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Law of Large Numbers for Random Dynamical Systems

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Abstract We consider random dynamical systems with randomly chosen jumps. The choice of deterministic dynamical system and jumps depends on a position. We prove the existence of an exponentially attractive invariant measure and the strong law of large numbers.

Keywords Dynamical systems · Law of large numbers · Invariant measure

Mathematics Subject Classification 60J25 · 60J75

1 Introduction

In the present paper we are concerned with the problem of proving the law of large numbers (LLN) for random dynamical systems.

The question of establishing the LLN for an additive functional of a Markov process is one of the most fundamental in probability theory and there exists a rich literature on the subject, see e.g. [21] and the citations therein. However, in most of the existing results, it is usually assumed that the process under consideration is stationary and its equilibrium state is stable in some sense, usually in the L^2, or total variation norm. Our stability condition is formulated in a weaker metric than the total variation distance.

The law of large numbers we study in this note was also considered in many papers. Our results are based on a version of the law of large numbers due to Shirikyan (see [23,24]). Recently Komorowski et al. [15] obtained the weak law of large numbers for the passive tracer model in a compressible environment and Walczuk studied Markov processes with the

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transfer operator having spectral gap in the Wasserstein metric and proved the LLN in the non-stationary case [30].

Random dynamical systems [6,8,10] take into consideration some very important and widely studied cases, namely dynamical systems generated by learning systems [1,11,13,19], Poisson driven stochastic differential equations [7,18,25,27], iterated function systems with an infinite family of transformations [17,28,29], random evolutions [4,22] and irreducible Markov systems [31], used for the computer modelling of different stochastic processes.

A large class of applications of such models, both in physics and biology, is worth mentioning here: the shot noise, the photo conductive detectors, the growth of the size of structural populations, the motion of relativistic particles, both fermions and bosons (see [3,14,16]), the generalized stochastic process introduced in the recent model of gene expression by Lipniacki et al. [20] see also [2,5,9].

A number of results have been obtained that claim an existence of an asymptotically stable, unique invariant measure for Markov processes generated by random dynamical systems for which the state space need not be locally compact. We consider random dynamical systems with randomly chosen jumps acting on a given Polish space (Y, ϱ).

The aim of this paper is to study stochastic processes whose paths follow deterministic dynamics between random times, jump times, at which they change their position randomly. Hence, we analyse stochastic processes in which randomness appears at times $\tau_0 < \tau_1 < \tau_2 < \ldots$ We assume that a point $x_0 \in Y$ moves according to one of the dynamical systems $T_i : \mathbb{R}_+ \times Y \to Y$ from some set $\{T_1, \ldots, T_N\}$. The motion of the process is governed by the equation $X(t) = T_i(t, x_0)$ until the first jump time τ_1. Then we choose a transformation $q_\theta : Y \to Y$ from a family $\{q_\theta : \theta \in \Theta = \{1, \ldots, K\}\}$ and define $x_1 = q_\theta(T_i(\tau_1, x_0))$. The process restarts from that new point x_1 and continues as before. This gives the stochastic process $\{X(t)\}_{t\geq 0}$ with jump times $\{\tau_1, \tau_2, \ldots\}$ and post jump positions $\{x_1, x_2, \ldots\}$. The probability determining the frequency with which the dynamical systems T_i are chosen is described by a matrix of probabilities $[p_{ij}]_{i,j=1}^N$, $p_{ij} : Y \to [0, 1]$. The maps q_θ are randomly chosen with place dependent distribution. Given a Lipschitz function $\psi : X \to \mathbb{R}$ we define

$$S_n(\psi) = \psi(x_0) + \cdots + \psi(x_n).$$

Our aim is to find conditions under which $S_n(\psi)$ satisfies law of large numbers. Our results are based on an exponential convergence theorem due to Ślęczka and Kapica (see [12]) and a version of the law of large numbers due to Shirikyan (see [23,24]).

2 Notation and Basic Definitions

Let (X, d) be a *Polish* space, i.e. a complete and separable metric space and denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. By $B_b(X)$ we denote the space of bounded Borel-measurable functions equipped with the supremum norm, $C_b(X)$ stands for the subspace of bounded continuous functions. Let $\mathcal{M}_{fin}(X)$ and $\mathcal{M}_1(X)$ be the sets of Borel measures on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{fin}(X)$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1(X)$. The elements of $\mathcal{M}_1(X)$ are called *probability measures*. The elements of $\mathcal{M}_{fin}(X)$ for which $\mu(X) \leq 1$ are called *subprobability measures*. By *supp* μ we denote the support of the measure μ. We also define

$$\mathcal{M}_1^L(X) = \left\{\mu \in \mathcal{M}_1(X) : \int_X L(x)\mu(dx) < \infty\right\}$$

where $L : X \to [0, \infty)$ is an arbitrary Borel measurable function and

$$\mathcal{M}_1^1(X) = \left\{ \mu \in \mathcal{M}_1(X) : \int_X d(\bar{x}, x)\mu(dx) < \infty \right\},$$

where $\bar{x} \in X$ is fixed. By the triangle inequality this family is independent of the choice of $\bar{x}$.

The space $\mathcal{M}_1(X)$ is equipped with the *Fortet-Mourier metric*:

$$\|\mu_1 - \mu_2\|_{FM} = \sup \left\{ |\int_X f(x)(\mu_1 - \mu_2)(dx)| : f \in \mathcal{F} \right\},$$

where

$$\mathcal{F} = \{ f \in C_b(X) : |f(x) - f(y)| \le d(x, y) \quad \text{and} \quad |f(x)| \le 1 \quad \text{for} \quad x, y \in X \}.$$

Let $P : B_b(X) \to B_b(X)$ be a *Markov operator*, i.e. a linear operator satisfying $P\mathbf{1}_X = \mathbf{1}_X$ and $Pf(x) \ge 0$ if $f \ge 0$. Denote by P^* the the dual operator, i.e operator $P^* : \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X)$ defined as follows

$$P^*\mu(A) := \int_X P\mathbf{1}_A(x)\mu(dx) \qquad \text{for} \qquad A \in \mathcal{B}_X.$$

We say that a measure $\mu_* \in \mathcal{M}_1(X)$ is *invariant* for P if

$$\int_X Pf(x)\mu_*(dx) = \int_X f(x)\mu_*(dx) \qquad \text{for every} \qquad f \in B_b(X)$$

or, alternatively, we have $P^*\mu_* = \mu_*$. An invariant measure μ is *attractive* if

$$\lim_{n\to\infty} \int_X P^n f(x)\,\mu(dx) = \int_X f(x)\,\mu(dx) \quad \text{for} \quad f \in C_b(X),\ \mu \in \mathcal{M}_1(X).$$

By $\{\mathbf{P}_x : x \in X\}$ we denote a *transition probability function* for P, i.e. a family of measures $\mathbf{P}_x \in \mathcal{M}_1(X)$ for $x \in X$, such that the map $x \mapsto \mathbf{P}_x(A)$ is measurable for every $A \in \mathcal{B}_X$ and

$$Pf(x) = \int_X f(y)\mathbf{P}_x(dy) \qquad \text{for} \qquad x \in X \quad \text{and} \quad f \in B_b(X)$$

or equivalently $P^*\mu(A) = \int_X \mathbf{P}_x(A)\mu(dx)$ for $A \in \mathcal{B}_X$ and $\mu \in \mathcal{M}_{fin}(X)$. We say that a vector $(p_1, \dots, p_N)$ where $p_i : Y \to [0, 1]$ is *a probability vector* if

$$\sum_{i=1}^{N} p_i(x) = 1 \quad \text{for} \quad x \in Y.$$

Analogously a matrix $[p_{ij}]_{i,j}$ where $p_{ij} : Y \to [0, 1]$ for $i, j \in \{1, \dots, N\}$ is *a probability matrix* if

$$\sum_{j=1}^{N} p_{ij}(x) = 1 \quad \text{for} \quad x \in Y \quad \text{and} \quad i \in \{1, \dots, N\}.$$

Definition 1 A coupling for $\{\mathbf{P}_x : x \in X\}$ is a family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of probability measures on $X \times X$ such that for every $B \in \mathcal{B}_{X^2}$ the map $X^2 \ni (x, y) \mapsto \mathbf{B}_{x,y}(B)$ is measurable and

$$\mathbf{B}_{x,y}(A \times X) = \mathbf{P}_x(A), \qquad \mathbf{B}_{x,y}(X \times A) = \mathbf{P}_y(A)$$

for every $x, y \in X$ and $A \in \mathcal{B}_X$.

In the following we assume that there exists a *subcoupling* for $\{\mathbf{P}_x : x \in X\}$, i.e. a family $\{\mathbf{Q}_{x,y} : x, y \in X\}$ of subprobability measures on X^2 such that the map $(x, y) \mapsto \mathbf{Q}_{x,y}(B)$ is measurable for every Borel $B \subset X^2$ and

$$\mathbf{Q}_{x,y}(A \times X) \leq \mathbf{P}_x(A) \quad \text{ and } \quad \mathbf{Q}_{x,y}(X \times A) \leq \mathbf{P}_y(A)$$

for every $x, y \in X$ and Borel $A \subset X$.

Measures $\{\mathbf{Q}_{x,y} : x, y \in X\}$ allow us to construct a coupling for $\{\mathbf{P}_x : x \in X\}$. Define on X^2 the family of measures $\{\mathbf{R}_{x,y} : x, y \in X\}$ which on rectangles $A \times B$ are given by

$$\mathbf{R}_{x,y}(A \times B) = \frac{1}{1 - \mathbf{Q}_{x,y}(X^2)}(\mathbf{P}_x(A) - \mathbf{Q}_{x,y}(A \times X))(\mathbf{P}_y(B) - \mathbf{Q}_{x,y}(X \times B)),$$

when $\mathbf{Q}_{x,y}(X^2) < 1$ and $\mathbf{R}_{x,y}(A \times B) = 0$ otherwise. A simple computation shows that the family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of measures on X^2 defined by

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y} \quad \text{for} \quad x, y \in X \tag{1}$$

is a coupling for $\{\mathbf{P}_x : x \in X\}$.

The following Theorem due to M. Ślęczka and R. Kapica (see [12]) will be used in the proof of Theorem 3 in Sect. 4.

Theorem 1 *Assume that a Markov operator P and transition probabilities $\{\mathbf{Q}_{x,y} : x, y \in X\}$ satisfy*

A0 *P is a* Feller operator, *i.e.* $P(C_b(X)) \subset C_b(X)$.
A1 *There exists a* Lapunov function *for P, i.e. continuous function $L : X \to [0, \infty)$ such that L is bounded on bounded sets,* $\lim_{x\to\infty} L(x) = +\infty$ *and for some* $\lambda \in (0, 1)$, $c > 0$

$$PL(x) \leq \lambda L(x) + c \qquad for \qquad x \in X.$$

A2 *There exist $F \subset X^2$ and $\alpha \in (0, 1)$ such that supp $\mathbf{Q}_{x,y} \subset F$ and*

$$\int_{X^2} d(u, v)\mathbf{Q}_{x,y}(du, dv) \leq \alpha d(x, y) \qquad for \qquad (x, y) \in F. \tag{2}$$

A3 *There exist $\delta > 0$, $l > 0$ and $\nu \in (0, 1]$ such that*

$$1 - \mathbf{Q}_{x,y}(X^2) \leq l d(x, y)^{\nu}$$

and

$$\mathbf{Q}_{x,y}(\{(u, v) \in X^2 : d(u, v) < \alpha d(x, y)\}) \geq \delta$$

for $(x, y) \in F$
A4 *There exist $\beta \in (0, 1)$, $\tilde{C} > 0$ and $R > 0$ such that for*

$$\kappa(\,(x_n, y_n)_{n\in\mathbb{N}_0}\,) = \inf\{n \in \mathbb{N}_0 : (x_n, y_n) \in F \quad and \quad L(x_n) + L(y_n) < R\}$$

we have

$$\mathbb{E}_{x,y}\beta^{-\kappa} \leq \tilde{C} \qquad whenever \qquad L(x) + L(y) < \frac{4c}{1 - \lambda},$$

where $\mathbb{E}_{x,y}$ denotes here the expectation with respect to the chain starting from (x, y) and with transition function $\{\mathbf{B}_{x,y} : x, y \in X\}$. Then operator P possesses a unique invariant measure $\mu_* \in \mathcal{M}_1^L(X)$, which is attractive in $\mathcal{M}_1(X)$. Moreover, there exist $q \in (0, 1)$ and $C > 0$ such that

$$\|P^{*n}\mu - \mu_*\|_{FM} \le q^n C(1 + \int_X L(x)\mu(dx)) \tag{3}$$

for $\mu \in \mathcal{M}_1^L(X)$ and $n \in \mathbb{N}$.

We will also need a version of the strong law of large numbers due to Shirikyan [23,24]. It is originally formulated for Markov chains on a Hilbert space, however analysis of the proof shows that it remains true for Polish spaces.

Theorem 2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a Polish space. Suppose that for a family of Markov chains $((X_n^x)_{n\ge 0}, \mathbb{P}_x)_{x\in X}$ on X with Markov operator $P : B_b(X) \to B_b(X)$ there exists a unique invariant measure $\mu_* \in \mathcal{M}_1(X)$, a continuous function $v : X \to \mathbb{R}_+$ and a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive numbers such that $\eta_n \to 0$ as $n \to \infty$ and*

$$||P^{*n}\delta_x - \mu_*||_{FM} \le \eta_n v(x) \quad \text{for} \quad x \in X.$$

If

$$C = \sum_{n=0}^{\infty} \eta_n < \infty$$

and there exists a continuous function $h : X \to \mathbb{R}_+$ such that

$$\mathbb{E}_x(v(X_n^x)) \le h(x) \quad \text{for} \quad x \in X, n \ge 0,$$

where $\mathbb{E}_x$ is the expectation with respect to $\mathbb{P}_x$, then for any $x \in X$ and any bounded Lipschitz function $f : X \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k^x) = \int_X f(y)\, \mu_*(dy)$$

$\mathbb{P}_x$ almost surely.

3 Random Dynamical Systems

Let (Y, ϱ) be a Polish space, $\mathbb{R}_+ = [0, +\infty)$ and $I = \{1, \ldots, N\}$, $\Theta = \{1, \ldots, K\}$, where N and K are given positive integers.

We are given a family of continuous functions $q_\theta : Y \to Y, \theta \in \Theta$ and a finite sequence of semidynamical systems $T_i : \mathbb{R}_+ \times Y \to Y, i \in I$, i.e.

$$T_i(s + t, x) = T_i(s, (T_i(t, x)), \quad T_i(0, x) = x \quad \text{for} \quad s, t \in \mathbb{R}_+, \ i \in I \ \text{ and } \ x \in Y,$$

the transformations $T_i : \mathbb{R}_+ \times Y \to Y, i \in I$ are continuous.

Let $p_i : Y \to [0, 1]$, $i \in I$, $\tilde{p}_\theta : Y \to [0, 1]$, $\theta \in \Theta$ be probability vectors and $[p_{ij}]_{i,j\in I}$, $p_{ij} : Y \to [0, 1]$, $i, j \in I$ be a matrix of probabilities . In the sequel we denote the system by (T, q, p).

Finally, let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{\tau_n\}_{n\geq 0}$ be an increasing sequence of random variables $\tau_n : \Omega \to \mathbb{R}_+$ with $\tau_0 = 0$ and such that the increments $\Delta\tau_n = \tau_n - \tau_{n-1}$, $n \in \mathbb{N}$, are independent and have the same density $g(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

The intuitive description of random dynamical system corresponding to the system (T, q, p) is the following.

For an initial point $x_0 \in Y$ we randomly select a transformation T_{i_0} from the set $\{T_1, \ldots, T_N\}$ in such a way that the probability of choosing T_{i_0} is equal to $p_{i_0}(x_0)$, and we define

$$X(t) = T_{i_0}(t, x_0) \quad \text{for} \quad 0 \leq t < \tau_1.$$

Next, at the random moment τ_1, at the point $T_{i_0}(\tau_1, x_0)$ we choose a jump q_θ from the set $\{q_1, \ldots, q_K\}$ with probability $\tilde{p}_\theta(T_{i_0}(\tau_1, x_0))$ and we define

$$x_1 = q_\theta(T_{i_0}(\tau_1, x_0)).$$

Finally, given x_n, $n \geq 1$, we choose T_{i_n} in such a way that the probability of choosing T_{i_n} is equal to $p_{i_{n-1}i_n}(x_n)$ and we define

$$X(t) = T_{i_n}(t - \tau_n, x_n) \quad \text{for} \quad \tau_n < t < \tau_{n+1}.$$

At the point $T_{i_n}(\Delta\tau_{n+1}, x_n)$ we choose q_{θ_n} with probability $\tilde{p}_{\theta_n}(T_{i_n}(\Delta\tau_{n+1}, x_n))$. Then we define

$$x_{n+1} = q_{\theta_n}(T_{i_n}(\Delta\tau_{n+1}, x_n)).$$

We obtain a piecewise-deterministic trajectory for $\{X(t)\}_{t\geq 0}$ with jump times $\{\tau_1, \tau_2, \ldots\}$ and post jump locations $\{x_1, x_2, \ldots\}$.

The above considerations may be reformulated as follows. Let $\{\xi_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ be sequences of random variables, $\xi_n : \Omega \to I$ and $\gamma_n : \Omega \to \Theta$, such that

$$\begin{aligned} &\mathbb{P}(\xi_0 = i | x_0 = x) = p_i(x), \\ &\mathbb{P}(\xi_n = k | x_n = x \quad \text{and} \quad \xi_{n-1} = i) = p_{ik}(x), \\ &\mathbb{P}(\gamma_n = \theta | T_{\xi_{n-1}}(\Delta\tau_n, x_{n-1}) = y) = \tilde{p}_\theta(y). \end{aligned} \tag{4}$$

Assume that $\{\xi_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are independent of $\{\tau_n\}_{n\geq 0}$ and that for every $n \in \mathbb{N}$ the variables $\gamma_1, \ldots, \gamma_{n-1}, \xi_1, \ldots, \xi_{n-1}$ are also independent.

Given an initial random variable ξ_0 the sequence of the random variables $\{x_n\}_{n\geq 0}$, $x_n : \Omega \to Y$, is given by

$$x_n = q_{\gamma_n}\big(T_{\xi_{n-1}}(\Delta\tau_n, x_{n-1})\big) \quad \text{for} \quad n = 1, 2, \ldots \tag{5}$$

and the stochastic process $\{X(t)\}_{t\geq 0}$, $X(t) : \Omega \to Y$, is given by

$$X(t) = T_{\xi_{n-1}}(t - \tau_{n-1}, x_{n-1}) \quad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots \tag{6}$$

It is easy to see that $\{X(t)\}_{t\geq 0}$ and $\{x_n\}_{n\geq 0}$ are not Markov processes. In order to use the theory of Markov operators we must redefine the processes $\{X(t)\}_{t\geq 0}$ and $\{x_n\}_{n\geq 0}$ in such a way that the redefined processes become Markov.

To this end, consider the space $Y \times I$ endowed with the metric d given by

$$d\big((x, i), (y, j)\big) = \varrho(x, y) + \varrho_d(i, j) \quad \text{for} \quad x, y \in Y, \ i, j \in I, \tag{7}$$

where ϱ_d is the discrete metric in I. Now define the process $\{\xi(t)\}_{t\geq 0}$, $\xi(t) : \Omega \to I$, by

$$\xi(t) = \xi_{n-1} \quad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots$$

Then the stochastic process $\{(X(t), \xi(t))\}_{t\geq 0}$, $(X(t), \xi(t)) : \Omega \to Y \times I$ has the required Markov property.

We will study the Markov process (post jump locations) $\{(x_n, \xi_n)\}_{n\geq 0}$, $(x_n, \xi_n) : \Omega \to Y \times I$.

Define the Markov operator $P : B_b(Y \times I) \to B_b(Y \times I)$

$$Pf(x,i) = \sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \lambda e^{-\lambda t} f\big(q_\theta\big(T_j(t,x)\big), j\big) p_{ij}(x)\tilde{p}_\theta\big(T_j(t,x)\big)\, dt. \tag{8}$$

Now consider the sequence of distributions

$$\overline{\mu}_n(A) = \mathbb{P}\big((x_n, \xi_n) \in A\big) \quad \text{for} \quad A \in \mathcal{B}(Y \times I),\ n \geq 0.$$

It is easy to see that

$$\overline{\mu}_{n+1} = P^*\overline{\mu}_n \quad \text{for} \quad n \geq 0,$$

where $P^*\mathcal{M}_1(Y \times I) \to \mathcal{M}_1(Y \times I)$ is the dual operator

$$P^*\mu(A) = \sum_{j\in I}\sum_{\theta\in\Theta}\int_{Y\times I}\int_0^{+\infty} \lambda e^{-\lambda t} 1_A\big(q_\theta\big(T_j(t,x)\big), j\big) p_{ij}(x)\tilde{p}_\theta\big(T_j(t,x)\big)\, dt\, \mu(dx, di). \tag{9}$$

In order to get the existence of an exponentially attractive invariant measure and the strong law of large numbers, we make the following assumptions on the system (T, q, p).

The transformations $T_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$ and $q_\theta : Y \to Y$, $\theta \in \Theta$, are continuous and there exists $x_* \in Y$ such that

$$\int_{\mathbb{R}_+} e^{-\lambda t}\varrho(q_\theta(T_j(t, x_*)), q_\theta(x_*))\, dt < \infty \quad \text{for} \quad j \in I, \quad \theta \in \Theta. \tag{10}$$

For the system (T, q, p) there are three constants $L \geq 1$, $\alpha \in \mathbb{R}$ and $L_q > 0$ such that

$$\sum_{j\in I} p_{ij}(y)\varrho(T_j(t,x), T_j(t,y)) \leq Le^{\alpha t}\varrho(x,y) \quad \text{for} \quad x, y \in Y,\ i \in I,\ t \geq 0 \tag{11}$$

and

$$\sum_{\theta\in\Theta} \tilde{p}_\theta(x)\varrho(q_\theta(x), q_\theta(y)) \leq L_q\varrho(x,y) \quad \text{for} \quad x, y \in Y. \tag{12}$$

We also assume that the functions $\tilde{p}_\theta$, $\theta \in \Theta$, and p_{ij}, $i, j \in I$, satisfy the following conditions

$$\begin{aligned} &\sum_{j\in I} |p_{ij}(x) - p_{ij}(y)| \leq L_p\varrho(x,y) \quad \text{for} \quad x, y \in Y,\ i \in I, \\ &\sum_{\theta\in\Theta} |\tilde{p}_\theta(x) - \tilde{p}_\theta(y)| \leq L_{\tilde{p}}\varrho(x,y) \quad \text{for} \quad x, y \in Y, \end{aligned} \tag{13}$$

where $L_p, L_{\tilde{p}} > 0$.

For $x, y \in Y$, $t \geq 0$ we define

$$\begin{aligned} &I_T(t,x,y) = \{j \in I : \varrho(T_j(t,x), T_j(t,y)) \leq Le^{\alpha t}\varrho(x,y)\} \\ &I_q(x,y) = \{\theta \in \Theta : \varrho(q_\theta(x), q_\theta(y)) \leq L_q\varrho(x,y)\} \end{aligned} \tag{14}$$

Assume that there are $p_0 > 0$, $q_0 > 0$ such that : for every $i_1, i_2 \in I$, $x, y \in Y$ and $t \geq 0$ we have

$$\sum_{j\in I_T(t,x,y)} p_{i_1 j}(x)p_{i_2 j}(y) > p_0,$$
$$\sum_{\theta\in I_q(x,y)} \tilde{p}_\theta(x)\tilde{p}_\theta(y) > q_0. \tag{15}$$

Remark 1 The condition (15) is satisfied if there are $i_0 \in I, \theta_0 \in \Theta$ such that

$$\varrho(T_{i_0}(t,x), T_{i_0}(t,y)) \le Le^{\alpha t}\varrho(x,y) \quad \text{for} \quad x,y\in Y,\ t\ge 0,$$
$$\varrho(q_{\theta_0}(x), q_{\theta_0}(y)) \le L_q\varrho(x,y) \quad \text{for} \quad x,y\in Y, \tag{16}$$

and

$$\inf_{i\in I}\inf_{x\in Y} p_{ii_0}(x) > 0,$$
$$\inf_{x\in Y} \tilde{p}_{\theta_0}(x) > 0. \tag{17}$$

4 The Main Theorem

Theorem 3 *Assume that system (T,p,q) satisfies conditions (10)–(15). If*

$$LL_q + \frac{\alpha}{\lambda} < 1. \tag{18}$$

then

(i) *there exists a unique invariant measure $\mu_* \in \mathcal{M}_1^1(Y\times I)$ for the process $(x_n,\xi_n)_{n\ge 0}$, which is attractive in $\mathcal{M}_1(Y\times I)$.*
(ii) *there exist $q\in(0,1)$ and $C>0$ such that for $\mu\in\mathcal{M}_1^1(Y\times I)$ and $n\in\mathbb{N}$*

$$||P^{*n}\mu - \mu_*||_{FM} \le q^n C\left(1+\int_Y \varrho(x,x_*)\,\mu(dx)\right),$$

where x_* is given by (10),
(iii) *the strong law of large numbers holds for the process $(x_n,\xi_n)_{n\ge 0}$ starting from $(x_0,\xi_0)\in Y\times I$, i.e. for every bounded Lipschitz function $f: Y\times I\to\mathbb{R}$ and every $x_0\in Y$ and $\xi_0\in I$ we have*

$$\lim_{n\in\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(x_k,\xi_k) = \int_{Y\times I} f(x,\xi)\,\mu_*(dx,d\xi)$$

$\mathbb{P}_{x_0,\xi_0}$ *almost surely.*

Proof of Theorem 3 We are going to verify assumptions of Theorem 1. Set $X = Y\times I$, $F = X\times X$ and define

$$\mathbf{Q}_{(x_1,i_1)(x_2,i_2)}(A) =$$
$$\sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty}\lambda e^{-\lambda t}\left\{p_{i_1 j}(x_1)\tilde{p}_\theta\big(T_j(t,x_1)\big)\wedge p_{i_2 j}(x_2)\tilde{p}_\theta\big(T_j(t,x_2)\big)\right\}$$
$$\times 1_A\Big(\big(q_\theta\big(T_j(t,x_1)\big),j\big),\big(q_\theta\big(T_j(t,x_2)\big),j\big)\Big)\,dt$$

for $A\subset X\times X$, where $a\wedge b$ stands for the minimum of a and b.

A0. The continuity of functions $p_{ij}, \tilde{p}_\theta, q_\theta$ implies that the operator P defined in (8) is a Feller operator.
A1. Define $L(x,i) = \varrho(x, x_*)$ for $(x,i) \in X$. By (8) we have

$$PL(x,i) \le \sum_{j\in I}\sum_{s\in\Theta}\int_0^{+\infty} \varrho(q_\theta\big(T_j(t,x)\big), q_\theta\big(T_j(t,x_*)\big))\lambda e^{-\lambda t} p_{ij}(x)\tilde{p}_\theta\big(T_j(t,x)\big)\,dt$$
$$+\sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \varrho(q_\theta\big(T_j(t,x_*)\big), q_\theta(x_*))\lambda e^{-\lambda t} p_{ij}(x)\tilde{p}_\theta\big(T_j(t,x)\big)\,dt$$
$$+\sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \varrho(q_\theta(x_*), x_*)\lambda e^{-\lambda t} p_{ij}(x)\tilde{p}_\theta\big(T_j(t,x)\big)\,dt.$$

Further, using (10), (11) and (12) we obtain

$$PL(x,i) \le aL(x,i) + b, \tag{19}$$

where

$$a = \frac{\lambda L L_q}{\lambda - \alpha},$$
$$b = \sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \lambda e^{-\lambda t}\varrho(q_\theta\big(T_j(t,x_*)\big), q_\theta(x_*))\,dt + \sum_{\theta\in\Theta}\varrho(q_\theta(x_*), x_*), \tag{20}$$

so L is a Lapunov function for P.
A2. Observe that by (7), (11) and (12) we have for $(x_1,i_1),(x_2,i_2) \in X$

$$\int_{X^2} d(u,v)\,\mathbf{Q}_{(x_1,i_1)(x_2,i_2)}(du,dv) =$$
$$\sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \lambda e^{-\lambda t}\Big\{p_{i_1 j}(x_1)\tilde{p}_\theta\big(T_j(t,x_1)\big) \wedge p_{i_2 j}(x_2)\tilde{p}_\theta\big(T_j(t,x_2)\big)\Big\} \times$$
$$\times\varrho(q_\theta\big(T_j(t,x_1)\big), q_\theta\big(T_j(t,x_2)\big))\,dt$$
$$\le \sum_{j\in I}\sum_{\theta\in\Theta}\int_0^{+\infty} \lambda e^{-\lambda t} p_{i_1 j}(x_1)\tilde{p}_\theta\big(T_j(t,x_1)\big)\varrho\Big(q_\theta\big(T_j(t,x_1)\big), q_\theta\big(T_j(t,x_2)\big)\Big)\,dt$$
$$\le \beta\varrho(x_1,x_2) \le \beta\, d\big((x_1,i_1),(x_2,i_2)\big)$$

with $\beta = \frac{\lambda L L_q}{\lambda-\alpha} < 1$ by (18).
A3. From (13) and (11) it follows that

$$1 - \sum_{j\in I}\sum_{\theta\in\Theta}\Big\{p_{i_1 j}(x_1)\tilde{p}_\theta\big(T_j(t,x_1)\big) \wedge p_{i_2 j}(x_2)\tilde{p}_\theta\big(T_j(t,x_2)\big)\Big\}$$
$$\le \sum_{j\in I}\sum_{\theta\in\Theta}|p_{i_1 j}(x_1)\tilde{p}_\theta\big(T_j(t,x_1)\big) - p_{i_2 j}(x_2)\tilde{p}_\theta\big(T_j(t,x_2)\big)|$$
$$\le \sum_{j\in I}\sum_{\theta\in\Theta} p_{i_1 j}(x_1)|\tilde{p}_\theta\big(T_j(t,x_1)\big) - \tilde{p}_\theta\big(T_j(t,x_2)\big)|$$
$$+\sum_{j\in I}\sum_{\theta\in\Theta}\tilde{p}_\theta\big(T_j(t,x_2)\big)|p_{i_1 j}(x_1) - p_{i_2 j}(x_2)|$$
$$\le LL_{\tilde{p}}e^{\alpha t}\varrho(x_1,x_2) + L_p\varrho(x_1,x_2) + 2N\varrho_d(i_1,i_2)$$

and consequently

$$1-\mathbf{Q}_{(x_1,i_1)(x_2,i_2)}(X^2)\le\Big(L_p+\frac{\lambda L L_{\tilde p}}{\lambda-\alpha}\Big)\varrho(x_1,x_2)+2N\varrho_d(i_1,i_2).$$

Fix $x_1,x_2\in Y$ and $i_1,i_2\in I$. Define $B=\{\big((u,j),(v,j)\big):\varrho(u,v)<\beta\varrho(x_1,x_2),\ j\in I\}$. If $\alpha\ge 0$ then there exists $t_*>0$ such that $LL_qe^{\alpha t}<\beta$ for $t<t_*$. Set $A=(0,t_*)$. If $\alpha<0$ then there exists $t_*>0$ such that $LL_qe^{\alpha t}<\beta$ for $t>t_*$. Set $A=(t_*,\infty)$. In both cases define $r=\int_A\lambda e^{-\lambda t}\,dt$. For all $x,y\in Y$, $t\in A$, $j\in I_T(t,x,y)$ and $\theta\in I_q\big(T_j(t,x),T_j(t,y)\big)$ we have

$$\big((q_\theta(T_j(t,x)),j),(q_\theta(T_j(t,y)),j)\big)\in B. \tag{21}$$

From (15) and (21) we obtain

$$\begin{aligned}
&\mathbf{Q}_{(x_1,i_1)(x_2,i_2)}(B)\\
&\ge\int_A\lambda e^{-\lambda t}\sum_{j\in I_T(t,x_1,x_2)}\ \sum_{\theta\in I_q(T_j(t,x),T_j(t,y))}\Big\{p_{i_1j}(x_1)\tilde p_\theta\big(T_j(t,x_1)\big)\\
&\quad\wedge p_{i_2j}(x_2)\tilde p_\theta\big(T_j(t,x_2)\big)\Big\}\\
&\quad\times 1_B\Big(\big(q_\theta\big(T_j(t,x_1)\big),j\big),\big(q_\theta\big(T_j(t,x_2),j\big)\big)\Big)\,dt\\
&=\int_A\lambda e^{-\lambda t}\sum_{j\in I_T(t,x_1,x_2)}\ \sum_{\theta\in I_q(T_j(t,x),T_j(t,y))}\Big\{p_{i_1j}(x_1)\tilde p_\theta\big(T_j(t,x_1)\big)\\
&\quad\wedge p_{i_2j}(x_2)\tilde p_\theta\big(T_j(t,x_2)\big)\Big\}\,dt\\
&\ge\int_A\lambda e^{-\lambda t}\sum_{j\in I_T(t,x_1,x_2)}\ \sum_{\theta\in I_q(T_j(t,x),T_j(t,y))}\Big\{p_{i_1j}(x_1)p_{i_2j}(x_2)\tilde p_\theta\big(T_j(t,x_1)\big)\\
&\quad\times\tilde p_\theta\big(T_j(t,x_2)\big)\Big\}\,dt\\
&>p_0q_0r>0,
\end{aligned}$$

so **A3** is satisfied. Since $F=X\times X$, assumption **A4** is trivially satisfied. From Theorem 1 we obtain (i) and (ii). Set $v(x,i)=C(\varrho(x,x_*)+1)$ and $h(x,i)=C(\varrho(x,x_*)+1+\frac{b}{1-a})$ for $x\in X$, $i\in I$, with a,b as in (20). Iterating (19) we obtain

$$\mathbb{E}_{x_0,\xi_0}(v(x_n,\xi_n))\le h(x_0,\xi_0)\quad\text{for}\quad x_0\in X,\ \xi_0\in I.$$

Application of Theorem 2 ends the proof.

The next result describing the asymptotic behavior of the process $(x_n)_{n\ge0}$ on Y is an obvious consequence of Theorem 3. Let $\tilde\mu_0$ be the distribution of the initial random vector x_0 and $\tilde\mu_n$ the distribution of x_n, i.e.

$$\tilde\mu_n(A)=\mathbb{P}(x_n\in A)\quad\text{for}\quad A\in\mathcal{B}_Y,\ n\ge1.$$

□

Theorem 4 *Under the hypotheses of Theorem 3 the following statements hold:*

(i) *there exists a measure $\tilde\mu_*\in\mathcal{M}_1^1(Y)$ such that for any $\tilde\mu_0$ the sequence $(\tilde\mu_n)_{n\ge0}$ converges weakly to $\tilde\mu_*$. Moreover, if*

$$\mathbb{P}(x_0\in A)=\tilde\mu_*(A)\quad\textit{for}\quad A\in\mathcal{B}_Y$$

then $\tilde\mu_n(A)=\tilde\mu_*(A)$ for $A\in\mathcal{B}_Y$ and $n\ge1$.

(ii) *there exist* $q \in (0, 1)$ *and* $C > 0$ *such that*

$$||\tilde{\mu}_n - \tilde{\mu}_*||_{FM} \le q^n C(1 + \int_Y \varrho(x, x_*)\, \tilde{\mu}_0(dx))$$

for any initial distribution $\tilde{\mu}_0 \in \mathcal{M}_1^1(Y)$ *and* $n \ge 1$.

(iii) *for any starting point* $x_0 \in Y$, $\xi_0 \in I$ *and any bounded Lipschitz function* f *on* Y

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k) = \int_Y f(x)\, \tilde{\mu}_*(dx)$$

$\mathbb{P}_{x_0,\xi_0}$ *almost surely.*

The examples below show that our model generalizes some important and widely studied objects, namely dynamical systems generated by iterated function systems [1,11,13,19] and Poisson driven stochastic differential equations [7,18,25,27].

Example 1 Iterated Function Systems.

Let $(Y, \|\cdot\|)$ be a separable Banach space. An iterated function system (IFS) consists of a sequence of continuous transformations

$$q_\theta : Y \to Y, \quad \theta = 1, \dots, K$$

and a probability vector

$$\tilde{p}_\theta : Y \to [0, 1], \quad \theta = 1, \dots, K.$$

Such a system is briefly denoted by $(q, \tilde{p})_K = (q_1, \dots, q_K, \tilde{p}_1, \dots, \tilde{p}_K)$. The action of an IFS can be roughly described as follows. We choose an initial point x_0 and we randomly select from the set $\Theta = \{1, \dots, K\}$ an integer θ_0 in such a way that the probability of choosing θ_0 is $\tilde{p}_{\theta_0}(x_0)$. If a number θ_0 is drawn, we define $x_1 = q_{\theta_0}(x_0)$. Having x_1 we select θ_1 in such a way that the probability of choosing θ_1 is $\tilde{p}_{\theta_1}(x_1)$. Now we define $x_2 = q_{\theta_1}(x_1)$ and so on.

An IFS is a particular example of a random dynamical system with randomly chosen jumps. Consider a dynamical system of the form $I = \{1\}$ and $T_1(t, x) = x$ for $x \in Y$, $t \in \mathbb{R}_+$. Moreover assume that $p_1(x) = 1$ and $p_{11}(x) = 1$ for $x \in Y$. Then we obtain an IFS $(q, \tilde{p})_K$.

Denoting by $\tilde{\mu}_n$, $n \in \mathbb{N}$, the distribution of x_n, i.e., $\tilde{\mu}_n(A) = \mathbb{P}(x_n \in A)$ for $A \in \mathcal{B}(Y)$, we define $\widetilde{P}^*$ as the transition operator such that $\tilde{\mu}_{n+1} = \widetilde{P}^* \tilde{\mu}_n$ for $n \in \mathbb{N}$. The transition operator corresponding to iterated function system $(q, \tilde{p})_K$ is given by

$$\widetilde{P}^* \mu(A) = \sum_{\theta\in\Theta} \int_Y 1_A\big(q_\theta(x)\big)\, \tilde{p}_\theta(x)\, \mu(dx) \quad \text{for} \quad A \in \mathcal{B}(Y),\ \ \mu \in \mathcal{M}_1(Y). \tag{22}$$

We assume (13) and (15) and take $\alpha = 0$ in (11). If

$$\sum_{\theta\in\Theta} \tilde{p}_\theta(x)\varrho(q_\theta(x), q_\theta(y)) \le L_q \varrho(x, y) \quad \text{for} \quad x, y \in Y.$$

with $L_q < 1$ then from Theorem 4 we obtain existence of an invariant measure $\mu_* \in \mathcal{M}_1^1(Y \times I)$ for the process $(x_n, \xi_n)_{n\ge0}$, which is attractive in $\mathcal{M}_1(Y \times I)$, exponentially attractive in $\mathcal{M}_1^1(Y \times I)$ and for which the strong law of large numbers holds (cf. [26]).

Remark 2 The convergence in Theorem 4 is weak and one cannot expect the strong one, i.e. in the total variation norm. Indeed, let $Y = \mathbb{R}$, $q_1(x) = x$ and $q_2(x) = 0$ for $x \in \mathbb{R}$. For every probabilistic vector (p_1, p_2) with $p_1 < 1$ condition $L_q < 1$ from the above Example is satisfied. Thus for every $\mu \in \mathcal{M}_1(Y)$ the sequence $\{\widetilde{P}^{*n}\mu\}_{n\geq 1}$ given by (22) converges weakly to $\mu_0 = \delta_0$. Obviously, the strong convergence does not hold.

Example 2 Poisson driven stochastic differential equation.

Consider a stochastic differential equation

$$dX(t) = a(X(t), \xi(t))dt + b(X(t))dp(t) \qquad \text{for} \quad t \geq 0$$

with the initial condition

$$X(0) = x_0,$$

where $a : Y \times I \to Y$, $b : Y \to Y$ are Lipschitz functions, $(Y, \|\cdot\|)$ is a separable Banach space, $\{p(t)\}_{t\geq)}$ is a Poisson process and $\{\xi(t)\}_{t\geq 0}$, $\xi(t) : \Omega \to I$ is a stochastic process describing random switching at random moments τ_n. Consider a sequence of random variables $\{x_n\}_{n\geq 0}$, $x_n : \Omega \to Y$ such that

$$\begin{aligned}
&x_n = q(T_{\xi(\tau_{n-1})}(\tau_n - \tau_{n-1}, x_{n-1})), \quad q(x) = x + b(x)\\
&\mathbb{P}\{\xi(0) = k | x_0 = x\} = p_k(x),\\
&\mathbb{P}\{\xi(\tau_n) = s | x_n = y,\ \xi(\tau_{n-1}) = i\} = p_{is}(y), \quad \text{for} \quad n = 1, \ldots\\
&\text{and}\\
&\xi(t) = \xi(\tau_{n-1}) \qquad \text{for} \quad \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \ldots
\end{aligned}$$

This is a particular example of continuous random dynamical systems where $q_\theta(x) = q(x)$, $\theta \in \{1, \ldots, K\}$, and for every $i \in I$, $T_i(t, x) = v_i(t)$ are the solutions of the unperturbed Cauchy problems

$$v_i'(t) = a(v_i(t), i) \qquad \text{and} \quad v_i(0) = x, \qquad x \in Y.$$

It is easy to check that $\mu_n = P^{*n}\mu$, where P^* is the transition operator corresponding to the above stochastic equation‘ and given by

$$P^*\mu(A) = \sum_{j\in I} \int_{Y\times I} \int_{\mathbb{R}_+} \lambda e^{-\lambda t} 1_A(q(T_j(t, x)), j) p_{ij}(x) dt d\mu(x, i)$$

for $A \in \mathcal{B}(Y \times I)$ and $\mu \in \mathcal{M}_1$.

Assume that there exist positive constants L_q, L and α such that

$$\|q(x) - q(y)\| \leq L_q \|x - y\|$$

and

$$\|T_{i_0}(t, x) - T_{i_0}(t, y)\| \leq Le^{\alpha t} \|x - y\|$$

for some $i_0 \in I$ such that $\inf_{i\in I} \inf_{x\in Y} p_{ii_0}(x) > 0$ and $\quad x, y \in Y, t \geq 0$. If

$$LL_q + \frac{\alpha}{\lambda} < 1.$$

then there exists a unique invariant measure $\mu_* \in \mathcal{M}_1^1(Y \times I)$ for the process $(x_n, \xi_n)_{n\geq 0}$, which is attractive in $\mathcal{M}_1(Y \times I)$, exponentially attractive in $\mathcal{M}_1^1(Y \times I)$ and the strong law of large numbers holds for the process $(x_n, \xi_n)_{n\geq 0}$ starting from $(x_0, \xi_0) \in Y \times I$, i.e. for every bounded Lipschitz function $f : Y \times I \to \mathbb{R}$ and every $x_0 \in Y$ and $\xi_0 \in I$ we have

$$\lim_{n\in\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k, \xi_k) = \int_{Y\times I} f(x, \xi)\, \mu_*(dx, d\xi)$$

$\mathbb{P}_{x_0,\xi_0}$ almost surely.

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