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A Nonlinear Transfer Operator Theorem

Mark Pollicott[1]

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Abstract In recent papers, Kenyon et al. (Ergod Theory Dyn Syst 32:1567–1584 2012), and Fan et al. (C R Math Acad Sci Paris 349:961–964 2011, Adv Math 295:271–333 2016) introduced a form of non-linear thermodynamic formalism based on solutions to a non-linear equation using matrices. In this note we consider the more general setting of Hölder continuous functions.

Keywords Ruelle operator theorem · Transfer operator · Thermodynamic formalism

1 Introduction

We first recall a classical result for matrices dating back to work of Perron (1907) and Frobenius (1912) (cf. [5], p. 53). A $k \times k$ matrix A is called *non-negative* if all the entries are non-negative real numbers and *aperiodic* if there exists $n > 0$ such that all entries of the nth power A^n are strictly positive.

Theorem 1.1 (Perron–Frobenius Theorem) *Let A be a non-negative aperiodic $k \times k$-matrix. There exists a unique positive maximal eigenvalue $\lambda > 0$ and a unique positive eigenvector $\underline{v}$ such that $A\underline{v} = \lambda\underline{v}$.*

We next recall a generalisation of the Perron–Frobenius Theorem to Banach spaces of functions. Let $\sigma : \Sigma \to \Sigma$ be a one-sided mixing subshift of finite type with alphabet $F = \{1, \cdots, k\}$ (i.e., there exists an aperiodic $k \times k$ matrix B with 0–1 entries such that $\Sigma = \{x = (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} F : x_n \in F, B(x_n, x_{n+1}) = 1, \forall n \geq 0\}$ and $(\sigma x)_n = x_{n+1}$. Given $0 < \theta < 1$, let $\mathcal{F}_\theta$ be the space of functions $f : \Sigma \to \mathbb{R}$ for which the semi-norm

Dedicated to David Ruelle on the occasion of his 80th Birthday.

✉ Mark Pollicott
mpollic@maths.warwick.ac.uk

1 Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

$$\|f\|_\theta = \sup_{n\geq 0} \frac{\mathrm{var}_n(f)}{\theta^n} < +\infty$$

is finite, where $\mathrm{var}_n(f) = \sup\{|f(x) - f(x')| : x_i = x_i' \text{ for } i = 0, \cdots, n-1\}$. In particular, $\mathcal{F}_\theta$ is a Banach space with respect to the norm $\|f\| = \|f\|_\theta + \|f\|_\infty$.

Definition 1.2 Let $\phi \in \mathcal{F}_\theta$. We can define a transfer operator $\mathcal{L}_\phi : \mathcal{F}_\theta \to \mathcal{F}_\theta$ by

$$\mathcal{L}_\phi \psi(x) = \sum_{\sigma y = x} e^{\phi(y)} \psi(y),$$

where $\psi \in \mathcal{F}_\theta$ and $x \in \Sigma$.

The following result of Ruelle is a cornerstone of the classical theory of thermodynamic formalism (cf.[10]).

Theorem 1.3 (Ruelle–Perron–Frobenius Theorem) *Let $\phi \in \mathcal{F}_\theta$.*

1. *There exists $\lambda = \lambda_\phi > 0$ and $\varphi = \varphi_\phi \in \mathcal{F}_\theta$ with $\psi > 0$ such that $\mathcal{L}_\phi \varphi = \lambda \varphi$;*
2. *any $\psi' \in \mathcal{F}_\theta$ with $\mathcal{L}_\phi \psi' = \lambda \psi'$ is necessarily a multiple of ψ; and*
3. *the dependences $\mathcal{F}_\theta \ni \phi \mapsto \lambda_\phi \in \mathbb{R}^+$ and $\mathcal{F}_\theta \ni \phi \mapsto \varphi_\phi \in \mathcal{F}_\theta$ are analytic.*

It is also known that, aside from the maximal eigenvalue λ, the rest of the spectrum of $\mathcal{L}_\phi : \mathcal{F}_\theta \to \mathcal{F}_\theta$ is contained in $\{z \in \mathbb{C} : |z| < \lambda_\phi\}$. In particular, part 3 of Theorem 1.3 then follows from part 1 by standard perturbation theory.

The value $P(\phi) := \log \lambda_\phi$ is called the *pressure* of the function $\phi \in \mathcal{F}_\theta$ [10]. In the case that $\phi(x) = \phi(x_0, x_1)$ depends only on the first two terms in $x = (x_n)_{n=0}^\infty$ then Theorem 1.3 reduces to Theorem 1.1, by taking $A(i, j) = \exp \phi(i, j)$. In this case, $\psi(x) = \psi(x_0)$ depends only on the first term and we can set $\underline{v} = (\psi(1), \ldots, \psi(k))$.

Recently, several authors introduced a particular non-linear version of Theorem 1.1 for matrices which is useful in the study of the dimension of certain sets in the theory of non standard ergodic averages (see section 2).

Theorem 1.4 (Kenyon–Peres–Solomyak, Fan–Schmeling–Wu) *Let B be a non-negative irreducible $k \times k$ matrix. There exists a unique positive vector $\underline{v}$ such that $B\underline{v} = \underline{v}^2$, where the entries of $\underline{v}^2$ are the square of those of $\underline{v}$, i.e., $(\underline{v})_i{}^2 = v_i$ for $i = 1, \cdots, k$.*

In the special case that A has entries which are natural numbers, this appears as Lemma 1.2 in [6]. A version of this for more general positive matrices appears as 4.1 in [2] (cf. also [3]) under very modest assumptions on the matrix. Other types of non-linear Perron–Frobenius Theorem appear in [7] and [8].

The following is our main result, which can be viewed either as a non-linear version of Theorem 1.3, or a generalisation of the Theorem 1.4 (at least for aperiodic matrices) from matrices to functions.

Theorem 1.5 (Main Theorem) *Let $\phi \in \mathcal{F}_\theta$.*

1. *There exists $\psi = \psi_\phi \in \mathcal{F}_\theta$ with $\psi > 0$ such that $\mathcal{L}_\phi \psi = \psi^2$;*
2. *for any $\psi' \in \mathcal{F}_\theta$ with $\mathcal{L}_\phi \psi' = \psi'^2$ and $\psi' > 0$ then $\psi' = \psi$; and*
3. *the dependence $\mathcal{F}_\theta \ni \phi \mapsto \psi_\phi \in \mathcal{F}_\theta$ is analytic providing ψ_ϕ is sufficiently close to the constant function $\lambda_\phi \mathbf{1}$ in norm.*

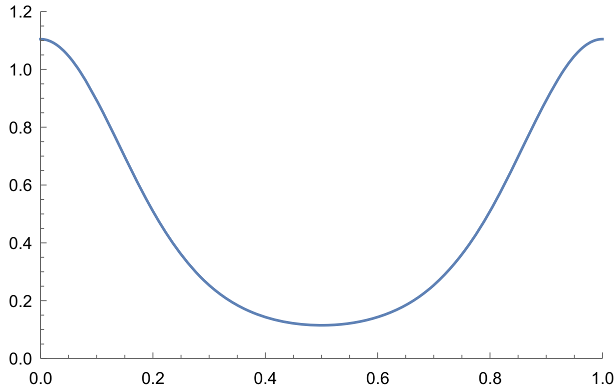

Fig. 1 A plot of $\psi_\phi(x)$ using the dyadic expansion $0 \le \sum_{n=0}^{\infty} \frac{(x_n-1)}{2^{n+1}} \le 1$ on the unit interval to represent $x \in \Sigma_2$

The result easily generalises to $\mathcal{L}_\phi \psi = \psi^q$, for any natural number $q \ge 2$. We consider only the case $q = 2$ to avoid introducing additional notation.

In the particular case that the function $\phi(x) = \phi(x_0, x_1)$ depends on only finitely many coordinates then Theorem 1.4 can be recovered as a corollary to Theorem 1.5.

Example 1.6 Let $\Sigma_2 = \{1, 2\}^{\mathbb{Z}_+}$ correspond to a full shift with $F = \{1, 2\}$. We can define a function $\phi : \Sigma_2 \to \mathbb{R}$ by

$$\phi(x) = -4 \sin\left(2\pi \sum_{n=0}^{\infty} \frac{(x_n - 1)}{2^{n+1}}\right)$$

and observe that $\phi \in \mathcal{F}_\theta$ for any $1/2 < \theta < 1$. By Theorem 1.5 (with the choice $q = 2$) there is a function ψ such that $\mathcal{L}_\phi \psi = \psi^2$. In Fig. 1 we plot a realisation of ψ using the dyadic expansion on the unit interval.

Remark 1.7 If $\phi \in \mathcal{F}_\theta$ then, as usual, by replacing ϕ by $\phi_1 = \phi + \log \psi_\phi - \log \psi_\phi \circ \sigma \in \mathcal{F}_\theta$, where ψ is the positive eigenfunction in Theorem 1.3, we can assume without loss of generality that $\psi_{\phi_1}(x) = \mathbf{1}$ is the constant function taking the value 1, i.e., $\mathcal{L}_{\phi_1}\mathbf{1} = \lambda\mathbf{1}$, where $\lambda = \lambda_\phi = \lambda_{\phi_1}$. In particular, for such special normalized functions ϕ_1 the function ψ Theorem 1.5 can easily be identified as $\psi = \psi_{\phi_1} = \lambda\mathbf{1}$, then we see that $\mathcal{L}_{\phi_1}\psi = \psi^2$. Furthermore, the hypothesis for analyticity in part 3 of Theorem 1.5 automatically holds.

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2 Background to Theorem 1.4

Although our main result (Theorem 1.5) is of independent interest, for the reader's benefit we will now give a brief description of the original application of its precursor (Theorem 1.4) which provided the motivation for its introduction.

Following [6] and [2,3] given a probability measure μ on Σ we can define a so-called multiplicative measure ν on $\Sigma = \{1, \ldots, k\}^{\mathbb{Z}^+}$, say, by writing $\Sigma = \prod_{j \text{ odd}} \Sigma_j$ where $\Sigma_j = \{1, \cdots, k\}^{\Lambda_j}$ and $\Lambda_j = \{j2^n : n \ge 0\}$, for $j = 1, 3, 5, \cdots$, which form a natural partition of $\mathbb{N}$ by $\mathbb{N} = \cup_{j \text{ odd}} \Lambda_j$. We can then define $\nu = \prod_j \mu$, in a natural sense. In [6] and [3] the authors consider the measure μ to be a (generalised) Markov measure defined in terms of the entries in the vector $\underline{v}$ in Theorem 1.4. The measure ν will typically not be σ-invariant but is still useful in studying the Hausdorff dimension of certain sets.

We can define the pointwise dimension of ν by

$$\dim_H(\nu) = -\lim_{n\to+\infty} \frac{1}{n} \log \nu\left([x_0, x_1, \cdots, x_{n-1}]\right) \text{ for a.e.}(\nu) x = (x_n) \in \Sigma$$

where $[x_0, x_1, \cdots, x_{n-1}] = \{y = (y_n) \in \Sigma : x_i = y_i, 0 \le i \le n-1\}$ is a cylinder set. Finally, by Proposition 2.3 of [6] we have that for any σ-ergodic measure ν the pointwise dimension is constant and takes the explicit value

$$\dim_H(\nu) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} H_\mu\left(\vee_{i=0}^{n-1} \sigma^{-i}\alpha\right)$$

where $\alpha = \{[1], \ldots, [k]\}$ is the standard partition into cylinders of length one; $\vee_{i=0}^{n-1}\sigma^{-i}\alpha$ is the usual refinement to a partition by cylinders of length n; and $H_\mu(\cdot)$ is the entropy for partitions [11].

The pointwise dimension is particularly useful in estimating the Hausdorff Dimension of sets (especially lower bounds via the usual mass distribution principle cf. [1], §4.2) associated to multiple ergodic theorems, as the following example illustrates.

Example 2.1 (Golden Mean Example [4,6]) Fan–Liao–Ma and Kenyon–Peres–Solomyak considered the golden mean example:

$$X = \left\{(x_n) \in \{0,1\}^{\mathbb{N}} : x_n x_{2n} = 0, \forall n \ge 1\right\},$$

with the usual metric

$$d_\theta(x, x') = \begin{cases} \theta^{N(x,x')} & \text{if } x \ne x' \\ 0 & \text{if } x = x'. \end{cases}$$

where $N(x, x') = \sup\{n \ge 0 : x_i = x_i' \text{ for } 0 \le i \le n\}$ (and $N(x, x') = 0$ if $x_0 \ne x_0'$).

In this case one can consider the matrix $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and the solution to $B\underline{v} = \underline{v}^2$, i.e., $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies $v_1^2 = v_1 + v_2$ and $v_2^2 = v_1$. We then have that μ is a Markov measure for $P = \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix}$, where $p^3 = (1-p)^2$, and finally $\dim_H(X) = -\log_2 p = 0 \cdot 81137\ldots$ which is strictly less than the Minkowski dimension $dim_M(X) = 0 \cdot 82429 \ldots$ [4,6].

3 Proof of Theorem 1.5

The proof of the existence of the fixed point is the more interesting part of the problem. The uniqueness and analyticity are then relatively easy to establish.

3.1 Existence of the Fixed Point

The existence can be shown by looking for a fixed point of a suitable map in the space

$$\Lambda_c := \left\{u : \Sigma \to [0,1] : u(x) \leq u(x')e^{cd_\theta(x,x')} \text{ for those } x, x' \in \Sigma \text{ with } x_0 = x_0'\right\}$$

where $c > 0$ and $d_\theta(\cdot,\cdot)$ is as defined in Example 2.1. We first note that $\Lambda_c \subset \mathcal{F}_\theta$ since for $u \in \Lambda_c$ and $x, x' \in \Sigma$ we can bound $u(x) - u(x') \leq \|u\|_\infty \left(e^{cd_\theta(x,x')} - 1\right) \leq Cd_\theta(x,x')$ for sufficiently large $C > 0$ and then interchanging x and x' gives that $\|u\|_\theta \leq C$ (cf. [9], p. 22).

We can now introduce a family of non-linear operators defined as follows:

Definition 3.1 For each $n \geq 1$ we can associate to $u \in \Lambda_c$ a new function $\mathcal{N}_n(u) : \Sigma \to \mathbb{R}$ defined by

$$\mathcal{N}_n(u)(x) = \left(\frac{\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)(x)}{\|\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)\|_\infty}\right)^{\frac{1}{2}}$$

where $\frac{1}{n}\mathbf{1}$ represents the function taking the constant value $\frac{1}{n}$.

Lemma 3.2 *We have that* $\mathcal{L}_\phi(\Lambda_c) \subset \Lambda_{c'}$ *for* $c' = (c + \|\phi\|_\theta)\theta$.

Proof Let $x, x' \in \Sigma$ with $x_0 = x_0'$. Assume that $d_\theta(x,x') = \theta^N$, for some $N \geq 0$, then $x_i = x_i'$ for $0 \leq i \leq N$ and $x_{N+1} \neq x_{N+1}'$. If $y \in \sigma^{-1}x$ then we denote by $y' \in \sigma^{-1}x'$ the corresponding sequence for which $y_0 = y_0'$, and thus we have that $d_\theta(y,y') = \theta^{N+1}$. Let $u \in \Lambda_c$ then we have that

$$\begin{aligned}\mathcal{L}_\phi u(x) &= \textstyle\sum_{\sigma y = x} e^{\phi(y)}u(y)\\ &\leq \textstyle\sum_{\sigma y = x} e^{\phi(y') + \|\phi\|_\theta \theta^{N+1}}\left(u(y')e^{c\theta^{N+1}}\right)\\ &\leq e^{(c+\|\phi\|_\theta)\theta d_\theta(x,x')} \textstyle\sum_{\sigma y' = x'} e^{\phi(y')}u(y')\\ &= e^{(c+\|\phi\|_\theta)\theta d_\theta(x,x')}\mathcal{L}_\phi u(x')\end{aligned}$$

where we have used that $d_\theta(y,y') = \theta^{N+1}$ and then since $u \in \Lambda_c$ we have that $u(y) \leq u(y')e^{c\theta^{N+1}}$. In particular, we have that $\mathcal{L}_\phi u(x) \leq e^{c'd_\theta(x,x')}\mathcal{L}_\phi u(x')$, i.e., $\mathcal{L}_\phi u \in \Lambda_{c'}$. □

Remark 3.3 By definition of Λ_c, we see that if $c' < c$ then $\Lambda_{c'} \subset \Lambda_c$ thus, providing c is sufficiently large, Lemma 3.2 gives $\mathcal{L}_\phi(\Lambda_c) \subset \Lambda_c$.

We can use the above lemma to deduce the following.

Lemma 3.4 *For* $c > 0$ *sufficiently large we have that* $\mathcal{N}_\phi(\Lambda_c) \subset \Lambda_c$.

Proof Let $u \in \Lambda_c$. For each $n \geq 1$ the constant function $\frac{1}{n}\mathbf{1} \in \Lambda_c$ and so by applying Lemma 3.2 to the new function $u + \frac{1}{n}\mathbf{1}$ we see that

$$\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)(x) \leq e^{c'd_\theta(x,x')}\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)(x'), \tag{3.1}$$

for all $x, x' \in \Sigma$ with $x_0 = x_0'$. Dividing both sides of (3.1) by $\|\mathcal{L}\left(u + \frac{1}{n}\mathbf{1}\right)\|_\infty > 0$ we have that

$$\frac{\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)(x)}{\|\mathcal{L}\left(u + \frac{1}{n}\mathbf{1}\right)\|_\infty} \leq e^{c'd_\theta(x,x')}\frac{\mathcal{L}_\phi\left(u + \frac{1}{n}\mathbf{1}\right)(x')}{\|\mathcal{L}\left(u + \frac{1}{n}\mathbf{1}\right)\|_\infty}. \tag{3.2}$$

Finally, since the values taken by both sides of (3.2) lie in the unit interval, taking square roots preserves this property with c' replaced by $c'/2$, i.e.,

$$\mathcal{N}_n(u)(x) = \left(\frac{\mathcal{L}_\phi\left(u+\frac{1}{n}\mathbf{1}\right)(x)}{\|\mathcal{L}_\phi\left(u+\frac{1}{n}\mathbf{1}\right)\|_\infty} \right)^{\frac{1}{2}} \leq e^{(c'/2)d_\theta(x,x')} \left(\frac{\mathcal{L}_\phi\left(u+\frac{1}{n}\mathbf{1}\right)(x')}{\|\mathcal{L}_\phi\left(u+\frac{1}{n}\mathbf{1}\right)\|_\infty} \right)^{\frac{1}{2}}$$
$$= e^{(c'/2)d_\theta(x,x')} \mathcal{N}_n(u)(x')$$

i.e., $N_n(u) \in \Lambda_{c'/2}$. Providing c is sufficiently large that $c > c'/2 = (c + \|\phi\|_\theta)\theta/2$ we have that $\Lambda_{c'/2} \subset \Lambda_c$ and the result follows. □

This now brings us to the existence of the fixed point for each of the operators $\mathcal{N}_n : \Lambda_c \to \Lambda_c$.

Lemma 3.5 *For each $n \geq 1$, there exists a non-trivial fixed point $\psi_n \in \Lambda_c$ such that $\mathcal{N}_n(\psi_n) = \psi_n$.*

Proof By the Arzela–Ascoli Theorem the space Λ_c is compact with respect to the norm $\|\cdot\|_\infty$. For each $n \geq 1$ and $c > 0$ sufficiently large the map $\mathcal{N}_n : \Lambda_c \to \Lambda_c$ is a continuous map on a compact convex subspace of $C^0(\Sigma)$ and we can apply the Schauder fixed point theorem to deduce that there is a fixed point $\psi_n \in \Lambda_c$ for $\mathcal{N}_n$. To see that ψ_n is not identically zero we need only observe that by the definition of $\mathcal{N}_n$ there exists $x^{(n)} \in \Sigma$ with $\mathcal{N}_n(\psi_n)(x^{(n)}) = 1$ and by construction $\psi_n(x^{(n)}) = \mathcal{N}_n(\psi_n)(x^{(n)}) = 1$. This completes the proof. □

We can now use the compactness of Λ_c with respect to $\|\cdot\|_\infty$ to deduce that $(\psi_n)_{n=1}^\infty$ has a $\|\cdot\|_\infty$-convergent subsequence. We denote the limit point by $\psi_0 \in \Lambda$ and observe that we have that $\mathcal{L}_\phi(\psi_0) = \lambda\psi_0^2$, where $\lambda = \|\mathcal{L}_\phi(\psi_0)\|_\infty$. Moreover, since we observed that by construction that $\|\psi_n\|_\infty = 1$, for each $n \geq 1$, we can deduce that $\|\psi_0\|_\infty = 1$ and thus, in particular, ψ_0 is non-zero. If we replace ψ_0 by $\psi = \lambda\psi_0$ then we finally get $\mathcal{L}_\phi(\psi) = \psi^2$, as required.

To see that $\psi > 0$, assume for a contradiction that there exists $x_0 \in \Sigma$ such that $\psi(x_0) = 0$. Since $\psi \geq 0$ we see from the identity $\mathcal{L}_\phi(\psi)(x_0) = \psi^2(x_0) = 0$, which implies that $\psi(y) = 0$ whenever $\sigma y = x_0$. Proceeding iteratively, we have that ψ vanishes on the set $\cup_{n=0}^\infty \sigma^{-n}x_0$, which is dense by the mixing hypothesis on $\sigma : \Sigma \to \Sigma$ (corresponding to the aperiodicity assumption on A). However, this contradicts that $\psi \neq 0$.

3.2 Uniqueness of the Positive Fixed Point

Assume for a contradiction that we had a second distinct non-trivial positive fixed point, i.e., $\mathcal{L}_\phi(\psi') = \psi'^2$ with $\psi' > 0$ and $\psi \neq \psi'$. We can then associate $\xi := \inf\{t > 0 : t\psi - \psi' \geq 0\}$ and thus, in particular, $\xi\psi \geq \psi'$. Observe that $\mathcal{L}_\phi(\xi\psi - \psi') = \xi\psi^2 - \psi'^2 \geq 0$, since $\mathcal{L}_\phi$ preserves positive functions. Since $\xi\psi^2 - \psi'^2 = (\sqrt{\xi}\psi + \psi')(\sqrt{\xi}\psi - \psi') \geq 0$ we deduce that $\sqrt{\xi}\psi - \psi' \geq 0$. In particular, this implies that $\xi \leq 1$, otherwise it contradicts the original definition of ξ.

Interchanging the roles of ψ and ψ' we can define $\xi' := \inf\{t > 0 : t\psi' - \psi \geq 0\}$ and and thus, in particular, $\xi'\psi' \geq \psi \geq 0$. A similar argument to the above shows that $\xi' \leq 1$. However, since we can then write $(\xi'\xi)\psi' \geq \xi\psi \geq \psi'$ this implies that $\xi = \xi' = 1$.

For the definition of ξ we can choose x_0 with $\psi(x_0) = \psi'(x_0)$. We can then write $\mathcal{L}_\phi(\psi - \psi')(x_0) = \psi^2(x_0) - \psi'^2(x_0) = (\psi(x_0) + \psi'(x_0))(\psi(x_0) - \psi'(x_0)) = 0$ which implies that $\psi(y) = \psi'(y)$ whenever $\sigma y = x_0$. Proceeding inductively we deduce that $\psi(y) = \psi'(y)$ on the dense set of $y \in \cup_{n=1}^\infty \sigma^{-n}x_0$, and this $\psi = \psi'$.

Remark 3.6 This simple argument doesn't rule out the possibility of another non-positive fixed point.

3.3 Analyticity

To show the analytic dependence of the solution we want to use the implicit function theorem applied to the function $G : \mathcal{F}_\theta \times \mathcal{F}_\theta \to \mathcal{F}_\theta$ defined by

$$G(\phi, \psi) = \mathcal{L}_\phi \psi - \psi^2.$$

In order to apply the implicit function theorem at $(\phi_0, \psi_0) \in \mathcal{F}_\theta \times \mathcal{F}_\theta$ with $\psi_0 > 0$, say, satisfying $G(\phi_0, \psi_0) = 0$ we need to show that $(D_2G)(\phi_0, \psi_0) : \mathcal{F}_\theta \to \mathcal{F}_\theta$ is invertible. An easy calculation gives that

$$(D_2G)(\phi_0, \psi_0) = \left(\mathcal{L}_{\phi_0} - 2\psi_0\right) : \mathcal{F}_\theta \to \mathcal{F}_\theta. \tag{3.3}$$

The spectral radius of any linear operator is the radius of the smallest disk (centred at the origin) containing the spectrum.

We recall the following result [9] which is also due to Ruelle.

Lemma 3.7 (Ruelle) *The operator $\mathcal{L}_{\phi_0} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ has spectral radius λ_{ϕ_0}.*

In particular, we see from Lemma 3.7 that $(\mathcal{L}_{\phi_0} - 2\lambda_{\phi_0}\mathbf{1})^{-1} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ is a bounded linear operator since $2\lambda_{\phi_0}$ is not in the spectrum of $\mathcal{L}_{\phi_0}$. We can write

$$\begin{aligned}\left(\mathcal{L}_{\phi_0} - 2\psi_0\right)^{-1} &= \left((\mathcal{L}_{\phi_0} - 2\lambda_{\phi_0})\mathbf{1} + 2(\lambda_{\phi_0}\mathbf{1} - \psi_0)\right)^{-1} \\ &= (\mathcal{L}_{\phi_0} - 2\lambda_{\phi_0}\mathbf{1})^{-1}\left(\sum_{n=0}^{\infty}\left(2(\lambda_{\phi_0}\mathbf{1} - \psi_0)(\mathcal{L}_{\phi_0} - 2\lambda_{\phi_0}\mathbf{1})^{-1}\right)^n\right)\end{aligned}$$

which exists and is a bounded linear operator provided $\|\lambda_{\phi_0}\mathbf{1} - \psi_0\| < \frac{1}{2\|(\mathcal{L}_{\phi_0} - 2\lambda_{\phi_0}\mathbf{1})^{-1}\|}$.

In particular, by (3.3) we see that $(D_2G)(\phi_0, \psi_0)$ is invertible and thus the implicit function theorem applies. This allows us to deduce the analytic dependence.

Remark 3.8 (The Tangent Operator) Closely related to this circle of ideas is the use of a standard technique in understanding the iterates of a non-linear operator in a neighbourhood of a fixed point. More precisely, we consider the first order approximation to $G(\phi_0, \cdot) : \psi \mapsto \mathcal{L}_{\phi_0}\psi - \psi^2$ where $\psi = \psi_0 + \epsilon\psi^{(1)} + o(\epsilon)$. A simple calculation gives that the tangent operator

$$\mathcal{T}_{\phi_0}\psi := \lim_{\epsilon \searrow 0} \frac{G(\phi_0, \psi_0 + \epsilon\psi^{(1)}) - G(\phi_0, \psi_0)}{\epsilon} = \mathcal{L}_{\phi_0}\psi^{(1)} - 2\psi_0\psi^{(1)}.$$

For definiteness, we can consider the specific case where we replace ϕ_0 by ϕ_1 as in Remark 1.7, then we see that the spectra $\mathrm{sp}(\mathcal{T}_{\phi_1})$ and $\mathrm{sp}(\mathcal{L}_{\phi_1})$ are simply related by $\mathrm{sp}(\mathcal{T}_{\phi_1}) = \mathrm{sp}(\mathcal{L}_{\phi_1}) - 2$. Thus, since $\lambda_{\phi_1} = 1$, by Lemma 3.7 the tangent operator $\mathcal{T}_{\phi_1}$ will have its spectra in the disk centred at -2 and of radius 1 (and thus outside the unit disk, except for the value -1). This suggests that the fixed point ψ_{ϕ_1} is locally unstable in a codimension one space under the iteration $G(\phi_1, \cdot)^n$

4 Measures

The classical transfer operator $\mathcal{L}_\phi$ plays an important role in the ergodic theory of equilibrium states associated to ϕ. More precisely, the equilibrium state is a fixed point for the dual $\mathcal{L}_{\phi_1}^*$

to the transfer operator $\mathcal{L}_{\phi_1}$ (satisfying $\mathcal{L}_{\phi_1}\mathbf{1} = \mathbf{1}$) for the associated function ϕ_1 (cf. Remark 1.6). Although there is no direct analogue of equilibrium states in the context of the nonlinear equations $\mathcal{L}_\phi \psi = \psi^2$ we have been studying, one can still use this identity to associate to the two functions ϕ and ψ a natural invariant measure.

Given a solution $\mathcal{L}_\phi \psi = \psi^2$ as in Theorem 1.5, we can consider the linear operator $\mathcal{M}_\phi : \mathcal{F}_\theta \to \mathcal{F}_\theta$ given by

$$(\mathcal{M}_\phi w)(x) = \sum_{\sigma y = x} e^{\phi(y)} \left(\frac{\psi}{\psi^2 \circ \sigma} \right)(y) w(y)$$

which then satisfies $\mathcal{M}_\phi \mathbf{1} = \mathbf{1}$ (cf. Remark 1.7). Since $\mathcal{M}_\phi$ is a transfer operator with a Hölder continuous potential, it is a consequence of the simplicity of the maximal positive eigenvalue for the operator in Theorem 1.3, and thus of its dual, that there is a unique σ-invariant probability measure μ such that $\mathcal{M}_\phi^* \mu = \mu$, i.e., $\int f d\mu = \int \mathcal{M}_\psi f d\mu$ for all $f \in C^0(\Sigma)$. This leads to a non-standard version of the variational principle.

Lemma 4.1 (Variational Principle for ϕ and ψ) *For any σ-invariant probability measure ν we have that*

$$h(\nu) + \int \phi d\nu - \int \log \psi d\nu \leq 0 \tag{4.1}$$

with equality if and only if $\nu = \mu$.

Proof Let $\phi_2 := \phi + \log \psi - 2 \log \psi \circ \sigma$ then since $\mathcal{M}_\phi = \mathcal{L}_{\phi_2}$ satisfies $\mathcal{L}_{\phi_2}\mathbf{1} = \mathbf{1}$ we see that $P(\phi_2) = 0$ and μ is the unique equilibrium state associated to ϕ_2, by Proposition 3.4 in [9]. Thus by the variational principle (Theorem 3.5 in [9]) we have

$$\begin{aligned} &h(\nu) + \int (\phi + \log \psi - 2(\log \psi) \circ \sigma)\, d\nu \\ &\quad = h(\nu) + \int \phi_2 d\nu \\ &\quad \leq P(\phi_2) = 0 = h(\mu) + \int \phi_2 d\mu \\ &\quad = h(\mu) + \int (\phi + \log \psi - 2(\log \psi) \circ \sigma)\, d\mu \end{aligned} \tag{4.2}$$

with equality if and only if $\mu = \nu$. By σ-invariance of the measures we have that $\int (\log \psi) \circ \sigma d\mu = \int \log \psi d\mu$ and $\int (\log \psi) \circ \sigma d\nu = \int \log \psi d\nu$ and thus (4.1) follows from (4.2). □

We consequently have a particularly simple expression for the entropy $h(\mu)$.

Corollary 4.2 *We can write*

$$h(\mu) = \int (\log \psi) d\mu - \int \phi d\mu.$$

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