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ERRATUM

Erratum to: The Asymptotic Distribution of Self-Normalized Triangular Arrays

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Abstract We correct and clarify some ambiguous statements in D. M. Mason (2005): The asymptotic distribution of self-normalized triangular arrays. *J. Theoret. Probab.*, **18**, 853–870.

Corrections and Clarifications of Mason (2005)

This note has two purposes. First is to correct some statements in the **Introduction and Statements of Results** of [4], and second is to provide the result given in Proposition [A] below, which clarifies a claim at the end of the proof of Theorem 2.

Our corrections are needed since it is not clear that (1.9) always implies (1.2). They are the following:

(i) On page 855, line 10, change the "further shows" to "further shows that under the setup of Proposition [A] in this note" and on line 13 change "(1.4)" to "(1.4) nondegenerate".
(ii) On page 855, line 14, replace "or equivalently (1.9) holds" with "$P(V > 0) = 1$".
(iii) On page 856, line 3, replace " Actually" with "Actually under the setup of Proposition [A] in this note".

We remark in passing that the statements in [4] about triangular arrays of the form $X_{1,n}, \ldots, X_{n,n}$, $n \geq 1$, are equally valid for triangular arrays of the form $X_{1,n_k}, \ldots, X_{n_k,n_k}$, $k \geq 1$, where $\{n_k\}_{k \geq 1}$ is an infinite subsequence of the positive integers. Also we point out that everywhere *triangular array of infinitesimal independent*

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random variables should be changed to *infinitesimal triangular array of independent random variables*, as stated in Proposition [A]. The word *infinitesimal* was everywhere put in the wrong place.

The following Proposition [A] and Remark 1 justify the claim towards the end of the proof of Theorem 2 on page 868 that says, "This means that every subsequential distributional limit random variable T must be of the form (1.10)." They should have been included in an appendix in the original paper.

Proposition [A] *Let $\{n_k\}_{k\geq 1}$ be an infinite subsequence of the positive integers and $X_{1,n_k}, \ldots, X_{n_k,n_k}$, $k \geq 1$, be an infinitesimal triangular array of independent random variables such that for each $k \geq 1$, $X_{1,n_k}, \ldots, X_{n_k,n_k}$ are i.i.d. X_{1,n_k}. Assume that for a necessarily infinitely divisible random variable U,*

$$\sum_{i=1}^{n_k} X_{i,n_k} \to_d U, \ \text{as } k \to \infty. \tag{1.1}$$

Then

$$\left(\sum_{i=1}^{n_k} X_{i,n_k}, \sum_{i=1}^{n_k} X_{i,n_k}^2\right) \to_d (U, V), \ \text{as } k \to \infty, \tag{1.2}$$

where the two dimensional infinitely divisible random vector (U, V) in (1.2) has the representation:

$$(U, V) =_d \left(b + W + \tau Z, S + \tau^2\right), \tag{1.3}$$

with b and $\tau \geq 0$ being suitable constants,

$$\begin{aligned} W = {} & \int_0^1 \varphi_1(s)\, dN_1(s) + \int_1^\infty \varphi_1(s)\, d\{N_1(s) - s\} \\ & - \int_0^1 \varphi_2(s)\, dN_2(s) - \int_1^\infty \varphi_2(s)\, d\{N_2(s) - s\} \end{aligned} \tag{1.4}$$

and

$$S = \int_0^\infty \varphi_1^2(s)\, dN_1(s) + \int_0^\infty \varphi_2^2(s)\, dN_2(s), \tag{1.5}$$

with N_1 and N_2 being independent right continuous Poisson processes on $[0, \infty)$ with rate 1, Z being a standard normal random variable independent of N_1 and N_2, and φ_1 and φ_2 being two left continuous, nonincreasing, nonnegative functions defined on $(0, \infty)$ satisfying for all $\delta > 0$,

$$\int_\delta^\infty \varphi_i^2(s)\, ds < \infty \quad \text{for } i = 1, 2. \tag{1.6}$$

Proof The proof that (1.1) implies (1.2) follows along very similar lines to that of Lemma 4 in [2]. To relieve the notational burden, in the following we shall write $n = n_k$. By parts (ii) and (iii) of Theorem 4.7 on page 61 of [1], the distributional convergence (1.1) implies that there exists a Lévy measure μ such that for every $\delta > 0$ such that $\mu\{-\delta, \delta\} = 0$,

$$\begin{aligned} w - \lim_{n\to\infty} \sum_{i=1}^{n} n\mathcal{L}\left(X_{i,n}\right) \mid (|x| > \delta) \\ = w - \lim_{n\to\infty} n\mathcal{L}\left(X_{1,n}\right) \mid (|x| > \delta) = \mu \mid (|x| > \delta)\,; \end{aligned} \tag{1.7}$$

and for some a_δ

$$\lim_{n\to\infty} ES_{n,\delta} = \lim_{n\to\infty} \left(nEX_{1,n,\delta}\right) = a_\delta, \tag{1.8}$$

where

$$S_{n,\delta} := \sum_{i=1}^{n} X_{i,n} 1\left\{\left|X_{i,n}\right| \le \delta\right\} =: \sum_{i=1}^{n} X_{i,n,\delta}. \tag{1.9}$$

(Note that we use here the notation of [1].) Now by part (i) of the same theorem, (1.1) also implies that for some $0 \le \sigma^2 < \infty$,

$$\lim_{\delta\searrow 0} \left\{ \begin{array}{c} \limsup_{n\to\infty} \\ \liminf_{n\to\infty} \end{array} \right\} \sum_{i=1}^{n} E\left(X_{i,n,\delta} - EX_{i,n,\delta}\right)^2 = \sigma^2. \tag{1.10}$$

Notice that

$$\sum_{i=1}^{n} E\left(X_{i,n,\delta} - EX_{i,n,\delta}\right)^2 = nEX_{1,n,\delta}^2 - n^{-1}\left(nEX_{1,n,\delta}\right)^2. \tag{1.11}$$

Further by (1.8) for every $\delta > 0$ such that $\mu\{-\delta, \delta\} = 0$, $n^{-1}\left(nEX_{1,n,\delta}\right)^2 \to 0$, which by (1.10) and (1.11), implies

$$\lim_{\delta\searrow 0} \left\{ \begin{array}{c} \limsup_{n\to\infty} \\ \liminf_{n\to\infty} \end{array} \right\} EV_{n,\delta} = \sigma^2, \tag{1.12}$$

where

$$V_{n,\delta} := \sum_{i=1}^{n} X_{i,n}^2 1\left\{\left|X_{i,n}\right| \le \delta\right\} = \sum_{i=1}^{n} X_{i,n,\delta}^2, \tag{1.13}$$

with

$$EV_{n,\delta} = nEX_{1,n,\delta}^2 = \int_{|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right). \tag{1.14}$$

Now let $\delta_m, m \ge 1$, be a sequence of constants converging to zero such that $0 < \delta_{m+1} < \delta_m < \delta_0 = \delta$, and $\mu\{-\delta_m, \delta_m\} = 0$, $m \ge 0$. Then for each $m \ge 1$, by (1.7) and $\mu\{-\delta_m, \delta_m\} = \mu\{-\delta, \delta\} = 0$,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{|x|\le\delta_m} nx^2 d\mathcal{L}\left(X_{1,n}\right) + \int_{\delta_m<|x|\le\delta} x^2 d\mu(x) \\
&\quad = \liminf_{n\to\infty} \int_{|x|\le\delta_m} nx^2 d\mathcal{L}\left(X_{1,n}\right) + \lim_{n\to\infty} \int_{\delta_m<|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right) \\
&\quad = \liminf_{n\to\infty} \int_{|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right) \le \limsup_{n\to\infty} \int_{|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right) \\
&\quad = \limsup_{n\to\infty} \int_{|x|\le\delta_m} nx^2 d\mathcal{L}\left(X_{1,n}\right) + \lim_{n\to\infty} \int_{\delta_m<|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right) \\
&\quad = \limsup_{n\to\infty} \int_{|x|\le\delta_m} nx^2 d\mathcal{L}\left(X_{1,n}\right) + \int_{\delta_m<|x|\le\delta} x^2 d\mu(x).
\end{aligned}$$

Now by letting $m \to \infty$, we see by (1.12) that

$$\lim_{n\to\infty} \int_{|x|\le\delta} nx^2 d\mathcal{L}\left(X_{1,n}\right) = \sigma^2 + \int_{0<|x|\le\delta} x^2 d\mu(x) =: b_\delta. \tag{1.15}$$

Moreover, we get from (1.15) that for every $k > 2$,

$$\lim_{\delta\searrow 0} \limsup_{n\to\infty} \int_{|x|\le\delta} n|x|^k d\mathcal{L}\left(X_{1,n}\right) = 0. \tag{1.16}$$

We now proceed as in the proof of Lemma 4 in [2]. We see using (1.16) that for any $\alpha, \beta \in \mathbb{R}$,

$$\lim_{\delta\searrow 0} \left\{ \begin{matrix} \limsup_{n\to\infty} \\ \liminf_{n\to\infty} \end{matrix} \right\} E\left(\alpha\left(S_{n,\delta} - ES_{n,\delta}\right) + \beta\left(V_{n,\delta} - EV_{n,\delta}\right)\right)^2 = \alpha^2\sigma^2. \tag{1.17}$$

Write $\rho = \mu \circ T^{-1}$, where $T(x) = \left(x, x^2\right)$. Clearly by (1.7) for every $\delta > 0$ such that $\mu\{-\delta, \delta\} = 0$

$$w - \lim_{n\to\infty} n\mathcal{L}\left(X_{1,n}, X_{1,n}^2\right) | \left(\|x\| > \delta\right) = \rho| \left(\|x\| > \delta\right). \tag{1.18}$$

Furthermore, by (1.8) and (1.15) we have for every $\delta > 0$ such that $\mu\{-\delta, \delta\} = 0$

$$\left(ES_{n,\delta}, EV_{n,\delta}\right) \to \left(a_\delta, b_\delta\right).$$

Thus by the central limit theorem in $\mathbb{R}^2$ on pp. 67–68 of [1] and arguing just as in [2] we get that (1.2) holds with (U, V) having characteristic function $E \exp(sU + tV) =$

$$\exp\left\{-\frac{\sigma^2 s^2}{2} + i\left(a_\delta s + \sigma^2 t\right) + \int \left(\exp\left(i\left(su + tu^2\right)\right) - 1 - isu1\left\{|u| \le \delta\right\}\right) d\mu(u)\right\}, \tag{1.19}$$

for any $\delta > 0$ such that $\mu\{-\delta, \delta\} = 0$. It can be shown using Proposition 5.7 in [3] that a pair of random variables (U, V) with this characteristic function has the distributional representation (1.3) where $\tau^2 = \sigma^2$ and b is a suitable constant. It is shown there how φ_1 and φ_2 are defined via the Lévy measure μ. □

Remark 1 We note that if X is in the centered Feller class with a_n an appropriate sequence of norming constants and $X_1, X_2, \ldots,$ are i.i.d. X, then for every subsequence of $\{n\}$ there exists a further subsequence $\{n_k\}$ such that the triangular array $X_{i,n_k} = X_i / a_{n_k}$, $1 \le i \le n_k$, $k \ge 1$, satisfies (1.1), with U nondegenerate, and thus (1.2) and (1.19) hold, as was pointed out in [2]. Also we mention that it can be inferred using the Theorem in [5] that necessarily "$P(V > 0) = 1$".

Remark 2 A special case of the Proposition 1 implies that for any triangular array $X_{1,n_k}, \ldots, X_{n_k,n_k}$, $k \ge 1$, satisfying its assumptions, and

$$\sum_{i=1}^{n_k} X_{i,n_k} \to_d N(0, \sigma^2), \quad \text{as } k \to \infty, \tag{1.20}$$

then

$$\sum_{i=1}^{n_k} X_{i,n_k}^2 \to_P \sigma^2, \quad \text{as } k \to \infty. \tag{1.21}$$

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References

1. Araujo, A., Giné, E.: The Central Limit Theorem for Real and Banach Valued Random Variables. Wiley Series in Probability and Mathematical Statistics. Wiley, New York (1980)
2. Giné, E., Mason, D.M.: On the LIL for self-normalized sums of IID random variables. J. Theoret. Probab. **11**, 351–370 (1998)
3. Maller, R., Mason, D.M.: Small-time compactness and convergence behavior of deterministically and self-normalised Lévy processes. Trans. Am. Math. Soc. **362**, 2205–2248 (2010)
4. Mason, D.M.: The asymptotic distribution of self-normalized triangular arrays. J. Theoret. Probab. **18**, 853–870 (2005)
5. Pruitt, W.E.: The class of limit laws for stochastically compact normed sums. Ann. Probab. **11**, 962–969 (1983)