} < +\infty.$$

We have

$$\Big|\sum_{k=1-j}^{n-j} c_k\Big| \leq \sum_{k=1-j}^{n-j} |c_k| \leq \left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right) \cdot \max_{1-j \leq k \leq n-j} |c_k|^{1-\alpha}, \tag{53}$$

hence

$$\begin{aligned}
\frac{1}{n} \sum_{j=-\infty}^{-j_n} \Big|\sum_{k=1-j}^{n-j} c_k\Big|^{\alpha-\gamma} &= \frac{1}{n} \sum_{j=-\infty}^{-j_n} \left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right) \frac{\big|\sum_{k=1-j}^{n-j} c_k\big|^{\alpha-\gamma}}{\left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right)} \\
&\leq \frac{1}{n} \sum_{j=-\infty}^{-j_n} \left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right) \frac{\max_{1-j \leq k \leq n-j} |c_k|^{(1-\alpha)(\alpha-\gamma)}}{\left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right)^{1-\alpha+\gamma}} \\
&\leq (K_+)^{1-\alpha+\gamma} \frac{1}{n} \sum_{j=-\infty}^{-j_n} \left(\sum_{k=1-j}^{n-j} |c_k|^\alpha\right) \to 0.
\end{aligned}$$

This is again more than needed. The proof of

$$\sum_{j=n+j_n}^{+\infty} \frac{|d_{n,j}|^\alpha}{a_n^\alpha} h\left(\frac{a_n}{|d_{n,j}|}\right) \to 0, \quad \text{as } n \to \infty.$$

goes the same way.

Example 4.11 If $\alpha < 1$,

$$|c_j| = \frac{1}{|j|^{1/\alpha} \log^{(1+\varepsilon)/\alpha} |j|}, \quad |j| \geq 3,$$

and $\{X_i\}$ is well-defined, then under the usual conditions

$$S_n(t) \underset{\mathcal{D}(S)}{\longrightarrow} A \cdot Z(t).$$

Remark 4.12 In our considerations, we search for conditions giving functional convergence of $\{S_n(t)\}$ with *the same normalization* as $\{Z_n(t)\}$ (by $\{a_n\}$). It is possible to provide examples of linear processes, which are convergent in the sense of finite dimensional distribution with different normalization. Moreover, it is likely that also in the heavy-tailed case one can obtain a complete description of the convergence of linear processes, as it is done by Peligrad and Sang [23] in the case of innovations belonging to the domain of attraction of a normal distribution. We conjecture that whenever the limit is a stable Lévy motion our functional approach can be adapted to the more general setting.

5 Some Complements

5.1 S-Continuous Functionals

A phenomenon of self-canceling oscillations, typical for the S topology, was described in Example 3.14. This example shows that *supremum cannot be* continuous in the S topology. In fact, supremum is lower semi-continuous with respect to S, as many other popular functionals—see [13], Corollary 2.10. On the other hand *addition is* sequentially continuous and this property was crucial in consideration given in Sect. 3.4.

Here is another positive example of an S-continuous functional.

Let μ be an atomless measure on $[0, 1]$ and let $h : \mathbb{R}^1 \to \mathbb{R}^1$ be a continuous function. Consider a smoothing operation $s_{\mu,h}$ on $\mathbb{D}([0, 1])$ given by the formula

$$s_{\mu,h}(x)(t) = \int_0^t h(x(s))\, \mathrm{d}\mu(s).$$

Then, $s_{\mu,h}(x)(\cdot)$ is a continuous function on $[0, 1]$ and a slight modification of the Proof of Proposition 2.15 in [13] shows that the mapping

$$(\mathbb{D}([0, 1]), S) \ni x \mapsto s_{\mu,h}(x) \in (\mathbb{C}([0, 1]), \|\cdot\|_\infty)$$

is continuous. In particular, if we set $\mu = \ell$ (the Lebesgue measure), $h(0) = 0$, $h(x) \geq 0$, and suppose that $x_n \longrightarrow_S 0$, then

$$\int_0^1 h(x_n(s))\,\mathrm{d}s \to 0.$$

In the case of linear processes, such functionals lead to the following result.

Corollary 5.1 *Under the conditions of Corollaries 4.1, 4.3, 4.6 or 4.9 we have for any $\beta > 0$*

$$\frac{1}{na_n^\beta}\sum_{k=1}^n \Big|\sum_{i=1}^k \left(\sum_j c_{i-j}Y_j\right) - AY_i\Big|^\beta \underset{\mathcal{P}}{\longrightarrow} 0.$$

Proof of Corollary 5.1 The expression to be analyzed has the form

$$\int_0^1 H_\beta\left(S_n(t) - A\cdot Z_n(t)\right)\,\mathrm{d}t,$$

where $H_\beta(x) = |x|^\beta$ and by (26)

$$S_n(t) - A\cdot Z_n(t) \underset{f.d.d.}{\longrightarrow} 0.$$

We have checked in the course of the Proof of Theorem 3.18, that $\{S_n\}$ is uniformly S-tight. By (3) $\{A\cdot Z_n\}$ is uniformly J_1-tight, hence also S-tight. Similarly as in the Proof of Proposition 3.16 we deduce that $\{S_n - A\cdot Z_n\}$ is uniformly S-tight. Now an application of Proposition 3.3 gives

$$S_n - A\cdot Z_n \underset{\mathcal{D}}{\longrightarrow} 0,$$

on the Skorokhod space $\mathbb{D}([0,1])$ equipped with the S topology.

5.2 An Example Related to Convergence in the M_1 Topology

In Introduction, we provided an example of a linear process ($c_0 = 1, c_1 = -1$) for which no Skorokhod's convergence is possible. In this example $A = 0$ and the limit is degenerate, what might suggest that another, more appropriate norming is applicable, under which the phenomenon disappears. Here, we give an example with a *non-degenerate limit* showing that in the general case M_1-convergence need not hold.

Example 5.2 Let $c_0 = \zeta > -c_1 = \xi > 0$. Then $X_j = \zeta Y_j - \xi Y_{j-1}$ and defining $Z_n(t)$ by (3) we obtain for $t \in [k/n, (k+1)/n)$

$$S_n(t) = \frac{1}{a_n}\sum_{j=1}^k X_j = \frac{1}{a_n}\left(\zeta Y_k - \xi Y_0\right) + (\zeta - \xi)Z_n((k-1)/n).$$

Clearly, the f.d.d. limit $\{(\zeta-\xi)Z(t)\}$ is non-degenerate. We will show that the sequence $\{S_n(t)\}$ is not uniformly M_1-tight and so cannot converge to $\{(\zeta-\xi)Z(t)\}$ in the M_1 topology.

For the sake of simplicity, let us assume that Y_j's are non-negative and

$$\mathbb{P}(Y_1 > x) = x^{-\alpha}, \quad x \geq 1,$$

with $\alpha < 1$. Then, we can choose $a_n = n^{1/\alpha}$. Consider sets

$$G_n = \bigcup_{j=0}^{n-1} \left\{Y_j > \varepsilon_n a_n, Y_{j+1} > \varepsilon_n a_n\right\}.$$

where $\varepsilon_n = n^{-1/(3\alpha)}$. Then,

$$\mathbb{P}(G_n) \leq (n+1)\mathbb{P}(Y_i > \varepsilon_n a_n)^2 = (n+1)\varepsilon_n^{-2\alpha}\left(n^{1/\alpha}\right)^{-2\alpha} \longrightarrow 0.$$

Notice that

$$\text{on } G_n^c \text{ there are no two consecutive values of } Y_j \text{ exceeding} \varepsilon_n a_n. \tag{54}$$

Let us define $Y_{n,j} = Y_j \mathbf{1}\{Y_j > \varepsilon_n a_n\}$ and set for $t \in [k/n, (k+1)/n)$

$$\widetilde{S}_n(t) = \frac{1}{a_n}\left(\zeta Y_{n,k} - \xi Y_{n,0}\right) + \frac{\zeta-\xi}{a_n}\sum_{j=1}^{k-1} Y_{n,j}.$$

We have by (61)

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,1]}\left|S_n(t)-\widetilde{S}_n(t)\right|\right] &\leq \frac{\zeta}{a_n}\sum_{j=0}^{n}\mathbb{E}\left[Y_j\mathbf{1}\{Y_j \leq \varepsilon_n a_n\}\right] \\ &\leq C_1\zeta\frac{(n+1)(\varepsilon_n a_n)^{1-\alpha}}{a_n} \to 0.\end{aligned}$$

It follows that $\{S_n(t)\}$ are uniformly M_1-tight if, and only if, $\{\widetilde{S}_n(t)\}$ are. Let $w^{M_1}(x,\delta)$ be given by (34). Since $\mathbb{P}\left(G_n^c\right) \to 1$ we have for any $\delta > 0$ and $\eta > 0$

$$\limsup_n \mathbb{P}\left(w^{M_1}(\widetilde{S}_n(\cdot),\delta) > \eta\right) = \limsup_n \mathbb{P}\left(\{w^{M_1}(\widetilde{S}_n(\cdot),\delta) > \eta\} \cap G_n^c\right).$$

And on G_n^c, by the property (54) and if $2/n < \delta$ we have

$$\omega(\widetilde{S}_n(\cdot),\delta) \geq \frac{1}{a_n}(\zeta-\xi)\max_j Y_{n,j}.$$

If $\eta/(\zeta - \xi) > \varepsilon_n$, then

$$\begin{aligned}
&\mathbb{P}\left((1/a_n)\max_j Y_{n,j} > \eta/(\zeta - \xi)\right)\\
&\quad = \mathbb{P}\left((1/a_n)\max_j Y_j > \eta/(\zeta - \xi)\right)\\
&\quad \longrightarrow 1 - \exp\left(-((\zeta - \xi)/\eta)^{\alpha}\right) = \theta > 0.
\end{aligned}$$

Hence for each $\delta > 0$

$$\liminf_n \mathbb{P}\left(w^{M_1}(\widetilde{S}_n(\cdot), \delta) > \eta\right) \geq \theta > 0,$$

and the sequence $\{\widetilde{S}_n(t)\}$ cannot be uniformly M_1-tight.

5.3 Linear Space of Convergent Linear Processes

We can explore the machinery of Sect. 4 to obtain a natural

Proposition 5.3 *We work under the assumptions of Theorem 2.1. Denote by $\mathcal{C}_Y$ the set of sequences $\{c_i\}_{i\in\mathbb{Z}}$ such that if*

$$X_i = \sum_{j\in\mathbb{Z}} c_j Y_{i-j}, \quad i \in \mathbb{Z},$$

then

$$S_n(t) = \frac{1}{a_n}\sum_{i=1}^{[nt]} X_i \underset{f.d.d.}{\longrightarrow} A \cdot Z(t),$$

with $A = \sum_{i\in\mathbb{Z}} c_i$.

Then $\mathcal{C}_Y$ is a linear subspace of $\mathbb{R}^{\mathbb{Z}}$.

Proof of Proposition 5.3 Closeness of $\mathcal{C}_Y$ under multiplication by a number is obvious. So let us assume that $\{c'_i\}$ and $\{c''_i\}$ are elements of $\mathcal{C}_Y$. By Theorem 2.1, we have to prove that

$$\begin{aligned}
&\sum_{j=-\infty}^{0} \mathbb{P}\left(\Big|\sum_{k=1-j}^{n-j}(c'_k + c''_k)\Big||Y_j| > a_n\right) \to 0, \quad \text{as } n \to \infty.\\
&\sum_{j=n+1}^{\infty} \mathbb{P}\left(\Big|\sum_{k=1-j}^{n-j}(c'_k + c''_k)\Big||Y_j| > a_n\right) \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{55}$$

But

$$\begin{aligned}
&\sum_{j=-\infty}^{0} \mathbb{P}\left(\Big| \sum_{k=1-j}^{n-j} (c'_k + c''_k) \Big| |Y_j| > a_n \right) \\
&\quad \leq \sum_{j=-\infty}^{0} \mathbb{P}\left(\Big| \sum_{k=1-j}^{n-j} c'_k \Big| |Y_j| + \Big| \sum_{k=1-j}^{n-j} c''_k \Big| |Y_j| > a_n \right) \\
&\quad \leq \sum_{j=-\infty}^{0} \mathbb{P}\left(\Big| \sum_{k=1-j}^{n-j} c'_k \Big| |Y_j| > a_n/2 \right) + \sum_{j=-\infty}^{0} \mathbb{P}\left(\Big| \sum_{k=1-j}^{n-j} c''_k \Big| |Y_j| > a_n/2 \right).
\end{aligned}$$

Now both terms tend to 0 by Remark 2.7. Identical reasoning can be used in the proof of the "dual" condition in (55).

5.4 Dependent Innovations

In the main results of the paper, we studied only *independent* innovations $\{Y_j\}$. It is however clear that the functional S-convergence can be obtained under much weaker assumptions. In order to apply crucial Proposition 3.16 we need only that

$$S_n(t) \underset{f.d.d.}{\longrightarrow} A \cdot Z(t),$$

and that

$$T_n^+ \underset{\mathcal{D}}{\longrightarrow} A_+ \cdot Z, \quad \text{and} \quad T_n^- \underset{\mathcal{D}}{\longrightarrow} A_- \cdot Z,$$

on the Skorokhod space $\mathbb{D}([0, 1])$ equipped with the M_1 topology. For the latter relations, Theorem 1 of [20] seems to be an ideal tool for associated sequences (see our Proposition 3.1). A variety of potential other possible examples is given in [27].

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Appendix

We provide two results of a technical character. The first one is well-known [1] and is stated here for completeness. Proposition 5.5 might be of independent interest.

Proposition 5.4 *Let $\{Y_j\}$ be an i.i.d.sequence satisfying (4), (7) and (8) and let $\{c_j\}$ be a sequence of numbers. Then the series $\sum_{j\in\mathbb{Z}} c_j Y_j$ is well-defined if, and only if,*

$$\sum_{j\in\mathbb{Z}} |c_j|^{\alpha} h(|c_j|^{-1}) < +\infty. \tag{56}$$

Proposition 5.5 *Let $\{Y_j\}$ be an i.i.d.sequence satisfying (4), (7) and (8). Consider an array $\{c_{n,j}\ ;\ n\in\mathbb{N},\ j\in\mathbb{Z}\}$ of numbers such that for each $n\in\mathbb{N}$*

$$\sum_{j\in\mathbb{Z}} |c_{n,j}|^{\alpha} h(|c_{n,j}|^{-1}) < +\infty. \tag{57}$$

Set $V_n = \sum_{j\in\mathbb{Z}} c_{n,j} Y_j$, $n\in\mathbb{N}$. Then

$$V_n \underset{\mathcal{P}}{\longrightarrow} 0 \tag{58}$$

if, and only if,

$$\sum_{j\in\mathbb{Z}} |c_{n,j}|^{\alpha} h(|c_{n,j}|^{-1}) \to 0, \quad \textit{as } n\to\infty. \tag{59}$$

In the proofs, we shall need some estimates which seem to be a part of the probabilistic folklore.

Lemma 5.6 *Assume that*

$$\mathbb{P}\left(|Y| > x\right) = x^{-\alpha} h(x),$$

where $h(x)$ is slowly varying at $x=\infty$.

(i) If $\alpha\in(0,2)$, then there exists a constant C_2, depending on α and the law of Y such that

$$\mathbb{E}\left[Y^2\mathbb{I}\left(|Y|\le x\right)\right] \le C_2 x^{2-\alpha} h(x), \quad x>0. \tag{60}$$

(ii) If $\alpha\in(0,1)$, then there exists a constant C_1, depending on α and the law of Y such that

$$\mathbb{E}\left[|Y|\mathbb{I}\left(|Y|\le x\right)\right] \le C_1 x^{1-\alpha} h(x), \quad x>0. \tag{61}$$

(iii) If $\alpha\in(1,2)$, then there is $x_0>0$, depending on the law of Y, such that

$$\mathbb{E}\left[|Y|\mathbb{I}\left(|Y|>x\right)\right] \le \mathbb{E}\left[|Y|\,\mathbb{I}\left(x\le x_0\right)\right] + \frac{2\alpha}{\alpha-1} x^{1-\alpha} h(x), \quad x>0. \tag{62}$$

Proof Take $\beta>\alpha$. Applying the direct half of Karamata's Theorem (Th. 1.5.11 [5]), we obtain

$$\mathbb{E}\left[|Y|^{\beta}\mathbb{I}\left(|Y|\le x\right)\right] = \beta\int_0^x t^{\beta-1}\mathbb{P}\left(|Y|>t\right)\,\mathrm{d}t - x^{\beta}\mathbb{P}\left(|Y|>t\right) \sim \frac{\alpha}{\beta-\alpha} x^{\beta-\alpha} h(x).$$

Hence there exists x_0 such that

$$\mathbb{E}\left[|Y|^{\beta}\mathbb{I}\left(|Y| \le x\right)\right] \le \frac{2\alpha}{\beta-\alpha}x^{\beta-\alpha}h(x), \quad x > x_0.$$

If $0 < x \le x_0$, then

$$\mathbb{E}\left[|Y|^{\beta}\mathbb{I}\left(|Y| \le x\right)\right] \le x^{\beta} = x^{\beta}\frac{x^{-\alpha}h(x)}{\mathbb{P}\left(|Y| > x\right)} \le \frac{1}{\mathbb{P}\left(|Y| > x_0\right)}x^{\beta-\alpha}h(x).$$

Setting $C_\beta = \max\{1/\mathbb{P}\left(|Y| > x_0\right), 2\alpha/(\beta-\alpha)\}$ one obtains both (60) and (61).

To get (62), we proceed similarly. First, by Karamata's Theorem

$$\mathbb{E}\left[|Y|\mathbb{I}\left(|Y| > x\right)\right] = \int_x^{\infty}\mathbb{P}\left(|Y| > t\right)\,\mathrm{d}t + x\mathbb{P}\left(|Y| > x\right) \sim \frac{\alpha}{\alpha-1}x^{1-\alpha}h(x),$$

Hence, for some x_0, we have

$$\mathbb{E}|Y|\mathbb{I}\left(|Y| > x\right) \le \frac{2\alpha}{\alpha-1}x^{1-\alpha}h(x), \quad x > x_0.$$

Since $\alpha > 1$, we have $\mathbb{E}\left[|Y|\right] < +\infty$ and (62) follows.

Proof of Proposition 6.1 We begin with specifying the conditions of the Kolmogorov Three Series Theorem in terms of our linear sequences. We have

$$\sum_{j\in\mathbb{Z}}\mathbb{P}\left(|c_jY_j| > 1\right) = \sum_{j\in\mathbb{Z}}\left(\frac{1}{|c_j|}\right)^{-\alpha}h(|c_j|^{-1}) = \sum_{j\in\mathbb{Z}}|c_j|^{\alpha}h(|c_j|^{-1}). \tag{63}$$

Applying (60) we obtain

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\mathbb{V}\mathrm{ar}\left((c_jY_j)\mathbb{I}\left(|c_jY_j| \le 1\right)\right) &\le \sum_{j\in\mathbb{Z}}\mathbb{E}\left[\left(c_jY_j\right)^2\mathbb{I}\left(|c_jY_j| \le 1\right)\right]\\ &= \sum_{j\in\mathbb{Z}}|c_j|^2\mathbb{E}\left[Y_j^2I\left(|Y_j| \le 1/|c_j|\right)\right]\\ &\le C_2\sum_{j\in\mathbb{Z}}|c_j|^2(1/|c_j|)^{2-\alpha}h(|c_j|^{-1})\\ &= C_2\sum_{j\in\mathbb{Z}}|c_j|^{\alpha}h(|c_j|^{-1}).\end{aligned} \tag{64}$$

Similarly, if $\alpha \in (0, 1)$, then by (61)

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left|\mathbb{E}\left[c_j Y_j \mathbb{I}\left(|c_j Y_j| \le 1\right)\right]\right| &\le \sum_{j\in\mathbb{Z}} |c_j| \mathbb{E}\left[|Y_j| \mathbb{I}\left(|Y_j| \le 1/|c_j|\right)\right] \\ &\le C_1 \sum_{j\in\mathbb{Z}} |c_j| (1/|c_j|)^{1-\alpha} h(|c_j|^{-1}) \\ &= C_1 \sum_{j\in\mathbb{Z}} |c_j|^{\alpha} h(|c_j|^{-1}).\end{aligned} \tag{65}$$

If $\alpha = 1$, then by the symmetry we have $\mathbb{E}\left[Y_j \mathbb{I}\left(|Y_j| \le a\right)\right] = 0$, $a > 0$, and the series of truncated expectations trivially vanishes

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\left[c_j Y_j \mathbb{I}\left(|c_j Y_j| \le 1\right)\right] = 0. \tag{66}$$

For $\alpha \in (1, 2)$ we have $\mathbb{E}\left[X_j\right] = 0$ and by (62)

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\mathbb{E}\left[c_j Y_j \mathbb{I}\left(|c_j Y_j| \le 1\right)\right]| &= \sum_{j\in\mathbb{Z}} |\mathbb{E}\left[c_j Y_j \mathbb{I}\left(|c_j Y_j| > 1\right)\right]| \\ &\le \sum_{j\in\mathbb{Z}} |c_j| \mathbb{E}\left[|Y_j| \mathbb{I}\left(|Y_j| > 1/|c_j|\right)\right] \\ &\le \mathbb{E}\left[|Y|\right] \max_{j\in\mathbb{Z}} |c_j| \#\{j\,;\ |c_j| \ge 1/x_0\} \\ &\quad + \frac{2\alpha}{\alpha - 1} \sum_{j\in\mathbb{Z}} |c_j| (1/|c_j|)^{1-\alpha} h(|c_j|^{-1})\end{aligned} \tag{67}$$

By (63)–(67) we obtain that $\sum_{j\in\mathbb{Z}} |c_j|^{\alpha} h(|c_j|^{-1}) < +\infty$ if, and only if, all the assumptions of the Three Series Theorem are satisfied. Hence $\sum_{j\in\mathbb{Z}} c_j Y_j$ is a.s. convergent if, and only if, (56) holds.

Proof of Proposition 6.2 By Proposition 5.4, all random variables $V_n = \sum_{j\in\mathbb{Z}} c_{n,j} Y_j$ are well-defined. Let us consider a decomposition of each V_n into a sum of another three (convergent!) series:

$$\begin{aligned}V_n &= \sum_{j\in\mathbb{Z}} \left(c_{n,j} Y_j I(|c_{n,j} Y_j| \le 1) - \mathbb{E}\left[c_{n,j} Y_j I(|c_{n,j} Y_j| \le 1)\right]\right) \\ &\quad + \sum_{j\in\mathbb{Z}} \mathbb{E}\left[c_{n,j} Y_j I(|c_{n,j} Y_j| \le 1)\right] \\ &\quad + \sum_{j\in\mathbb{Z}} c_{n,j} Y_j I(|c_{n,j} Y_j| > 1) \\ &= V_{n,1} + V_{n,2} + V_{n,3}.\end{aligned}$$

By (64), we have

$$\mathbb{V}\mathrm{ar}\left(V_{n,1}\right) \le C_2 \sum_{j\in\mathbb{Z}} |c_{n,j}|^{\alpha} h(|c_{n,j}|^{-1}) \to 0, \quad \text{as } n \to \infty,$$

if (59) holds. Similarly $V_{n,2} \to 0$ by (65)–(67). Finally, we have

$$\begin{aligned}\mathbb{P}\left(V_{n,3} \neq 0\right) &\le \mathbb{P}\left(\bigcup_{j\in\mathbb{Z}} \{|c_{n,j}Y_j| > 1\}\right)\\ &\le \sum_{j\in\mathbb{Z}} \mathbb{P}\left(|c_{n,j}Y_j| > 1\right)\\ &= \sum_{j\in\mathbb{Z}} |c_{n,j}|^{\alpha} h(|c_{n,j}|^{-1}) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

We have proved the sufficiency part of Proposition 5.5.

To prove the "only if" part, we show first that $V_n \longrightarrow_{\mathcal{P}} 0$ implies uniform infinitesimality of the coefficients, that is

$$\sup_{j\in\mathbb{Z}} |c_{n,j}| \to 0, \text{ as } n \to \infty. \tag{68}$$

Let $\{\bar{Y}_j\}$ be an independent copy of $\{Y_j\}$. If $\bar{V}_n = \sum_{j\in\mathbb{Z}} c_{n,j}\bar{Y}_j$, then also $V_n - \bar{V}_n \longrightarrow_{\mathcal{P}} 0$ and these are series of *symmetric* random variables. For each n select some arbitrary $j_n \in \mathbb{Z}$ and consider decomposition into *independent symmetric* random variables

$$V_n - \bar{V}_n = c_{n,j_n}(Y_{j_n} - \bar{Y}_{j_n}) + \sum_{j\in\mathbb{Z}, j\neq j_n} c_{n,j}(Y_j - \bar{Y}_j) = W_n + \widetilde{W}_n.$$

Since $\{V_n - \bar{V}_n\}_{n\in\mathbb{N}}$ is uniformly tight, so is $\{W_n\}_{n\in\mathbb{N}}$ (it follows from the Lévy–Ottaviani inequality, see e.g., Proposition 1.1.1 in [18]). Since the law of $Y_j - \bar{Y}_j$ is non-degenerate, we obtain

$$\sup_n |c_{n,j_n}| < +\infty.$$

If along some subsequence n', we would have $c_{n',j_{n'}} \to c \neq 0$, then for some $\theta \in \mathbb{R}^1$

$$\lim_{n'\to\infty} \mathbb{E}\left[e^{i\theta W_{n'}}\right] = |\mathbb{E}\left[e^{i\theta c Y}\right]|^2 < 1.$$

It follows that also

$$\lim_{n'\to\infty} \mathbb{E}\left[e^{i\theta(V_{n'} - \bar{V}_{n'})}\right] = \lim_{n'\to\infty} \mathbb{E}\left[e^{i\theta W_{n'}}\right] \mathbb{E}\left[e^{i\theta \widetilde{W}_{n'}}\right] < 1.$$

This is in contradiction with $V_n - \bar{V}_n \longrightarrow_{\mathcal{P}} 0$. Hence $c = 0$, $c_{n,j_n} \to 0$ and since j_n was chosen arbitrary, (68) follows.

Now let us choose k_n such that both

$$\sum_{|j|>k_n} c_{n,j} Y_j \underset{\mathcal{P}}{\longrightarrow} 0, \quad \text{as } n \to \infty,$$

and

$$\sum_{|j|>k_n} \mathbb{P}\left(|c_{n,j} Y_j| > 1\right) \to 0, \quad \text{as } n \to \infty.$$

Then $\{X_{n,j} = c_{n,j} Y_j\,;\, |j| \leq k_n, n \in \mathbb{N}\}$ is an *infinitesimal* array of row-wise independent random variables, with row sums convergent in probability to zero. Applying the general central limit theorem (see e.g., Theorem 5.15 in [15]), we obtain

$$\sum_{|j|\leq k_n} \mathbb{P}\left(|X_{n,j}| > 1\right) = \sum_{|j|\leq k_n} \mathbb{P}\left(|c_{n,j} Y_j| > 1\right) = \sum_{|j|\leq k_n} |c_{n,j}|^{\alpha} h(|c_{n,j}|^{-1}) \to 0.$$

This completes the Proof of Proposition 5.5.

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