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Rank One Non-Hermitian Perturbations of Hermitian β-Ensembles of Random Matrices

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Abstract We provide a tridiagonal matrix model and compute the joint eigenvalue density of a rank one non-Hermitian perturbation of a random matrix from the Gaussian or Laguerre β-ensemble.

Keywords Non-Hermitian random matrices · β-Ensembles · Resonances · Jacobi matrices

1 Introduction

The energy Hamiltonian of a closed quantum system is usually modelled by a Hermitian random matrix H. The Hamiltonian of this system after coupling it to the outer world via s open channels is modelled by the so-called effective Hamiltonian[1]

$$H_{eff} = H + i\Gamma, \tag{1.1}$$

where $\Gamma \geq \mathbf{0}$ is a rank s positive semi-definite Hermitian matrix that is independent of H. The eigenvalues of H_{eff} are the mathematical model for the *resonances*, which are the long-lived decaying states of our open quantum system.

In this paper we are concerned with the exact joint distribution of these eigenvalues when there is one open channel ($s = 1$), and H is a Gaussian or Laguerre (Wishart) orthogonal/unitary/symplectic random matrix. Γ may be deterministic or random with a given distribution function. We obtain tridiagonal models (in the spirit of Dumitriu–Edelman [2]) and compute the joint eigenvalue distribution for any $\beta > 0$, not merely $\beta = 1, 2, 4$ (Theorems 3 and 4).

The joint eigenvalue law for non-Hermitian perturbations of Laguerre ensembles has not been addressed in the literature before (however, see [11] for a related topic), while the joint

[1] In the physics literature it is more common to take $H - i\Gamma$, which can be reduced to our case by a simple symmetry.

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eigenvalue law for non-Hermitian perturbations of Gaussian ensembles has been studied in the physics literature by numerous authors: Ullah [19] (for the case $\beta = 1$), Sokolov–Zelevinsky [15] ($\beta = 1$), Stöckmann–Šeba [17] ($\beta = 1, 2$), Fyodorov–Khoruzhenko [5] ($\beta = 2$). The present paper provides a rigorous derivation of this law which works for any $\beta > 0$ and for any choice of Γ—deterministic or random. More importantly, our approach can be applied to other models, e.g., perturbations of Laguerre β-ensembles (done in this paper); of chiral Gaussian β-ensembles; multiplicative perturbations of Gaussian and Laguerre β-ensembles (to be explored in a forthcoming paper). We also expect that the tridiagonal matrix models proposed here will be useful for establishing asymptotic properties of these "weakly non-Hermitian" ensembles. Finally, we note that our methods can provide matrix models (namely, *block* Jacobi matrices with independent (matrix-valued) Jacobi coefficients) for higher order perturbations $s \geq 2$ as well, which could prove to be useful for computing their eigenvalue density (for the case $\beta = 2$, $s \geq 2$, Fyodorov–Khoruzhenko [5] provide another approach). The solution to this matrix-valued eigenvalue problem is currently beyond our reach. We leave this as a challenging open problem.

The asymptotic analysis of the weakly non-Hermitian ensembles are of high interest in the mathematics and physics literature and have been studied in [3,4,6,14], see also [11,12]. The numerous physical applications of such random matrices can be found in the review papers [6,7,10].

The important cornerstones of our proofs are the Dumitriu–Edelman Hermitian matrix models [2], and the Arlinskiĭ–Tsekanovskiĭ result [1] on the spectral analysis of (deterministic) Jacobi matrices.

2 Preliminaries

2.1 Gaussian and Laguerre Ensembles

Definition 1 Denote by $N(0,\sigma)$, $N(0,\sigma\mathbf{I}_2)$, and $N(0,\sigma\mathbf{I}_4)$ the real, complex, and quaternionic normal random variables (r.v.) with variance $\beta\sigma^2$ ($\beta = 1, 2, 4$, respectively).

Denote by χ_k^2 ($k > 0$) a real r.v. with p.d.f. $\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$. Denote by χ_k ($k > 0$) a square root of a χ_k^2 r.v., and $\tilde{\chi}_k$ ($k > 0$) to be $\frac{1}{\sqrt{2}}\chi_k$.

Definition 2 Let Y be an $n \times n$ matrix with independent identically distributed (i.i.d.) entries chosen from $N(0,1)$, $N(0,\mathbf{I}_2)$, or $N(0,\mathbf{I}_4)$. Then we say that $X = \frac{1}{2}(Y + Y^*)$ belongs to the Gaussian orthogonal/unitary/symplectic ensemble, respectively. We denote it by GOE_n, GUE_n, GSE_n, respectively.

Definition 3 Let Y be an $m \times n$ matrix with i.i.d. entries chosen from $N(0,1)$, $N(0,\mathbf{I}_2)$, or $N(0,\mathbf{I}_4)$. Then we say that the $n \times n$ matrix $X = Y^*Y$ belongs to the Laguerre (Wishart) orthogonal/unitary/symplectic ensemble, respectively. We denote it by $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$, respectively.

2.2 Tridiagonalization of Hermitian Matrices

Let H be an $n \times n$ Hermitian matrix. Denote $\mathbf{e}_j$ to be the j-th standard vector in $\mathbb{C}^n$, that is, having 1 in its j-th entry and 0 everywhere else. Let $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^*\mathbf{y}$, the usual inner product in $\mathbb{C}^n$. Let us apply the Gram–Schmidt orthogonalization procedure in $\mathbb{C}^n$ to the sequence of vectors $\mathbf{e}_1, H\mathbf{e}_1, H^2\mathbf{e}_1, \ldots, H^{k-1}\mathbf{e}_1$, where $k = \dim\operatorname{span}\{H^j\mathbf{e}_1 : j \geq 0\}$. Note that

$1 \le k \le n$. After normalization we obtain an orthonormal sequence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in $\mathbb{C}^n$. If $k < n$, then we choose an arbitrary unit vector $\mathbf{v}_{k+1}$ in $\mathbb{C}^n \ominus \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and repeat the procedure but with $\mathbf{v}_{k+1}$ instead of $\mathbf{e}_1$. By repeating this procedure finitely many times more if necessary and combining all the resulting vectors together, we obtain an orthonormal basis $\{\mathbf{v}_j\}_{j=1}^n$ of $\mathbb{C}^n$.

Standard arguments (see, e.g., [16, Sect 1.3]) show that the matrix of H in the basis $\{\mathbf{v}_j\}_{j=1}^n$ is tridiagonal. In other words, if we form unitary matrix S with $\{\mathbf{v}_j\}_{j=1}^n$ as its columns, then $S^* H S = \mathcal{J}$, where

$$\mathcal{J} = S^* H S = \begin{pmatrix} b_1 & a_1 & 0 & & \\ a_1 & b_2 & a_2 & \ddots & \\ 0 & a_2 & b_3 & \ddots & 0 \\ & \ddots & \ddots & \ddots & a_{n-1} \\ & & 0 & a_{n-1} & b_n \end{pmatrix}, \quad a_j \ge 0, b_n \in \mathbb{R}. \tag{2.1}$$

We call matrices of the form (2.1) Jacobi, and the coefficients $\{a_j, b_j\}$—their Jacobi coefficients. For a future reference, observe that

$$S\mathbf{e}_1 = S^* \mathbf{e}_1 = \mathbf{e}_1 \tag{2.2}$$

since $\mathbf{v}_1 = \mathbf{e}_1$ in the Gram–Schmidt procedure. Note that in the tridiagonalization procedure above, if $\dim \operatorname{span}\{H^j \mathbf{e}_1 : j \ge 0\} = k < n$, then $a_j > 0$ for $1 \le j \le k-1$, and $a_k = 0$, i.e., $\mathcal{J}$ becomes a direct sum of Jacobi matrices of smaller sizes.

2.3 Matrix Models for Gaussian and Laguerre Ensembles

Now let us apply the tridiagonalization procedure from the previous section to a random matrix from a Gaussian or a Laguerre ensemble. This is the idea of Dumitriu–Edelman [2] (see also Trotter's [18]).

If H is from GOE_n, GUE_n, or GSE_n, then $\mathbf{e}_1$ is a cyclic vector for H with probability 1. Therefore we obtain (2.1) with $a_j > 0$ for all $1 \le j \le n-1$.

The same is true for a random matrix H from $LOE_{(m,n)}$, $LUE_{(m,n)}$, or $LSE_{(m,n)}$, but only if $m \ge n$. If $m < n$, then with probability 1, $\dim \operatorname{span}\{H^j \mathbf{e}_1 : j \ge 0\} = m+1 \le n$, and $\mathbb{C}^n \ominus \operatorname{span}\{H^j \mathbf{e}_1 : j \ge 0\} \subseteq \ker H$, so that the resulting Jacobi matrix (2.1) that we obtain has $a_{m+1} = \dots = a_{n-1} = 0$, $b_{m+2} = \dots = b_n = 0$. In other words, we have that $\mathcal{J}$ is the direct sum of an $(m+1) \times (m+1)$ Jacobi matrix and the $(n-m-1) \times (n-m-1)$ zero matrix. The proof of this case can be done by following the Dumitriu–Edelman [2] arguments.

Lemma 1 (Dumitriu–Edelman [2]) *Let H be a random matrix taken from GOE_n, GUE_n, or GSE_n ensemble. There exists a (random) unitary matrix S satisfying (2.2) such that $SHS^* = \mathcal{J}$ is tridiagonal (2.1), where*

$$a_j \sim \tilde{\chi}_{\beta(n-j)}, \qquad 1 \le j \le n-1,$$
$$b_j \sim N(0,1), \qquad 1 \le j \le n,$$

where $\beta = 1, 2, 4$ for GOE_n, GUE_n, GSE_n, respectively.

Lemma 2 (Dumitriu–Edelman [2]) *Let H be a random matrix taken from $LOE_{(m,n)}$, $LUE_{(m,n)}$, or $LSE_{(m,n)}$ ensemble. There exists a (random) unitary matrix S satisfying (2.2) such that $SHS^* = \mathcal{J} = B^* B$ is tridiagonal (2.1), where*

$$B = \begin{pmatrix} x_1 & y_1 & 0 & & \\ 0 & x_2 & y_2 & \ddots & \\ 0 & 0 & x_3 & \ddots & 0 \\ & \ddots & \ddots & \ddots & y_{n-1} \\ & & 0 & 0 & x_n \end{pmatrix}, \quad \textit{with} \tag{2.3}$$

(i) *If* $m \geq n$*:*

$$x_j \sim \chi_{\beta(m-j+1)}, \qquad 1 \leq j \leq n,$$
$$y_j \sim \chi_{\beta(n-j)}, \qquad 1 \leq j \leq n-1;$$

(ii) *If* $m \leq n-1$*:*

$$x_j \sim \begin{cases} \chi_{\beta(m-j+1)}, & \textit{if } 1 \leq j \leq m, \\ 0, & \textit{if } m+1 \leq j \leq n, \end{cases}$$
$$y_j \sim \begin{cases} \chi_{\beta(n-j)}, & \textit{if } 1 \leq j \leq m, \\ 0, & \textit{if } m+1 \leq j \leq n-1; \end{cases}$$

where $\beta = 1, 2, 4$ *for* $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$*, respectively.*

Remarks 1. For GSE_n and $LSE_{(m,n)}$ every entry is quaternionic, so all the instances of $\mathbb{C}$ in the arguments above should be replaced with the algebra of quaternions. The resulting coefficients a_j, b_j, x_j, y_j in Lemmas 1 and 2 are quaternionic too, but with the i, j, and k parts equal to zero.

2. It is worth reminding the reader that the random matrix S in Lemmas 1 and 2 is statistically independent of $\mathcal{J}$.

2.4 β-Ensembles

The tridiagonal matrix ensembles from Lemmas 1 and 2 make sense for any $\beta > 0$, not merely for $\beta = 1, 2, 4$. They are called the Gaussian β-ensemble $G\beta E_n$ and the Laguerre β-ensemble $L\beta E_{(m,n)}$, respectively.

2.5 Spectral Measures of Gaussian and Laguerre β-Ensembles

By the Riesz representation theorem, for any Hermitian matrix H there exists a probability measure μ (called the spectral measure) satisfying

$$\langle \mathbf{e}_1, H^k \mathbf{e}_1 \rangle = \int_{\mathbb{R}} x^k d\mu(x), \quad \text{for all} \quad k \geq 0. \tag{2.4}$$

In fact, any Hermitian can be unitarily diagonalized, so that we can write $H = UDU^*$, where D is the diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of H on the diagonal, and the columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of U are the corresponding orthonormal eigenvectors of H. This easily implies (2.4) with

$$\mu(x) = \sum_{j=1}^{n} w_j \delta_{\lambda_j}, \quad \text{where} \quad w_j = |\langle \mathbf{e}_1, \mathbf{u}_j \rangle|^2. \tag{2.5}$$

Here δ_λ is the Dirac measure at λ. The support of μ consists of $\leq n$ points.

As our matrix H is random, its spectral measure is random too. The joint law of w_j's and λ_j's in (2.5) will be referred to as the law of the spectral measure of H.

Because of (2.2), the laws of the spectral measures of H and of its Jacobi form $\mathcal{J}$ coincide, that is, H and $\mathcal{J}$ have identically distributed eigenvalues λ_j's and eigenweights w_j's. In particular, laws of the spectral measures of GOE_n and $G\beta E_n$ with $\beta = 1$ coincide; laws of the spectral measures of GUE_n and $G\beta E_n$ with $\beta = 2$ coincide; laws of the *quaternion-valued* spectral measures of GSE_n and $G\beta E_n$ with $\beta = 4$ (viewed as a matrix with purely-real quaternion entries) coincide. The analogous statements hold true for the Laguerre case.

Laws of the spectral measures for $G\beta E_n$ and $L\beta E_{(m,n)}$ with $m \ge n$ have been computed in [2], see Lemmas 3 and 4 below. We also need the spectral measure of $L\beta E_{(m,n)}$ when $m < n$, which we compute in Proposition 1 below.

Lemma 3 (Dumitriu–Edelman [2]) *For any $\beta > 0$, the spectral measure of a random matrix from the $G\beta E_n$-ensemble is (2.5) with the joint distribution*

$$\tfrac{1}{g_{\beta,n}} \prod_{j=1}^n e^{-\lambda_j^2/2} \prod_{1\le j<k\le n} |\lambda_j - \lambda_k|^\beta d\lambda_1 \dots d\lambda_n \times \tfrac{1}{c_{\beta,n}} \prod_{j=1}^n w_j^{\beta/2-1} dw_1 \dots dw_{n-1}, \tag{2.6}$$

where

$$\sum_{j=1}^n w_j = 1; \quad w_j > 0, \quad 1 \le j \le n; \quad \lambda_j \in \mathbb{R}, \tag{2.7}$$

$$g_{\beta,n} = (2\pi)^{n/2} \prod_{j=1}^n \frac{\Gamma(1+\beta j/2)}{\Gamma(1+\beta/2)}, \quad c_{\beta,n} = \frac{\Gamma(\beta/2)^n}{\Gamma(\beta n/2)}. \tag{2.8}$$

Lemma 4 (Dumitriu–Edelman [2]) *For any $m \ge n$ and $\beta > 0$, the spectral measure of a random matrix from the $L\beta E_{(m,n)}$-ensemble is (2.5) with the joint distribution*

$$\begin{aligned} &\tfrac{1}{h_{\beta,n,a}} \prod_{j=1}^n \lambda_j^{\beta a/2} e^{-\lambda_j/2} \prod_{1\le j<k\le n} |\lambda_j - \lambda_k|^\beta d\lambda_1 \dots d\lambda_n \\ &\times \Gamma(\beta n/2) \prod_{j=1}^n \frac{w_j^{\beta/2-1}}{\Gamma(\beta/2)} dw_1 \dots dw_{n-1}, \end{aligned} \tag{2.9}$$

where $a = m - n + 1 - 2/\beta$ and

$$\sum_{j=1}^n w_j = 1; \quad w_j > 0, \quad 1 \le j \le n; \quad \lambda_j > 0, \tag{2.10}$$

$$h_{\beta,n,a} = 2^{n(a\beta/2+1+(n-1)\beta/2)} \prod_{j=1}^n \frac{\Gamma(1+\beta j/2)\Gamma(1+\beta a/2 + \beta(j-1)/2)}{\Gamma(1+\beta/2)}, \tag{2.11}$$

Proposition 1 *If $m \le n-1$ and $\beta > 0$, the spectral measure of a random matrix from the $L\beta E_{(m,n)}$ is*

$$\mu(x) = w_0\delta_0 + \sum_{j=1}^m w_j \delta_{\lambda_j}, \tag{2.12}$$

with the joint distribution

$$\frac{1}{h_{\beta,m,a}} \prod_{j=1}^{m} \lambda_j^{\beta a/2} e^{-\lambda_j/2} \prod_{1\le j<k\le m} |\lambda_j - \lambda_k|^{\beta} d\lambda_1 \dots d\lambda_m$$
$$\times \frac{w_0^{\beta(n-m)/2-1}}{\Gamma(\beta(n-m)/2)} \times \Gamma(\beta n/2) \prod_{j=1}^{m} \frac{w_j^{\beta/2-1}}{\Gamma(\beta/2)} dw_1 \dots dw_m, \tag{2.13}$$

where $a = n - m + 1 - 2/\beta$*;* $h_{\beta,m,a}$ *is as in* (2.11)*; and*

$$\sum_{j=0}^{m} w_j = 1; \quad w_j > 0, \quad 0 \le j \le m; \quad \lambda_j > 0. \tag{2.14}$$

Let us denote the normalization constant for w_j*'s as*

$$d_{\beta,m,n} = \frac{\Gamma(\beta(n-m)/2)\Gamma(\beta/2)^m}{\Gamma(\beta n/2)}. \tag{2.15}$$

Proof Let us first deal with $\beta = 1$ case. The distribution of the eigenvalues of a matrix H from $LOE_{(m,n)}$ is well-known. Let its eigenvalues be $\lambda_1 > \dots > \lambda_m > 0 = 0 = \dots = 0$ ($n - m$ zeros). Now choose an orthonormal system of (real) eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of H corresponding to these eigenvalues, respectively. We pick each $\mathbf{u}_j$ at random uniformly from the set of all possible choices. Since for any $n \times n$ orthogonal matrix O, the matrix $O^T H O$ also belongs to $LOE_{(m,n)}$, we can see that: $\mathbf{u}_1$ is uniformly distributed on the unit sphere $\{\mathbf{u} \in \mathbb{R}^n : ||\mathbf{u}|| = 1\}$; and for any $1 \le j \le n$, the vector $\mathbf{u}_j$ conditionally on $\mathbf{u}_1, \dots, \mathbf{u}_{j-1}$ is uniformly distributed on the subset of this unit sphere orthogonal to $\mathbf{u}_1, \dots, \mathbf{u}_{j-1}$. So the matrix consisting of $\mathbf{u}_1, \dots, \mathbf{u}_n$ as its columns is a Haar distributed orthogonal matrix (see, e.g., [9, Prop. 2.2(a)]). Then its first row $(v_1, \dots, v_n)$ is distributed uniformly on the unit sphere $\{\mathbf{u} \in \mathbb{R}^n : ||\mathbf{u}|| = 1\}$. Now recalling (2.5), we obtain that $w_j = v_j^2$, $1 \le j \le m$, and $w_0 = v_{m+1}^2 + \dots + v_n^2$. Now one can apply arguments from the proof of [8, Cor. A.2] (note that $dw_j = 2w_j^{1/2} dv_j$) to see that the joint distribution of $w_1, \dots, w_m$ is proportional to $w_0^{(n-m-2)/2} \prod_{j=1}^m w_j^{-1/2} dw_1 \dots dw_m$.

This allows us to compute the Jacobian for the change of variables from $\{x_j, y_j\}_{j=1}^m$ in (2.3) to $\{\lambda_j, w_j\}_{j=1}^m$. Why is this change of variables bijective? By Favard's theorem (see, e.g., [16, Thms. 1.3.2–1.3.3]), there is 1-to-1 correspondence between all $(m+1) \times (m+1)$ Jacobi matrices (2.1) with $a_j > 0$ $(1 \le j \le m)$ and all probability measures supported on $m+1$ distinct points. This means there is 1-to-1 correspondence between all positive semi-definite $(m+1) \times (m+1)$ Jacobi matrices $\mathcal{J}$ with $a_j > 0$ $(1 \le j \le m)$, $\det \mathcal{J} = 0$ and all probability measures supported on $m+1$ points of the form (2.12), (2.14). By semi-definiteness, any such $\mathcal{J}$ can be Cholesky factorized $\mathcal{J} = B^* B$ with B upper-triangular with non-negative entries on the diagonal. Since $\mathcal{J}$ is tridiagonal, this $(m+1) \times (m+1)$ matrix B must be two-diagonal as in (2.3) with $x_j \ge 0$, $1 \le j \le m+1$. Since $\det \mathcal{J} = 0$, we must have that $x_j = 0$ for at least one $1 \le j \le m+1$. But since all $a_j > 0$, we obtain that $x_{m+1} = 0$, $x_j > 0$ for $1 \le j \le m$, and $y_j > 0$, $1 \le j \le m$. Conversely, any $(m+1) \times (m+1)$ matrix B (2.3) with $x_j > 0$, $y_j > 0$ for $1 \le j \le m$ and $x_{m+1} = 0$ leads to a positive semi-definite $(m+1) \times (m+1)$ Jacobi matrix $\mathcal{J}$ with $\det \mathcal{J} = 0$ and $a_j > 0$ $(1 \le j \le m)$.

Using the matrix model in Lemma 2 (case $m < n$) and the distribution (2.13) that we proved for $\beta = 1$, we obtain that the Jacobian is proportional (let us ignore the normalizing constants for now) to

$$\det \frac{\partial(x_1,\ldots,x_m,y_1,\ldots,y_m)}{\partial(\lambda_1,\ldots,\lambda_m,w_1,\ldots,w_m)} \propto \prod_{j=1}^m x_j^{-m+j} e^{x_j^2/2} \prod_{j=1}^m y_j^{-n+j+1} e^{y_j^2/2}$$
$$\times\, w_0^{\frac{n-m}{2}-1} \prod_{j=1}^m w_j^{-\frac{1}{2}} \prod_{j=1}^m \lambda_j^{\frac{n-m-1}{2}} e^{-\frac{\lambda_j}{2}} \prod_{1\le j<k\le m} |\lambda_j-\lambda_k|.$$

Now taking the specified in Lemma 2(ii) joint distribution of $\{x_1,\ldots,x_m,y_1,\ldots,y_m\}$ for $L\beta E_{(m,n)}$, $m<n$, applying the the above Jacobian, and using the identities from Lemma 5 below, one obtains (2.13), up to a normalization. Finally, note that $h_{\beta,m,a}$ is the right normalization constant for the eigenvalues in (2.13) by Lemma 4. The normalization constant $d_{\beta,m,n}$ can be computed by evaluating the Dirichlet integral, see, e.g., [8, Cor. A.4]. □

Lemma 5 *The following identities hold:*

$$\prod_{j=1}^m x_j^{m-j+1} y_j^{m-j+1} = \prod_{j=0}^m w_j^{1/2} \prod_{1\le j<k\le m} |\lambda_j-\lambda_k| \prod_{j=1}^m \lambda_j, \tag{2.16}$$

$$\prod_{j=1}^m y_j^2 = w_0 \prod_{j=1}^m \lambda_j. \tag{2.17}$$

Proof (2.16) follows immediately by noting that $x_j y_j = a_j$, $1\le j\le m$, and then applying [2, Lemma 2.7]. Note the clash of notations: their n is our $m+1$, their $\{b_1,\ldots,b_{n-1}\}$, $\{\lambda_1,\ldots,\lambda_n\}$, and $\{q_1^2,\ldots,q_n^2\}$ are ours $\{a_m,\ldots,a_1\}$, $\{\lambda_1,\ldots,\lambda_m,0\}$, and $\{w_1,\ldots,w_m,w_0\}$, respectively. To prove (2.17), we use theory of orthogonal polynomials, see, e.g., [16]. By combining [16, Prop. 3.2.8] and [16, Prop. 2.3.12] we get

$$w_0 = -\lim_{z\to 0}\langle \mathbf{e}_1, z(\mathcal{J}-z)^{-1}\mathbf{e}_1\rangle = \lim_{z\to 0} \tfrac{z q_{m+1}(z)}{p_{m+1}(z)} = \tfrac{q_{m+1}(0)}{p'_{m+1}(0)},$$

where p_j's and q_j's are the orthonormal polynomials associated to $\mathcal{J}$ of the first and second kind, respectively (in order to define p_{m+1} and q_{m+1} we need a_{m+1} which we take to be an arbitrary positive number). By [16, Thm. 1.2.4], $p_{m+1}(z) = \left(\prod_{j=1}^{m+1} a_j^{-1}\right)\det(z-\mathcal{J})$, so $p'_{m+1}(0) = (-1)^m \prod_{j=1}^{m+1} a_j^{-1} \prod_{j=1}^m \lambda_j$. Using the Wronskian relation [16, Prop. 3.2.3] and $p_{m+1}(0)=0$ (since 0 is an eigenvalue of $\mathcal{J}$), we obtain $q_{m+1}(0) = 1/(a_{m+1}p_m(0))$. Finally, $p_m(z) = \left(\prod_{j=1}^m a_j^{-1}\right)\det(z-\mathcal{J}_{m\times m})$, where $\mathcal{J}_{m\times m}$ is the $m\times m$ top left corner of $\mathcal{J}$. Recall that $\mathcal{J} = B^*B$. It is easy to see that $\mathcal{J}_{m\times m} = B^*_{m\times m}B_{m\times m}$, where $B_{m\times m}$ is the $m\times m$ top left corner of B. Therefore $p_m(0) = (\prod_{j=1}^m a_j^{-1})\det(-B^*_{m\times m}B_{m\times m}) = (-1)^m(\prod_{j=1}^m a_j^{-1})\prod_{j=1}^m x_j^2$. Combining this all together with $a_j = x_j y_j$, $1\le j\le m$, we obtain (2.17). □

3 Rank One Perturbations: Location of the Eigenvalues

Let us discuss all attainable configurations of eigenvalues of rank one perturbations of (deterministic) Hermitian matrices. Part (i) of the following statement is certainly well-known (see, e.g., [1,11]), but (ii) and (iii) seem to be new.

For the rest of the paper let $\mathbb{C}_+ := \{z\in\mathbb{C} : \operatorname{Im} z > 0\}$.

Theorem 1 *Let H_{eff} be as in (1.1), where $H = H^*$, $\Gamma \ge \mathbf{0}$, $\operatorname{rank}\Gamma = 1$. Choose any $\mathbf{w}\in\operatorname{Ran}\Gamma$, $\mathbf{w}\ne 0$, and let $k = \dim\operatorname{span}\{H^j\mathbf{w} : j\ge 0\}$. Then:*

(i) *H_{eff} has k complex eigenvalues in $\mathbb{C}_+$ and $n-k$ real eigenvalues (counted with their algebraic multiplicities).*
(ii) *If $H > \mathbf{0}$, then the k complex eigenvalues $\{z_j\}_{j=1}^k$ of H_{eff} belong to the set $\{(z_j)_{j=1}^k \in (\mathbb{C}_+)^k : \sum_{j=1}^k \operatorname{Arg} z_j < \frac{\pi}{2}\}$, and every such a configuration may occur.*
(iii) *If $H \geq \mathbf{0}$ and $\det H = 0$, then the k complex eigenvalues $\{z_j\}_{j=1}^k$ of H_{eff} belong to the set $\{(z_j)_{j=1}^k \in (\mathbb{C}_+)^k : \sum_{j=1}^k \operatorname{Arg} z_j \leq \frac{\pi}{2}\}$, and every such a configuration may occur.*

Remark Using similar ideas one can prove the analogue for the case when H is not positive semi-definite, but has s negative eigenvalues. The k complex eigenvalues (the other $n-k$ being real) of H_{eff} then belong to $\left\{(z_j)_{j=1}^k \in (\mathbb{C}_+)^k : \frac{\pi}{2} + \pi(s-1) < \sum_{j=1}^k \operatorname{Arg} z_j \leq \frac{\pi}{2} + \pi s\right\}$, and every such a configuration may occur.

The proof relies on the following uniqueness+existence result for Jacobi matrices. We use n in (i) and $m+1$ in (ii) as the size of our matrices in order to be consistent with what follows later.

Proposition 2 *For $l > 0$, let*

$$\mathcal{J}_l = \mathcal{J} + il I_{1\times 1}, \tag{3.1}$$

where $I_{1\times 1}$ is the matrix with $(1,1)$-entry equal to 1 and 0 everywhere else.

(i) *Let $\mathcal{J}$ be an $n\times n$ positive definite (real) Jacobi matrix (2.1) with $a_j > 0$, $j = 1, \ldots, n-1$. Eigenvalues of $\mathcal{J}_l$, counting algebraic multiplicities, belong to*

$$\left\{ (z_j)_{j=1}^n \in (\mathbb{C}_+)^n : \sum_{j=1}^n \operatorname{Arg} z_j < \frac{\pi}{2} \right\}. \tag{3.2}$$

Moreover, for every configuration of n points from (3.2) there exists a unique matrix $\mathcal{J}_l$ of the form above with such a system of eigenvalues.

(ii) *Let $\mathcal{J}$ be an $(m+1)\times(m+1)$ positive semi-definite (real) Jacobi matrix (2.1) with $a_j > 0$, $j = 1, \ldots, m$, satisfying $\det \mathcal{J} = 0$. Eigenvalues of $\mathcal{J}_l$, counting with their algebraic multiplicities, belong to*

$$\left\{ (z_j)_{j=1}^{m+1} \in (\mathbb{C}_+)^{m+1} : \sum_{j=1}^{m+1} \operatorname{Arg} z_j = \frac{\pi}{2} \right\}. \tag{3.3}$$

Moreover, for every configuration of $m+1$ points from (3.3) there exists a unique matrix $\mathcal{J}_l$ of the form above with such a system of eigenvalues.

We will prove Proposition 2 in Sect. 5.2.

Proof of Theorem 1 Since $\Gamma \geq \mathbf{0}$, we can diagonalize $\Gamma = U(lI_{1\times 1})U^*$, where $l > 0$ and U is unitary. We may assume $\mathbf{w} = Ue_1$. Then $H_{eff} = U(U^*HU + ilI_{1\times 1})U^*$. Applying the tridiagonalization procedure from Sect. 2.2, we can reduce U^*HU to the Jacobi form (2.1): $U^*HU = S\mathcal{J}S^*$ with S unitary. Note that $k = \dim \operatorname{span}\{H^j \mathbf{w} : j \geq 0\} = \dim \operatorname{span}\{(U^*HU)^j \mathbf{e}_1 : j \geq 0\}$, so $a_j > 0$ for $1 \leq j \leq k-1$, $a_k = 0$ (see Sect. 2.2). Therefore $\mathcal{J}$ is a direct sum of a $k \times k$ Jacobi matrix with positive a_j's and some other $(n-k)\times(n-k)$ Jacobi matrix. Because of (2.2), $S^* I_{1\times 1} S = I_{1\times 1}$ and therefore

$$H_{eff} = US(\mathcal{J} + ilI_{1\times 1})S^*U^*. \tag{3.4}$$

Part (i) now follows from [1]. Part (ii) follows from Proposition 2(i). For the case (iii), $\det \mathcal{J} = 0$, but it might happen that the zero eigenvalue of $\mathcal{J}$ is an eigenvalue either of the $k \times k$ or $(n-k) \times (n-k)$ submatrix of $\mathcal{J}$. Thus either Proposition 2(i) or (ii) applies and finishes the proof. □

4 Rank One Perturbations: Tridiagonal Matrix Models

Let H be an $n \times n$ matrix from one of the six ensembles GOE_n, $LOE_{m\times n}$ ($\beta = 1$); GUE_n, $LUE_{m\times n}$ ($\beta = 2$); GSE_n, $LSE_{m\times n}$ ($\beta = 4$). Let H_{eff} be as in (1.1), where $\Gamma = (\Gamma_{jk})_{j,k=1}^n$ is an $n \times n$ positive definite (deterministic or random) matrix with real (if $\beta = 1$), complex (if $\beta = 2$), or quaternionic (if $\beta = 4$) entries. We assume that Γ is independent of H and has rank 1 (for the case $\beta = 4$, the (right) rank is viewed over quaternions, see, e.g., [13]).

Since $\Gamma \geq \mathbf{0}$, we can write $\Gamma = U(lI_{1\times 1})U^*$, where U is orthogonal, unitary, or unitary symplectic for $\beta = 1, 2, 4$, respectively (for quaternion diagonalization, see, e.g., [13, Thm. 5.3.6]). Since the Hilbert–Schmidt norm should be preserved, we see that $l = ||\Gamma||_{HS} = \left(\sum_{j,k=1}^n |\Gamma_{jk}|^2\right)^{1/2}$.

Then $H_{eff} = U(U^*HU + ilI_{1\times 1})U^*$, where U is independent of H. From Definitions 2 and 3, it is clear that U^*HU belongs to the same ensemble as H. Therefore we can apply the tridiagonalization procedure from Sect. 2.2 to reduce U^*HU to the Dumitriu–Edelman form: $U^*HU = S\mathcal{J}S^*$ with $\mathcal{J}$ as in Lemma 1 or 2, and S unitary satisfying $S^*I_{1\times 1}S = I_{1\times 1}$ (by 2.2), so (3.4) holds. We proved

Theorem 2 (Matrix model for rank one non-Hermitian perturbations of Gaussian and Laguerre ensembles) *Let H be taken from one of the six ensembles GOE_n, GUE_n, GSE_n, $LOE_{m\times n}$, $LUE_{m\times n}$, $LSE_{m\times n}$. Suppose the (deterministic or random) matrix Γ is independent of H and $\Gamma \geq \mathbf{0}$, $\operatorname{rank}\Gamma = 1$. Then $H_{eff} = H + i\Gamma$ is unitarily equivalent to*

$$\mathcal{J} + ilI_{1\times 1} \tag{4.1}$$

where $\mathcal{J}$ is as in Lemma 1 or 2, respectively, and $l = ||\Gamma||_{HS} = (\sum_{j,k=1}^n |\Gamma_{jk}|^2)^{1/2}$ is independent of $\mathcal{J}$.

Remark This tridiagonal matrix ensemble (4.1) makes sense for any $\beta > 0$.

5 Rank One Perturbations: Joint Eigenvalue Distribution

5.1 Perturbations of Gaussian β-Ensembles

Theorem 3 *Fix a deterministic $l > 0$, and for any $\beta > 0$ let $\mathcal{J}$ be from $G\beta E_n$ ensemble. Then the eigenvalues of $\mathcal{J}_l$, (4.1), are distributed on $\{(z_j) \in (\mathbb{C}_+)^n : \sum_{j=1}^n \operatorname{Im} z_j = l\}$ according to*

$$\frac{1}{h_{\beta,n}} e^{-\frac{1}{2}\sum_{j=1}^n \operatorname{Re}(z_j^2)} \times \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j<k} |z_j - z_k|^2$$
$$\times\, l^{-\frac{\beta n}{2}+1} e^{-\frac{l^2}{2}} d^2 z_1 \ldots d^2 z_{n-1} d(\operatorname{Re} z_n), \tag{5.1}$$

where d^2z stands for the 2-dimensional Lebesgue measure on $\mathbb{C}$; and

$$h_{\beta,n} = 2^{n(\beta/2-1)} g_{\beta,n} c_{\beta,n}, \tag{5.2}$$

where $g_{\beta,n}$ and $c_{\beta,n}$ are as in (2.8).

Remarks 1. In view of Theorem 2, distribution (5.1) with $\beta = 1, 2, 4$ is the eigenvalue distribution of rank one perturbations of GOE_n, GUE_n, GSE_n, respectively.

2. If we suppose that $l > 0$ is random (independent of $\mathcal{J}_l$) with a distribution γ, then the expression in (5.1) should be viewed as the conditional distribution of z_j's given l. The joint distribution of z_j's and l is therefore equal to the product of (5.1) and $d\gamma(l)$. In the special case when γ is absolutely continuous $d\gamma(l) = F(l)dl$, we get that the eigenvalues of $\mathcal{J}_l$ are distributed on $\{(z_j) \in (\mathbb{C}_+)^n : \sum_{j=1}^n \operatorname{Im} z_j \in \operatorname{supp}(F)\}$ according to

$$\frac{1}{h_{\beta,n}} e^{-\frac{1}{2}\sum_{j=1}^n \operatorname{Re}(z_j^2)} \times \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j<k} |z_j - z_k|^2 \times l^{-\frac{\beta n}{2}+1} e^{-\frac{l^2}{2}} F(l) d^2 z_1 \dots d^2 z_n, \tag{5.3}$$

where $l = \sum_{j=1}^n \operatorname{Im} z_j$.

Proof By Theorem 1(i), each of the eigenvalues $z_1, \dots, z_n$ lies in $\mathbb{C}_+$. Moreover, by the result of Arlinskiĭ–Tsekanovskiĭ [1, Thm. 5.1], the mapping

$$\begin{aligned} \{a_j\}_{j=1}^{n-1}, \{b_j\}_{j=1}^n &\mapsto z_1, \dots, z_n \\ (0, \infty)^{n-1} \times \mathbb{R}^n &\to (\mathbb{C}_+)^n \end{aligned} \tag{5.4}$$

is one-to-one and onto the set $\{(z_j) \in (\mathbb{C}_+)^n : \sum_{j=1}^n \operatorname{Im} z_j = l\}$ (see 5.17 below). Then so is the mapping $\{\lambda_j\}_{j=1}^n, \{w_j\}_{j=1}^{n-1} \mapsto z_1, \dots, z_n$, where μ (2.5) is the spectral measure of $\mathcal{J}$. Let us compute the Jacobian of this transformation.

Lemma 6

$$\left| \det \frac{\partial\, (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_1, \dots, \operatorname{Im} z_{n-1})}{\partial\, (\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1})} \right| = l^{n-1} \prod_{j<k} \frac{|\lambda_j - \lambda_k|^2}{|z_j - z_k|^2}. \tag{5.5}$$

Proof Let $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle = \sum_{j=1}^n \frac{w_j}{\lambda_j - z}$. Denote the characteristic polynomial as $\sum_{j=0}^n \kappa_j z^j = \det(z - \mathcal{J}_l) = \prod_{j=1}^n (z - z_j)$, where $\kappa_n = 1$. Let us first compute the Jacobian of the transformation of $\operatorname{Re}\kappa_0, \dots, \operatorname{Re}\kappa_{n-1}, \operatorname{Im}\kappa_0, \dots, \operatorname{Im}\kappa_{n-2}$ with respect to $\lambda_1, \dots, \lambda_n, w_1, \dots, w_{n-1}$. Note that $\operatorname{Im}\kappa_{n-1} = -\sum_{j=1}^n \operatorname{Im} z_j = -l$ is fixed.

Observe that

$$\sum_{j=0}^n \kappa_j z^j = \det(z - \mathcal{J}) \det(I - (z - \mathcal{J})^{-1} i l I_{1\times 1}) = (1 + i l m(z)) \prod_{j=1}^n (z - \lambda_j). \tag{5.6}$$

By taking the real parts for $z \in \mathbb{R}$, and then using analytic continuation, we obtain

$$\frac{1}{2} \prod_{j=1}^n (z - z_j) + \frac{1}{2} \prod_{j=1}^n (z - \bar{z}_j) = \sum_{j=0}^n (\operatorname{Re}\kappa_j) z^j = \prod_{j=1}^n (z - \lambda_j). \tag{5.7}$$

This implies that the Jacobian submatrix $\frac{\partial(\operatorname{Re}\kappa_0, \dots, \operatorname{Re}\kappa_{n-1})}{\partial(w_1, \dots, w_{n-1})}$ is equal to the $n \times (n-1)$ zero matrix, while

$$\left| \det \frac{\partial\, (\operatorname{Re}\kappa_0, \dots, \operatorname{Re}\kappa_{n-1})}{\partial\, (\lambda_1, \dots, \lambda_n)} \right| = \prod_{j<k} |\lambda_j - \lambda_k|. \tag{5.8}$$

Thus we just need to evaluate $|\det \frac{\partial(\operatorname{Im}\kappa_0,\dots,\operatorname{Im}\kappa_{n-2})}{\partial(w_1,\dots,w_{n-1})}|$, regarding λ_j's as constants.

The imaginary parts of (5.6) for $z \in \mathbb{R}$ give

$$\sum_{j=0}^{n-1} (\operatorname{Im}\kappa_j) z^j = lm(z) \prod_{j=1}^{n} (z-\lambda_j) = -l \sum_{j=1}^{n} w_j \prod_{\substack{1\le k\le n\\ k\ne j}} (z-\lambda_k)$$

$$= -l \left[\sum_{j=1}^{n-1} w_j(\lambda_j-\lambda_n) \prod_{\substack{1\le k\le n-1\\ k\ne j}} (z-\lambda_k) \right] - l \prod_{k=1}^{n-1} (z-\lambda_k) \qquad (5.9)$$

Denote the polynomial in the square brackets as $s(z) = \sum_{j=0}^{n-2} s_j z^j$. Then by (5.9),

$$\det \frac{\partial\,(\operatorname{Im}\kappa_0,\dots,\operatorname{Im}\kappa_{n-2})}{\partial\,(s_0,\dots,s_{n-2})} = (-l)^{n-1}. \qquad (5.10)$$

Now note that $s(z)$ can be rewritten as

$$s(z) = \sum_{j=1}^{n-1} \widetilde{w}_j \prod_{\substack{1\le k\le n-1\\ k\ne j}} \frac{z-\lambda_k}{\lambda_j-\lambda_k},$$

where

$$\widetilde{w}_j = w_j(\lambda_j-\lambda_n) \prod_{\substack{1\le k\le n-1\\ k\ne j}} (\lambda_j-\lambda_k). \qquad (5.11)$$

One can now recognize that $s(z)$ is the interpolating polynomial $s(\lambda_k) = \widetilde{w}_k$ for $k = 1,\dots,n-1$. This implies

$$\left|\det \frac{\partial\,(\widetilde{w}_1,\dots,\widetilde{w}_{n-1})}{\partial\,(s_0,\dots,s_{n-2})}\right| = \prod_{1\le j<k\le n-1} |\lambda_j-\lambda_k|. \qquad (5.12)$$

Finally, from (5.11),

$$\det \frac{\partial\,(\widetilde{w}_1,\dots,\widetilde{w}_{n-1})}{\partial\,(w_1,\dots,w_{n-1})} = \prod_{j=1}^{n-1} (\lambda_j-\lambda_n) \prod_{1\le j<k\le n-1} |\lambda_j-\lambda_k|^2. \qquad (5.13)$$

Combining (5.10), (5.12), (5.13), we get

$$\left|\det \frac{\partial\,(\operatorname{Im}\kappa_0,\dots,\operatorname{Im}\kappa_{n-2})}{\partial\,(w_1,\dots,w_{n-1})}\right| = l^{n-1} \prod_{1\le j<k\le n} |\lambda_j-\lambda_k|. \qquad (5.14)$$

Using (5.8), we get

$$\left|\det \frac{\partial\,(\operatorname{Re}\kappa_0,\dots,\operatorname{Re}\kappa_{n-1},\operatorname{Im}\kappa_0,\dots,\operatorname{Im}\kappa_{n-2})}{\partial\,(\lambda_1,\dots,\lambda_n,w_1,\dots,w_{n-1})}\right| = l^{n-1} \prod_{1\le j<k\le n} |\lambda_j-\lambda_k|^2. \qquad (5.15)$$

Finally, observe that if we have to restriction on κ_j's and z_j's, then

$$\prod_{j<k}|z_j - z_k|^2 = \left|\det \frac{\partial\left(\operatorname{Re}\kappa_0,\ldots,\operatorname{Re}\kappa_{n-1},\operatorname{Im}\kappa_0,\ldots,\operatorname{Im}\kappa_{n-1}\right)}{\partial\left(\operatorname{Re}z_1,\ldots,\operatorname{Re}z_n,\operatorname{Im}z_1,\ldots,\operatorname{Im}z_n\right)}\right|$$
$$= \left|\det \frac{\partial\left(\operatorname{Re}\kappa_0,\ldots,\operatorname{Re}\kappa_{n-1},\operatorname{Im}\kappa_0,\ldots,\operatorname{Im}\kappa_{n-1}\right)}{\partial\left(\operatorname{Re}z_1,\ldots,\operatorname{Re}z_n,\operatorname{Im}z_1,\ldots,\operatorname{Im}z_{n-1},\operatorname{Im}\kappa_{n-1}\right)}\right|$$
$$= \left|\det \frac{\partial\left(\operatorname{Re}\kappa_0,\ldots,\operatorname{Re}\kappa_{n-1},\operatorname{Im}\kappa_0,\ldots,\operatorname{Im}\kappa_{n-2}\right)}{\partial\left(\operatorname{Re}z_1,\ldots,\operatorname{Re}z_n,\operatorname{Im}z_1,\ldots,\operatorname{Im}z_{n-1}\right)}\right|.$$

The first equality is the standard fact; second equality comes from the change of variables $\operatorname{Im}\kappa_{n-1} = -\sum_{j=1}^n \operatorname{Im} z_j$; the last equality comes from Laplace expansion for the determinants (under the condition $\operatorname{Im}\kappa_{n-1} = \text{const}$).

Combining the last Jacobian with (5.15), we obtain the statement of the lemma. □

The joint distribution of $\{\lambda_j\}_{j=1}^n, \{w_j\}_{j=1}^{n-1}$ is

$$\frac{1}{g_{\beta,n}c_{\beta,n}}\prod_{j<k}|\lambda_j-\lambda_k|^\beta \prod_{j=1}^n e^{-\lambda_j^2/2}\prod_{j=1}^n w_j^{\beta/2-1}d\lambda_1\ldots d\lambda_n dw_1\ldots dw_{n-1}.$$

Using this and Lemma 6, we obtain that the distribution of z_j's is

$$\frac{1}{g_{\beta,n}c_{\beta,n}}l^{-(n-1)}\prod_{j<k}|\lambda_j-\lambda_k|^{\beta-2}\prod_{j=1}^n e^{-\lambda_j^2/2}\prod_{j=1}^n w_j^{\beta/2-1}\prod_{j<k}|z_j-z_k|^2 d^2z_1\ldots d(\operatorname{Re}z_n). \tag{5.16}$$

Note that

$$l = -\operatorname{Im}\kappa_{n-1} = \sum_{j=1}^n \operatorname{Im} z_j, \tag{5.17}$$
$$\sum_{j=1}^n \lambda_j = \sum_{j=1}^n \operatorname{Re} z_j, \tag{5.18}$$
$$\sum_{j\neq k}\lambda_j\lambda_k = \sum_{j\neq k}\operatorname{Re}(z_j z_k). \tag{5.19}$$

The first equation comes from (5.9), while the latter two follow from (5.7). Then

$$\sum_{j=1}^n \lambda_j^2 = \left(\sum_{j=1}^n \operatorname{Re} z_j\right)^2 - \sum_{j\neq k}\operatorname{Re}(z_j z_k) = \sum_{j=1}^n (\operatorname{Re} z_j)^2 + \sum_{j\neq k}(\operatorname{Im} z_j)(\operatorname{Im} z_k)$$
$$= \sum_{j=1}^n \operatorname{Re}\left(z_j\right)^2 + l^2. \tag{5.20}$$

Finally, from (5.6),

$$-ilw_j = il\operatorname*{Res}_{z=\lambda_j} m(z) = \operatorname*{Res}_{z=\lambda_j}\prod_{k=1}^n \frac{z-z_k}{z-\lambda_k} = \frac{\prod_{k=1}^n(\lambda_j - z_k)}{\prod_{k\neq j}(\lambda_j-\lambda_k)}, \tag{5.21}$$

so

$$\prod_{j=1}^{n} w_j = (\tfrac{i}{l})^n \frac{\prod_{j,k}(\lambda_j - z_k)}{\prod_{j<k}|\lambda_j-\lambda_k|^2} = (\tfrac{i}{l})^n \tfrac{1}{2^n} \frac{\prod_{j,k}(\bar{z}_j - z_k)}{\prod_{j<k}|\lambda_j-\lambda_k|^2} = \tfrac{1}{(2l)^n} \frac{\prod_{j,k}|\bar{z}_j - z_k|}{\prod_{j<k}|\lambda_j-\lambda_k|^2}, \tag{5.22}$$

where we used (5.7) with $z = z_k, k = 1, \ldots, n$. Combining (5.17), (5.20), (5.22) with (5.16), we obtain (5.1). □

Example

Since Γ in Theorem 2 has rank 1, we can decompose it as $\Gamma = L^*L$, where $L = (l_{1j})_{j=1}^n$ is an $1 \times n$ matrix. Assuming the entries l_{1j} of L are independent and normal $N(0, \sigma \mathbf{I}_\beta)$, then $l = \sum_{j=1}^n |l_{1j}|^2 \sim \sigma^2 \chi^2_{\beta n}$, that is l is distributed on $(0, \infty)$ according to $F(l)dl$ with $F(l) = \frac{1}{(\sqrt{2}\sigma)^{\beta n}\Gamma(\beta n/2)} l^{\beta n/2-1} e^{-l/(2\sigma^2)}$. In this special case, eigenvalues $\{z_1, \ldots, z_n\}$ are distributed on $(\mathbb{C}_+)^n$ according to

$$\frac{1}{(\sqrt{2}\sigma)^{\beta n}\Gamma(\beta n/2) c_{\beta,n} g_{\beta,n}} e^{-\frac{1}{2}\sum_{j=1}^n \mathrm{Re}(z_j^2)} \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j<k} |z_j - z_k|^2$$
$$\times e^{-\frac{l^2}{2} - \frac{l}{2\sigma^2}} d^2 z_1 \ldots d^2 z_n. \tag{5.23}$$

5.2 Perturbations of Laguerre β-Ensembles

Proof of Proposition 2 We use the same notation as in the previous section: let z_j's be the eigenvalues of $\mathcal{J}_l$; let λ_j's and w_j's be the eigenvalues and eigenweights of the spectral measure of $\mathcal{J}$ (which is of the form (2.5) with (2.10) for the case (i) and (2.12) with (2.14) for the case (ii)). By [1], $z_j \in \mathbb{C}_+$ for every j.

Consider now case (i). Equations (5.7) and (5.9) imply

$$\mathrm{Re}\, s_k(z_1, \ldots, z_n) = s_k(\lambda_1, \ldots, \lambda_n), \quad k = 1, 2, \ldots, n; \tag{5.24}$$

$$\mathrm{Im}\, s_k(z_1, \ldots, z_n) = l \sum_{j=1}^n w_j s_{k-1}(\{\lambda_t\}_{t \neq j}), \quad k = 1, 2, \ldots, n, \tag{5.25}$$

where $s_0 := 1$, and s_k $(k \geq 1)$ is the k-th elementary symmetric polynomial

$$s_k(z_1, \ldots, z_n) := \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} z_{j_1} \ldots z_{j_k}. \tag{5.26}$$

Since for each j, $\lambda_j > 0$, $w_j > 0$, $l > 0$, we obtain that $z_1, \ldots, z_n$ must belong to

$$\left\{ (z_j)_{j=1}^n \in (\mathbb{C}_+)^n : s_k(z_1, \ldots, z_n) \in Q_1, \quad k = 1, 2, \ldots, n \right\}, \tag{5.27}$$

where $Q_1 := \{z : 0 < \mathrm{Arg} z < \pi/2\}$. Conversely, take a collection of points from (5.27). Since it belongs to $(\mathbb{C}_+)^n$, we know from [1, Thm. 5.1] that there exists a unique matrix of the form $\mathcal{J} + il I_{1\times 1}$ with $l > 0$ and $a_j > 0$, $j = 1, \ldots, n-1$. Equation (5.7) along with the positivity of (5.24) implies that $\lambda_1, \ldots, \lambda_n$ are the real roots of the polynomial $\prod_{j=1}^n (z - \lambda_j)$ with alternating signs of the coefficients. By Descartes' rule of signs, such a polynomial cannot have negative zeros. This means that all λ_j's are positive. Therefore (5.27) is precisely the space of all possible eigenvalue configurations of H_{eff}. Let us now show that it coincides with (3.2).

It is elementary that (3.2) is a subset of (5.27). To see the converse, take any sequence from (5.27). Since $s_n(z_1, \dots, z_n) = z_1 z_2 \dots z_n \in Q_1$, we must have that

$$0 + 2k\pi < \mathrm{Arg} z_1 + \mathrm{Arg} z_2 + \cdots + \mathrm{Arg} z_n < \pi/2 + 2k\pi \tag{5.28}$$

for some integer $k \ge 0$. We already know that these $z_1, \dots, z_n$ are the eigenvalues of $\mathcal{J} + ilI_1$, where $\mathcal{J}$ is *positive definite*. Let us now fix $\mathcal{J}$ and view $z_1, \dots, z_n$ as functions of $l \ge 0$ only. Each of these functions is continuous and never passes through 0. For any $0 < l < \infty$, we have (5.28) for some k. But when $l = 0$ the sum of the arguments is zero. By continuity $k = 0$ for any l, i.e., (5.27) = (3.2).

To deal with the case (ii), we use similar arguments with $m + 1$ instead of n and $\lambda_1, \dots, \lambda_m, 0$ as the eigenvalues (with $\lambda_j > 0$, $j = 1, \dots, m$). Then Eqs. (5.24) and (5.25) imply that the eigenvalues $z_1, \dots, z_{m+1}$ of $\mathcal{J} + ilI_{1\times1}$ belong to

$$\Big\{(z_j)_{j=1}^{m+1} \in (\mathbb{C}_+)^{m+1} : s_{m+1}(z_1, \dots, z_{m+1}) \in i\mathbb{R}_+; \\ s_k(z_1, \dots, z_{m+1}) \in Q_1, \quad k = 1, 2, \dots, m\Big\}, \tag{5.29}$$

where $\mathbb{R}_+ = \{z \in \mathbb{R} : z > 0\}$. Conversely, by [1, Thm. 5.1], any configuration of point from (5.29) coincides with eigenvalues of some $\mathcal{J} + ilI_{1\times1}$, $l > 0$. The eigenvalues $\lambda_1, \dots, \lambda_{m+1}$ of $\mathcal{J}$ satisfy $s_k(\lambda_1, \dots, \lambda_{m+1}) > 0$ for $k = 1, \dots, m$ and $s_{m+1}(\lambda_1, \dots, \lambda_{m+1}) = 0$. This implies $\lambda_j > 0$ for all j except for one zero eigenvalue.

Finally, let us show that (5.29) coincides with (3.3). The inclusion (3.3)⊆(5.29) is easy. Conversely, take any configuration $\{z_j\}_{j=1}^{m+1}$ from (5.29). By the above, these points are the eigenvalues of some $\mathcal{J} + ilI_{1\times1}$ with $l > 0$, where $\mathcal{J}$ has eigenvalues $\{0, \lambda_1, \dots, \lambda_m\}$ with $\lambda_j > 0$ for $1 \le j \le m$. Since $s_{m+1} \in i\mathbb{R}_+$ in (5.29), we have

$$\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 + \ldots + \mathrm{Arg}\, z_{m+1} = \pi/2 + 2k\pi \tag{5.30}$$

for some integer $k \ge 0$. After reordering, we can assume that $z_j \to \lambda_j$, $1 \le j \le m$, and $z_{m+1} \to 0$ when $l \to 0$ (while $\mathcal{J}$ is fixed). Therefore $\mathrm{Arg}\, z_j \to 0$ as $l \to 0$ for $1 \le j \le m$, while $0 \le \mathrm{Arg}\, z_{m+1} \le \pi/2$ for any l. This proves that $k = 0$, and so (5.29) ⊆ (3.3), finishing the proof. □

In the next theorem we compute the joint distribution of eigenvalues of rank one perturbations of the Laguerre β-ensembles.

Theorem 4 *Fix a deterministic $l > 0$, and for any $\beta > 0$ and any integer $m, n > 0$, let $\mathcal{J}$ be the $n \times n$ matrix from $L\beta E_{(m,n)}$ ensemble.*

(i) *If $m \ge n$, then the eigenvalues $\{z_1, \dots, z_n\}$ of $\mathcal{J}_l = \mathcal{J} + ilI_{1\times1}$ are distributed on*

$$\left\{(z_j)_{j=1}^n \in (\mathbb{C}_+)^n : \sum_{j=1}^n \mathrm{Arg}\, z_j < \tfrac{\pi}{2}, \sum_{j=1}^n \mathrm{Im}\, z_j = l\right\} \tag{5.31}$$

according to

$$\frac{1}{q_{\beta,n,a,l}} \prod_{j,k=1}^n |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{j<k} |z_j - z_k|^2 \\ \times e^{-\frac{1}{2}\sum_{j=1}^n \mathrm{Re}\, z_j} \left(\mathrm{Re} \prod_{j=1}^n z_j\right)^{\frac{\beta a}{2}} d^2 z_1 \dots d^2 z_{n-1} d(\mathrm{Re}\, z_n), \tag{5.32}$$

where $a = m - n + 1 - 2/\beta$ *and*

$$q_{\beta,n,a,l} = 2^{n(\beta/2-1)} h_{\beta,n,a} c_{\beta,n} l^{\frac{\beta n}{2}-1},$$

where $h_{\beta,n,a}$ *and* $c_{\beta,n}$ *are as in* (2.11) *and* (2.8).

(ii) *If* $m \le n-1$, *then the* $m+1$ *nonzero eigenvalues of* $\mathcal{J}_l = \mathcal{J} + il I_{1\times 1}$ *are distributed on*

$$\left\{ (z_j)_{j=1}^{m+1} \in (\mathbb{C}_+)^{m+1} : \sum_{j=1}^{m+1} \operatorname{Arg} z_j = \tfrac{\pi}{2}, \ \sum_{j=1}^{m+1} \operatorname{Im} z_j = l \right\} \tag{5.33}$$

according to

$$\frac{1}{t_{\beta,m,n,l}} \prod_{j,k=1}^{m+1} |z_j - \bar{z}_k|^{\frac{\beta}{2}-1} \prod_{1\le j<k\le m+1} |z_j - z_k|^2$$
$$\times\, e^{-\frac{1}{2}\sum_{j=1}^{m+1} \operatorname{Re} z_j} \prod_{j=1}^{m+1} |z_j|^{\frac{\beta(n-m-1)}{2}} \left(\operatorname{Re} \prod_{j=1}^{m} z_j \right)^{-1} d^2 z_1 \dots d^2 z_m, \tag{5.34}$$

where

$$t_{\beta,m,n,l} = (m+1) 2^{(m+1)(\beta/2-1)} h_{\beta,m,a} d_{\beta,m,n} l^{\frac{\beta n}{2}-1}, \tag{5.35}$$

where $a = n - m + 1 - 2/\beta$, *and* $h_{\beta,m,a}$ *and* $d_{\beta,m,n}$ *are as in* (2.11) *and* (2.15).

Remarks 1. Distributions (5.32) and (5.34) with $\beta = 1, 2, 4$ are the eigenvalue distribution of rank one perturbations of $LOE_{(m,n)}$, $LUE_{(m,n)}$, $LSE_{(m,n)}$, respectively.

2. In (ii), z_{m+1} is determined from $z_1, \dots, z_m$ because of (5.33).
3. Similarly to the remark 2 after Theorem 3, we can also assume that $l > 0$ is random (independent of $\mathcal{J}_l$) with a distribution γ. Then (5.32) and (5.34) are the conditional distributions of z_j's given l. The joint distribution of z_j's and l is then equal to the product with $d\gamma(l)$ and can be calculated as in the case of Gaussian ensembles above.

Proof (i) We can take the known joint distribution of the eigenvalues λ_j's and the eigenweights w_j's (see Lemma 4) and change the variables to z_j's (by Proposition 2(i) it is one-to-one and onto (5.31), so the Jacobian (5.5) applies). Using (5.22), (5.17), (5.18), (5.24) (with $k = n$), we obtain the resulting distribution (5.32).

(ii) By Proposition 2(ii), the map from the spectral measures of the form (2.12), (2.14) to the eigenvalues of $\mathcal{J} + il I_{1\times1}$: $\lambda_1, \dots, \lambda_m, w_1, \dots, w_m \mapsto z_1, \dots, z_{m+1}$ is one-to-one and onto (5.33) (if we impose some natural ordering on λ_j's and z_j's; we will remove it in the end of the proof). Its Jacobian is different from (5.5) computed earlier. Similar to the notation in the proof of Lemma 6, let $m(z) = \langle e_1, (\mathcal{J} - z)^{-1} e_1 \rangle = -\frac{w_0}{z} + \sum_{j=1}^m \frac{w_j}{\lambda_j - z}$ and $\sum_{j=0}^{m+1} \kappa_j z^j = \det(z - \mathcal{J}_l) = \prod_{j=1}^{m+1} (z - z_j)$, where $\kappa_{m+1} = 1$. Because of $\det \mathcal{J} = 0$, we obtain $\operatorname{Re} \kappa_0 = 0$. Following similar reasoning as in the proof of Lemma 6, we first obtain the value of the Jacobian

$$\left| \det \frac{\partial (\operatorname{Re} \kappa_1, \dots, \operatorname{Re} \kappa_m, \operatorname{Im} \kappa_0, \dots, \operatorname{Im} \kappa_{m-1})}{\partial (\lambda_1, \dots, \lambda_m, w_1, \dots, w_m)} \right| = l^m \prod_{j=1}^m \lambda_j \prod_{1\le j<k\le m} |\lambda_j - \lambda_k|^2. \tag{5.36}$$

Since $\operatorname{Re}(z_1 \dots z_{m+1}) = (-1)^{m+1} \operatorname{Re} \kappa_0 = 0$ and $\operatorname{Im} \kappa_m = -\sum_{j=1}^{m+1} \operatorname{Im} z_j = -l$, we have that z_{m+1} is determined by $z_1, \dots, z_m$. Therefore we have a one-to-one map

$\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ taking $z_1, \ldots, z_m$ to $\operatorname{Re}\kappa_1, \ldots, \operatorname{Re}\kappa_m, \operatorname{Im}\kappa_0, \ldots, \operatorname{Im}\kappa_{m-1}$. We need its Jacobian on the manifold $\operatorname{Re}(z_1 \ldots z_{m+1}) = 0$, $\sum_{j=1}^{m+1} \operatorname{Im} z_j = l$.
If we have no restrictions on z_j's or κ_j's, then

$$\prod_{1\le j<k\le m+1} |z_j - z_k|^2 = \left|\det \frac{\partial\,(\operatorname{Re}\kappa_0, \operatorname{Im}\kappa_0, \ldots, \operatorname{Re}\kappa_m, \operatorname{Im}\kappa_m)}{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_{m+1}, \operatorname{Im} z_{m+1})}\right|$$
$$= \left|\det \frac{\partial\,(\operatorname{Re}\kappa_0, \operatorname{Im}\kappa_0, \ldots, \operatorname{Re}\kappa_m, \operatorname{Im}\kappa_m)}{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \operatorname{Im} z_m, \operatorname{Re}\kappa_0, \operatorname{Im}\kappa_m)}\right|$$
$$\times \left|\det \frac{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \operatorname{Im} z_m, \operatorname{Re}\kappa_0, \operatorname{Im}\kappa_m)}{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_{m+1}, \operatorname{Im} z_{m+1})}\right|$$

The last determinant is equal to $|\operatorname{Re}(z_1 \ldots z_m)|$, so

$$\frac{\prod_{1\le j<k\le m+1} |z_j - z_k|^2}{|\operatorname{Re}(z_1 \ldots z_m)|} = \left|\det \frac{\partial\,(\operatorname{Re}\kappa_0, \operatorname{Im}\kappa_0, \ldots, \operatorname{Re}\kappa_m, \operatorname{Im}\kappa_m)}{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \operatorname{Im} z_m, \operatorname{Re}\kappa_0, \operatorname{Im}\kappa_m)}\right|$$
$$= \left|\det \frac{\partial\,(\operatorname{Re}\kappa_1, \ldots, \operatorname{Re}\kappa_m, \operatorname{Im}\kappa_0, \ldots, \operatorname{Im}\kappa_{m-1})}{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \operatorname{Im} z_m)}\right|,$$

where in the last determinant we are assuming that $\operatorname{Re}\kappa_0 = \text{const}$ and $\operatorname{Im}\kappa_m = \text{const}$. Combining this with (5.36), we get that on $\operatorname{Re}\kappa_0 = 0$, $\operatorname{Im}\kappa_m = -l$,

$$\left|\det \frac{\partial\,(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \operatorname{Im} z_m)}{\partial\,(\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_m)}\right| = l^m \left|\operatorname{Re} \prod_{j=1}^m z_j\right| \prod_{j=1}^m \lambda_j \frac{\prod_{1\le j<k\le m} |\lambda_j - \lambda_k|^2}{\prod_{1\le j<k\le m+1} |z_j - z_k|^2}. \tag{5.37}$$

Repeating the arguments from (5.21) and (5.22), we obtain

$$w_0 = \frac{\prod_{j=1}^{m+1} |z_j|}{l \prod_{j=1}^m |\lambda_j|}, \quad \text{and} \quad \prod_{j=1}^m w_j = \frac{1}{l^m 2^{m+1}} \frac{\prod_{j,k=1}^{m+1} |z_j - \bar{z}_k|}{\prod_{j=1}^{m+1} |z_j| \prod_{j=1}^m |\lambda_j| \prod_{j<k} |\lambda_j - \lambda_k|^2}.$$

Finally, just as in (i), we still have $\sum_{j=1}^m \lambda_j = \sum_{j=1}^{m+1} \operatorname{Re} z_j$.
Now, starting from the joint distribution of $\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_m$ (see Proposition 1), applying the Jacobian (5.37), and using these substitutions (note that terms with $\prod |\lambda_j|$ cancel out in the process), we arrive at the distribution (5.34). Note that the factor $(m+1)$ in (5.35) comes from removing the ordering of z_j's and λ_j's (there are $(m+1)!$ of permutations for $\{z_j\}_{j=1}^{m+1}$, and only $m!$ for $\{\lambda_j\}_{j=1}^m$).

□

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