

A Strong Metric Subregularity Analysis of Nonsmooth Mappings Via Steepest Displacement Rate

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Abstract In this paper, a systematic study of the strong metric subregularity property of mappings is carried out by means of a variational tool, called steepest displacement rate. With the aid of this tool, a simple characterization of strong metric subregularity for multifunctions acting in metric spaces is formulated. The resulting criterion is shown to be useful for establishing stability properties of the strong metric subregularity in the presence of perturbations, as well as for deriving various conditions, enabling one to detect such a property in the case of nonsmooth mappings. Some of these conditions, involving several nonsmooth analysis constructions, are then applied in studying the isolated calmness property of the solution mapping to parameterized generalized equations.

Keywords Strong metric subregularity · Steepest descent rate · Sharp minimality · Isolated calmness · Injectivity constant · First-order ϵ -approximation · Outer prederivative · Generalized equation

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1 Introduction

Several remarkable advances in optimization have been made possible in recent years thanks to a deepened understanding of stability properties of multifunctions. In fact,

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their study has gained a well-recognized place within modern variational analysis. Among the properties of multifunctions mainly applied in optimization and related topics, those describing a Lipschitzian behaviour play a crucial role. Under this category falls metric regularity, which is most likely the best known and widely employed. Nonetheless, it turns out that metric regularity is not strictly requested in certain circumstances, while its work can be done by a weaker property called metric subregularity, of course at a lower price in terms of problem assumptions. Consider, for instance, the algebraic characterization of the tangent space to a manifold, which is defined by an equation expressed by a smooth mapping. According to a standard argument, this is the key tool for deriving the Euler–Lagrange multiplier rule in nonlinear optimization, often presented among the consequences of the celebrated Lyusternik’s theorem (see [1–3]). In order to establish the nontrivial inclusion (the kernel of the derivative is contained in the tangent space), metric regularity is usually invoked, even though the mere metric subregularity would be enough. As another example, consider the exact penalization principle for constrained optimization problem with Lipschitz objective function (see [4, 5]). It happens that this principle can be invoked, provided that a certain error bound inequality is valid, and for the latter circumstance, the metric subregularity of the constraining mapping is enough. A further important application concerns the subdifferential calculus theory within nonsmooth analysis, where metric subregularity provides for the best qualification conditions in basic calculus rules (see [6, 7]).

All of this contributed to raise a large interest in metric subregularity, on which a dedicated vast literature does now exist (see [8–20] and references therein).

The main drawback of metric subregularity is its lack of robustness under (even small) perturbations. More precisely, it has been observed that such property happens to be broken if adding to a metrically subregular mapping (even single-valued and smooth) a Lipschitz term, yet with a small Lipschitz constant. This well-known phenomenon explains the difficulty in employing perturbation schemes, when studying criteria for detecting metric subregularity. In this paper, a systematic study is proposed of a special variant of metric subregularity, called strong metric subregularity, which is known to exhibit a notable robustness quality, while keeping rather low requirements in comparison with metric regularity. In particular, the present study concentrates on sufficient conditions for the strong metric subregularity of (possibly) nonsmooth mappings. The analysis of this topic is performed by making use of a variational tool called steepest displacement rate that enables to formulate a general criterion already in a metric space setting. Such an approach leads to a unifying scheme of analysis and emphasizes the connection of the property under study with the notion of local sharp minimality. Strong metric subregularity along with its stability properties and related infinitesimal characterizations have been considered recently by several authors. For an account on various aspects of the emerging theory of strong metric subregularity, the reader may refer to [9], whereas for applications to stability properties of generalized equations arising in concrete models, one may refer to [21–23].

The contents of the paper are exposed according to the following structure. In Sect. 2, after the notion of strong metric subregularity is presented and its reformulation in terms of isolated calmness property for the inverse mapping is recalled, the notion of steepest displacement rate is introduced and exploited to establish the

basic characterization. The connection with local sharp minimality is also discussed, while several situations arising in different topics are illustrated, aimed at providing motivations for the interest in the main subject of the paper. In Sect. 3, two relevant manifestations of the robustness behaviour of strong metric subregularity are embedded in the framework of the steepest displacement rate analysis. In Sect. 4, several known tools of nonsmooth analysis are combined with the main criterion in order to obtain conditions for the strong metric subregularity of nonsmooth mappings. Some of these results are then applied in Sect. 5 to investigate the isolated calmness property of the solution mapping to parameterized generalized equations, with base and field term. A final section is reserved for general comments on the exposed achievements.

2 Strong Metric Subregularity and Its Equivalent Reformulations

Let us start with recalling the main properties under study. This will be done in a metric space setting, which is the natural environment where the Lipschitzian analysis of stability of multifunctions can be conducted. In a metric space (X, d) , the distance from a point $x \in X$ to a subset $S \subseteq X$ is denoted by $\text{dist}(x, S)$, with the convention that $\text{dist}(x, \emptyset) = +\infty$, while $B(x, r)$ denotes the closed ball with centre x and radius r .

Definition 2.1 (i) A set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is called *metrically subregular* at $(\bar{x}, \bar{y}) \in \text{graph } F$ if there exist $\kappa \geq 0$ and $r > 0$ such that

$$\text{dist}\left(x, F^{-1}(\bar{y})\right) \leq \kappa \text{dist}(\bar{y}, F(x)), \quad \forall x \in B(\bar{x}, r). \tag{1}$$

Denote by

$$\text{subreg } F(\bar{x}, \bar{y}) = \inf\{\kappa \geq 0 : \exists r > 0 \text{ satisfying (1)}\}$$

the *modulus of subregularity* of F at (\bar{x}, \bar{y}) . Whenever F is single-valued, the simpler notation $\text{subreg } F(\bar{x})$ will be used.

(ii) A set-valued mapping $F : X \rightrightarrows Y$ between metric spaces is called *strongly metrically subregular* at $(\bar{x}, \bar{y}) \in \text{graph } F$ if F is metrically subregular at (\bar{x}, \bar{y}) and, in addition, \bar{x} is an isolated point of $F^{-1}(\bar{y})$ or, equivalently, if there exist $\kappa \geq 0$ and $r > 0$ such that

$$d(x, \bar{x}) \leq \kappa \text{dist}(\bar{y}, F(x)), \quad \forall x \in B(\bar{x}, r). \tag{2}$$

Roughly speaking, whereas the well-known metric regularity property of a mapping F near $(\bar{x}, \bar{y}) \in \text{graph } F$ can be viewed as a quantitative form of local solvability for the inclusion $y \in F(x)$, the strong metric subregularity corresponds to a quantitative form of local uniqueness for the solution \bar{x} to the particular inclusion $\bar{y} \in F(x)$. The independence of these two properties is illustrated in the next example.

Example 2.1 Let $X = Y = \mathbb{R}$ be equipped with its usual Euclidean metric structure. Consider the mapping $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F_1(x) = \begin{cases} [0, 1/2[, & \text{if } x = 0, \\ [1, +\infty[, & \text{otherwise.} \end{cases}$$

Clearly, F_1 is strongly metrically subregular at $(0, 0)$, with $\text{subreg } F_1(0, 0) = 0$, but it fails to be metrically regular near the same point. Notice that F_1 has not closed graph and it is not upper hemicontinuous at 0.

In the same setting, consider the mapping $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F_2(x) = [x, +\infty[$. This multifunction is metrically regular near $(0, 0)$, but it is not strongly metrically subregular at the same point.

The basic tool of analysis in use throughout the present section is introduced in the next definition.

Definition 2.2 (i) Given a function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on a metric space and an element $\bar{x} \in \text{dom } \varphi$, the value (possibly infinite)

$$\varphi^\downarrow(\bar{x}) = \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x})}{d(x, \bar{x})}$$

is called the *steepest descent rate* of φ at \bar{x} .

(ii) Let $F : X \rightrightarrows Y$ be a set-valued mapping between metric spaces and let $(\bar{x}, \bar{y}) \in \text{graph } F$. The (nonnegative, possibly infinite) quantity

$$|F|^\downarrow(\bar{x}, \bar{y}) = \text{dist}(\bar{y}, F(\cdot))^\downarrow(\bar{x})$$

is called the *steepest displacement rate* of F at (\bar{x}, \bar{y}) .

Remark 2.1 The steepest descent rate is evidently connected with the notion of (strong) slope, even though they are not completely equivalent in describing the variational behaviour of a given function. The employment of such kind of tools in connection with variational problems is witnessed since [24], whereas their application to non-differentiable optimization goes back at least to [25]. Starting with Demyanov, the steepest descent rate became steadily exploited for formulating optimality conditions in metric space settings, as a starting point for more involved nonsmooth analysis constructions (see [26–31]). The use of the distance function from images of a given multifunction to characterize its Lipschitzian properties follows the spirit of [32].

A first basic characterization of strong metric subregularity is established next as a positivity condition on the steepest displacement rate of a given multifunction.

Proposition 2.1 *A set-valued mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $(\bar{x}, \bar{y}) \in \text{graph } F$ if and only if*

$$|F|^\downarrow(\bar{x}, \bar{y}) > 0. \tag{3}$$

Moreover, whenever inequality (3) holds true, it results in

$$\text{subreg } F(\bar{x}, \bar{y}) = \frac{1}{|F|^\downarrow(\bar{x}, \bar{y})}, \tag{4}$$

with the convention that $1/+\infty = 0$.

Proof Necessity: according to Definition 2.1(ii), corresponding to an arbitrary $\kappa > \text{subreg } F(\bar{x}, \bar{y})$ there exists $r > 0$ such that

$$\frac{\text{dist}(\bar{y}, F(x))}{d(x, \bar{x})} \geq \frac{1}{\kappa}, \quad \forall x \in B(\bar{x}, r) \setminus \{\bar{x}\},$$

whence

$$|F|^\downarrow(\bar{x}, \bar{y}) \geq \frac{1}{\kappa}.$$

This evidently implies inequality (3) and, by arbitrariness of κ , the inequality

$$\text{subreg } F(\bar{x}, \bar{y}) \geq \frac{1}{|F|^\downarrow(\bar{x}, \bar{y})}. \tag{5}$$

Sufficiency: according to Definition 2.2(ii), in the case $|F|^\downarrow(\bar{x}, \bar{y}) = +\infty$, for every $\eta > 0$ there exists $r_\eta > 0$ such that

$$\frac{1}{\eta} \text{dist}(\bar{y}, F(x)) \geq d(x, \bar{x}), \quad \forall x \in B(\bar{x}, r_\eta).$$

Thus F is strongly metrically subregular at (\bar{x}, \bar{y}) with $\text{subreg } F(\bar{x}, \bar{y}) \leq 1/\eta$, which leads to $\text{subreg } F(\bar{x}, \bar{y}) = 0$. In the case $|F|^\downarrow(\bar{x}, \bar{y}) < +\infty$, for an arbitrary $\epsilon \in]0, |F|^\downarrow(\bar{x}, \bar{y})[$, there is $r_\epsilon > 0$ such that

$$\frac{1}{|F|^\downarrow(\bar{x}, \bar{y}) - \epsilon} \text{dist}(\bar{y}, F(x)) \geq d(x, \bar{x}), \quad \forall x \in B(\bar{x}, r_\epsilon),$$

which shows that F is strongly metrically subregular at (\bar{x}, \bar{y}) and, by arbitrariness of ϵ , leads to

$$\text{subreg } F(\bar{x}, \bar{y}) \leq \frac{1}{|F|^\downarrow(\bar{x}, \bar{y})}. \tag{6}$$

Then, as inequalities (5) and (6) are both valid now, one obtains (4), thereby completing the proof. □

Remark 2.2 Let us recall that, after [33], an element $\bar{x} \in \text{dom } \varphi$ is said to be a *local sharp minimizer* of a function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on a metric space if there exist positive ζ and r such that

$$\varphi(x) \geq \varphi(\bar{x}) + \zeta d(x, \bar{x}), \quad \forall x \in B(\bar{x}, r).$$

Clearly, \bar{x} is a local sharp minimizer of φ if and only if $\varphi^\downarrow(\bar{x}) > 0$. Thus, on account of Definition 2.2 and of Proposition 2.1, a set-valued mapping F is strongly metrically subregular at (\bar{x}, \bar{y}) if and only if \bar{x} is a local sharp minimizer of the displacement function $x \mapsto \text{dist}(\bar{y}, F(x))$. Notice that the positivity of the steepest descent rate of a function is a circumstance essentially connected, in more structured settings, with nonsmoothness (see [34]).

Another characterization of the main property under study for a multifunction $F : X \rightrightarrows Y$ can be obtained through the following stability behaviour of its inverse $F^{-1} : Y \rightrightarrows X$, i.e. $F^{-1}(y) = \{x \in X : y \in F(x)\}$.

Definition 2.3 (i) A set-valued mapping $G : X \rightrightarrows Y$ between metric spaces is called *calm* at $(\bar{x}, \bar{y}) \in \text{graph } G$ if there exists $\kappa \geq 0$ and $r > 0$ such that

$$\sup_{y \in G(x) \cap B(\bar{y}, r)} \text{dist}(y, G(\bar{x})) \leq \kappa d(x, \bar{x}), \quad \forall x \in B(\bar{x}, r). \tag{7}$$

Denote by

$$\text{clm } G(\bar{x}, \bar{y}) = \inf\{\kappa \geq 0 : \exists r > 0 \text{ satisfying (7)}\}$$

the *calmness modulus* of G at (\bar{x}, \bar{y}) ($\text{clm } G(\bar{x})$ whenever G is a single-valued mapping).

- (ii) A set-valued mapping $G : X \rightrightarrows Y$ between metric spaces is said to have the *isolated calmness property* at $(\bar{x}, \bar{y}) \in \text{graph } G$ if G is calm at (\bar{x}, \bar{y}) and, in addition, \bar{y} is an isolated point of $G(\bar{x})$.
- (iii) A function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is called *calm from below* at $\bar{x} \in \text{dom } \varphi$ if

$$\varphi^\downarrow(\bar{x}) > -\infty.$$

Isolated calmness seems to have made its first formal appearance in [35], where it was called “upper-Lipschitz property at a point” (see also [9]).

Theorem 2.1 ([9]) *A set-valued mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $(\bar{x}, \bar{y}) \in \text{graph } F$ if and only if F^{-1} has the isolated calmness property at (\bar{y}, \bar{x}) . In this case, it holds*

$$\text{clm } F^{-1}(\bar{y}, \bar{x}) = \text{subreg } F(\bar{x}, \bar{y}).$$

Below several situations, connecting different topics of optimization and variational analysis, are illustrated, where the strong metric subregularity naturally emerges. A

further relevant motivation for being interested in strong metric subregularity has to do with the analysis of the solution mapping to generalized equations. This topic will be discussed in Sect. 5.

Example 2.2 From Remark 2.2, it should be clear that every scalar function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined in a metric space is strongly metrically subregular at each of its local sharp minimizers (if any). As an obvious consequence, the related epigraphical set-valued mapping $F_\varphi : X \rightrightarrows \mathbb{R}$, defined as

$$F_\varphi(x) = [\varphi(x), +\infty[,$$

is strongly metrically subregular at $(\bar{x}, \varphi(\bar{x}))$, whenever \bar{x} is a local sharp minimizer of φ . It is worth noting that if φ is calm from below at \bar{x} , then it can be perturbed in such a way to have the point \bar{x} as a local sharp minimizer. Indeed, if for some $l > 0$ it is $\varphi^\downarrow(\bar{x}) > -l$, then function $\varphi + ld(\cdot, \bar{x})$ admits a local sharp minimizer at \bar{x} . Now, for a lower semicontinuous (henceforth, for short, l.s.c.) proper function $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined in a complete metric space, the set of all points at which φ is calm from below is large enough. In fact, as a direct consequence of the Ekeland variational principle, it is possible to prove that such set is dense in $\text{dom } \varphi$. All of this should show that it is not difficult to generate situations where strong metric subregularity appears.

Example 2.3 Let $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ denote the space of all linear bounded operators between two normed spaces, having null vector $\mathbf{0}$. Given $\Lambda \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, its *injectivity constant* is defined as

$$\alpha(\Lambda) = \inf_{\|u\|=1} \|\Lambda u\|$$

(see, for instance, [36]). By linearity, for any pair $(\bar{x}, \bar{y}) \in \mathbf{X} \times \mathbf{Y}$, with $\bar{y} = \Lambda\bar{x}$, one finds

$$|\Lambda|^\downarrow(\bar{x}, \bar{y}) = |\Lambda|^\downarrow(\mathbf{0}, \mathbf{0}) = \liminf_{x \rightarrow \mathbf{0}} \frac{\|\Lambda x\|}{\|x\|} = \alpha(\Lambda). \tag{8}$$

While any bounded linear operator Λ is known to be metric subregular at each point of its graph, according to Proposition 2.1 it is strongly metrically subregular iff $\alpha(\Lambda) > 0$ and

$$\text{subreg } \Lambda(\bar{x}, \bar{y}) = \text{subreg } \Lambda(\mathbf{0}, \mathbf{0}) = \frac{1}{\alpha(\Lambda)}.$$

Notice that, whenever \mathbf{X} and \mathbf{Y} are finite-dimensional spaces, $\alpha(\Lambda) > 0$ holds iff $\text{Ker } \Lambda = \Lambda^{-1}(\mathbf{0}) = \{\mathbf{0}\}$, that is iff Λ is injective. This fact fails to remain true in abstract normed space. Consider, for instance, the identity operator $\text{Id} : \ell^1 \rightarrow \ell^\infty$ (the immersion of ℓ^1 into ℓ^∞), which is injective, and define for each $n \in \mathbb{N}$ the elements $x^n = \{x_k^n\} \in \ell^1$ as follows

$$x_k^n = \begin{cases} 1/n, & \text{for } 1 \leq k \leq n, \\ 0, & \text{for } k \geq n + 1, \end{cases} \quad n \in \mathbb{IN}.$$

It is clear that $\|x^n\|_{\ell^1} = 1$, whereas $\|x^n\|_{\ell^\infty} = 1/n$ for every $n \in \mathbb{IN}$. Consequently, one has

$$\alpha(\text{Id}) = \inf_{\|u\|_{\ell^1}=1} \|\text{Id } u\|_{\ell^\infty} \leq \inf_{n \in \mathbb{IN}} \|x^n\|_{\ell^\infty} = 0,$$

so Id is not strongly metrically subregular (anywhere). On the other hand, it is clear that $\text{Id} : \ell^1 \rightarrow \ell^1$ is strongly metrically subregular, with $\text{subreg } \text{Id}(x) = 1$, for every $x \in \ell^1$.

Recall that the injectivity constant is connected with the Banach constant of linear operators, through transposition. Namely, given $\Lambda(\mathbf{X}, \mathbf{Y})$, one defines

$$\beta(\Lambda) = \alpha(\Lambda^\top),$$

where Λ^\top stands for the transpose of Λ . This allows one to link the notion of strong metric subregularity with that of metric regularity, which is well known to amount to openness (at a linear rate) in the case of linear operators. More precisely, whenever Λ is open, one has $0 < \beta(\Lambda) = \alpha(\Lambda^\top)$, so that Λ^\top is strongly metrically subregular. For more details, see [36]. The current example helps also to illustrate the fact that, whereas the appearance of sharp minimality for a given function is a symptom of nonsmoothness, strong metric subregularity is a property that may happen to take place for very nice (even linear) mappings.

Example 2.4 Let $\varphi : \mathbf{X} \rightarrow \mathbb{IR} \cup \{+\infty\}$ be a proper, l.s.c. convex function defined on a Banach space \mathbf{X} , whose dual is indicated by \mathbf{X}^* . Let us denote by $\partial\varphi(\bar{x})$ the subdifferential of φ at $\bar{x} \in \text{dom } \varphi$ in the sense of convex analysis. Generalizing a previous result valid in Hilbert spaces, in [37] it has been proved that the set-valued mapping $\partial\varphi : \mathbf{X} \rightrightarrows \mathbf{X}^*$ is strongly metrically subregular at $(\bar{x}, \bar{x}^*) \in \text{graph } \partial\varphi$ if and only if there exist positive γ and r such that

$$\varphi(x) \geq \varphi(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle + \gamma \|x - \bar{x}\|^2, \quad \forall x \in B(\bar{x}, r),$$

where $\langle \cdot, \cdot \rangle : \mathbf{X}^* \times \mathbf{X} \rightarrow \mathbb{IR}$ denotes the duality pairing \mathbf{X}^* and \mathbf{X} . In particular, in the case of a (global) minimizer \bar{x} of φ , $\partial\varphi$ is strongly metrically subregular at $(\bar{x}, \mathbf{0}^*)$, where $\mathbf{0}^*$ stands for the null vector of \mathbf{X}^* , iff

$$\varphi(x) \geq \varphi(\bar{x}) + \gamma \|x - \bar{x}\|^2, \quad \forall x \in B(\bar{x}, r).$$

The last inequality formalizes a variational behaviour known as *quadratic growth condition*, which has been studied in connection with second-order sufficient conditions in nonlinear programming (see [38]). Notice that, if a function admits a sharp minimizer, it satisfies the quadratic growth condition around that point, but the converse may not be true. Similar characterizations of various metric regularity properties have been

recently extended to the Mordukhovich subdifferential mapping (see [39]). Investigations by means of second-order variational analysis tools revealed that they are also interrelated to the tilt stability of local minimizer (see [39,40]).

Remark 2.3 From Example 2.4, it is possible to see at once that if φ is a proper, l.s.c. convex function, whose subdifferential mapping is strongly metrically subregular at $(\bar{x}, \mathbf{0}^*)$, where $\bar{x} \in \text{dom } \varphi$ is a minimizer of φ , then \bar{x} turns out to be Tykhonov well-posed, namely every minimizing sequence $\{x_n\}$ of φ converges to \bar{x} . Now, it is worth noting that the notion of strong metric subregularity generalizes, yet in a local form, such a behaviour to solutions of equations/inclusions. More precisely, if a set-valued mapping $F : X \rightrightarrows Y$, defining with \bar{y} the inclusion $\bar{y} \in F(x)$, is strongly metrically subregular at (\bar{x}, \bar{y}) , then for every sequence $\{y_n\}$ in Y , with $y_n \rightarrow \bar{y}$ as $n \rightarrow \infty$, and for every sequence $\{x_n\}$ in X of solutions of the inclusions with data perturbed $y_n \in F(x)$, according to (2) one finds

$$d(x_n, \bar{x}) \leq \kappa \text{dist}(\bar{y}, F(x_n)) \leq \kappa d(\bar{y}, y_n),$$

so $x_n \rightarrow \bar{x}$, provided that the elements of $\{x_n\}$ fall in a proper neighbourhood of \bar{x} .

3 Perturbation Stability

As it happens for other Lipschitzian properties of multifunctions, a method for establishing criteria or conditions for the validity of strong metric subregularity consists in analysing its stability in the presence of perturbations. Two results of this type are presented in what follows, which are both proved through the criterion discussed in Sect. 2.

Theorem 3.1 *Let $F : X \rightrightarrows Y$ be a set-valued mapping between metric spaces and let $g : Z \rightarrow X$ a mapping defined on a metric space. Let $\bar{z} \in Z$ and let $(g(\bar{z}), \bar{y}) \in \text{graph } F$. Suppose that:*

- (i) g is continuous at \bar{z} and strongly metrically subregular at \bar{z} ;
- (ii) F is strongly metrically subregular at $(g(\bar{z}), \bar{y})$.

Then, their composition $F \circ g : Z \rightrightarrows Y$ is strongly metrically subregular at (\bar{z}, \bar{y}) , and it results in

$$\text{subreg}(F \circ g)(\bar{z}, \bar{y}) \leq \text{subreg } g(\bar{z}) \cdot \text{subreg } F(g(\bar{z}), \bar{y}).$$

Proof Set $\bar{x} = g(\bar{z})$. Since F is strongly metrically subregular at (\bar{x}, \bar{y}) , corresponding to an arbitrary $\kappa_F > \text{subreg } F(\bar{x}, \bar{y})$, there exists $r > 0$ such that

$$\text{dist}(\bar{y}, F(x)) \geq \frac{1}{\kappa_F} d(x, \bar{x}), \quad \forall x \in B(\bar{x}, r). \tag{9}$$

Owing to hypothesis (i), corresponding to an arbitrary $\kappa_g > \text{subreg } g(\bar{x})$, there exists $\delta > 0$ such that, up to a further reduction of its value, if needed,

$$d(\bar{x}, g(z)) \geq \frac{1}{\kappa_g} d(z, \bar{z}) \quad \text{and} \quad g(z) \in B(\bar{x}, r), \quad \forall z \in B(\bar{z}, \delta). \quad (10)$$

From inequalities (10) and (9), one obtains

$$\frac{\text{dist}(\bar{y}, F(g(z)))}{d(z, \bar{z})} \geq \frac{d(g(z), \bar{x})}{\kappa_F d(z, \bar{z})} \geq \frac{1}{\kappa_F \kappa_g}, \quad \forall z \in B(\bar{z}, \delta) \setminus \{\bar{z}\},$$

whence $|F \circ g|^\downarrow(\bar{z}, \bar{y}) \geq (\kappa_F \kappa_g)^{-1} > 0$ follows. To complete the proof, it remains to apply Proposition 2.1. \square

The following example shows that the continuity assumption on the inner mapping cannot be dropped out, in general.

Example 3.1 Let $Z = X = Y = \mathbb{R}$ be endowed with the usual Euclidean metric structure. Consider the functions $g: \mathbb{R} \rightarrow \mathbb{R}$ and $F: \mathbb{R} \rightarrow \mathbb{R}$, defined, respectively, by

$$g(z) = \begin{cases} 0, & \text{if } z = 0, \\ 2, & \text{otherwise,} \end{cases} \quad \text{and} \quad F(x) = \begin{cases} |x|, & \text{if } |x| \leq 1, \\ 2 - |x|, & \text{otherwise.} \end{cases}$$

Here $\bar{z} = \bar{x} = \bar{y} = 0$. It is evident that both g and F have a sharp minimizer at 0, whereas their composition $F \circ g \equiv 0$ does not.

Strong metric subregularity is not preserved under composition of set-valued mappings, as shown by the next counterexample.

Example 3.2 In the same setting as in Example 3.1, let $G: \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$G(z) = \begin{cases} \mathbb{R}, & \text{if } z = 0, \\ \{2\}, & \text{otherwise,} \end{cases}$$

and let F be as in the previous example. As one readily checks, G is strongly metrically subregular at $(0, 0)$. If composing G and F , one finds

$$(F \circ G)(z) = \begin{cases}] - \infty, 1], & \text{if } z = 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is easily seen that $F \circ G$ fails to be strongly metrically subregular at $(0, 0)$. Notice that multifunction G , the inner term of the composition, is upper hemicontinuous at 0.

For the next result, a slightly more structured setting is needed.

Theorem 3.2 *Let $F : X \rightrightarrows Y$ be a set-valued mapping defined on a metric space X and taking values in a linear metric space Y , whose metric is shift invariant. If F is strongly metrically subregular at $(\bar{x}, \bar{y}) \in \text{graph } F$, then for any mapping $g : X \rightarrow Y$ such that $\text{subreg } F(\bar{x}, \bar{y}) \cdot \text{clm } g(\bar{x}) < 1$, the set-valued mapping $F + g$ is strongly metrically subregular at $(\bar{x}, \bar{y} + g(\bar{x}))$ and it results in*

$$\text{subreg } (F + g)(\bar{x}, \bar{y} + g(\bar{x})) \leq \frac{\text{subreg } F(\bar{x}, \bar{y})}{1 - \text{subreg } F(\bar{x}, \bar{y}) \cdot \text{clm } g(\bar{x})}. \tag{11}$$

Proof Notice that, by virtue of the shift invariance of the metric on Y , for any $x \in X$ one has

$$\text{dist}(\bar{y}, F(x)) \leq \text{dist}(\bar{y} + g(\bar{x}), F(x) + g(x)) + d(g(x), g(\bar{x})).$$

Consequently, one obtains

$$\begin{aligned} |F + g|^\downarrow(\bar{x}, \bar{y} + g(\bar{x})) &\geq \liminf_{x \rightarrow \bar{x}} \frac{\text{dist}(\bar{y}, F(x)) - d(g(x), g(\bar{x}))}{d(x, \bar{x})} \\ &\geq |F|^\downarrow(\bar{x}, \bar{y}) - \limsup_{x \rightarrow \bar{x}} \frac{d(g(x), g(\bar{x}))}{d(x, \bar{x})} \\ &\geq \frac{1}{\text{subreg } F(\bar{x}, \bar{y})} - \text{clm } g(\bar{x}) > 0. \end{aligned}$$

The strong metric subregularity of $F + g$ at $(\bar{x}, \bar{y} + g(\bar{x}))$ follows at once by the characterization provided in Proposition 2.1, whereas the estimate (11) is a straightforward consequence of (4). □

Remark 3.1 (i) The result provided by Theorem 3.2 on the persistence of strong metric subregularity under calm additive perturbations can be found in [9] (see Theorem 3I.6), formulated for multifunctions acting in finite-dimensional spaces, with a different proof. It is worth noting that, since neither the Ekeland variational principle nor the convergence of iteration procedures are used in the proof of Theorem 3.2, metric completeness plays no role in the above robustness phenomenon. Instead, it may be viewed as a direct consequence of a stability behaviour for the local sharp minimality called *superstability*, which was observed already by Polyak (see [33]). Essentially, it means that a point preserves its local minimality even in the presence of additive calm perturbations. This makes the robustness of strong metric subregularity different from the corresponding behaviour of metric regularity, requiring on the one hand metric completeness and on the other hand the Lipschitz property of perturbations (see [9,41]). A result similar to Theorem 3.2, obtained in a more specific context, appears in [23] (see Theorem 3).

(ii) Note that the shift invariance assumption on the metric of Y is not actually restrictive. Indeed, a result due to Kakutani ensures that any linear metric space can be equivalently remetrized by a shift-invariant metric (see Theorem 2.2.11 in [42]).

4 Strong Metric Subregularity of Nonsmooth Mappings

The main subject of this paper is the strong metric subregularity of (possibly) nonsmooth mappings $f : \mathbf{X} \rightarrow \mathbf{Y}$. To deal with them, throughout this section $(\mathbf{X}, \|\cdot\|)$ and $(\mathbf{Y}, \|\cdot\|)$ are supposed to be normed (vector) spaces. The (closed) unit ball and the unit sphere in any normed space are indicated by \mathbf{B} and \mathbf{S} , respectively, whereas, in the case of dual spaces, they are indicated by \mathbf{B}^* and \mathbf{S}^* , respectively.

4.1 A Criterion Via First-Order ϵ -Approximations

Differentiability is a wise combination of linearity and approximation. The approach of analysis considered in this subsection relies on the employment of positively homogeneous (for short, p.h.) mappings as an appealing substitute of derivatives (that are linear operators), while calmness replaces the classical convergence of the remainder term. To do so, set

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = \{h : \mathbf{X} \rightarrow \mathbf{Y} : \text{p.h. and continuous at } \mathbf{0}\}.$$

Definition 4.1 Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be a mapping between normed spaces, let $\bar{x} \in \mathbf{X}$ and let $\epsilon > 0$. A mapping $h \in \mathcal{H}(\mathbf{X}, \mathbf{Y})$ is said to be a *first-order ϵ -approximation* of f at \bar{x} if

$$\text{clm}(f - h(\cdot - \bar{x}))(\bar{x}) < \epsilon.$$

Remark 4.1 Whenever h is a first-order ϵ -approximation of f at \bar{x} , the mapping $f(\bar{x}) + h(\cdot - \bar{x})$ is a special case of what is called in [9] an “estimator”. Of course, first-order ϵ -approximation is a nonsmooth analysis notion, which allows to include (Fréchet) differentiability. Indeed, note that if f is Fréchet differentiable at \bar{x} , with derivative $Df(\bar{x}) \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, then $Df(\bar{x})$ is a first-order ϵ -approximation of f at \bar{x} , for every $\epsilon > 0$. P.h. functions and mappings, or some special classes of them, have been utilized as a rough material for constructing generalized derivatives since the very birth of nonsmooth analysis (see, for instance, [43, 44]). On the other hand, the idea of studying properties of nonlinear mappings by means of “approximate differentials”, which avoid differentiability assumptions, precedes even nonsmooth analysis (see, for instance, [45]).

After having replaced linear operators with p.h. mappings, the next step consists in extending to $\mathcal{H}(\mathbf{X}, \mathbf{Y})$ the definition of injectivity constant, by letting

$$\alpha_0(h) = \inf_{\|u\|=1} \|h(u)\|.$$

Remark 4.2 By applying Proposition 2.1, it is readily seen that $h \in \mathcal{H}(\mathbf{X}, \mathbf{Y})$ is strongly metrically subregular at $\mathbf{0}$ if and only if it holds $\alpha_0(h) > 0$. It is to be noted, however, that, in contrast with the linear case, such a characterization is not valid for the strong metric subregularity of h at every point of \mathbf{X} . Consider, for instance, the

norm function $\|\cdot\| : \mathbf{X} \rightarrow \mathbb{R}$, with the dimension of \mathbf{X} being greater than 1 (possibly infinite). Clearly, it is $\alpha_0(\|\cdot\|) = 1$. Taking any element \bar{u} in the unit sphere \mathbf{S} of \mathbf{X} , one finds

$$\|\cdot\|^\downarrow(\bar{u}) = \liminf_{x \rightarrow \bar{u}} \frac{\| \|x\| - \|\bar{u}\| \|}{\|x - \bar{u}\|} \leq \sup_{\delta > 0} \inf_{x \in \mathbf{S} \cap B(\bar{u}, \delta) \setminus \{\bar{u}\}} \frac{\| \|x\| - \|\bar{u}\| \|}{\|x - \bar{u}\|} = 0,$$

and hence $\|\cdot\|$ fails to be strongly metrically subregular at \bar{u} , even though it is so at $\mathbf{0}$.

Theorem 4.1 *Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be a mapping between normed spaces and let $\bar{x} \in \mathbf{X}$. If f is first-order ϵ -approximated at \bar{x} by $h \in \mathcal{H}(\mathbf{X}, \mathbf{Y})$, with $\alpha_0(h) > \epsilon$, then f is strongly metrically subregular at \bar{x} , and*

$$\text{subreg } f(\bar{x}) \leq \frac{1}{\alpha_0(h) - \epsilon}.$$

Vice versa, if f is strongly metrically subregular at \bar{x} , then for any mapping $h \in \mathcal{H}(\mathbf{X}, \mathbf{Y})$ first-order ϵ -approximating f at \bar{x} , with $\epsilon < \text{subreg } f(\bar{x})$, it results in

$$\alpha_0(h) \geq \frac{1}{\text{subreg } f(\bar{x}) - \epsilon},$$

so h is strongly metrically subregular at $\mathbf{0}$.

Proof According to Definition 4.1, for both the assertions of the thesis the respective hypotheses imply

$$\text{clm } (f - h(\cdot - \bar{x}))(\bar{x}) = \text{clm } (h(\cdot - \bar{x}) - f)(\bar{x}) < \epsilon.$$

To prove the first assertion, it suffices to apply Theorem 3.2, with F and g given by

$$F(x) = f(\bar{x}) + h(x - \bar{x}) \quad \text{and} \quad g(x) = f(x) - f(\bar{x}) - h(x - \bar{x}),$$

respectively, and to observe that F is strongly metrically subregular at \bar{x} iff h is so at $\mathbf{0}$, while $\text{clm } g(\bar{x}) = \text{clm } (f - h(\cdot - \bar{x}))(\bar{x})$.

Analogously, to prove the second assertion, it suffices to apply once again Theorem 3.2, now with

$$F(v) = f(\bar{x} + v) + f(\bar{x}) \quad \text{and} \quad g(v) = h(v) - f(\bar{x} + v) - f(\bar{x}).$$

Indeed, f is strongly metrically subregular at \bar{x} iff F is so at $\mathbf{0}$, whereas

$$\text{clm } g(\mathbf{0}) = \text{clm } (h(\cdot - \bar{x}) - f)(\bar{x}).$$

The quantitative estimates complementing the thesis are direct consequences of inequality (11). □

As a special case of Theorem 4.1, it is possible to derive the following criterion for smooth mappings.

Corollary 4.1 *A Fréchet differentiable mapping $f : \mathbf{X} \rightarrow \mathbf{Y}$ between normed spaces is strongly metrically subregular at $\bar{x} \in \mathbf{X}$ if and only if $\alpha(Df(\bar{x})) > 0$ and*

$$\text{subreg } f(\bar{x}) \leq \frac{1}{\alpha(Df(\bar{x}))}.$$

In particular, if \mathbf{X} and \mathbf{Y} are finite-dimensional spaces, f is strongly metrically subregular at \bar{x} if and only if

$$\text{Ker } Df(\bar{x}) = \{\mathbf{0}\}.$$

4.2 A Sufficient Condition Via Outer ϵ -Prederivative

In order to introduce the next tool of nonsmooth analysis to be used, recall that a set-valued mapping $F : \mathbf{X} \rightrightarrows \mathbf{Y}$ is said to be p.h. if $\mathbf{0} \in F(\mathbf{0})$ and $F(tx) = tF(x)$ for all $x \in \mathbf{X}$ and $t > 0$. In [46] p.h. set-valued mappings have been used to define a notion of generalized derivative. Below, a generalization of it, which seems to be adequate for the purposes of the present analysis, is introduced.

Definition 4.2 Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be a mapping between normed spaces, let $\bar{x} \in \mathbf{X}$ and let $\mathcal{A} : \mathbf{X} \rightrightarrows \mathbf{Y}$ a p.h. homogeneous set-valued mapping. Given $\epsilon > 0$, \mathcal{A} is said to be an *outer (Fréchet) ϵ -prederivative* of f at \bar{x} if there exists $\delta > 0$ and a function $r : \delta\mathbf{B} \rightarrow [0, \epsilon]$ such that

$$f(\bar{x} + v) - f(\bar{x}) \in \mathcal{A}(v) + r(v)\|v\|\mathbf{B}, \quad \forall v \in \delta\mathbf{B}. \tag{12}$$

Remark 4.3 Recall that, according to [46], \mathcal{A} is said to be an outer prederivative of f at \bar{x} if (12) holds true with a function $r : \delta\mathbf{B} \rightarrow [0, \epsilon]$ such that $\lim_{v \rightarrow \mathbf{0}} r(v) = 0$. In such an event, \mathcal{A} is an outer ϵ -prederivative of f at \bar{x} , for every $\epsilon > 0$. It is worth noting that set-valued homogeneous approximations of single-valued mappings have been recently utilized in connection with primal estimates of metric regularity of single-valued mappings (see [47]).

Given a p.h. set-valued mapping $\mathcal{A} : \mathbf{X} \rightrightarrows \mathbf{Y}$, to detect its strong metric subregularity at $(\mathbf{0}, \mathbf{0})$, it seems to be natural to introduce an injectivity constant notion as follows

$$\alpha(\mathcal{A}) = \inf_{\|u\|=1} \text{dist}(\mathbf{0}, \mathcal{A}(u)).$$

In the light of Proposition 2.1, it is readily seen that \mathcal{A} is strongly metrically subregular at $(\mathbf{0}, \mathbf{0})$ if and only if $\alpha(\mathcal{A}) > 0$ and, upon this condition, it is

$$\text{subreg } \mathcal{A}(\mathbf{0}, \mathbf{0}) = \frac{1}{\alpha(\mathcal{A})}.$$

By employing the above nonsmooth analysis tools, one can establish the following sufficient condition for strong metric subregularity.

Theorem 4.2 *Suppose that a mapping $f : \mathbf{X} \rightarrow \mathbf{Y}$ between normed spaces admits an outer ϵ -prederivative $\mathcal{A} : \mathbf{X} \rightrightarrows \mathbf{Y}$ at \bar{x} , such that $\alpha(\mathcal{A}) > \epsilon$. Then, f is strongly metrically subregular at \bar{x} and*

$$\text{subreg } f(\bar{x}) \leq \frac{1}{\alpha(\mathcal{A}) - \epsilon}.$$

If, in particular, \mathcal{A} is an outer prederivative of f at \bar{x} , then stricter estimate holds

$$\text{subreg } f(\bar{x}) \leq \frac{1}{\alpha(\mathcal{A})}.$$

Proof Let positive ϵ, δ and $r : \delta\mathbf{B} \rightarrow [0, \epsilon]$ be as in Definition 4.2. Then one obtains for every $x \in B(\bar{x}, \delta)$

$$\begin{aligned} \|f(\bar{x}) - f(x)\| &= d(\mathbf{0}, f(x) - f(\bar{x})) \\ &\geq \text{dist}(\mathbf{0}, \mathcal{A}(x - \bar{x}) + r(x - \bar{x})\|x - \bar{x}\|\mathbf{B}), \end{aligned} \tag{13}$$

On the other hand, observe that for any $v \in \delta\mathbf{B} \setminus \{\mathbf{0}\}$ it is

$$\begin{aligned} \text{dist}(\mathbf{0}, \mathcal{A}(v/\|v\|) + r(v)\mathbf{B}) &= \inf_{y \in \mathcal{A}(v/\|v\|), u \in \mathbf{B}} \|y + r(v)u\| \\ &\geq \inf_{y \in \mathcal{A}(v/\|v\|), u \in \mathbf{B}} [\|y\| - |r(v)|\|u\|] \\ &\geq \text{dist}(\mathbf{0}, \mathcal{A}(v/\|v\|)) - r(v). \end{aligned}$$

Therefore, letting $\bar{y} = f(\bar{x})$, from inequality (13), it follows

$$\begin{aligned} |f|^\downarrow(\bar{x}, \bar{y}) &\geq \liminf_{x \rightarrow \bar{x}} \frac{\text{dist}(\mathbf{0}, \mathcal{A}(x - \bar{x}) + r(x - \bar{x})\|x - \bar{x}\|\mathbf{B})}{\|x - \bar{x}\|} \\ &\geq \liminf_{x \rightarrow \bar{x}} \text{dist}\left(\mathbf{0}, \mathcal{A}\left(\frac{x - \bar{x}}{\|x - \bar{x}\|}\right) + r(x - \bar{x})\mathbf{B}\right) \\ &\geq \inf_{\|u\|=1} \text{dist}(\mathbf{0}, \mathcal{A}(u)) - \limsup_{x \rightarrow \bar{x}} r(x - \bar{x}) = \inf_{\|u\|=1} \text{dist}(\mathbf{0}, \mathcal{A}(u)) - \epsilon. \end{aligned}$$

Thus, to show the first assertion in the thesis, it suffices to apply the characterization stated in Proposition 2.1, along with the estimate (4). For the second assertion, on account of Remark 4.3 and of the last inequalities, one has that

$$|f|^\downarrow(\bar{x}, \bar{y}) \geq \alpha(\mathcal{A}) - \epsilon, \quad \forall \epsilon \in]0, \alpha(\mathcal{A})[.$$

which leads immediately to the estimate to be proved. □

Remark 4.4 Among the p.h. set-valued mappings that can be used as prederivatives, one can consider in particular those generated by a convex weakly closed set of linear operators. In other terms, given a set $\mathcal{U} \subseteq \mathcal{L}(\mathbf{X}, \mathbf{Y})$ convex and closed with respect to the weak topology, let

$$\mathcal{A}(x) = \{y \in \mathbf{Y} : y = \Lambda x, \Lambda \in \mathcal{U}\}.$$

According to [46], this is an example of fan. In this case, one has

$$\alpha(\mathcal{A}) = \inf_{\Lambda \in \mathcal{U}} \alpha(\Lambda).$$

Notice that, whenever f is Fréchet differentiable at \bar{x} , Definition 4.2 applies with $\mathcal{A}(v) = \{Df(\bar{x})v\}$ and $r(v) = \|f(\bar{x} + v) - f(\bar{x}) - Df(\bar{x})v\|/\|v\|$, being now $\mathcal{U} = \{Df(\bar{x})\}$. Therefore, the sufficient part of Corollary 4.1 can be achieved also from Theorem 4.2.

4.3 A Scalarization Approach

Let $(\mathbf{Y}, \|\cdot\|)$ be now a normed space, which is partially ordered by a relation $\leq_{\mathbf{Y}}$ or, equivalently, by a convex cone $\mathbf{Y}_+ \subseteq \mathbf{Y}$, in the sense that

$$y_1 \leq_{\mathbf{Y}} y_2 \quad \text{iff} \quad y_2 - y_1 \in \mathbf{Y}_+.$$

In this setting, a mapping $f : \mathbf{X} \rightarrow \mathbf{Y}$ is said to be \mathbf{Y}_+ -convex if

$$f(tx_1 + (1 - t)x_2) \leq_{\mathbf{Y}} tf(x_1) + (1 - t)f(x_2), \quad \forall t \in [0, 1], \quad \forall x_1, x_2 \in \mathbf{X}.$$

\mathbf{Y}_+ -convex mappings are found quite easily in nature. For instance, if $f : \mathbf{X} \rightarrow \mathbb{R}^m$ is given by $f(x) = (f_1(x), \dots, f_m(x))$, with each function $f_i : \mathbf{X} \rightarrow \mathbb{R}$ being convex, then f is \mathbb{R}_+^m -convex, where $\mathbb{R}_+^m = \{y \in \mathbb{R}^m : y_i \geq 0, i = 1, \dots, m\}$. One immediately sees that if f is \mathbf{Y}_+ -convex and if $y^* \in \mathbf{Y}_+^* = \{y^* \in \mathbf{Y}^* : \langle y^*, y \rangle \geq 0, \forall y \in \mathbf{Y}_+\}$, then each scalar function $y^* \circ f : \mathbf{X} \rightarrow \mathbb{R}$ is convex.

Since the scalarization approach to strong metric subregularity exploits the connection of that property with the sharp minimality of scalarized terms, the following characterization of a sharp minimizer for a convex function will be useful in the sequel.

Lemma 4.1 *Let $\varphi : \mathbf{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. An element $\bar{x} \in \text{dom } \varphi$ is a (global) sharp minimizer of φ if and only if $\mathbf{0}^* \in \text{int } \partial\varphi(\bar{x})$. Moreover, it results in*

$$\varphi^\downarrow(\bar{x}) = \sup \{ \rho > 0 : \rho \mathbf{B}^* \subseteq \partial\varphi(\bar{x}) \}.$$

Proof Necessity: suppose point \bar{x} to be a sharp minimizer of φ and take $\epsilon \in (0, \varphi^\downarrow(\bar{x}))$. Then, setting $\rho = \varphi^\downarrow(\bar{x}) - \epsilon$, one gets

$$\frac{\varphi(x) - \varphi(\bar{x})}{\|x - \bar{x}\|} \geq \rho \geq \left\langle x^*, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle, \quad \forall x^* \in \rho\mathbf{B}^*, \forall x \in \mathbf{X} \setminus \{\bar{x}\}.$$

This means that $\rho\mathbf{B}^* \subseteq \partial\varphi(\bar{x})$ and, by arbitrariness of ϵ , that $\varphi^\downarrow(\bar{x}) \leq \sup\{\rho > 0 : \rho\mathbf{B}^* \subseteq \partial\varphi(\bar{x})\}$.

Sufficiency: from the definition of subgradient of φ at \bar{x} , it is possible to deduce

$$\frac{\varphi(x) - \varphi(\bar{x})}{\|x - \bar{x}\|} \geq \sup_{x^* \in \partial\varphi(\bar{x})} \left\langle x^*, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle, \quad \forall x \in \mathbf{X} \setminus \{\bar{x}\}.$$

Since by hypothesis there exists $\rho > 0$ such that $\rho\mathbf{B}^* \subseteq \partial\varphi(\bar{x})$, it is true that

$$\sup_{x^* \in \partial\varphi(\bar{x})} \langle x^*, u \rangle \geq \sup_{x^* \in \rho\mathbf{B}^*} \langle x^*, u \rangle = \rho, \quad \forall u \in \mathbf{X} : \|u\| = 1.$$

From this and the previous inequality, it is possible to conclude that

$$\varphi^\downarrow(\bar{x}) \geq \rho > 0.$$

Actually, this shows that $\varphi^\downarrow(\bar{x}) \geq \sup\{\rho > 0 : \rho\mathbf{B}^* \subseteq \partial\varphi(\bar{x})\}$. The proof is complete. □

To formulate the next condition for the strong metric subregularity of a \mathbf{Y}_+ -convex mapping $f : \mathbf{X} \rightarrow \mathbf{Y}$ at $\bar{x} \in \mathbf{X}$, define

$$\varrho(f)(\bar{x}) = \sup \{ \rho > 0 : \rho\mathbf{B}^* \subseteq \partial(y^* \circ f)(\bar{x}), y^* \in \mathbf{S}^* \cap \mathbf{Y}_+^* \}.$$

Theorem 4.3 *Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be a mapping between normed spaces, with \mathbf{Y} partially ordered by a cone \mathbf{Y}_+ , and let $\bar{x} \in \mathbf{X}$. Suppose that f is \mathbf{Y}_+ -convex and*

$$\mathbf{0}^* \in \bigcup_{y^* \in \mathbf{S}^* \cap \mathbf{Y}_+^*} \text{int } \partial(y^* \circ f)(\bar{x}).$$

Then f is strongly metrically subregular at \bar{x} . Moreover, one has

$$\text{subreg } f(\bar{x}) \leq \frac{1}{\varrho(f)(\bar{x})}.$$

Proof From the well-known dual representation of a norm

$$\|v\| = \sup_{y^* \in \mathbf{B}^*} \langle y^*, v \rangle = \sup_{y^* \in \mathbf{S}^*} \langle y^*, v \rangle,$$

one obtains

$$\begin{aligned} \frac{\|f(\bar{x}) - f(x)\|}{\|x - \bar{x}\|} &= \sup_{y^* \in \mathbf{S}^*} \left\langle y^*, \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \right\rangle \\ &\geq \sup_{y^* \in \mathbf{S}^* \cap \mathbf{Y}_+^*} \frac{(y^* \circ f)(x) - (y^* \circ f)(\bar{x})}{\|x - \bar{x}\|}, \quad \forall x \in \mathbf{X} \setminus \{\bar{x}\}. \end{aligned}$$

Since by hypothesis there exist $\rho > 0$ and $y_0^* \in \mathbf{S}^* \cap \mathbf{Y}_+^*$ such that $\rho \mathbf{B}^* \subseteq \partial(y_0^* \circ f)(\bar{x})$ and function $y_0^* \circ f$ is convex, then in the light of Lemma 4.1, \bar{x} is a sharp minimizer of $y_0^* \circ f$. Consequently, setting $\bar{y} = f(\bar{x})$, from the last inequality one has

$$|f|^\downarrow(\bar{x}, \bar{y}) \geq (y_0^* \circ f)^\downarrow(\bar{x}) \geq \rho.$$

The proof of all assertions in the thesis is therefore completed by applying Proposition 2.1. □

Theorem 4.3 demonstrates a typical use of the scalarization method in the presence of convexity assumption. It should be clear that this method extends its potential far beyond convexity and can be employed in combination with more general subdifferential constructions. For example, by utilizing the (Fréchet) regular subdifferential, defined as

$$\widehat{\partial}\varphi(\bar{x}) = \left\{ x^* \in \mathbf{X}^* : \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}$$

(for more details, the reader is referred to [2,41,48–50]), the following finite-dimensional generalization of Lemma 4.1 has been proved in [34] (see Theorem 4 therein).

Lemma 4.2 *Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and $\bar{x} \in \text{dom } \varphi$. Then $\varphi^\downarrow(\bar{x}) > 0$ if and only if $\mathbf{0}^* \in \text{int } \widehat{\partial}\varphi(\bar{x})$.*

The above lemma allows one to obtain the next result valid for mappings defined in a finite-dimensional space.

Theorem 4.4 *Given a mapping $f : \mathbb{R}^n \rightarrow \mathbf{Y}$ and $\bar{x} \in \mathbb{R}^n$, if*

$$\mathbf{0}^* \in \bigcup_{y^* \in \mathbf{S}^*} \text{int } \widehat{\partial}(y^* \circ f)(\bar{x}),$$

then f is strongly metrically subregular at \bar{x} .

Proof The thesis can be achieved through the same argument as in the proof of Theorem 4.3. Indeed, by hypothesis there exists $y_0^* \in \mathbf{S}^*$ such that $\mathbf{0}^* \in \text{int } \widehat{\partial}(y_0^* \circ f)(\bar{x})$. In the light of Lemma 4.2, this implies

$$(y_0^* \circ f)^\downarrow(\bar{x}) > 0. \tag{14}$$

Since it is

$$\begin{aligned} \frac{\|f(\bar{x}) - f(x)\|}{\|x - \bar{x}\|} &= \sup_{y^* \in \mathbf{S}^*} \left\langle y^*, \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \right\rangle \\ &\geq \frac{(y_0^* \circ f)(x) - (y_0^* \circ f)(\bar{x})}{\|x - \bar{x}\|}, \quad \forall x \in \mathbf{X} \setminus \{\bar{x}\}, \end{aligned}$$

by virtue of inequality (14), it follows

$$|f|^\downarrow(\bar{x}, \bar{y}) \geq (y_0^* \circ f)^\downarrow(\bar{x}) > 0.$$

Proposition 2.1 allows one to complete the proof. □

5 An Application

An application of the above exposed ideas and techniques is going to be illustrated now, which concerns the stability behaviour of solution mappings to generalized equations. A generalized equation is a rather general problem that is able to provide a proper framework for studying several specific issues in mathematical analysis, having or not having a variational nature. Among the others, let us mention optimality conditions in constrained or unconstrained optimization, various types of constraint systems, variational inequalities and complementarity problems, equilibrium problems and differential inclusions.

Here, parameterized generalized equations are considered that can be formalized as follows

$$\mathbf{0} \in f(p, x) + T(x), \tag{GE}$$

where $f : P \times \mathbf{X} \rightarrow \mathbf{Y}$ (sometimes referred to as the base of (GE)) and $T : \mathbf{X} \rightrightarrows \mathbf{Y}$ (referred to as the field) are the problem data. (P, d) is a metric space where the parameters vary, whereas $(\mathbf{X}, \|\cdot\|)$ and $(\mathbf{Y}, \|\cdot\|)$ are supposed to be normed vector spaces. The solution mapping associated with (GE) is the (generally) set-valued mapping implicitly defined by

$$S_{f,T}(p) = \{x \in \mathbf{X} : \mathbf{0} \in f(p, x) + T(x)\}.$$

In this context, an issue of interest is how to certify and to quantify a certain stability behaviour of $S_{f,T}$, near a solution $\bar{x} \in S_{f,T}(\bar{p})$. More precisely, here the stability behaviour quantitatively described by the isolated calmness property is investigated. This amounts to establish for solutions to (GE) lying near a reference one a reaction, which is (directly) proportional to the parameter variations. Following the spirit of classical and more recent implicit function theorems, this question is approached by analysing a simplified variant of (GE), called *approximated generalized equation* (AGE), on which the main regularity assumption is made. Of course, a (AGE) can be defined in several ways, depending on the features of the problem data. In what

follows, dealing with a nonsmooth analysis setting, the use of an adaptation of the outer ϵ -prederivative is proposed.

Definition 5.1 Given $\epsilon > 0$, a p.h. set-valued mapping $\mathcal{A} : \mathbf{X} \rightrightarrows \mathbf{Y}$ is said to be a *partial outer ϵ -prederivative* of a mapping $f : P \times \mathbf{X} \longrightarrow \mathbf{Y}$ at (\bar{p}, \bar{x}) , *uniformly with respect to p* , if there exist positive δ and ζ and a function $r : P \times \delta\mathbf{B} \longrightarrow [0, \epsilon]$ such that

$$f(p, x) \in f(p, \bar{x}) + \mathcal{A}(x - \bar{x}) + r(p, x - \bar{x})\|x - \bar{x}\|\mathbf{B}, \quad \forall x \in B(\bar{x}, \delta), \forall p \in B(\bar{p}, \zeta).$$

Now, assuming that the base f of (GE) admits, for some $\epsilon > 0$, as a partial outer ϵ -prederivative at (\bar{p}, \bar{x}) a mapping \mathcal{A} , one can associate with (GE) an approximated generalized equation defined by

$$\mathbf{0} \in f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x). \tag{AGE}$$

It turns out that the strong metric subregularity of the mapping $\mathcal{A} + T$ is a key assumption to guarantee the isolated calmness property of $S_{f,T}$, as below stated.

Theorem 5.1 *With reference to a generalized equation (GE), let $\bar{x} \in S_{f,T}(\bar{p})$. Suppose the data of (GE) to satisfy the following assumptions:*

- (i) *function $f(\cdot, \bar{x})$ is calm at \bar{p} with modulus $\text{clm } f(\cdot, \bar{x})(\bar{p})$;*
- (ii) *f admits a partial outer ϵ -prederivative \mathcal{A} at (\bar{p}, \bar{x}) , uniformly with respect to p ;*
- (iii) *the set-valued mapping $x \rightrightarrows f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)$ is strongly metrically subregular at $(\bar{x}, \mathbf{0})$, with modulus $\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0})$, such that*

$$\epsilon \cdot \text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) < 1. \tag{15}$$

Then, $S_{f,T}$ has the isolated calmness property at (\bar{p}, \bar{x}) and the below estimate holds

$$\text{clm } S_{f,T}(\bar{p}, \bar{x}) \leq \frac{\text{clm } f(\cdot, \bar{x})(\bar{p}) \cdot \text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0})}{1 - \epsilon \cdot \text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0})}. \tag{16}$$

Proof Take an arbitrary η such that

$$0 < \eta < \frac{1}{\epsilon} - \text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}),$$

which is possible by virtue of condition (15). By the assumption (i), corresponding to η there exists $\zeta > 0$ such that

$$\|f(p, \bar{x}) - f(\bar{p}, \bar{x})\| \leq (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p}), \quad \forall p \in B(\bar{p}, \zeta).$$

By the assumption (ii), there exist $\tilde{\zeta} \in]0, \zeta[$, $\delta > 0$ and a function $r : P \times \delta\mathbf{B} \rightarrow [0, \epsilon]$ such that

$$f(p, x) \in f(p, \bar{x}) + \mathcal{A}(x - \bar{x}) + r(p, x - \bar{x})\|x - \bar{x}\|\mathbf{B}, \forall x \in B(\bar{x}, \delta), \forall p \in B(\bar{p}, \tilde{\zeta}).$$

Consequently, one obtains

$$\begin{aligned} & \text{dist}(\mathbf{0}, f(p, x) + T(x)) \\ & \geq \text{dist}(\mathbf{0}, f(p, \bar{x}) + \mathcal{A}(x - \bar{x}) + r(p, x - \bar{x})\|x - \bar{x}\|\mathbf{B} + T(x)) \\ & \geq \text{dist}(\mathbf{0}, f(p, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)) - \epsilon\|x - \bar{x}\| \\ & \geq \text{dist}(\mathbf{0}, f(\bar{p}, \bar{x}) + (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p})\mathbf{B} + \mathcal{A}(x - \bar{x}) + T(x)) \\ & \quad - \epsilon\|x - \bar{x}\| \\ & \geq \text{dist}(\mathbf{0}, f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)) - (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p}) - \epsilon\|x - \bar{x}\| \end{aligned}$$

for every $x \in B(\bar{x}, \delta)$ and $p \in B(\bar{p}, \tilde{\zeta})$, wherefrom it follows

$$\begin{aligned} \text{dist}(\mathbf{0}, f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)) & \leq \text{dist}(\mathbf{0}, f(p, x) + T(x)) \\ & \quad + (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p}) + \epsilon\|x - \bar{x}\|. \end{aligned}$$

Now, according to assumption (iii), since \bar{x} is evidently a solution to (AGE), corresponding to $\eta > 0$ there exists $\tilde{\delta} \in (0, \delta)$ such that

$$\begin{aligned} \|x - \bar{x}\| & \leq (\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta)\text{dist}(\mathbf{0}, f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)) \\ & \leq (\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta) \left(\text{dist}(\mathbf{0}, f(p, x) + T(x)) \right. \\ & \quad \left. + (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p}) + \epsilon\|x - \bar{x}\| \right) \end{aligned}$$

and hence

$$\begin{aligned} (1 - \epsilon(\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta))\|x - \bar{x}\| & \leq (\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta) \\ & \quad \times \left(\text{dist}(\mathbf{0}, f(p, x) + T(x)) + (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)d(p, \bar{p}) \right) \end{aligned}$$

for every $x \in B(\bar{x}, \tilde{\delta})$ and $p \in B(\bar{p}, \tilde{\zeta})$. As a consequence, whenever it is $x \in S_{f,T}(p) \cap B(\bar{x}, \tilde{\delta})$, it results in

$$\|x - \bar{x}\| \leq \frac{(\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta) \cdot (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)}{(1 - \epsilon(\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta))} d(p, \bar{p}),$$

for every $p \in B(\bar{p}, \tilde{\zeta})$. The last inequality shows that $S_{f,T}$ has the isolated calmness property at (\bar{p}, \bar{x}) with

$$\text{clm } S_{f,T}(\bar{p}, \bar{x}) \leq \frac{(\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta) \cdot (\text{clm } f(\cdot, \bar{x})(\bar{p}) + \eta)}{(1 - \epsilon(\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) + \eta))}.$$

From the last inequality and the arbitrariness of η , it is possible to deduce the estimate in the thesis, thereby completing the proof. \square

Corollary 5.1 *With reference to a generalized equation (GE), let $\bar{x} \in S_{f,T}(\bar{p})$. Suppose the data of (GE) to satisfy the following assumptions:*

- (i) *function $f(\cdot, \bar{x})$ is calm at \bar{p} with modulus $\text{clm } f(\cdot, \bar{x})(\bar{p})$;*
- (ii') *f has a partial outer prederivative \mathcal{A} at (\bar{p}, \bar{x}) , uniformly with respect to p ;*
- (iii) *the set-valued mapping $x \rightrightarrows f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)$ is strongly metrically subregular at $(\bar{x}, \mathbf{0})$, with modulus $\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0})$.*

Then $S_{f,T}$ has the isolated calmness property at (\bar{p}, \bar{x}) and the stricter estimate

$$\text{clm } S_{f,T}(\bar{p}, \bar{x}) \leq \text{clm } f(\cdot, \bar{x})(\bar{p}) \cdot \text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) \tag{17}$$

holds.

Proof The thesis can be easily achieved by applying Theorem 5.1. Recall indeed that with assumption (ii') being valid, \mathcal{A} is an outer partial ϵ -prederivative of f at (\bar{p}, \bar{x}) , for any $\epsilon > 0$. Then, it suffices to observe that, since ϵ can be taken arbitrarily “small”, condition (15) is fulfilled independently of the value of $\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0})$. \square

Remark 5.1 A result quite close to Theorem 5.1, called “implicit mapping theorem with strong metric subregularity”, can be found in [9] (Theorem 3I.12). Instead of prederivatives, partial estimators (i.e. first-order ϵ -approximations) are employed there. In this regard, it is must be noted that the technique of proof in Theorem 5.1 can be readily adapted to derive a version of it, employing partial first-order ϵ -approximations of the base term. Characterizations of the isolated calmness property of solution mappings in the specific context of generalized equations arising in parametric cone programming have been recently established in [51, 52]. These results have been achieved following a different approach, which relies on the Levy’s full characterization of isolated calmness for closed-graph multifunctions between finite-dimensional spaces (see [53]). Such an approach, requiring involved elements of second-order variational analysis, has been devised on the base of the specific structure of generalized equations expressing optimality conditions in cone programming and seems not to be easily embedded in the theory resulting from Theorem 5.1.

Theorem 5.1 reduces the study of the isolated calmness property of $S_{f,T}$ to the certification of the strong metric subregularity of the set-valued mapping defining (AGE). The latter question is expected to be easier to be faced than a direct study of $S_{f,T}$, inasmuch as the former set-valued mapping is explicitly defined in terms of problem data or their approximation, while $S_{f,T}$ can be hardly calculated in practice.

Besides, in some special case, the study of the strong metric subregularity of the set-valued mapping defining (AGE) may happen to be particularly simple. Let us consider, as an example, the case in which the field term T happens to be single-valued near \bar{x} .

Corollary 5.2 *Let $\bar{x} \in S_{f,T}(\bar{p})$ be a solution to (GE). Suppose that:*

- (i) *function $f(\cdot, \bar{x})$ is calm at \bar{p} with modulus $\text{clm } f(\cdot, \bar{x})(\bar{p})$;*
- (ii) *f admits a partial outer ϵ -prederivative \mathcal{A} at (\bar{p}, \bar{x}) , uniformly with respect to p ;*
- (iii) *T is single-valued near \bar{x} and calm at \bar{x} , with modulus $\text{clm } T(\bar{x})$;*
- (iv) *the following condition holds*

$$\alpha(\mathcal{A}) - \text{clm } T(\bar{x}) > \epsilon. \tag{18}$$

Then $S_{f,T}$ has the isolated calmness property at (\bar{p}, \bar{x}) and the following modulus estimate holds

$$\text{clm } S_{f,T}(\bar{p}, \bar{x}) \leq \frac{\text{clm } f(\cdot, \bar{x})(\bar{p})}{\alpha(\mathcal{A}) - \text{clm } T(\bar{x}) - \epsilon}.$$

Proof Observe first of all that $\mathcal{A}(\cdot - \bar{x})$ is strongly metrically subregular at $(\bar{x}, \mathbf{0})$ iff \mathcal{A} is so at $(\mathbf{0}, \mathbf{0})$, and one has

$$\text{subreg } \mathcal{A}(\cdot - \bar{x})(\bar{x}, \mathbf{0}) = \text{subreg } \mathcal{A}(\mathbf{0}, \mathbf{0}).$$

Under the current assumptions, it is possible to apply Theorem 3.2, with $F = \mathcal{A}(\cdot - \bar{x})$ and $g = f(\bar{p}, \bar{x}) + T$. Indeed, it is clear that

$$\text{clm } (f(\bar{p}, \bar{x}) + T)(\bar{x}) = \text{clm } T(\bar{x}),$$

so, in force of condition (18), it holds

$$\text{subreg } \mathcal{A}(\cdot - \bar{x})(\bar{x}, \mathbf{0}) \cdot \text{clm } (f(\bar{p}, \bar{x}) + T)(\bar{x}) = \frac{\text{clm } T(\bar{x})}{\alpha(\mathcal{A})} < 1.$$

Consequently, the set-valued mapping $x \rightrightarrows f(\bar{p}, \bar{x}) + \mathcal{A}(x - \bar{x}) + T(x)$ turns out to be strongly metrically subregular at $(\bar{x}, \mathbf{0})$, with

$$\text{subreg } (\mathcal{A} + T)(\bar{x}, \mathbf{0}) \leq \frac{1}{\alpha(\mathcal{A}) - \text{clm } T(\bar{x})}.$$

One is therefore in a position to apply Theorem 5.1, as the validity of condition (15) is ensured by the assumption (18). Thus the proof is complete. □

In the remaining part of this section, while continuing to assume T to be single-valued, a further result is presented, which can be obtained via the scalarization approach.

Theorem 5.2 *With reference to a generalized equation (GE), let $\bar{x} \in S_{f,T}(\bar{p})$. Suppose that:*

- (i) *T is single-valued near \bar{x} and calm at \bar{x} , with modulus $\text{clm } T(\bar{x})$;*
- (ii) *$f(\cdot, x)$ is calm at \bar{p} , uniformly with respect to x near \bar{x} , with modulus $\text{clm } f(\cdot, x)(\bar{p})$;*
- (iii) *the space $(\mathbf{Y}, \|\cdot\|)$ is partially ordered by a cone \mathbf{Y}_+ and the mapping $f(\bar{p}, \cdot)$ is \mathbf{Y}_+ -convex;*
- (iv) *both the conditions*

$$\mathbf{0}^* \in \bigcup_{y^* \in \mathbf{S}^* \cap \mathbf{Y}_+^*} \text{int } \partial(y^* \circ f)(\bar{x}) \tag{19}$$

and

$$\frac{\text{clm } T(\bar{x})}{\varrho(f(\bar{p}, \cdot))(\bar{x})} < 1 \tag{20}$$

hold true.

Then $S_{f,T}$ has the isolated calmness property at (\bar{p}, \bar{x}) and the following modulus estimate holds

$$\text{clm } S_{f,T}(\bar{p}, \bar{x}) \leq \frac{\text{clm } f(\cdot, x)(\bar{p})}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})}.$$

Proof Hypothesis (iii) and condition (19) allow one to apply Theorem 4.3 to the mapping $f(\bar{p}, \cdot)$. According to it, $f(\bar{p}, \cdot)$ turns out to be strongly metrically subregular at \bar{x} and it results in

$$\text{subreg } f(\bar{p}, \cdot)(\bar{x}) \leq \frac{1}{\varrho(f(\bar{p}, \cdot))(\bar{x})}.$$

Now, since owing to condition (20), it is

$$\text{subreg } f(\bar{p}, \cdot)(\bar{x}) \cdot \text{clm } T(\bar{x}) < 1,$$

Theorem 3.2 guarantees that the mapping $x \mapsto f(\bar{p}, x) + T(x)$ is strongly metrically subregular at $(\bar{x}, \mathbf{0})$ and that it results in

$$\text{subreg } (f(\bar{p}, \cdot) + T)(\bar{x}, \mathbf{0}) \leq \frac{1}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})}.$$

This means that, corresponding to $\eta > 0$, there exists $r > 0$ such that

$$\|x - \bar{x}\| \leq \frac{(1 + \eta)\|f(\bar{p}, x) + T(x)\|}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})}, \quad \forall x \in B(\bar{x}, r).$$

By taking account of hypothesis (i), one has that for some $\zeta > 0$ and $\tilde{r} \in (0, r)$ it holds

$$\begin{aligned} \|x - \bar{x}\| &\leq \frac{(1 + \eta)\|f(\bar{p}, x) - f(p, x)\| + \|f(p, x) + T(x)\|}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})} \\ &\leq \frac{(1 + \eta)[(\text{clm } f(\cdot, x)(\bar{p}) + \eta)d(p, \bar{p}) + \|f(p, x) + T(x)\|]}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})} \end{aligned}$$

for every $x \in B(\bar{x}, \tilde{r})$ and $p \in B(\bar{p}, \zeta)$. Thus, if taking $x \in B(\bar{x}, \tilde{r}) \cap S_{f,T}(p)$, one finds

$$\|x - \bar{x}\| \leq \frac{(1 + \eta)(\text{clm } f(\cdot, x)(\bar{p}) + \eta)}{\varrho(f(\bar{p}, \cdot))(\bar{x}) - \text{clm } T(\bar{x})} d(p, \bar{p}), \quad \forall p \in B(\bar{p}, \zeta).$$

The last inequality shows that $S_{f,T}$ fulfils the isolated calmness property at (\bar{p}, \bar{x}) , and by arbitrariness of η , it allows one to achieve the asserted modulus estimation. \square

6 Conclusions

The approach of analysis proposed in this paper shows that several techniques for detecting strong metric subregularity of nonsmooth mappings can be derived from a unique elementary criterion, based on the notion of steepest displacement rate, which can be formulated already in a metric space setting. This criterion, besides providing a unifying scheme of analysis with transparent proofs, emphasizes the variational nature of the property under study. Optimization (especially, nondifferentiable optimization) is well recognized as a field where many results and constructions of set-valued analysis are fruitfully applied. The findings of the present study should contribute to make it evident that, symmetrically, nondifferentiable optimization can provide useful insights and methods for investigating properties of multifunctions; some of them not necessarily related to extremum problems. This seems to agree with the very spirit of the Euler’s variational faith.

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