The Boltzmann Equation for Bose–Einstein Particles: Regularity and Condensation

Xuguang Lu

Abstract We study regularity and finite time condensation of distributional solutions of the space-homogeneous and velocity-isotropic Boltzmann equation for Bose–Einstein particles for the hard sphere model. Global in time existence of distributional solutions had been proven before. Here we prove that the equation is locally and can be globally (in time) well-posed for the class of distributional solutions having finite moment of the negative order $-1/2$, and solutions in this class with regular initial data are mild solutions in their regularity time-intervals. By observing a necessary condition on the initial data for the absence of condensation at some finite time, we also propose a sufficient condition on the initial data for the occurrence of condensation at all large time, and then using a positivity of a partial collision integral we prove further that the critical time of condensation can be strictly positive.

1 Introduction

It has been known that the time evolution of a dilute Bose gas and the formation of Bose–Einstein condensation can be described by the Boltzmann equation for Bose–Einstein particles. Derivations of this important semiclassical model can be found in [3–5, 17, 19, 24]. The present paper is a continuation of our previous work [15] on the study of space-homogeneous solutions of the equation for the hard sphere model:

$$\frac{\partial}{\partial t} f = \int_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{|(\mathbf{v} - \mathbf{v}_*) \cdot \omega|}{(4\pi)^2} [f' f'_*(1 + f + f_*) - f f_*(1 + f' + f'_*)] \mathrm{d}\omega \mathrm{d}\mathbf{v}_*. \quad (1.1)$$

X. Lu (✉)

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China
e-mail: xglu@math.tsinghua.edu.cn

Here the solution $f = f(\mathbf{v}, t) \geq 0$ is the number density of particles at time t with the velocity $\mathbf{v}$; $f_* = f(\mathbf{v}_*, t)$, $f' = f(\mathbf{v}', t)$, $f'_* = f(\mathbf{v}'_*, t)$, and $\mathbf{v}$, $\mathbf{v}_*$ and $\mathbf{v}'$, $\mathbf{v}'_*$ are velocities of two particles before and after their collision:

$$\mathbf{v}' = \mathbf{v} - ((\mathbf{v} - \mathbf{v}_*) \cdot \omega)\omega, \quad \mathbf{v}'_* = \mathbf{v}_* + ((\mathbf{v} - \mathbf{v}_*) \cdot \omega)\omega, \quad \omega \in \mathbb{S}^2$$

which conserves the kinetic energy

$$|\mathbf{v}'|^2 + |\mathbf{v}'_*|^2 = |\mathbf{v}|^2 + |\mathbf{v}_*|^2. \tag{1.2}$$

The main difficulty in investigating Eq.(1.1) comes from the cubic terms $ff'f'_*$, $f_*f'f'_*$, etc., because their collision integrals (even with a strong cutoff on the collision kernel) over $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$ in the usual weak form (i.e. involving smooth test functions) are unbounded on a set of non-isotropic functions in $\{0 \leq f \in C_c^\infty(\mathbb{R}^3) \mid \|f\|_{L^1(\mathbb{R}^3)} \leq 1\}$ ([13]). So far the results about global existence, convergence to equilibrium, singular behavior, kinetics of Bose–Einstein condensation, numerical analysis, etc. of solutions of Eq.(1.1) and its modifications have been mainly concerned with isotropic solutions, see e.g. [1,2,6–8,11,13,14,18,20–23]. Recent works [9,10,15] also considered blow up and condensation in finite time for such solutions (see below).

By velocity translation one can assume that the mean velocity is zero so that the isotropic solution $f(\mathbf{v}, t)$ can be written as $f(|\mathbf{v}|^2/2, t)$. Accordingly, let $\epsilon, \epsilon_*, \epsilon', \epsilon'_*$ stand for $|\mathbf{v}|^2/2, |\mathbf{v}_*|^2/2, |\mathbf{v}'|^2/2, |\mathbf{v}'_*|^2/2$ respectively. Then (1.2) implies that $\epsilon, \epsilon', \epsilon'_*$ are independent variables and $\epsilon_* = \epsilon' + \epsilon'_* - \epsilon$, and this 3-degree of freedom is just enough for the total integration of the cubic isotropic term $ff'f'_*$ over $\mathbb{R}^3_+$. As in the previous work [15], we are only interested in such solutions that have finite mass and energy, i.e.

$$\int_{\mathbb{R}^3} (1 + |\mathbf{v}|^2/2) f(|\mathbf{v}|^2/2, t) \mathrm{d}\mathbf{v} = 4\pi\sqrt{2} \int_{\mathbb{R}_+} (1 + \epsilon) f(\epsilon, t)\sqrt{\epsilon}\mathrm{d}\epsilon < \infty.$$

It will be more convenient to write $\epsilon, \epsilon_*, \epsilon', \epsilon'_*$ as x, x_*, y, z respectively. Then introducing

$$w(x, y, z) = \frac{\min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\}}{\sqrt{x}}, \quad x > 0, y \geq 0, z \geq 0, x_* = (y + z - x)_+ \tag{1.3}$$

where $(u)_+ = \max\{u, 0\}$, and using the formula (see e.g. [15])

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{|(\mathbf{v} - \mathbf{v}_*) \cdot \omega|}{(4\pi)^2} \Phi\left(|\mathbf{v}|^2/2, |\mathbf{v}'|^2/2, |\mathbf{v}'_*|^2/2\right) \mathrm{d}\omega \mathrm{d}\mathbf{v}_* = \int_{\mathbb{R}^2_+} w(x, y, z)\Phi(x, y, z)\mathrm{d}y\mathrm{d}z \tag{1.4}$$

for $\mathbf{v} \in \mathbb{R}^3 \setminus \{0\}$, $x = |\mathbf{v}|^2/2$, the equation (1.1) for isotropic solutions becomes

$$\frac{\partial}{\partial t} f(x, t) = Q(f)(x, t) \tag{1.5}$$

where $Q(f)(x, t) = Q(f(\cdot, t))(x)$ is the isotropic version of the collision integral in (1.1):

$$Q(f)(x) = \int_{\mathbb{R}^2_+} w(x, y, z)[f'f'_*(1 + f + f_*) - ff_*(1 + f' + f'_*)]\mathrm{d}y\mathrm{d}z, \tag{1.6}$$

$$f = f(x), \quad f_* = f(x_*), \quad f' = f(y), \quad f'_* = f(z). \tag{1.7}$$

As usual, in order to study kinetic equations, the first step is to extend the concept of classical solutions to that of mild solutions:

Definition 1.1 Let $f(x, t)$ be a nonnegative measurable function on $\mathbb{R}_+ \times [0, T_\infty)$ $(0 < T_\infty \leq \infty)$. We say that $f(\cdot, t)$ is a mild solution of Eq.(1.5) on the time-interval $[0, T_\infty)$ if f satisfies

(i) $\sup_{t \in [0,T]} \int_{\mathbb{R}_+} (1 + x) f(x, t) \sqrt{x} \mathrm{d}x < \infty \quad \forall 0 < T < T_\infty$,

(ii) there is a null set $Z \subset \mathbb{R}_+$ (i.e. $\mathrm{mes}(Z) = 0$) such that for all $x \in \mathbb{R}_+ \backslash Z$ and all $t \in [0, T_\infty)$, $\int_0^t \mathrm{d}\tau \int_{\mathbb{R}_+^2} w(x, y, z)[f' f_*'(1 + f + f_*)) + f f_*(1 + f' + f_*')]\mathrm{d}y\mathrm{d}z < \infty$ and

$$f(x, t) = f_0(x) + \int_0^t Q(f)(x, \tau)\mathrm{d}\tau.$$

Here $f_0 = f(\cdot, 0)$ denotes the initial datum of $f(\cdot, t)$.

By definition, a mild solution describes only "regular" behavior of a Bose gas in a time-interval in which there is no condensation. In order to cover the whole time of evolution, in particular in order to describe the formation, nucleation, and growth of Bose–Einstein condensation, the class of mild solutions has to be extended, and any extension should include at least the low temperature equilibria $\mathrm{d}F_{\mathrm{be}}(x) = f_{\mathrm{be}}(x)\sqrt{x}\mathrm{d}x + N_0\delta(x)\mathrm{d}x$, where $f_{\mathrm{be}}(x) = (e^{x/\kappa} - 1)^{-1}, \kappa > 0, N_0 \geq 0$, and $\delta(x)$ is the Dirac delta function concentrating at $x = 0$. Some authors extend mild solution $f(\cdot, t)$ to the solution $(f(\cdot, t), n(t))$ of an equation system where $n(t) \geq 0$ is the coefficient of the delta function, i.e., $n(t)$ describes the condensate, see e.g. [22,23] on the kinetics and structural analysis of $(f(\cdot, t), n(t))$ including nucleation of the condensate. See also [1,2] for the global in time existence of such solutions for the case of low temperature and $n(0) > 0$. In this paper we (as before) extend mild solutions to distributional solutions ([13,15]) and we will use the existence result of distributional solutions to study both mild solutions and condensation.

Multiplying by a bounded smooth function $\varphi(x)$ to Eq. (1.5) and taking integration over $\mathbb{R}_{\geq 0}$ with respect to the weighted Lebesgue measure $\sqrt{x}\mathrm{d}x$, and then replacing the density element $f(x, t)\sqrt{x}\mathrm{d}x$ with a measure element $\mathrm{d}F_t(x)$, we obtain a weak form (1.8) of Eq. (1.5) which leads to the definition of distributional solutions ([15]):

Definition 1.2 Let $\{F_t\}_{t \geq 0} \subset \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$. We say that $\{F_t\}_{t \geq 0}$, or simply F_t, is a distributional solution of Eq. (1.5) on the time-interval $[0, \infty)$ if

(i) $\sup_{t \in [0,T]} \int_{\mathbb{R}_{\geq 0}} (1 + x)\mathrm{d}F_t(x) < \infty$ for all $T \in (0, \infty)$,

(ii) for every $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$, the function $t \mapsto \int_{\mathbb{R}_{\geq 0}} \varphi(x)\mathrm{d}F_t(x)$ belongs to $C^1([0, \infty))$,

(iii) for every $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_{\geq 0}} \varphi \mathrm{d}F_t = \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi]\mathrm{d}^2 F_t + \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi]\mathrm{d}^3 F_t \quad \forall t \in [0, \infty). \tag{1.8}$$

Here $d^2 F = d(F \otimes F) = dF(y)dF(z)$, $d^3 F = d(F \otimes F \otimes F) = dF(x)dF(y)dF(z)$,

$$\mathcal{J}[\varphi](y, z) = \frac{1}{2} \int_0^{y+z} \mathcal{K}[\varphi](x, y, z)\sqrt{x}\,dx, \quad \mathcal{K}[\varphi](x, y, z) = W(x, y, z)\Delta\varphi(x, y, z),$$

$$\Delta\varphi(x, y, z) = \varphi(x) + \varphi(x_*) - \varphi(y) - \varphi(z), \quad x_* = (y + z - x)_+,$$

$$W(x, y, z) = \frac{\sqrt{x}w(x, y, z)}{\sqrt{xyz}} = \frac{\min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\}}{\sqrt{xyz}} \quad \text{if} \quad x, y, z > 0,$$

$$W(x, y, z) = \frac{1}{\max\{\sqrt{yz}, \sqrt{xz}, \sqrt{xy}\}} \quad \text{if} \quad xyz = 0, \ x_* \max\{yz, xz, xy\} > 0,$$

$$W(x, y, z) = 0 \qquad \text{others.}$$

Remark 1.1 (a) The spaces of test functions and Borel measures appeared above are taken as follows

$$C_b^{1,1}(\mathbb{R}_{\geq 0}) = \left\{\varphi \in C^1(\mathbb{R}_{\geq 0}) \mid \varphi \in C_b(\mathbb{R}_{\geq 0}), \ \frac{d}{dx}\varphi \in C_b(\mathbb{R}_{\geq 0}) \cap \text{Lip}(\mathbb{R}_{\geq 0})\right\},$$

$$\mathcal{B}_k(\mathbb{R}_{\geq 0}) = \left\{F \in \mathcal{B}(\mathbb{R}_{\geq 0}) \mid \|F\|_k := \int_{\mathbb{R}_{\geq 0}} (1 + x)^k d|F|(x) < \infty\right\},$$

$$\mathcal{B}_k^+(\mathbb{R}_{\geq 0}) = \left\{F \in \mathcal{B}_k(\mathbb{R}_{\geq 0}) \mid F \geq 0\right\}, \quad k \geq 0; \quad \mathcal{B}^+(\mathbb{R}_{\geq 0}) := \mathcal{B}_0^+(\mathbb{R}_{\geq 0}),$$

where $C_b(\mathbb{R}_{\geq 0})$ is the class of bounded continuous on $\mathbb{R}_{\geq 0}$, $\text{Lip}(\mathbb{R}_{\geq 0})$ is the class of functions satisfying Lipschitz condition on $\mathbb{R}_{\geq 0}$, and $\mathcal{B}(\mathbb{R}_{\geq 0}) \equiv \mathcal{B}_0(\mathbb{R}_{\geq 0})$ is the class of signed real Borel measures F on $\mathbb{R}_{\geq 0}$ satisfying $\int_{\mathbb{R}_{\geq 0}} d|F|(x) < \infty$ where $|F|$ is the total variation of F.

(b) Distributional solutions can be also defined on finite time-intervals by replacing $[0, \infty)$ with $[0, T]$ or $[0, T_\infty)$. But this is not very necessary because it is different from mild solutions that the time-intervals of existence of distributional solutions can be always $[0, \infty)$, see Theorem 1.1 below.

(c) Under the the condition (i) in Definition 1.2, the collision integrals $t \mapsto \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi]d^2 F_t$, $t \mapsto \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi]d^3 F_t$ are well defined on $[0, \infty)$ for all $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$ (see [15]), and the Corollary of Lemma 4 in [13] insures that if $t \mapsto \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_t(x)$ is continuous on $[0, \infty)$ for every $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$, so are $t \mapsto \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi]d^2 F_t$, $t \mapsto \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi]d^3 F_t$ on $[0, \infty)$. Therefore under the condition (i), the conditions (ii)–(iii) in Definition 1.2 are equivalent to the following (ii)′–(iii)′:

(ii)′ for every $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$, the function $t \mapsto \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_t(x)$ is continuous on $[0, \infty)$;

(iii)′ for every $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$ and every $t \in [0, \infty)$

$$\int_{\mathbb{R}_{\geq 0}} \varphi dF_t = \int_{\mathbb{R}_{\geq 0}} \varphi dF_0 + \int_0^t \left(\int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi]d^2 F_\tau + \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi]d^3 F_\tau\right) d\tau. \tag{1.9}$$

Before introducing our main results and some recent relevant works, let us recall some terminologies.

1.1 Moments

Including negative orders, moments for a positive Borel measure F on $\mathbb{R}_{\geq 0}$ are defined by

$$M_p(F) = \int_{\mathbb{R}_{\geq 0}} x^p \mathrm{d}F(x), \qquad p \in (-\infty, \infty). \tag{1.10}$$

Here for the case $p < 0$ we adopt the convention $0^p = (0^+)^p = \infty$, and we recall that $\infty \cdot 0 = 0$. Then it should be noted that

$$M_p(F) < \infty \quad \text{and} \quad p < 0 \quad \Longrightarrow \quad F(\{0\}) = 0. \tag{1.11}$$

Moments of orders 0, 1 correspond to the mass and energy and are particularly denoted as

$$N(F) = M_0(F), \quad E(F) = M_1(F). \tag{1.12}$$

Moments for a nonnegative measurable function f on $\mathbb{R}_+$ are defined in consistence with the case of measures: $M_p(f) = M_p(F)$ with $\mathrm{d}F(x) = f(x)\sqrt{x}\mathrm{d}x$, i.e.

$$M_p(f) = \int_{\mathbb{R}_+} x^p f(x)\sqrt{x}\mathrm{d}x, \qquad p \in (-\infty, \infty). \tag{1.13}$$

We also denote $N(f) = M_0(f)$, $E(f) = M_1(f)$. And notice that $M_{-1/2}(f) = \int_{\mathbb{R}_+} f(x)\mathrm{d}x$.

1.2 Weighted L^1 Spaces

We denote by $L^1(\mathbb{R}_+, \varrho(x)\mathrm{d}x)$ the space of measurable functions f on $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} |f(x)|\varrho(x)\mathrm{d}x < \infty$, where $\varrho(x)$ is a given nonnegative measurable function on $\mathbb{R}_+$. For the case $\varrho(x) \equiv 1$ we denote $L^1(\mathbb{R}_+) = L^1(\mathbb{R}_+, \mathrm{d}x)$.

1.3 Kinetic Temperature

Let $N = N(F_0)$, $E = E(F_0)$. If m is the mass of one particle, then $m4\pi\sqrt{2}N$, $m4\pi\sqrt{2}E$ are total mass and kinetic energy of the particle system per unite space volume. The kinetic temperature $\overline{T}$ and the critical temperature $\overline{T}_c$ are defined by (see e.g.[13] and references therein)

$$\overline{T} = \frac{2}{3k_{\mathrm{B}}} \cdot \frac{mE}{N}, \qquad \frac{\overline{T}}{\overline{T}_c} = \frac{(2\pi)^{1/3}[\zeta(3/2)]^{5/3}}{3\zeta(5/2)} \cdot \frac{E}{N^{5/3}} \tag{1.14}$$

where k_{B} is the Boltzmann constant, $\zeta(\cdot)$ is the Riemann zeta function.

1.4 Conservative Solutions

F_t is called a conservative distributional solution if it conserves both mass and energy, i.e., $N(F_t) = N(F_0)$, $E(F_t) = E(F_0)$ for all $t \in [0, \infty)$. Similarly f is called conservative mild solution on the time-interval $[0, T_\infty)$ if $N(f(t)) = N(f_0)$, $E(f(t)) = E(f_0)$ for all $t \in [0, T_\infty)$. Since, by definition of $\mathcal{J}[\varphi]$ and $\mathcal{K}[\varphi]$, $\mathcal{J}[\varphi] = \mathcal{K}[\varphi] = 0$ for $\varphi(x) \equiv 1$ and $\varphi(x) \equiv x$, and since the constant $\varphi(x) \equiv 1$ belongs to the test function space while the function $\varphi(x) \equiv x$ does not, it follows from Definition 1.2 that every distributional solution *automatically* conserves the mass, but does not *automatically* conserve the energy. In general,

for every distributional solution F_t, the energy $t \mapsto E(F_t)$ is non-decreasing on $[0, \infty)$ ([13]). On the other hand, in practice, most of distributional solutions obtained by weak convergence of conservative approximate solutions satisfy the energy inequality $E(F_t) \leq E(F_0) (\forall t \geq 0)$. These two together imply that the conservation of energy actually holds for such solutions. Throughout this paper we always assume that $N(F_0) > 0$, $N(f_0) > 0$. We also assume that $E(F_0) > 0$ when dealing with conservative distributional solutions. This is because if $E(F_0) = 0$, then the conservation of mass and energy imply that $F_t \equiv F_0$ is a Dirac mass concentrating at $x = 0$, which is a trivial equilibrium. Basic result to be used in the paper on the existence and conservation law of distributional solutions is the following

Theorem 1.1 ([13,15])

(a) Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$. Then there exists a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0, and F_t has the moment production:

$$M_p(F_t) \leq C_p(1 + 1/t)^{2(p-1)}, \quad \forall t > 0, \quad \forall p \geq 1. \tag{1.15}$$

Here $0 < C_p = C_{N,E,p} < \infty$ depends only on $N = N(F_0)$, $E = E(F_0)$ and p.

(b) For every distributional solution F_t of Eq. (1.5) on $[0, \infty)$, the energy $t \mapsto E(F_t)$ is non-decreasing on $[0, \infty)$. Consequently if $E(F_t) \leq E(F_0)$ for all $t \geq 0$, then $E(F_t) \equiv E(F_0)$.

This theorem is in fact stated in Theorems 3–4 of [13] for distributional solutions $\bar{F}_t$ defined in [13, p. 1611], see also Definition 5.1 in [15]. It is proved in [15] that the two definitions of distributional solutions are equivalent and the corresponding solutions F_t and $\bar{F}_t$ determine to each other through the functional representation:

$$4\pi\sqrt{2} \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_t(x) = \int_{\mathbb{R}_{\geq 0}} \varphi(r^2/2)d\bar{F}_t(r) \tag{1.16}$$

for all $t \geq 0$ and all $\varphi \in C_b(\mathbb{R}_{\geq 0})$. By monotone convergence theorem, (1.16) holds also for all $t \geq 0$ and all $0 \leq \varphi \in C(\mathbb{R}_{\geq 0})$. Correspondingly, N, E and $\bar{N} = \int_{\mathbb{R}_{\geq 0}} d\bar{F}_0(r)$, $\bar{E} = \frac{1}{2}\int_{\mathbb{R}_{\geq 0}} r^2 d\bar{F}_0(r)$ have the relation $\bar{N} = 4\sqrt{2}N$, $\bar{E} = 4\sqrt{2}E$. The conservative distributional solution $\bar{F}_t$ obtained in [13] has the moment production: $\int_{\mathbb{R}_{\geq 0}} r^{2p} d\bar{F}_t(r) \leq \bar{C}_p(1+1/t)^{2(p-1)}$ for all $t > 0$, $p \geq 1$, where $\bar{C}_p = \bar{C}_{\bar{N},\bar{E},p} < \infty$ depends only on $\bar{N}$, $\bar{E}$, and p. If we let the initial datum $\bar{F}_0$ of $\bar{F}_t$ be defined by F_0 through (1.16) for $t = 0$, and let F_t be defined by $\bar{F}_t$ through (1.16), then F_t is a conservative distributional solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0. Since $M_p(F_t) = \frac{1}{4\sqrt{2}}\int_{\mathbb{R}_{\geq 0}}(r^2/2)^p d\bar{F}_t(r)$, it follows that F_t also satisfies (1.15).

1.5 Regular-Singular Decomposition

Let F_t be a distributional solution F_t of Eq. (1.5) on $[0, \infty)$. According to measure theory, F_t at every time t can be uniquely decomposed as the regular part $f(\cdot, t) \in L^1(\mathbb{R}_+, \sqrt{x}dx)$ and the singular part $\mu_t \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ with respect to the Lebesgue measure: there is a Borel null set $Z_t \subset \mathbb{R}_{\geq 0}$ (i.e. mes$(Z_t) = 0$) such that

$$dF_t(x) = f(x, t)\sqrt{x}dx + d\mu_t(x), \quad \mu_t(\mathbb{R}_{\geq 0}\backslash Z_t) = 0. \tag{1.17}$$

We say that F_t is regular at time t if $dF_t(x) = f(x, t)\sqrt{x}dx$, i.e. $\mu_t = 0$. In this case, $f(\cdot, t)$ is called the density or density function of F_t at time t. Similarly F_t is called regular on a

time-interval I if $dF_t(x) = f(x, t)\sqrt{x}dx$ for all $t \in I$, i.e. $\mu_t = 0$ for all $t \in I$. We say that F_t has condensation at time t if $F_t(\{0\}) > 0$. It is generally believed that if $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ is regular, then there exists a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 such that for any $t \in [0, \infty)$ the singular part μ_t (if not zero) will be only a condensate, i.e. $d\mu_t(x) = n(t)\delta(x)dx$, where $n(t) = F_t(\{0\})$ is the amount of condensation as mentioned above. But this has not been proven yet. The main difficulty is that while smooth test functions φ kill the singularity of the collision kernel $W(x, y, z)$ so that the collision integrals are convergent, they also suppress the singular behavior of F_t near the origin and thus the direct capture of condensation becomes difficult. We then have to consider some indirect way: we carefully study regular behavior of F_t (this part has its own interests) and by assuming F_t in some cases has certain regularity in a time-interval we hope to find some contradiction so as to obtain some information about condensation.

1.6 Main Results

For convenience of statement, results introduced below are only concerned with conservative solutions. Details and other results will be given in the following sections.

1.6.1 Strong Solutions and Regularity

The definition of strong solutions will be given in Sect. 3. We observe that an important condition that implies the existence of strong solutions and regularity is $M_{-1/2}(F_0) < \infty$. Recall that if F_t is regular and $f(\cdot, t)$ is its density, then $M_{-1/2}(F_t) = \int_{\mathbb{R}_+} f(x, t)dx$. In view of the structure of $Q(f)(x)$ and L^∞-bounds $0 \leq w(x, y, z) \leq 1$, the integrability $f(\cdot, t) \in L^1(\mathbb{R}_+)$ is easily produced from a closed form of L^1-inequalities. For the case $M_{-1/2}(F_0) = \infty$ and F_0 is regular, if (for instance) the density f_0 of F_0 satisfies $f_0(x) \sim \text{const.}x^{\alpha-3/2}$ ($x \to 0+$) with $0 < \alpha < 1/2$, then the solution F_t cannot be regular on any time-interval $I \subset (0, \infty)$ (see below), but for the critical case $\alpha = 1/2$, for instance if $f_0(x) \leq \text{const.}x^{-1}$ ($\forall x > 0$), then F_t can be regular on a time-interval. Main results on strong solutions and regularity are as follows:

- Let F_t be a conservative distributional solution on $[0, \infty)$ with the initial datum F_0 satisfying $M_{-1/2}(F_0) < \infty$. Then there exists the largest $0 < T_{\max} \leq \infty$ such that $M_{-1/2}(F_t) < \infty$ for all $t \in [0, T_{\max})$, and F_t is a unique strong solutions of Eq. (1.5) on $[0, T_{\max})$. Furthermore if $M_{-1/2}(F_0)$ is small enough as compared with $[N(F_0)E(F_0)]^{1/4}$, for instance $M_{-1/2}(F_0) \leq \frac{1}{80}[N(F_0)E(F_0)]^{1/4}$, then $T_{\max} = \infty$. In particular if F_0 is regular and f_0 is its density, then F_t is also regular on $[0, T_{\max})$ and its density $f(\cdot, t)$ is the unique conservative mild solution of Eq. (1.5) on $[0, T_{\max})$ satisfying $f \in C([0, T_{\max}); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$. Finally if in addition $f_0 \in L^\infty(\mathbb{R}_+)$, then $f(\cdot, t) \in L^\infty(\mathbb{R}_+)$ for all $t \in [0, T_{\max})$ including the case $T_{\max} = \infty$.
- Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ be regular with density f_0 satisfying $f_0(x) \leq Kx^{-1}$ ($\forall x > 0$) for some $K > 0$. Then there exist an explicit constant $T_K > 0$ and a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 such that F_t is a regular strong solution on $[0, T_K]$ and its density $f(x, t)$ satisfies $f(x, t) \leq 5Kx^{-1}$ for all $(x, t) \in \mathbb{R}_+ \times [0, T_K]$.

As one will see in Sect. 3 that the relative smallness $M_{-1/2}(F_0) \leq \frac{1}{80}[N(F_0)E(F_0)]^{1/4}$ belongs to the case of very high temperature, $\overline{T}/\overline{T}_c >> 1$. For the case of low temperature, $\overline{T}/\overline{T}_c < 1$, we do not know whether the intervals of t satisfying $M_{-1/2}(F_t) < \infty$ (or

$f(\cdot, t) \in L^1(\mathbb{R}_+))$ are only finite, but Escobedo and Velázquez proved in [9,10] that this is the case for mild solutions with initial data $f_0 \in L^\infty(\mathbb{R}_+, (1+x)^\gamma)(\gamma > 3)$ if the blow up is taken in the $L^\infty(\mathbb{R}_+)$ norm, see Theorem 1.2 below for details.

1.6.2 Condensation in Finite Time

The main result is based on the propagation of condensation

$$F_t(\{0\}) \geq F_s(\{0\})e^{\int_s^t [L_\varphi(F_\tau) - M_{1/2}(F_\tau)]d\tau} \qquad \forall 0 \leq s < t < \infty \qquad (1.18)$$

(see [15], where $t \mapsto L_\varphi(F_t)$ is a positive function) and the consideration for Lebesgue derivatives $\underline{D}_\alpha(F_0), \overline{D}_\alpha(F_0)$ of the initial data F_0 at the origin $x = 0$, where

$$\underline{D}_\alpha(F) = \liminf_{\varepsilon \to 0^+} \frac{F([0, \varepsilon])}{\varepsilon^\alpha}, \quad \overline{D}_\alpha(F) = \limsup_{\varepsilon \to 0^+} \frac{F([0, \varepsilon])}{\varepsilon^\alpha}, \quad \alpha > 0 \qquad (1.19)$$

and in case $\underline{D}_\alpha(F) = \overline{D}_\alpha(F)$ we define $D_\alpha(F) = \underline{D}_\alpha(F) = \overline{D}_\alpha(F)$. Notice that if initially $F_0(\{0\}) > 0$, then (1.18) implies $F_t(\{0\}) > 0$ for all $t \geq 0$. Thus the most interesting case for the occurrence of condensation is $F_0(\{0\}) = 0$ as satisfied by all regular initial data.

For the case $0 < \alpha < 1/2$, we have proved in [15] that if $\underline{D}_\alpha(F_0) > 0$, then $F_t(\{0\}) > 0$ for all $t > 0$. A typical example is the regular measure $dF_0(x) = f_0(x)\sqrt{x}dx$ whose density f_0 satisfies $f_0(x) \sim \text{const.}x^{\alpha-3/2} (x \to 0+)$. This example shows also that F_t is not regular for every $t > 0$.

For the case $\alpha > 1/2$, it is easily seen that the bigness of $\underline{D}_\alpha(F_0)$ does not destroy the smallness of $M_{-1/2}(F_0)$; we will show in Example 7.1 that there is a large class of regular initial data F_0 satisfying $D_\alpha(F_0) = \infty$ such that F_t are regular for all $t \in [0, \infty)$ hence there are no condensation at all.

For the critical case $\alpha = 1/2$, an important referential example is the equilibrium $F_t \equiv F_{be}$ at the critical temperature, i.e., F_{be} is regular with density $f_{be}(x) = (e^{x/\kappa} - 1)^{-1} \sim \kappa x^{-1} (x \to 0+)$ hence $D_{1/2}(F_{be}) = 2\kappa > 0$. Of course there is no condensation. However by computing we find $D_{1/2}(F_{be}) = 1.403\ldots[N(F_{be})E(F_{be})]^{1/4}$ which indicates that $D_{1/2}(F_{be})$ is not large as compared with $[N(F_{be})E(F_{be})]^{1/4}$. This motives us to establish the following theorem:

- Let $N = N(F_0)$, $E = E(F_0)$. If $[\underline{D}_{1/2}(F_0)\overline{D}_{1/2}(F_0)]^{1/2}$ is sufficiently large as compared with $(NE)^{1/4}$, then the condensation still occurs in finite time. For instance if

$$[\underline{D}_{1/2}(F_0)\overline{D}_{1/2}(F_0)]^{1/2} > 213(NE)^{1/4} \qquad (1.20)$$

then $F_t(\{0\}) > 0$ for all $t \geq \frac{3}{4}(NE)^{-1/2}$.

Notice that since $\underline{D}_{1/2}(F_0), \overline{D}_{1/2}(F_0)$ are determined only by local behavior of F_0 near the origin, they have almost no influence on macroscopic quantities such as mass, energy and temperature; we will give a large class of initial data F_0 which satisfy (1.20) so that condensation occurs in finite time, but the corresponding temperature can be arbitrarily high/low (see Example 7.2).

1.6.3 Critical Time of Condensation

Suppose a distributional solution F_t has condensation in finite time and let $t_c = \inf\{t \geq 0 \mid F_t(\{0\}) > 0\}$. It is easily seen from the condensate propagation (1.18) that the time-sets

of non-condensation and condensation are separated by the single point t_c, i.e.

$$F_t(\{0\}) = 0 \quad \forall\, t \in [0, t_c); \quad F_t(\{0\}) > 0 \quad \forall\, t \in (t_c, \infty). \tag{1.21}$$

We call t_c the critical time of condensation of F_t. Notice that the case $t_c = 0$ is possible even if $F_0(\{0\}) = 0$ as shown above. For that case, (1.21) is understood as just the second inequality appears. From (1.18) one sees also that the function $t \mapsto F_t(\{0\}) \exp\{\int_0^t M_{1/2}(F_\tau)\mathrm{d}\tau\}$ is monotone non-decreasing on $[0, \infty)$, which implies that $t \mapsto F_t(\{0\})$ has both left and right limits at every $t \in (0, \infty)$. We also proved in [15] that $t \mapsto F_t(\{0\})$ is at least right-continuous on $[0, \infty)$. In particular it holds $\lim_{t \to t_c+} F_t(\{0\}) = F_{t_c}(\{0\})$ which means that the phase transition is at least right-continuous, i.e. there is no jump starting from t_c in the forward direction. Our second result on condensation is about the strict positivity of t_c, which should be studied since in general the condensation of a dilute Bose gas takes place only after a shorter or longer time.

- If a regular measure $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ satisfies (1.20) (with $N = N(F_0), E = E(F_0)$) and its density satisfies $f_0(x) \leq Kx^{-1}$ ($\forall\, x > 0$) for some constant $K > 0$, then there exist an explicit constant $0 < T_K < \frac{3}{4}(NE)^{-1/2}$ and a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0, such that $t_c \in [T_K, \frac{3}{4}(NE)^{-1/2}]$.

In Example 7.3 we will show that the regular data F_0 in the above result exist extensively.

We emphasize that the inequality (1.20) and similar ones given in Theorem 6.1 are only sufficient conditions for the occurrence of condensation in finite time. For the case where F_0 does not satisfy (1.20), in particular for the case where F_0 is regular and its density f_0 is bounded (which implies $D_{1/2}(F_0) = 0$), one has to consider different methods, see Theorem 1.2 below.

1.7 Recent Progress

Escobedo and Velázquez recently obtained important results on the blow up and condensation in finite time ([9,10]). We summarize them as follows using our notations and the definition of critical time t_c in (1.21).

Theorem 1.2 ([9,10])

(I) (Local Well Posedness [9]) For any $\gamma > 3$ and any $0 \leq f_0 \in L^\infty(\mathbb{R}_+, (1 + x)^\gamma) :=$ $\{g \in L^\infty(\mathbb{R}_+) \mid \text{ess sup}_{x \in \mathbb{R}_+} |g(x)|(1 + x)^\gamma < \infty\}$, there exist $0 < T_{\max} \leq \infty$ and a unique conservative mild solution $f \in L_{\text{loc}}^\infty([0, T_{\max}); L^\infty(\mathbb{R}_+, (1 + x)^\gamma))$ of Eq. (1.5), with the initial datum f_0. And $T_{\max}$ is the maximal existence time in the sense that

$$T_{\max} < \infty \implies \|f(t)\|_{L^\infty(\mathbb{R}_+)} \to \infty \quad as \quad t \nearrow T_{\max}. \tag{1.22}$$

(II) (For Arbitrary Temperature [9]) There exists a universal constant $\theta_* > 0$ such that for all $N > 0, E > 0, \nu > 0$, there exist $\rho_0 = \rho_0(N, E, \nu) > 0, K^* = K^*(N, E, \nu) > 0, T_0 = T_0(N, E, \nu) > 0$ depending only on N, E, ν such that the following properties hold:

(II.1) If $0 \leq f_0 \in L^\infty(\mathbb{R}_+, (1 + x)^\gamma)$ with $\gamma > 3$ satisfies $N(f_0) = N, E(f_0) = E$ and

$$\sup_{0 < \rho \leq \rho_0} \min\left\{ \inf_{0 < \varepsilon \leq \rho} \frac{1}{\nu\varepsilon^{3/2}} \int_0^\varepsilon f_0(x)\sqrt{x}\,\mathrm{d}x, \ \frac{1}{K^*\rho^{\theta_*}} \int_0^\rho f_0(x)\sqrt{x}\,\mathrm{d}x \right\} \geq 1, \tag{1.23}$$

then the maximal existence time $T_{\max}$ of f obtained in part (I) *must be finite:* $T_{\max} < T_0$, *hence f satisfies the L^∞- blow up* (1.22).

(II.2) *If the initial datum F_0 of F_t satisfies $N(F_0) = N$, $E(F_0) = E$, and*

$$\sup_{0 < \rho \leq \rho_0} \min \left\{ \inf_{0 < \varepsilon \leq \rho} \frac{F_0([0, \varepsilon])}{\nu \varepsilon^{3/2}}, \frac{F_0([0, \rho])}{K^* \rho^{\theta_*}} \right\} \geq 1, \qquad (1.24)$$

then F_t has condensation in finite time and $t_c \in [0, T_0]$. Furthermore if in addition F_0 is regular with density $f_0 \in L^\infty(\mathbb{R}_+, (1 + x)^\gamma)(\gamma > 3)$, then $t_c \in [T_{\max}, T_0]$. [Note: it is pointed out in [9] that the first condition in (1.24) does not imply the second since θ_* might be small.]

(III) *(For Low Temperature [10]). Let N, E be the mass and energy of the conservative solutions appeared below and let $\overline{T}, \overline{T}_c$ be the kinetic temperature and critical temperature defined in (1.14). Then under only the low temperature condition $\overline{T}/\overline{T}_c < 1$, if t_0 in the time-interval of existence is large enough, then the conditions (1.23), (1.24) are satisfied for $\widetilde{f}_0(x) = f(x, t_0)$ and $\widetilde{F}_0 = F_{t_0}$ respectively and thus one has:*

(III.1) *If f obtained in part* (I) *satisfies $\overline{T}/\overline{T}_c < 1$, then $T_{\max} < \infty$ and f satisfies the L^∞-blow up* (1.22).

(III.2) *If F_0 satisfies $\overline{T}/\overline{T}_c < 1$, then F_t has condensation in finite time and there exists $T_0 = T_0(N, E) > 0$ depending only on N, E such that $t_c \in [0, T_0]$. Furthermore if F_0 is also regular with density $f_0 \in L^\infty(\mathbb{R}_+, (1 + x)^\gamma)) (\gamma > 3)$, then $t_c \in [T_{\max}, T_0]$.*

Part (III.2) of the theorem is a fundamental result in low temperature kinetic theory; it demonstrates that the occurrence of condensation in finite time does not depend on any local information of a conservative solution if the temperature is low enough. An application of part (III.2) will be given in Example 7.4 for dealing with a class of initial data F_0 which do not satisfy (1.20). Part (II.2) of Theorem 1.2 is the key result and is more closer to practical cases since it includes certain regular measures whose densities are bounded. Initial data satisfying all conditions of part (II.1) hence part (II.2) have been constructed in [9]. As for comparison between part (II.2) and our result, the main difference is that our condition (1.20) is purely local (near the origin) while (1.24) is not; and there is no implication relation between (1.20) and (1.24) except the subtle case $\theta_* = 1/2$. In fact, on the one hand, it is obvious that the condition (1.24) dose not imply (1.20) because the latter excludes all such regular measures whose densities are bounded near the origin. On the other hand, we will show in Example 7.5 that $\theta_* \leq 1/2$, and if $\theta_* < 1/2$, then the condition (1.20) does not imply (1.24); if $\theta_* = 1/2$ and $K^* > 214(NE)^{1/4}$, the non-implication still holds. We stress however that both part (II.2) and our results on condensation in finite time are proved for arbitrary temperature, which may be helpful for the study of Bose–Einstein condensation at room temperature. Finally we note as pointed out at the end of Remark 2.2 of [15] that the practical lifetime of condensation $F_t(\{0\}) > 0$ is "finite" since $\lim_{t \to \infty} F_t(\{0\}) = 0$ for high temperature $\overline{T}/\overline{T}_c \geq 1$, but it remains unclear whether $\liminf_{t \to \infty} F_t(\{0\}) > 0$ holds true for low temperature $\overline{T}/\overline{T}_c < 1$.

The rest of the paper is organized as follows. In Sect. 2 we collect and prove some basic properties of collision integrals and propose suitable approximate solutions which will be used here to prove some special properties of solutions. In Sect. 3 we prove the local and global existence of strong solutions and establish stability estimates. In Sect. 4 we prove the regularity of distributional solutions and the global existence of mild solutions. Section 5 is devoted to the proof of regularity of certain distributional solutions whose initial data are regular measures with densities satisfying $f_0(x) \leq Kx^{-1}$. In Sect. 6 we study the condensation in finite time. By observing a necessary condition on the initial datum for the

absence of condensation at some finite time, we propose some sufficient conditions as (1.20) for the occurrence of condensation in finite time, and this together with the result of Sect. 5 enables us to prove the existence and the strict positivity of the critical time t_c for a class of solutions. The examples mentioned above are given in Sect. 7.

Throughout this paper, $\mathbb{R}_+$ stands for either $\mathbb{R}_{\geq 0} := [0, \infty)$ or $\mathbb{R}_{>0} := (0, \infty)$ and they are only used to denote domains of x in order to distinguish intervals of t.

2 Basic Properties of Collision Integrals and Approximate Solutions

First of all we note that the space $\mathcal{B}(\mathbb{R}^n_{\geq 0})$ of finite Borel measures on $\mathbb{R}^n_{\geq 0}$ can be treated as a subspace of $\mathcal{B}(\mathbb{R}^n)$ by zero-extension: $\bar{\mu}(E) = \mu(\mathbb{R}^n_{\geq 0} \cap E)$ for all Borel sets $E \subset \mathbb{R}^n$. Thus many properties that hold for $\mathcal{B}(\mathbb{R}^n)$ also hold for $\mathcal{B}(\mathbb{R}^n_{\geq 0})$.

Basic properties of collision integrals of measures proved below will be used in Sect. 3 to study strong solutions. Roughly speaking, "strong" means that the smooth test functions can be replaced by bounded Borel functions, i.e., there is no need of the cancelation effect resulted from the symmetric difference $\Delta\varphi(x, y, z)$ of smooth functions (see also (3.6) below). Thus we have to establish integrability conditions with respect to each single term in the decompositions

$$\mathcal{J}[\varphi] = \mathcal{J}^+[\varphi] - \mathcal{J}^-[\varphi], \quad \mathcal{K}[\varphi] = \mathcal{K}^+[\varphi] - \mathcal{K}^-[\varphi],$$

where

$$\mathcal{J}^+[\varphi](y, z) = \frac{1}{2} \int_0^{y+z} \mathcal{K}^+[\varphi](x, y, z) \sqrt{x} \, dx, \quad \mathcal{J}^-[\varphi](y, z) = \frac{\varphi(y) + \varphi(z)}{2} \mathcal{J}^+[1](y, z),$$

$$\mathcal{K}^+[\varphi](x, y, z) = W(x, y, z)[\varphi(x) + \varphi(x_*)], \quad \mathcal{K}^-[\varphi](x, y, z) = W(x, y, z)[\varphi(y) + \varphi(z)]$$

and φ is a locally bounded Borel function on $\mathbb{R}_{\geq 0}$. To do this we consider measure spaces

$$\mathcal{B}_{p,1}(\mathbb{R}_{\geq 0}) = \{F \in \mathcal{B}_1(\mathbb{R}_{\geq 0}) \mid M_p(|F|) < \infty\}, \quad \mathcal{B}^+_{p,1}(\mathbb{R}_{\geq 0}) = \mathcal{B}_{p,1}(\mathbb{R}_{\geq 0}) \cap \mathcal{B}^+(\mathbb{R}_{\geq 0}).$$

It is easily seen that

$$p < q < 0 \implies \mathcal{B}_{p,1}(\mathbb{R}_{\geq 0}) \subset \mathcal{B}_{q,1}(\mathbb{R}_{\geq 0}), \quad \mathcal{B}^+_{p,1}(\mathbb{R}_{\geq 0}) \subset \mathcal{B}^+_{q,1}(\mathbb{R}_{\geq 0}).$$

Let us define

$$M_{p,q}(|F|) = M_p(|F|) + M_q(|F|), \quad -\infty < p, q < \infty.$$

As usual the notation $F_1 \otimes F_2 \otimes \cdots \otimes F_n$ stands for the product measure of $F_1, F_2, \ldots, F_n$.

In this paper we denote by $L^\infty_0(\mathbb{R}_{\geq 0})$ the set of bounded Borel functions on $\mathbb{R}_{\geq 0}$. In general, for any $k \geq 0$, we denote by $L^\infty_{-k}(\mathbb{R}_{\geq 0})$ the set of Borel functions φ on $\mathbb{R}_{\geq 0}$ satisfying

$$\|\varphi\|_{L^\infty_{-k}} := \sup_{x \in \mathbb{R}_{\geq 0}} |\varphi(x)|(1 + x)^{-k} < \infty.$$

Lemma 2.1 (a) *Let* $F, G \in \mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})$ *with* $k \in [0, 1]$. *Then for any* $\varphi \in L^\infty_{-k}(\mathbb{R}_{\geq 0})$ *satisfying* $\|\varphi\|_{L^\infty_{-k}} \leq 1$ *we have*

$$\int_{\mathbb{R}^2_{\geq 0}} |\mathcal{J}^\pm[\varphi]| \, d(|F| \otimes |G|) \leq \|F\|_{k+1/2} \|G\|_{k+1/2}. \tag{2.1}$$

(b) Let $F, G, H \in \mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})$. Then for any $\varphi \in L_0^{\infty}(\mathbb{R}_{\geq 0})$ satisfying $\|\varphi\|_{L_0^{\infty}} \leq 1$ we have

$$\int_{\mathbb{R}_{\geq 0}^3} |\mathcal{K}^{\pm}[\varphi]| d(|F| \otimes |G| \otimes |H|) \leq 2M_{-1/3}(|F|)M_{-1/3}(|G|)M_{-1/3}(|H|). \quad (2.2)$$

In general, for any $\varphi \in L_{-k}^{\infty}(\mathbb{R}_{\geq 0})$ satisfying $\|\varphi\|_{L_{-k}^{\infty}} \leq 1$ with $k \in [0, 1]$, we have

$$\int_{\mathbb{R}_{\geq 0}^3} |\mathcal{K}^{\pm}[\varphi]| d(|F| \otimes |G| \otimes |H|) \leq 2M_{-1/3}(|F|)M_{-1/3,k-1/3}(|G|)M_{-1/3,k-1/3}(|H|)$$

$$(2.3)$$

and if assuming further that $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$, then

$$\int_{\mathbb{R}_{\geq 0}^3} |\mathcal{K}^{\pm}[\varphi]| d(|F| \otimes |G| \otimes |H|) \leq 2a(F, G, H) \min\{\|F\|_k, \|G\|_k, \|H\|_k\} \quad (2.4)$$

where

$$a(F, G, H) := [M_{-1/2,1/2}(|F|) + M_{-1/2,1/2}(|G|) + M_{-1/2,1/2}(|H|)]^2. \quad (2.5)$$

Proof By definition of $W(x, y, z)$ we compute

$$\mathcal{J}^+[1](y, z) = \int_0^{y+z} W(x, y, z)\sqrt{x} dx = \frac{y \wedge z}{3\sqrt{y \vee z}} \mathbf{1}_{\{y \vee z > 0\}} + \sqrt{y \vee z}. \quad (2.6)$$

Here $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$; $\mathbf{1}_{\{a>b\}} = 1$ if $a > b$; $\mathbf{1}_{\{a>b\}} = 0$ if $a \leq b$. From (2.6) we have

$$\mathcal{J}^+[1](y, z) \leq \sqrt{y} + \sqrt{z} \leq \sqrt{1+y}\sqrt{1+z}.$$

In general, using $x + x_* = y + z$ for $W(x, y, z) > 0$ and using $1 + y + z \leq (1 + y)(1 + z)$ we see that if $k \in [0, 1]$ and $|\varphi(x)| \leq (1 + x)^k$ $(\forall x \in \mathbb{R}_{\geq 0})$, then

$$|\mathcal{J}^{\pm}[\varphi](y, z)| \leq (1 + y)^{k+1/2}(1 + z)^{k+1/2}, \quad (2.7)$$

$$|\mathcal{K}^{\pm}[\varphi](x, y, z)| \leq 2(1 + y^k + z^k)W(x, y, z). \quad (2.8)$$

The inequality (2.1) follows from (2.7).

Next we note that $F, G, H \in \mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})$ implies $(|F| \otimes |G| \otimes |H|)(\mathbb{R}_{\geq 0}^3 \setminus \mathbb{R}_{>0}^3) = 0$ so that $\int_{\mathbb{R}_{\geq 0}^3} \{\cdots\} d(|F| \otimes |G| \otimes |H|) = \int_{\mathbb{R}_{>0}^3} \{\cdots\} d(|F| \otimes |G| \otimes |H|)$. And by definition of $W(x, y, z)$ we have

$$W(x, y, z) \leq \min\left\{\frac{1}{(xyz)^{1/3}}, \frac{1}{\sqrt{yz}}, \frac{1}{\sqrt{xz}}, \frac{1}{\sqrt{xy}}\right\} \quad \forall (x, y, z) \in \mathbb{R}_{>0}^3. \quad (2.9)$$

From this and elementary calculations we deduce

$$\int_{\mathbb{R}^3_{\geq 0}} W(x, y, z) d(|F| \otimes |G| \otimes |H|) \leq M_{-1/3}(|F|) M_{-1/3}(|G|) M_{-1/3}(|H|), \tag{2.10}$$

$$\int_{\mathbb{R}^3_{\geq 0}} (1 + y^k + z^k) W d(|F| \otimes |G| \otimes |H|) \leq M_{-1/3}(|F|) M_{-1/3,k-1/3}(|G|) M_{-1/3,k-1/3}(|H|), \tag{2.11}$$

and, assuming further that $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$,

$$\int_{\mathbb{R}^3_{\geq 0}} (1 + y^k + z^k) W d(|F| \otimes |G| \otimes |H|) \leq a(F, G, H) \min\{\|F\|_k, \|G\|_k, \|H\|_k\}. \tag{2.12}$$

Thus (2.2) follows from (2.10), while (2.3), (2.4) follow from (2.8) and (2.11)–(2.12). □

According to Lemma 2.1 we can define Borel measures $\mathcal{Q}_2^{\pm}(F, G) \in \mathcal{B}_k(\mathbb{R}_{\geq 0})$ for $F, G \in \mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})$ ($k \in [0, 1]$) and $\mathcal{Q}_3^{\pm}(F, G, H) \in \mathcal{B}_1(\mathbb{R}_{\geq 0})$ for $F, G, H \in \mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})$ through Riesz representation theorem by

$$\int_{\mathbb{R}_{\geq 0}} \varphi(x) d\mathcal{Q}_2^{\pm}(F, G)(x) = \int_{\mathbb{R}^2_{\geq 0}} \mathcal{J}^{\pm}[\varphi] d(F \otimes G), \tag{2.13}$$

$$\int_{\mathbb{R}_{\geq 0}} \varphi(x) d\mathcal{Q}_3^{\pm}(F, G, H)(x) = \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}^{\pm}[\varphi] d(F \otimes G \otimes H) \tag{2.14}$$

for all $\varphi \in C_b(\mathbb{R}_{\geq 0})$. It is obvious that $(F, G) \mapsto \mathcal{Q}_2^{\pm}(F, G)$ and $(F, G, H) \mapsto \mathcal{Q}_3^{\pm}(F, G, H)$ are bounded bilinear and trilinear operators from $[\mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})]^2$ to $\mathcal{B}_k(\mathbb{R}_{\geq 0})$ ($k \in [0, 1]$) and from $[\mathcal{B}_{-1/3,1}(\mathbb{R}_{\geq 0})]^3$ to $\mathcal{B}_1(\mathbb{R}_{\geq 0})$ respectively, and

$$\|\mathcal{Q}_2^{\pm}(F, G)\|_k \leq \|F\|_{k+1/2} \|G\|_{k+1/2}, \tag{2.15}$$

$$\|\mathcal{Q}_3^{\pm}(F, G, H)\|_0 \leq 2 M_{-1/3}(|F|) M_{-1/3}(|G|) M_{-1/3}(|H|), \tag{2.16}$$

$$\|\mathcal{Q}_3^{\pm}(F, G, H)\|_k \leq 2 M_{-1/3}(|F|) M_{-1/3,k-1/3}(|G|) M_{-1/3,k-1/3}(|H|), \tag{2.17}$$

$$\|\mathcal{Q}_3^{\pm}(F, G, H)\|_k \leq 2 a(F, G, H) \min\{\|F\|_k, \|G\|_k, \|H\|_k\}. \tag{2.18}$$

Here in the third inequality (2.18) we assume further that $F, G, H \in \mathcal{B}_{-1/2,1}(\mathbb{R}_{\geq 0})$ so that $a(F, G, H) < \infty$. Notice that the equalities (2.13), (2.14) hold also for all $\varphi \in L_0^{\infty}(\mathbb{R}_{\geq 0})$.

In connecting with the equation Eq. (1.8) we define

$$\mathcal{Q}_2^{\pm}(F) = \mathcal{Q}_2^{\pm}(F, F), \quad \mathcal{Q}_2(F) = \mathcal{Q}_2^+(F) - \mathcal{Q}_2^-(F),$$
$$\mathcal{Q}_3^{\pm}(F) = \mathcal{Q}_3^{\pm}(F, F, F), \quad \mathcal{Q}_3(F) = \mathcal{Q}_3^+(F) - \mathcal{Q}_3^-(F),$$
$$\mathcal{Q}(F) = \mathcal{Q}_2(F) + \mathcal{Q}_3(F).$$

We then deduce from

$$F \otimes F - G \otimes G = \frac{1}{2}(F - G) \otimes (F + G) + \frac{1}{2}(F + G) \otimes (F - G),$$

$$\mathcal{Q}_2^{\pm}(F) - \mathcal{Q}_2^{\pm}(G) = \frac{1}{2}\mathcal{Q}_2^{\pm}(F - G, F + G) + \frac{1}{2}\mathcal{Q}_2^{\pm}(F + G, F - G),$$

and (2.15) that for all $F, G \in \mathcal{B}_{k+1/2}(\mathbb{R}_{\geq 0})$ (with $k \in [0, 1]$)

$$\|\mathcal{Q}_2^\pm(F) - \mathcal{Q}_2^\pm(G)\|_k \leq \|F + G\|_{k+1/2}\|F - G\|_{k+1/2}. \tag{2.19}$$

Similarly we deduce from

$$F \otimes F \otimes F - G \otimes G \otimes G = (F - G) \otimes F \otimes F + G \otimes (F - G) \otimes F + G \otimes G \otimes (F - G),$$
$$\|\mathcal{Q}_3^\pm(F) - \mathcal{Q}_3^\pm(G)\|_k \leq \|\mathcal{Q}_3^\pm(F - G, F, F)\|_k + \|\mathcal{Q}_3^\pm(G, F - G, F)\|_k + \|\mathcal{Q}_3^\pm(G, G, F - G)\|_k$$

and (2.16), (2.18) that

$$\|\mathcal{Q}_3^\pm(F) - \mathcal{Q}_3^\pm(G)\|_0 \leq 2[M_{-1/3}(|F|) + M_{-1/3}(|G|)]^2 M_{-1/3}(|F - G|), \tag{2.20}$$
$$\|\mathcal{Q}_3^\pm(F) - \mathcal{Q}_3^\pm(G)\|_k \leq b(F, G)\|F - G\|_k, \quad k \in [0, 1] \tag{2.21}$$

where for the inequality (2.21) we assume that $F, G \in \mathcal{B}_{-1/2,1}^+(\mathbb{R}_{\geq 0})$ so that

$$b(F, G) := 36[M_{-1/2,1/2}(|F|) + M_{-1/2,1/2}(|G|)]^2 < \infty. \tag{2.22}$$

Now we turn to the original form of collision integrals, say the case of functions. The following lemma collects some estimates for the study of mild solutions.

Lemma 2.2 *Let $w(x, y, z)$ be given by (1.3) and let f be a nonnegative measurable function on $\mathbb{R}_+$. Then for any $x > 0, y \geq 0, z \geq 0$ we have*

$$\int_{\mathbb{R}_+} w(x, y, z)\mathrm{d}x \leq 2\sqrt{yz}, \tag{2.23}$$

$$\int_{\mathbb{R}_+} w(x, y, z)f(x_*) \begin{Bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{Bmatrix} \leq M_{-1/2}(f), \tag{2.24}$$

$$M_{1/2}(f) \leq \int_{\mathbb{R}_+^2} w(x, y, z)f(x_*)\mathrm{d}y\mathrm{d}z \leq \sqrt{x}N(f) + M_{1/2}(f). \tag{2.25}$$

Consequently (with the notation (1.7))

$$\int_{\mathbb{R}_+^3} w(x, y, z)[f'f_*'(1 + f + f_*) + ff_*(1 + f' + f_*')]\mathrm{d}x\mathrm{d}y\mathrm{d}z \leq 4M_{1/2}(f)M_{-1/2}(f)$$

$$+ 4[M_{-1/2}(f)]^3 \tag{2.26}$$

Proof (2.23) is obvious by definition of $w(x, y, z)$ and $x_* = (y + z - x)_+$. (2.24) follows from the reflection and translation for the variable $y + z - x$ with respect to x, y, z respectively. Similarly, to prove (2.25) we use translation to get

$$\int_{\mathbb{R}_+^2} w(x, y, z)f(x_*)\mathrm{d}y\mathrm{d}z = \int_{\mathbb{R}_+} I(x, y)f(y)\mathrm{d}y$$

and the inner integral $I(x, y)$ is calculated, with $z_* = (x + y - z)_+$,

$$I(x, y) = \int_0^{x+y} \frac{\min\{\sqrt{x}, \sqrt{y}, \sqrt{z_*}, \sqrt{z}\}}{\sqrt{x}}\mathrm{d}z = \frac{1}{\sqrt{x}}\left(\frac{1}{3}(x \wedge y)^{3/2} + \sqrt{x \wedge y}(x \vee y)\right)$$

from which we have $y \leq I(x, y) \leq \sqrt{xy} + y$. This gives the lower and upper bound estimates in (2.25). The inequality (2.26) follows from (2.23)-(2.25), $0 \leq w(x, y, z) \leq 1$, and $[N(f)]^2 \leq M_{1/2}(f)M_{-1/2}(f)$. By the way, the lemma can be also proved by using classical argument: for instance using (1.4) and $x_* = y + z - x$, $|\mathbf{v}'|^2 + |\mathbf{v}'_*|^2 - |\mathbf{v}|^2 = |\mathbf{v}_*|^2$ with $\frac{1}{2}|\mathbf{v}|^2 = x$ we compute

$$\int\limits_{\mathbb{R}^2_+} w(x, y, z) f(x_*) \mathrm{d}y\mathrm{d}z = \frac{1}{(4\pi)^2} \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2} |\langle \mathbf{v} - \mathbf{v}_*, \omega \rangle| f(|\mathbf{v}_*|^2/2) \mathrm{d}\omega \mathrm{d}\mathbf{v}_*$$

$$= \frac{2\pi}{(4\pi)^2} \int\limits_{\mathbb{R}^3} \frac{|\mathbf{v}_* - \mathbf{v}| + |\mathbf{v}_* + \mathbf{v}|}{2} f(|\mathbf{v}_*|^2/2) \mathrm{d}\mathbf{v}_* \geq \frac{2\pi}{(4\pi)^2} \int\limits_{\mathbb{R}^3} |\mathbf{v}_*| f(|\mathbf{v}_*|^2/2) \mathrm{d}\mathbf{v}_*$$

which is equal to $M_{1/2}(f)$. $\square$

We next observe a new positivity of the collision integrals, see Lemma 2.3 below, which enables us to establish Theorem 5.1. In order to prove Theorem 5.1 we have to use suitable approximate solutions $\{f^n\}_{n=1}^\infty$. These are mild solutions of approximate equations of Eq. (1.5) defined by replacing $Q(f)$ with $Q_n(f)$, i.e., for each $n \geq 1$,

$$f^n(x, t) = f_0(x) + \int\limits_0^t Q_n(f^n)(x, \tau)\mathrm{d}\tau, \tag{2.27}$$

$$Q_n(f)(x) = \int\limits_{\mathbb{R}^2_+} w_n(x, y, z)[f'f'_*(1 + f + f_*)) - ff_*(1 + f' + f'_*)]\mathrm{d}y\mathrm{d}z \tag{2.28}$$

where $w_n(x, y, z)$ is given by $\sqrt{x}w_n(x, y, z) = S_n(x, x_*; y, z)$ and $S_n(\cdot, \cdot; \cdot, \cdot)$ is a well-constructed cutoff for the original function $\sqrt{x}w(x, y, z) = \min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\}$, satisfying the following properties (2.33)-(2.39). Two types of such cutoffs and the corresponding cutoffs $W_n(x, y, z)$ of $W(x, y, z)$ can be chosen as follows:

$$S_n(x, x_*; y, z) = \min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\} \min\left\{1, n\max\{x \wedge x_*, y \wedge z\}\right\}, \tag{2.29}$$

$$W_n(x, y, z) = W(x, y, z) \min\left\{1, n\max\{x \wedge x_*, y \wedge z\}\right\}; \tag{2.30}$$

$$S_n(x, x_*; y, z) = \min\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\} \min\left\{1, n\min\{x, x_*, y, z\}\right\}, \tag{2.31}$$

$$W_n(x, y, z) = W(x, y, z) \min\left\{1, n\min\{x, x_*, y, z\}\right\}.. \tag{2.32}$$

It is easily verified that both cutoffs (2.29)–(2.30) and (2.31)–(2.32) satisfy

$$S_n(x, x_*; y, z) \leq n \min\left\{\sqrt{xyz}, \sqrt{x_*yz}, \sqrt{xx_*y}, \sqrt{xx_*z}\right\}, \quad (x, y, z) \in \mathbb{R}^3_{\geq 0} \tag{2.33}$$

$$0 \leq W(x, y, z) - W_n(x, y, z) \leq \left(\frac{1}{n}\right)^{\frac{1}{2}-\alpha} \frac{(\min\{x, x_*, y, z\})^\alpha}{\sqrt{xyz}}, \quad (x, y, z) \in \mathbb{R}^3_{>0} \tag{2.34}$$

(for $0 \leq \alpha < 1/2$) and preserve the collisional symmetries on $\mathbb{R}^4_{\geq 0}$:

$$S(\varepsilon, \varepsilon_*; \varepsilon', \varepsilon'_*) = S(\varepsilon_*, \varepsilon; \varepsilon', \varepsilon'_*) = S(\varepsilon', \varepsilon'_*; \varepsilon_*, \varepsilon) = S(\varepsilon', \varepsilon'_*; \varepsilon, \varepsilon_*), \tag{2.35}$$

$$S(\varepsilon, \varepsilon_*; \varepsilon', \varepsilon'_*) = 0 \quad \text{if} \quad (\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \in \mathbb{R}^4_{\geq 0} \backslash \mathbb{R}^4_{>0}. \tag{2.36}$$

From (2.33) we have for all $0 \leq f, g, h \in L^1(\mathbb{R}_+, (1+x)\sqrt{x}\,\mathrm{d}x)$

$$\int\limits_{\mathbb{R}^3_+} S_n(x, x_*; y, z) \left\{ \begin{array}{c} f(x)g(y)h(z) \\ f(x_*)g(y)h(z) \end{array} \right\} \mathrm{d}x\mathrm{d}y\mathrm{d}z \leq nN(f)N(g)N(h), \tag{2.37}$$

$$\int\limits_{\mathbb{R}^3_+} x S_n(x, x_*; y, z) f(x)g(y)h(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \leq nE(f)N(g)N(h) \tag{2.38}$$

$$\int\limits_{\mathbb{R}^3_+} x S_n(x, x_*; y, z) f(x_*)g(y)h(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \leq nN(f)[E(g)N(h) + N(g)E(h)] \tag{2.39}$$

where the last inequality is due to $x \leq y + z$ when $S_n(x, x_*; y, z) \neq 0$.

In order to prove the existence of conservative mild solutions f^n, one also needs a further cutoff, for instance $S_{n,k}(x, x_*; y, z) = S_n(x, x_*; y, z)(1+\frac{1}{k}\max\{\sqrt{x}, \sqrt{x_*}, \sqrt{y}, \sqrt{z}\})^{-1}$ ($k \geq 1$), which is only used for dealing with the quadratic terms in (2.28) so as to prove the finiteness and conservation of energy. Let $w_{n,k}(x, y, z)$ be defined by $\sqrt{x}w_{n,k}(x, y, z) = S_{n,k}(x, x_*; y, z)$ and let $Q_{n,k}(f)$ correspond to the kernel $w_{n,k}(x, y, z)$. Then for every $n, k \geq 1$, using the same argument as in [12] it is easily proved that the equation $\frac{\mathrm{d}}{\mathrm{d}t} f = Q_{n,k}(f)$ with the initial datum f_0 has a conservative mild solution $f^{n,k}(x, t)$ on $\mathbb{R}_+ \times [0, \infty)$. Then, for every fixed $n \geq 1$, following the same argument in [12] (proving the L^1-weak compactness of the sequence $\{f^{n,k}(x, t)\sqrt{x}\}_{k=1}^{\infty}$, etc.) we obtain a conservative mild solution $f^n(x, t)$ to the equation $\frac{\mathrm{d}}{\mathrm{d}t} f = Q_n(f)$ on $\mathbb{R}_+ \times [0, \infty)$ with the initial datum f_0. Another method for proving the existence of f^n is to go back to the vector version $\mathbf{v} \mapsto f^n(|\mathbf{v}|^2/2, t)$ and use the first or the second cutoffs:

$B_n(\mathbf{v}, \mathbf{v}_*, \omega) = (4\pi)^{-2}|(\mathbf{v} - \mathbf{v}_*) \cdot \omega| \min\{1, n \max\{(|\mathbf{v}|^2/2) \wedge (|\mathbf{v}_*|^2/2), (|\mathbf{v}'|^2/2) \wedge (|\mathbf{v}'_*|^2/2)\}\}$ or $B_n(\mathbf{v}, \mathbf{v}_*, \omega) = (4\pi)^{-2}|(\mathbf{v} - \mathbf{v}_*) \cdot \omega| \min\{1, n \min\{|\mathbf{v}|^2/2, |\mathbf{v}_*|^2/2, |\mathbf{v}'|^2/2, |\mathbf{v}'_*|^2/2\}\}$. Thanks to the identities (1.2),(1.4), and the inequalities (2.37)–(2.39), the cutoff kernels $B_n(\mathbf{v}, \mathbf{v}_*, \omega)$ have all main properties presented in [12], and thus by just checking the proofs in [12], one obtains a conservative mild isotropic solution $f^n(|\mathbf{v}|^2/2, t)$ on $\mathbb{R}^3 \times [0, \infty)$ with the initial datum $f_0(|\mathbf{v}|^2/2)$.

Now let $\mathcal{J}_n[\varphi], \mathcal{K}_n[\varphi], \mathcal{J}_n^{\pm}[\varphi], \mathcal{K}_n^{\pm}[\varphi]$ be defined as $\mathcal{J}[\varphi], \mathcal{K}[\varphi], \mathcal{J}^{\pm}[\varphi], \mathcal{K}^{\pm}[\varphi]$ by replacing $W(x, y, z)$ with $W_n(x, y, z)$. Then from (2.37)–(2.39) and the collisional symmetries (2.35)–(2.36) we see that all integrals appeared below are absolutely convergent and thus the weak form (1.9) for the approximate mild solutions f^n is rigorously written (denoting $\mathrm{d}F_t^n(x) = f^n(x, t)\sqrt{x}\,\mathrm{d}x$)

$$\int\limits_{\mathbb{R}_{\geq 0}} \varphi \mathrm{d}F_t^n = \int\limits_{\mathbb{R}_{\geq 0}} \varphi \mathrm{d}F_0 + \int_0^t \mathrm{d}\tau \int\limits_{\mathbb{R}^2_{\geq 0}} \mathcal{J}_n[\varphi]\mathrm{d}^2 F_\tau^n + \int_0^t \mathrm{d}\tau \int\limits_{\mathbb{R}^3_{\geq 0}} \mathcal{K}_n[\varphi]\mathrm{d}^3 \mathrm{d}F_\tau^n \tag{2.40}$$

for all $t \geq 0$ and all $\varphi \in L_0^{\infty}(\mathbb{R}_{\geq 0})$. It should be noted that it is the first cutoff (2.29)–(2.30) that can be used to prove the existence of distributional solutions for general initial data. This is because the first cutoff (2.29)–(2.30) keeps the possibility of condensation at origin and insures the pointwise convergence on the whole $\mathbb{R}^3_{\geq 0}$:

$$\lim_{n \to \infty} |W(x, y, z) - W_n(x, y, z)| = 0 \quad \forall (x, y, z) \in \mathbb{R}^3_{\geq 0}. \tag{2.41}$$

And in fact we also have, for all $\varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$ (denoting $D^2\varphi(x) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x)$),

$$|\mathcal{K}[\varphi](x, y, z) - \mathcal{K}_n[\varphi](x, y, z)| \leq \frac{\|D^2\varphi\|_{L^\infty}}{\sqrt{n}} \max\{\sqrt{y}, \sqrt{z}\}, \qquad \forall\, (x, y, z) \in \mathbb{R}_{\geq 0}^3$$

which is enough for proving the existence of distributional solutions F_t as did in [13].

The second cutoff (2.31)–(2.32) kills the possibility of condensation at the origin, and it does not satisfy (2.41): for instance

$$\lim_{n\to\infty} |W(0, y, z) - W_n(0, y, z)| = W(0, y, z) \neq 0 \qquad \forall\, y > 0, z > 0.$$

However if the initial datum F_0 is regular and its density f_0 has only weaker singularity at the origin (for instance $f_0(x) \leq Kx^{-1}$ $(x > 0)$), the second cutoff (2.31)–(2.32) and the corresponding approximate solutions f^n are good ones for proving the existence of some expected solutions on a finite time-interval $[0, T]$ (see the proof of Theorem 5.1). This is because the second cutoff (2.31)–(2.32) as well as the original kernel has the following positivity:

Lemma 2.3 *Let $w(x, y, z)$ be given by (1.3) and let $w_n(x, y, z)$ be the second cutoff of $w(x, y, z)$:*

$$w_n(x, y, z) = w(x, y, z) \min\left\{1, \, n \min\{x, x_*, y, z\}\right\}.$$

Then for any $0 \leq f \in L^1(\mathbb{R}_+, \sqrt{x}\mathrm{d}x)$ (i.e. $N(f) < \infty$) we have

$$\int_{\mathbb{R}_+^2} w(x, y, z)[f(x_*)(f(y) + f(z)) - f(y)f(z)]\mathrm{d}y\mathrm{d}z \geq 0 \qquad \forall\, x > 0, \qquad (2.42)$$

$$\int_{\mathbb{R}_+^2} w_n(x, y, z)[f(x_*)(f(y) + f(z)) - f(y)f(z)]\mathrm{d}y\mathrm{d}z \geq 0 \qquad \forall\, x > 0. \qquad (2.43)$$

Proof First we note that the above integrals make sense since $(y, z) \mapsto w(x, y, z)f(y)f(z)$ is integrable, see Lemma 5.1 below. The proofs for (2.42) and (2.43) are the same. Let us prove (2.43). Recall the second cutoff (2.31), $w_n(x, y, z) = \frac{1}{\sqrt{x}} S_n(x, x_*; y, z)$ with $x_* = (y+z-x)_+$. Using translation of variables $y+z-x \to z$, $y+z-x \to y$ for the integrals of $S_n(x, x_*; y, z)f(x_*)f(y)$, $S_n(x, x_*; y, z)f(x_*)f(z)$ respectively, we compute for any $x > 0$

$$\int_{\mathbb{R}_+^2} w_n(x, y, z)[f(x_*)(f(y) + f(z)) - f(y)f(z)]\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{\sqrt{x}} \int_{\mathbb{R}_+^2} [S_n(x, z; y, y_*) + S_n(x, y; z, z_*) - S_n(x, x_*; y, z)]f(y)f(z)\mathrm{d}y\mathrm{d}z$$

where $y_* = (x + z - y)_+$, $z_* = (x + y - z)_+$. Now by definition of the second cutoff $S_n(\cdot, \cdot; \cdot, \cdot)$ in (2.31) and by checking three cases $0 < x \leq y \wedge z$, $y \wedge z \leq x \leq y \vee z$, and $y \vee z \leq x$, we obtain

$$S_n(x, z; y, y_*) + S_n(x, y; z, z_*) - S_n(x, x_*; y, z)$$
$$\geq \min\{\sqrt{y}(1 \wedge (ny)), \sqrt{z}(1 \wedge (nz)), \sqrt{y_*}(1 \wedge (ny_*)), \sqrt{z_*}(1 \wedge (nz_*))\} \geq 0.$$

This proves the lemma. □

3 Strong Solutions and Stability

On the basis of existence of distributional solutions (Theorem 1.1), our strong solutions is directly defined from the class of distributional solutions.

Definition 3.1 Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$. Let $0 < T_\infty \leq \infty$. We say that F_t is a strong solution of Eq. (1.5) on $[0, T_\infty)$ if it satisfies the following (i)-(iii):

(i) $t \mapsto F_t$ belongs to $C^1([0, T_\infty); \mathcal{B}_0(\mathbb{R}_{\geq 0}))$,
(ii) $t \mapsto \mathcal{Q}_2^\pm(F_t), t \mapsto \mathcal{Q}_3^\pm(F_t)$ belong to $C([0, T_\infty); \mathcal{B}_0(\mathbb{R}_{\geq 0}))$, and
(iii)

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = \mathcal{Q}(F_t) \quad in \quad (\mathcal{B}_0(\mathbb{R}_{\geq 0}), \|\cdot\|_0) \qquad \forall\, t \in [0, T_\infty). \tag{3.1}$$

Besides, if F_t also conserves the energy on $[0, T_\infty)$, then F_t is also called a conservative strong solution of Eq. (1.5) on $[0, T_\infty)$.
Strong solutions can be also defined on a finite closed time-interval by replacing $[0, T_\infty)$ with $[0, T]$ for $0 < T < \infty$.

Remark 3.1 (a) Since $\frac{\mathrm{d}}{\mathrm{d}t} F_t$ and $\int_a^b \mathcal{Q}(F_t)\mathrm{d}t$ as finite Borel measures are defined with the norm $\|\cdot\|_0$ of $\mathcal{B}_0(\mathbb{R}_{\geq 0})$, this gives explicit representations $(\frac{\mathrm{d}}{\mathrm{d}t} F_t)(E) = \frac{\mathrm{d}}{\mathrm{d}t} F_t(E)$, $(\int_a^b \mathcal{Q}(F_t)\mathrm{d}t)(E) = \int_a^b \mathcal{Q}(F_t)(E)\mathrm{d}t$ for all Borel sets $E \subset \mathbb{R}_{\geq 0}$. Here we recall that $\lim_{n\to\infty} \|\mu_n - \mu\|_0 = 0$ is equivalent to the uniform convergence $\lim_{n\to\infty} \sup_{E \subset \mathbb{R}_{\geq 0}} |\mu_n(E) - \mu(E)| = 0$.
(b) Under the condition (ii), the conditions (i), (iii) are equivalent to the integral equation:

$$F_t = F_0 + \int\limits_0^t \mathcal{Q}(F_\tau)\mathrm{d}\tau \qquad \forall\, t \in [0, T_\infty) \tag{3.2}$$

where the integral is taken as the Riemann integral defined with the norm $\|\cdot\|_0$. This then implies that, under the condition (ii), the integral equation (3.2) is equivalent to its dual form:

$$\int\limits_{\mathbb{R}_{\geq 0}} \varphi\mathrm{d}F_t = \int\limits_{\mathbb{R}_{\geq 0}} \varphi\mathrm{d}F_0 + \int\limits_0^t \mathrm{d}\tau \int\limits_{\mathbb{R}_{\geq 0}} \varphi\mathrm{d}\mathcal{Q}(F_\tau) \qquad \forall\, \varphi \in L_0^\infty(\mathbb{R}_{\geq 0}) \tag{3.3}$$

for all $t \in [0, T_\infty)$. In fact, the equation in (3.3) holds for all $\varphi \in C_b(\mathbb{R}_{\geq 0})$ and thus it holds for all $\varphi \in L_0^\infty(\mathbb{R}_{\geq 0})$.

Systematic results on strong solutions can be obtained for the case $M_{-1/2}(F_0) < \infty$. A special case of $M_{-1/2}(F_0) = \infty$, $M_{-1/3}(F_0) < \infty$ is considered in Sect. 5. We now begin with the following

Lemma 3.1 *Let $F \in \mathcal{B}_{1/2}^+(\mathbb{R}_{\geq 0})$ and let $0 \leq \varphi \in C_b^{1,1}(\mathbb{R}_{\geq 0})$ be convex on $\mathbb{R}_{\geq 0}$. Then*

$$\int\limits_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi]\mathrm{d}^2 F \geq -M_{1/2}(F) \int\limits_{\mathbb{R}_{\geq 0}} \varphi\mathrm{d}F, \quad \int\limits_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi]\mathrm{d}^3 F \geq 0. \tag{3.4}$$

Proof The second inequality in (3.4) has been proven in [15]. To prove the first one, it suffices to prove

$$\mathcal{J}[\varphi](y, z) \geq -\frac{1}{2}(\varphi(y)\sqrt{z} + \varphi(z)\sqrt{y}), \qquad \forall\, y, z \geq 0. \tag{3.5}$$

First, it is easily seen from the assumption on φ that φ is non-increasing on $\mathbb{R}_{\geq 0}$. Next we have

$$\Delta\varphi(x, y, z) = (x - y)(x - z)\int_0^1\int_0^1 (D^2\varphi)(\xi_{s,t})\mathrm{d}s\mathrm{d}t \tag{3.6}$$

for all $x, y, z \geq 0$ satisfying $x \leq y + z$ where $\xi_{s,t} = y + z - x + s(x - z) + t(x - y)$, $D^2\varphi(x) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x)$. Now given any $y, z \geq 0$. By symmetry $\mathcal{J}[\varphi](y, z) = \mathcal{J}[\varphi](z, y)$ we may assume $y \leq z$. The convexity of φ and (3.6) imply $\Delta\varphi(x, y, z) \geq 0$ for $x \in [0, y] \cup [z, y + z]$ and so by definition of $\mathcal{J}[\varphi]$

$$\mathcal{J}[\varphi](y, z) \geq \frac{1}{2}\int_y^z \sqrt{x}\,W(x, y, z)\Delta\varphi(x, y, z)\mathrm{d}x.$$

From this and $\varphi \geq 0$ we see that to prove (3.5) we can assume $y < z$. By definition of $W(x, y, z)$ and the non-increase of φ we deduce for all $x \in (y, z)$ that $W(x, y, z) = \frac{1}{\sqrt{xz}}$ and $\varphi(x_*) \geq \varphi(z)$ and so $\Delta\varphi(x, y, z) \geq -\varphi(y)$. Thus

$$\mathcal{J}[\varphi](y, z) \geq \frac{1}{2\sqrt{z}}\int_y^z \Delta\varphi(x, y, z)\mathrm{d}x \geq -\frac{1}{2\sqrt{z}}\varphi(y)(z - y) \geq -\frac{1}{2}\varphi(y)\sqrt{z}$$

and so (3.5) holds true. □

In order to estimate the moment $M_p(F)$ of negative order $p < 0$ for $F \in \mathcal{B}^+(\mathbb{R}_{\geq 0})$ we will often use a smooth approximation:

$$M_p^{(\varepsilon)}(F) := \int_{\mathbb{R}_{\geq 0}} (\varepsilon + x)^p\mathrm{d}F(x), \quad \varepsilon > 0; \quad \lim_{\varepsilon\to 0^+} M_p^{(\varepsilon)}(F) = M_p(F) \tag{3.7}$$

where the limit is due to the monotone convergence theorem.

A fact that will be frequently used for distributional solutions F_t is that the function $t \mapsto M_{1/2}(F_t)$ is continuous on $[0, \infty)$ and $M_{1/2}(F_t) \leq \sqrt{N(F_t)E(F_t)}$ for all $t \in [0, \infty)$. Another fact to be used is the following "monotone non-decrease" of the moment $t \mapsto M_p(F_t)$ for $p < 0$.

Lemma 3.2 *Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$. Then for any $p < 0$, the function $t \mapsto M_p(F_t)e^{\int_0^t M_{1/2}(F_\tau)\mathrm{d}\tau}$ is monotone non-decreasing on $[0, \infty)$, i.e.,*

$$M_p(F_t) \leq M_p(F_T)e^{\int_t^T M_{1/2}(F_\tau)\mathrm{d}\tau} \qquad \forall\, 0 \leq t \leq T < \infty \tag{3.8}$$

including the possible case $M_p(F_t) = \infty$ for some (or all) $t \in [0, \infty)$.

Proof Let $M_p^{(\varepsilon)}(F_t)$ be defined in (3.7). Applying Eq. (1.8) and Lemma 3.1 to the smooth convex function $x \mapsto \varphi_\varepsilon(x) = (\varepsilon + x)^p$ $(\varepsilon > 0)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_p^{(\varepsilon)}(F_t) \geq \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\varphi_\varepsilon] \mathrm{d}^2 F_t \geq -M_{1/2}(F_t) M_p^{(\varepsilon)}(F_t), \quad t \geq 0.$$

This implies that $t \mapsto M_p^{(\varepsilon)}(F_t) e^{\int_0^t M_{1/2}(F_\tau) \mathrm{d}\tau}$ is monotone non-decreasing on $[0, \infty)$, i.e.

$$M_p^{(\varepsilon)}(F_t) \leq M_p^{(\varepsilon)}(F_T) e^{\int_t^T M_{1/2}(F_\tau) \mathrm{d}\tau} \qquad \forall \, 0 \leq t \leq T < \infty.$$

Letting $\varepsilon \to 0^+$ and using the limit (3.7) we obtain (3.8). $\qquad \square$

Lemma 3.3 *Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$ and let $M_{-1/2}^{(\varepsilon)}(F_t)$ be defined in (3.7) with $p = -1/2$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{-1/2}^{(\varepsilon)}(F_t) \leq 2N^2 - M_{1/2}(F_t) M_{-1/2}^{(\varepsilon)}(F_t) + [M_{-1/2}^{(\varepsilon)}(F_t)]^3 \qquad \forall \, t \in [0, \infty). \quad (3.9)$$

Proof Let $\varphi(x) = (\varepsilon + x)^{-1/2}$ $(\varepsilon > 0)$. Applying the differential equation (1.8) we see that to prove (3.9) it suffices to prove that for any $x, y, z \geq 0$

$$\mathcal{J}[\varphi](y, z) \leq 2 - \frac{1}{2} [\varphi(y) + \varphi(z)] \sqrt{y \vee z}, \quad \mathcal{K}[\varphi](x, y, z) \leq \varphi(x) \varphi(y) \varphi(z). \quad (3.10)$$

By definition of $\mathcal{J}^{\pm}[\varphi](y, z)$ and (2.6) it is easily deduced $\mathcal{J}^+[\varphi](y, z) \leq 2$ and $\mathcal{J}^-[\varphi](y, z) \geq \frac{1}{2} [\varphi(y) + \varphi(z)] \sqrt{y \vee z}$. This gives the first inequality in (3.10). To prove the second one in (3.10) we can assume that $\mathcal{K}[\varphi](x, y, z) > 0$ for the given $x, y, z \geq 0$. In this case we have $x_* = y + z - x > 0$ and $(y - x)(z - x) > 0$, the latter is due to the convexity of φ and (3.6). Let us denote $a = \sqrt{\varepsilon + x}, b = \sqrt{\varepsilon + x_*}, c = \sqrt{\varepsilon + y}, d = \sqrt{\varepsilon + z}$. Then

$$\Delta\varphi(x, y, z) = \frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d} = \varphi(x)\varphi(x_*)\varphi(y)\varphi(z)[(a + b)cd - (c + d)ab]$$

and $c^2 d^2 - a^2 b^2 = (y - x)(z - x) > 0$ i.e. $cd > ab$ which together with $a^2 + b^2 = c^2 + d^2$ implies $a + b < c + d$ and $cd + ab \leq a^2 + b^2$. Thus, by computing derivative with respect to ε, we conclude that the function $\varepsilon \mapsto (a + b)cd - (c + d)ab$ is decreasing on $[0, \infty)$ and so

$$\Delta\varphi(x, y, z) \leq \varphi(x)\varphi(x_*)\varphi(y)\varphi(z)[(\sqrt{x} + \sqrt{x_*})\sqrt{y}\sqrt{z} - (\sqrt{y} + \sqrt{z})\sqrt{x}\sqrt{x_*}].$$

Then it is easily deduced from definition of $W(x, y, z)$ that

$$W(x, y, z)\varphi(x_*)[(\sqrt{x} + \sqrt{x_*})\sqrt{y}\sqrt{z} - (\sqrt{y} + \sqrt{z})\sqrt{x}\sqrt{x_*}] \leq 1.$$

This gives the second inequality in (3.10) by definition of $K[\varphi](x, y, z)$. $\qquad \square$

Lemma 3.4 *Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 satisfying $M_{-1/2}(F_0) < \infty$ and let*

$$T_{F_.,\max} = \sup\{t \in [0, \infty) \mid M_{-1/2}(F_t) < \infty\}. \quad (3.11)$$

Then

$$T_{F.,\max} \geq T_{F_0} := \frac{1}{2}[N^{2/3} + M_{-1/2}(F_0)]^{-2} > 0, \quad N := N(F_0), \tag{3.12}$$

$$\sup_{0 \leq t \leq T} M_{-1/2}(F_t) \leq M_{-1/2}(F_T) e^{\int_0^T M_{1/2}(F_\tau)\mathrm{d}\tau} < \infty \quad \forall\, T \in (0, T_{F.,\max}), \tag{3.13}$$

$$M_{-1/2}(F_t) \leq \frac{N^{2/3} + M_{-1/2}(F_0)}{\sqrt{1 - t/T_{F_0}}} - N^{2/3} \quad \forall\, t \in [0, T_{F_0}), \tag{3.14}$$

$$if \ \ t_0 := T_{F.,\max} < \infty, \ \ then \ \ \lim_{t \to t_0-} M_{-1/2}(F_t) = M_{-1/2}(F_{t_0}) = \infty. \tag{3.15}$$

Proof Let $M_{-1/2}^{(\varepsilon)}(F_t)$ be defined in (3.7) with $\varepsilon \in (0, N^{2/3})$. By writing $1 = (\varepsilon + x)^{1/4}(\varepsilon + x)^{-1/4}$ and using Cauchy-Schwarz inequality we have $N^2 \leq (\sqrt{\varepsilon}N + M_{1/2}(F_t))M_{-1/2}^{(\varepsilon)}(F_t)$ so that

$$2N^2 - M_{1/2}(F_t)M_{-1/2}^{(\varepsilon)}(F_t) + [M_{-1/2}^{(\varepsilon)}(F_t)]^3 \leq [N^{2/3} + M_{-1/2}^{(\varepsilon)}(F_t)]^3.$$

If we define $u_\varepsilon(t) = N^{2/3} + M_{-1/2}^{(\varepsilon)}(F_t)$, then we deduce from (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon(t) \leq [u_\varepsilon(t)]^3, \quad t \geq 0.$$

This gives $(u_\varepsilon(t))^{-2} \geq (u_\varepsilon(0))^{-2} - 2t$ for all $t \geq 0$. Since $u_\varepsilon(0) \leq N^{2/3} + M_{-1/2}(F_0)$, it follows from definition of T_{F_0} that

$$M_{-1/2}^{(\varepsilon)}(F_t) \leq \frac{N^{2/3} + M_{-1/2}(F_0)}{\sqrt{1 - t/T_{F_0}}} - N^{2/3} \quad \forall\, t \in [0, T_{F_0}).$$

Letting $\varepsilon \to 0^+$ we conclude from the limit (3.7) that $F_t \in \mathcal{B}_{-1/2,1}^+(\mathbb{R}_{\geq 0})$ for all $t \in [0, T_{F_0})$ and (3.14) holds true. This also proves (3.12).

Next let $\mathcal{T}(F.) = \{t \in [0, \infty) \,|\, M_{-1/2}(F_t) < \infty\}$. We will prove that $\mathcal{T}(F.) = [0, T_{F.,\max})$. First, it is easily seen from the monotone inequality (3.8) and $T_{F.,\max} = \sup \mathcal{T}(F.)$ that $\mathcal{T}(F.)$ is an interval and $[0, T_{F.,\max}) \subset \mathcal{T}(F.)$ which also implies (3.13) by (3.8). Next we prove $T_{F.,\max} \notin \mathcal{T}(F.)$. Suppose to the contrary that $t_0 := T_{F.,\max} \in \mathcal{T}(F.)$. Then $t_0 < \infty$ and $M_{-1/2}(F_{t_0}) < \infty$ and so applying the above result to the distributional solution $t \mapsto \widetilde{F}_t := F_{t+t_0}$ of Eq. (1.5) on $[0, \infty)$ with the initial datum $\widetilde{F}_0 = F_{t_0}$ we conclude that $T_{\widetilde{F}.} > 0$ and $[0, T_{\widetilde{F}.}) \subset \mathcal{T}(\widetilde{F}.)$. So $M_{-1/2}(F_{t+t_0}) = M_{-1/2}(\widetilde{F}_t) < \infty$ for all $t \in [0, T_{\widetilde{F}.})$, i.e. $M_{-1/2}(F_t) < \infty$ for all $t \in [t_0, t_0 + T_{\widetilde{F}.})$. This implies $[0, t_0 + T_{\widetilde{F}.}) \subset \mathcal{T}(F.)$ hence $t_0 + T_{\widetilde{F}.} \leq \sup \mathcal{T}(F.) = t_0$ which contradicts $T_{\widetilde{F}.} > 0$. Thus we must have $T_{F.,\max} \notin \mathcal{T}(F.)$. Since $T_{F.,\max} = \sup \mathcal{T}(F.)$, it follows that $\mathcal{T}(F.) \subset [0, T_{F.,\max})$ and thus $\mathcal{T}(F.) = [0, T_{F.,\max})$.

To prove the blow up (3.15) we denote again $t_0 := T_{F.,\max} < \infty$. We have proved that $t_0 \notin \mathcal{T}(F.)$, i.e. $M_{-1/2}(F_{t_0}) = \infty$. Using the continuity of $t \mapsto M_{-1/2}^{(\varepsilon)}(F_t)$ on $[0, \infty)$ we have

$$M_{-1/2}^{(\varepsilon)}(F_{t_0}) = \liminf_{t \to t_0-} M_{-1/2}^{(\varepsilon)}(F_t) \leq \liminf_{t \to t_0-} M_{-1/2}(F_t) \quad \forall\, \varepsilon > 0.$$

Letting $\varepsilon \to 0^+$ we conclude from the limit (3.7) that $M_{-1/2}(F_{t_0}) \leq \liminf_{t \to t_0-} M_{-1/2}(F_t)$ and so (3.15) holds true since $M_{-1/2}(F_{t_0}) = \infty$. $\qquad\square$

Proposition 3.1 *Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 satisfying $M_{-1/2}(F_0) < \infty$. Then F_t is a strong solution of Eq. (1.5) on $[0, T_{F.,\max})$.*

Proof Take any $T \in (0, T_{F.,\max})$. Using the estimates (2.15), (2.18) for $k = 1/2$ we have $\|\mathcal{Q}_2^\pm(F_t)\|_{1/2}, \|\mathcal{Q}_3^\pm(F_t)\|_{1/2} \le C_T$ for all $t \in [0, T]$, where $C_T < \infty$ depends only on $\sup_{t \in [0,T]} M_{-1/2}(F_t)$ and $\sup_{t \in [0,T]} \|F_t\|_1$. From this and the integral equation (1.9) which also reads

$$\int_{\mathbb{R}_{\ge 0}} \varphi \mathrm{d}(F_t - F_s) = \int_s^t \mathrm{d}\tau \int_{\mathbb{R}_{\ge 0}} \varphi \mathrm{d}\mathcal{Q}(F_\tau) \quad \forall \varphi \in C_b^{1,1}(\mathbb{R}_{\ge 0}) \tag{3.16}$$

we obtain $\|F_t - F_s\|_0 \le C_T |t - s|$ for all $t, s \in [0, T]$. Since $\|F_t - F_s\|_{1/2} \le \|F_t - F_s\|_1^{1/2} \|F_t - F_s\|_0^{1/2}$ by Cauchy-Schwarz inequality, it follows that $t \mapsto F_t$ also belongs to $C([0, T_{F.,\max}); \mathcal{B}_{1/2}(\mathbb{R}_{\ge 0}))$ and thus we conclude from (2.19)–(2.21) with $k = 0$ that $t \mapsto \mathcal{Q}_2^\pm(F_t), t \mapsto \mathcal{Q}_3^\pm(F_t)$ hence $t \mapsto \mathcal{Q}(F_t)$ all belong to $C([0, T_{F.,\max}); \mathcal{B}_0(\mathbb{R}_{\ge 0}))$. Next for any $T \in (0, T_{F.,\max})$, using smooth approximation it is easily deduced that (3.16) with $s = 0$ and $t \in [0, T]$ holds for all bounded Borel functions φ on $\mathbb{R}_{\ge 0}$, in particular it holds for all characteristic functions $\varphi(x) = \mathbf{1}_E(x)$ of Borel sets $E \subset \mathbb{R}_{\ge 0}$. Therefore F_t satisfies the integral equation (3.2) and so, according to the equivalent definition of strong solutions discussed in Remark 3.1, F_t is a strong solutions of Eq. (1.5) on $[0, T_{F.,\max})$. □

Now we are going to establish the stability estimate for conservative strong solutions. The method is similar to those for the space homogeneous measure-valued solutions of the classical Boltzmann equation (see e.g. [16]). Since we have here the cubic term $\mathcal{Q}_3(F_t)$ which determines the Bose–Einstein model, we would like to present a complete proof.

Let $F, G, H \in \mathcal{B}(\mathbb{R}_{\ge 0}^n)$. Recall that the inequality $F \le G$ means $F(E) \le G(E)$ for all Borel sets $E \subset \mathbb{R}_{\ge 0}^n$. Let $h(x)$ be the sign function of H, i.e., h is a real Borel function satisfying $[h(x)]^2 \equiv 1$ and $\mathrm{d}H(x) = h(x)\mathrm{d}|H|(x)$, which is equivalent to $h(x)\mathrm{d}H(x) = \mathrm{d}|H|(x)$. If we define $\kappa(x) = \frac{1}{2}(1 + h(x))$ and $H^+ = \frac{1}{2}(H + |H|)$, then $H^+ \ge 0$ and $\kappa(x)\mathrm{d}H(x) = \mathrm{d}H^+(x)$.

Lemma 3.5 *Let $F \in \mathcal{B}_{-1/2,1}^+(\mathbb{R}_{\ge 0}) \cap \mathcal{B}_{3/2}^+(\mathbb{R}_{\ge 0})$, $G \in \mathcal{B}_{-1/2,1}^+(\mathbb{R}_{\ge 0})$ and $H = F - G$. Let $\kappa : \mathbb{R}_{\ge 0} \mapsto \{0, 1\}$ be the Borel function such that $\kappa(\cdot)\mathrm{d}H = \mathrm{d}H^+$. Then for all $n \ge 1$*

$$\int_{\mathbb{R}_{\ge 0}^2} (1 + x \wedge n)\kappa(x)\mathrm{d}(\mathcal{Q}(F) - \mathcal{Q}(G)) \le \|F\|_{1/2}\|F\|_{3/2} + 2\|F\|_{3/2}\|H\|_0$$

$$+ 2\|F\|_1\|H\|_{1/2} + 2b(F, G)\|H\|_1, \quad (3.17)$$

$$\limsup_{n \to \infty} \int_{\mathbb{R}_{\ge 0}^2} (1 + x \wedge n)\kappa(x)\mathrm{d}(\mathcal{Q}(F) - \mathcal{Q}(G)) \le 2\|F\|_{3/2}\|H\|_0 + 2\|F\|_1\|H\|_{1/2}$$

$$+ 2b(F, G)\|H\|_1 \quad (3.18)$$

where $b(F, G)$ is given in (2.22).

Proof Let $\varphi_n(x) = (1 + x \wedge n)\kappa(x)$. Since $x \mapsto \varphi_n(x)$ is bounded, there is no problem of integrability in the proof. We have

$$\int_{\mathbb{R}_{\ge 0}^2} \varphi_n \mathrm{d}(\mathcal{Q}(F) - \mathcal{Q}(G)) \le \mathcal{Q}_{2,n}^+ - \mathcal{Q}_{2,n}^- + 2b(F, G)\|H\|_1, \tag{3.19}$$

$$\mathcal{Q}_{2,n}^+ := \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}^+[\varphi_n]\mathrm{d}(F \otimes F - G \otimes G), \quad \mathcal{Q}_{2,n}^- := \int_{\mathbb{R}_{\geq 0}^2} \varphi_n(y)\mathcal{J}^+[1]\mathrm{d}(F \otimes F - G \otimes G)$$

where we have used (2.21) for $k = 1$ to get $\|\mathcal{Q}_3(F) - \mathcal{Q}_3(G)\|_1 \leq 2b(F, G)\|H\|_1$. Recall that $W(x, y, z) > 0$ implies $x + x_* = y + z$ so that $\varphi_n(x) + \varphi_n(x_*) \leq 2 + x + x_* = 2 + y + z$. This gives $\mathcal{J}^+[\varphi_n](y, z) \leq (1 + \frac{y+z}{2})\mathcal{J}^+[1](y, z)$ and so

$$\begin{aligned}
\mathcal{Q}_{2,n}^+ &\leq \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}^+[\varphi_n]\mathrm{d}(F \otimes F - G \otimes G)^+ \\
&\leq \int_{\mathbb{R}_{\geq 0}^2} \left(1 + \frac{y+z}{2}\right)\mathcal{J}^+[1](y, z)\mathrm{d}(F \otimes F - G \otimes G)^+ \\
&= \int_{\mathbb{R}_{\geq 0}^2} (1 + y)\mathcal{J}^+[1](y, z)\mathrm{d}(F \otimes F - G \otimes G)^+
\end{aligned} \tag{3.20}$$

where the last equality is due to the symmetry $\mathcal{J}^+[1](y, z) = \mathcal{J}^+[1](z, y)$ and the equality

$$\int_{\mathbb{R}_{\geq 0}^2} \psi(y, z)\mathrm{d}(F \otimes F - G \otimes G)^+ = \int_{\mathbb{R}_{\geq 0}^2} \psi(z, y)\mathrm{d}(F \otimes F - G \otimes G)^+$$

which holds at least for all bounded Borel functions ψ on $\mathbb{R}_{\geq 0}^2$ (see e.g. Lemma 5.2 of [16]) and thus it holds for all nonnegative Borel functions on $\mathbb{R}_{\geq 0}^2$ by monotone convergence. Since

$$(F \otimes F - G \otimes G)^+ \leq F \otimes H^+ + H^+ \otimes G,$$

$1 + y = (y - n)_+ + 1 + y \wedge n$, and

$$\int_{\mathbb{R}_{\geq 0}^2} (y - n)_+\mathcal{J}^+[1]\mathrm{d}(F \otimes F - G \otimes G)^+ \leq \|F\|_{1/2} \int_{\mathbb{R}_{\geq 0}} (y - n)_+\sqrt{1 + y}\mathrm{d}F(y) =: e_n$$

it follows from (3.20) that

$$\mathcal{Q}_{2,n}^+ \leq e_n + \int_{\mathbb{R}_{\geq 0}^2} (1 + y \wedge n)\mathcal{J}^+[1]\mathrm{d}(F \otimes H^+) + \int_{\mathbb{R}_{\geq 0}^2} (1 + y \wedge n)\mathcal{J}^+[1]\mathrm{d}(H^+ \otimes G). \tag{3.21}$$

For the negative part $\mathcal{Q}_{2,n}^-$, recalling $\varphi_n(y) = (1 + y \wedge n)\kappa(y)$ we have

$$\varphi_n(y)\mathrm{d}(F \otimes F - G \otimes G) = \varphi_n(y)\mathrm{d}(F \otimes H) + (1 + y \wedge n)\mathrm{d}(H^+ \otimes G),$$

$$\mathcal{Q}_{2,n}^- = \int_{\mathbb{R}_{\geq 0}^2} \varphi_n(y)\mathcal{J}^+[1]\mathrm{d}(F \otimes H) + \int_{\mathbb{R}_{\geq 0}^2} (1 + y \wedge n)\mathcal{J}^+[1]\mathrm{d}(H^+ \otimes G). \tag{3.22}$$

The common terms in the right hand sides of (3.21)–(3.22) cancel each other in $\mathcal{Q}_{2,n}^+ - \mathcal{Q}_{2,n}^-$ and thus using $\mathcal{J}^+[1](y, z) \leq \sqrt{y} + \sqrt{z}$ we obtain

$$\mathcal{Q}_{2,n}^+ - \mathcal{Q}_{2,n}^- \leq e_n + 2\|F\|_{3/2}\|H\|_0 + 2\|F\|_1\|H\|_{1/2}. \tag{3.23}$$

Finally by definition of e_n and the assumption on F we have $e_n \leq \|F\|_{1/2}\|F\|_{3/2}, n = 1, 2, 3, \ldots$; and $\lim_{n\to\infty} e_n = 0$ by dominated convergence theorem. The lemma then follows from these and (3.19), (3.23). $\qquad\square$

For any given $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ we define a function $\Psi_{F_0}(\varepsilon)$ on $[0, \infty)$ by

$$\Psi_{F_0}(\varepsilon) = \varepsilon + \sqrt{\varepsilon} + \int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty} x\,\mathrm{d}F_0(x), \quad \varepsilon > 0; \quad \Psi_{F_0}(0) = 0. \tag{3.24}$$

Here $\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}$ can be understood as either $\int_{(\frac{1}{\sqrt{\varepsilon}}, \infty)}$ or $\int_{[\frac{1}{\sqrt{\varepsilon}}, \infty)}$.

Theorem 3.1 *Let F_t, G_t be conservative distributional solutions to Eq. (1.5) on $[0, \infty)$ with their initial data F_0, G_0 satisfying $M_{-1/2}(F_0) < \infty$, $M_{-1/2}(G_0) < \infty$. Let $T_{F_{\cdot},\max}$, $T_{G_{\cdot},\max}$ be defined in (3.11) for F_t, G_t respectively. Then for any $T \in (0, T_{F_{\cdot},\max} \wedge T_{G_{\cdot},\max})$*

$$\|F_t - G_t\|_1 \leq C\Psi_{F_0}(\|F_0 - G_0\|_1)e^{ct} \quad \forall\, t \in [0, T] \tag{3.25}$$

where $\Psi_{F_0}(\cdot)$ is defined in (3.24) and $C = C_T$, $c = c_T$ are finite positive constants depending only on $N(F_0)$, $E(F_0)$, $\sup_{0 \leq t \leq T} M_{-1/2}(F_t)$, and $\sup_{0 \leq t \leq T} M_{-1/2}(G_t)$ with $T \in (0, T_{F_{\cdot},\max} \wedge T_{G_{\cdot},\max})$.

In particular if $F_0 = G_0$, then $T_{F_{\cdot},\max} = T_{G_{\cdot},\max} := T_{\max}$ and $F_t = G_t$ for all $t \in [0, T_{\max})$.

Proof The proof is divided into five steps. First of all, according to Proposition 3.1, F_t, G_t are strong solutions on $[0, T_{F_{\cdot},\max})$ and $[0, T_{G_{\cdot},\max})$ respectively. In Steps 1–3 we assume that one of the two solutions, e.g. F_t, has the moment production (1.15) for all $t \in (0, T_{F_{\cdot},\max})$. The existence of such F_t is assured by Theorem 1.1. Let us denote

$$H_t = F_t - G_t.$$

By conservation of mass we have $\|F_t \pm G_t\|_1 \leq \|F_0\|_1 + \|G_0\|_1$ for all $t \geq 0$. So if $\|H_0\|_1 \geq 1$, then $\|H_t\|_1 \leq 2\|F_0\|_1 + \|H_0\|_1 \leq (2\|F_0\|_1 + 1)\|H_0\|_1$ for all $t \geq 0$. Therefore to prove (3.25) we can assume $\|H_0\|_1 < 1$. Given any $T \in (0, T_{F_{\cdot},\max} \wedge T_{G_{\cdot},\max})$ and $s \in (0, T)$.

Step 1: We prove that

$$\|H_t\|_0 \leq \|H_0\|_0 + C \int_0^t \|H_\tau\|_1 \mathrm{d}\tau, \quad t \in [0, T], \tag{3.26}$$

$$\|H_t\|_1 \leq \|H_s\|_1 + C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0 \mathrm{d}\tau + C \int_s^t \|H_\tau\|_1 \mathrm{d}\tau, \quad t \in [s, T]. \tag{3.27}$$

Here and below the constant $0 < C_0 < \infty$ depends only on $N(F_0)$ and $E(F_0)$, while the constants $0 < C, c < \infty$ depend only on $N(F_0)$, $E(F_0)$, $\sup_{0 \leq t \leq T} M_{-1/2}(F_t)$, and $\sup_{0 \leq t \leq T} M_{-1/2}(G_t)$. And C_0, C may have different value in different places. Also recall that $\|F_0\|_1 = N(F_0) + E(F_0)$.

The inequality (3.26) follows from $H_t = H_0 + \int_0^t [\mathcal{Q}(F_\tau) - \mathcal{Q}(G_\tau)]\mathrm{d}\tau$ and the estimates (2.19), (2.21) for $k = 0$. To prove (3.27) we first use $|H_t| = G_t - F_t + 2(H_t)^+$ and the conservation of mass and energy to write

$$\|H_t\|_1 = \|G_s\|_1 - \|F_s\|_1 + 2\|(H_t)^+\|_1, \quad t \geq s. \tag{3.28}$$

Let $x \mapsto \kappa_t(x) \in \{0, 1\}$ be the Borel function on $\mathbb{R}_{\geq 0}$ such that $\kappa_t \mathrm{d}H_t = \mathrm{d}(H_t)^+$. Since $t \mapsto \mathcal{Q}(F_t) - \mathcal{Q}(G_t)$ belongs to $C([0, T_{F_\cdot,\max} \wedge T_{G_\cdot,\max}); \mathcal{B}_0(\mathbb{R}_{\geq 0}))$, applying Lemma 5.1 of [16] to the measure equation $H_t = H_s + \int_s^t (\mathcal{Q}(F_\tau) - \mathcal{Q}(G_\tau))\mathrm{d}\tau$, $t \in [s, T_{F_\cdot,\max} \wedge T_{G_\cdot,\max})$, we have

$$\int_{\mathbb{R}_{\geq 0}} \varphi(x)\mathrm{d}(H_t)^+ = \int_{\mathbb{R}_{\geq 0}} \varphi(x)\mathrm{d}(H_s)^+ + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}_{\geq 0}} \varphi(x)\kappa_\tau(x)\mathrm{d}(\mathcal{Q}(F_\tau) - \mathcal{Q}(G_\tau))$$

for all $t \in [s, T]$ and all $\varphi \in L_0^\infty(\mathbb{R}_{\geq 0})$. In particular we have

$$\int_{\mathbb{R}_{\geq 0}} (1 + x \wedge n)\mathrm{d}(H_t)^+ \leq \|(H_s)^+\|_1 + \int_s^t \mathrm{d}\tau \int_{\mathbb{R}_{\geq 0}} (1 + x \wedge n)\kappa_\tau(x)\mathrm{d}(\mathcal{Q}(F_\tau) - \mathcal{Q}(G_\tau)).$$

Next applying (1.15) with $p = 3/2$ we see that the function $t \mapsto \|F_t\|_{3/2} \leq C_0(1 + 1/t)$ is integrable on $[s, T]$ and thus we deduce from Lemma 3.5 and the reverse Fatou's Lemma that

$$\limsup_{n \to \infty} \int_s^t \mathrm{d}\tau \int_{\mathbb{R}_{\geq 0}} (1 + x \wedge n)\kappa_\tau(x)\mathrm{d}(\mathcal{Q}_2(F_\tau) - \mathcal{Q}_2(G_\tau))(x)$$

$$\leq C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0 \mathrm{d}\tau + C \int_s^t \|H_\tau\|_1 \mathrm{d}\tau.$$

Here the second constant C is obtained by using the conservation of mass and energy which gives upper bounds $M_{1/2}(F_t) + M_{1/2}(G_t) \leq \frac{1}{2}\|F_0\|_1 + \frac{1}{2}\|G_0\|_1 \leq \|F_0\|_1 + \frac{1}{2}$, etc. Letting $n \to \infty$ we conclude

$$\|(H_t)^+\|_1 \leq \|(H_s)^+\|_1 + C_0 \int_s^t (1 + 1/\tau)\|H_\tau\|_0 \mathrm{d}\tau + C \int_s^t \|H_\tau\|_1 \mathrm{d}\tau.$$

This together with (3.28) and $\|G_s\|_1 - \|F_s\|_1 + 2\|(H_s)^+\|_1 = \|H_s\|_1$ gives (3.27).

Step 2: We prove that for any $R \geq 1$

$$\|H_t\|_1 \leq 5R\|H_0\|_1 + CRt + 2 \int_{x > R} x\mathrm{d}F_0(x), \quad t \in [0, T]. \tag{3.29}$$

In fact using $|H_t| = G_t - F_t + 2(H_t)^+$ and conservation of mass and energy we have

$$\|H_t\|_1 \leq \|H_0\|_1 + 4R\|H_t\|_0 + 2 \int_{x > R} x\mathrm{d}F_t(x) \tag{3.30}$$

and applying (3.3) to the bounded function $\psi(x) = \mathbf{1}_{\{x \leq R\}}x$ we deduce

$$\int\limits_{x>R} x\mathrm{d}F_t = E(F_0) - \int\limits_{\mathbb{R}_{\geq 0}} \mathbf{1}_{\{x \leq R\}} x\mathrm{d}F_t = \int\limits_{x>R} x\mathrm{d}F_0 - \int\limits_0^t \mathrm{d}\tau \int\limits_{\mathbb{R}_{\geq 0}} \mathbf{1}_{\{x \leq R\}} x\mathrm{d}\mathcal{Q}(F_\tau)$$

$$\leq \int\limits_{x>R} x\mathrm{d}F_0 + R \int\limits_0^t \|\mathcal{Q}(F_\tau)\|_0 \mathrm{d}\tau$$

$$\leq \int\limits_{x>R} x\mathrm{d}F_0 + CRt.$$

This together with (3.30) and $\|H_t\|_0 \leq \|H_0\|_0 + Ct$ (by (3.26)) yields (3.29).

Step 3: If $T \leq \|H_0\|_1$, we take $R = \frac{1}{\sqrt{\|H_0\|_1}}$ and use (3.29) to get

$$\|H_t\|_1 \leq C\Big(\sqrt{\|H_0\|_1} + \int\limits_{x > \frac{1}{\sqrt{\|H_0\|_1}}} x\mathrm{d}F_0(x)\Big) \leq C\Psi_{F_0}(\|H_0\|_1), \quad t \in [0, T].$$

Suppose now $\|H_0\|_1 < T$ and let $\varepsilon > 0$ satisfy $\|H_0\|_1 \leq \varepsilon < T \wedge 1$. Taking $R = \frac{1}{\sqrt{\varepsilon}}$ and using (3.29) we have

$$\|H_t\|_1 \leq C\sqrt{\varepsilon} + 2 \int\limits_{x > \frac{1}{\sqrt{\varepsilon}}} x\mathrm{d}F_0(x) \leq C\Psi_{F_0}(\varepsilon), \quad \forall t \in [0, \varepsilon]. \tag{3.31}$$

In particular this inequality holds for $t = \varepsilon$. Thus using (3.27) for $s = \varepsilon$ gives

$$\|H_t\|_1 \leq C\Psi_{F_0}(\varepsilon) + C_0 \int\limits_\varepsilon^t (1 + 1/\tau)\|H_\tau\|_0 \mathrm{d}\tau + C \int\limits_\varepsilon^t \|H_\tau\|_1 \mathrm{d}\tau, \quad t \in [\varepsilon, T \wedge 1] \tag{3.32}$$

Next, using (3.26) and $\|H_0\|_0 \leq \varepsilon \leq t \leq 1$,

$$\int\limits_\varepsilon^t (1 + 1/\tau)\|H_\tau\|_0 \mathrm{d}\tau \leq 2\varepsilon \log(1/\varepsilon) + 2C \int\limits_\varepsilon^t \frac{1}{\tau} \int\limits_0^\tau \|H_u\|_1 \mathrm{d}u \mathrm{d}\tau$$

$$\leq 2\sqrt{\varepsilon} + 2C \int\limits_0^t \|H_u\|_1 |\log u| \mathrm{d}u, \quad t \in [\varepsilon, T \wedge 1]. \tag{3.33}$$

This together with (3.32) and (3.31) gives

$$\|H_t\|_1 \leq C\Psi_{F_0}(\varepsilon) + C \int\limits_0^t (1 + |\log \tau|)\|H_\tau\|_1 \mathrm{d}\tau, \quad t \in [0, T \wedge 1]. \tag{3.34}$$

By Gronwall lemma we then obtain

$$\|H_t\|_1 \leq C\Psi_{F_0}(\varepsilon) \exp\Big(C \int\limits_0^t (1 + |\log \tau|)\mathrm{d}\tau\Big) \leq C\Psi_{F_0}(\varepsilon), \quad t \in [0, T \wedge 1]. \tag{3.35}$$

Now if $T \leq 1$, then (3.25) follows from (3.35). Suppose $T > 1$. Then (3.35) holds for all $t \in [0, 1]$. In particular $\|H_1\|_1 \leq C\Psi_{F_0}(\varepsilon)$. On the other hand from (3.27) with $s = 1$ we

have $\|H_t\|_1 \le \|H_1\|_1 + c \int_1^t \|H_\tau\|_1 d\tau$ for all $t \in [1, T]$ and so $\|H_t\|_1 \le \|H_1\|_1 e^{c(t-1)} \le C\Psi_{F_0}(\varepsilon) e^{ct}$ for all $t \in [1, T]$ by Gronwall Lemma. This together with the estimate for $t \in [0, 1]$ leads to

$$\|H_t\|_1 \le C\Psi_{F_0}(\varepsilon) e^{ct}, \quad t \in [0, T]. \tag{3.36}$$

Finally if $\|H_0\|_1 > 0$ then taking $\varepsilon = \|H_0\|_1$ in (3.36) gives (3.25). If $\|H_0\|_1 = 0$, then in (3.36) letting $\varepsilon \to 0^+$ we conclude $\|H_t\|_1 = 0$ for all $t \in [0, T]$ and thus (3.25) still holds true. This proves (3.25) for the case where F_t has the moment production (1.15) for all $t \in (0, T_{F.,\max})$.

Step 4: We prove that if any two conservative distributional solutions F_t, G_t stated in the theorem satisfy $F_t = G_t$ for all $t \in [0, T_{F.,\max} \wedge T_{G.,\max})$, then $T_{F.,\max} = T_{G.,\max}$.

Suppose this is not true and let for instance $t_0 := T_{F.,\max} < T_{G.,\max}$. Then $t_0 < \infty$ and, by Lemma 3.5, $M_{-1/2}(F_{t_0}) = \infty$ but $M_{-1/2}(G_{t_0}) < \infty$. Let $M_{-1/2}^{(\varepsilon)}(F_t)$, $M_{-1/2}^{(\varepsilon)}(G_t)$ be defined in (3.7). By the assumption $F_t = G_t$ for all $t \in [0, t_0)$ we have $M_{-1/2}^{(\varepsilon)}(F_t) = M_{-1/2}^{(\varepsilon)}(G_t)$ for all $t \in [0, t_0)$. Since $t \mapsto M_{-1/2}^{(\varepsilon)}(F_t)$, $t \mapsto M_{-1/2}^{(\varepsilon)}(G_t)$ are continuous on $[0, \infty)$, letting $t \nearrow t_0$ gives $M_{-1/2}^{(\varepsilon)}(F_{t_0}) = M_{-1/2}^{(\varepsilon)}(G_{t_0})$, and then letting $\varepsilon \to 0^+$ and using the limit (3.7) leads to $M_{-1/2}(F_{t_0}) = M_{-1/2}(G_{t_0}) < \infty$ which contradicts $M_{-1/2}(F_{t_0}) = \infty$. Thus we must have $T_{F.,\max} = T_{G.,\max}$.

Step 5: Let G_t be any conservative distributional solution of Eq. (1.5) on $[0, \infty)$ having the same initial datum $G_0 = F_0$ of F_t where F_t is used in Steps 1–3. Using (3.25) we conclude $G_t = F_t$ for all $t \in [0, T_{F.,\max} \wedge T_{G.,\max})$ and thus Step 4 insures that $T_{F.,\max} = T_{G.,\max} := T_{\max}$ hence $G_t = F_t$ for all $t \in [0, T_{\max})$. In particular G_t also has the moment production (1.15) for all $t \in (0, T_{\max})$. This proves that the moment condition added on F_t in Steps 1–3 is indeed satisfied for every conservative distributional solution satisfying the condition in the theorem and thus the stability estimate (3.25) holds true for any two conservative distributional solutions F_t, G_t stated in the theorem.

Finally let F_t, G_t be given in the theorem and suppose $F_0 = G_0$. Then (3.25) implies that $F_t = G_t$ for all $t \in [0, T_{F.,\max} \wedge T_{G.,\max})$ and Step 4 insures that $T_{F.,\max} = T_{G.,\max} := T_{\max}$ hence $F_t = G_t$ for all $t \in [0, T_{\max})$. This finishes the proof of the theorem. $\square$

Now we are going to prove the global existence of strong solutions for a class of initial data. We will use the Hölder inequality of moments for $F \in \mathcal{B}^+(\mathbb{R}_{\ge 0})$ (with $F(\mathbb{R}_{>0}) > 0$)

$$M_r(F) \le [M_p(F)]^{\frac{q-r}{q-p}} [M_q(F)]^{\frac{r-p}{q-p}}, \quad -\infty < p < r < q < \infty \tag{3.37}$$

and the following lemma:

Lemma 3.6 *Let F_t be a conservative distributional solution of Eq. (1.5) on $[0, \infty)$ satisfying the moment production (1.15), and let $N = N(F_0)$, $E = E(F_0)$. Then for all $t > 0$*

$$M_{1/2}(F_t) \ge \frac{1}{6} \left[1 - \exp\left(-(\sqrt{NE} + N^3/E)t \right) \right] \frac{NE}{\sqrt{NE} + N^3/E}. \tag{3.38}$$

Proof By conservation of energy and (3.37) we have $E \le [M_{1/2}(F_t)]^{2/3} [M_2(F_t)]^{1/3}$, i.e.

$$M_{1/2}(F_t) \ge \frac{E^{3/2}}{\sqrt{M_2(F_t)}} \quad \forall t > 0. \tag{3.39}$$

Thus to prove (3.38) we need only to prove that $M_2(F_t)$ has the corresponding upper bound. To do this we must use the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2(F_t) = \int_{\mathbb{R}^2_{\geq 0}} \mathcal{J}[\varphi] \mathrm{d}^2 F_t + \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \mathrm{d}^3 F_t \qquad \forall\, t > 0 \qquad (3.40)$$

where $\varphi(x) = x^2$. Of course this φ does not belong to the test function space $C^{1,1}(\mathbb{R}_{\geq 0})$, but thanks to the moment production (1.15) we are able to prove that (3.40) does hold rigourously. First, from the moment production (1.15) we have $\sup_{t \geq s} \|F_t\|_p < \infty$ for all $s > 0$ and $p \geq 1$ and thus using Corollary of Lemma 4 in [13] to $\{F_t\}_{t \geq s}$ we conclude that the collision integrals $t \mapsto \int_{\mathbb{R}^2_{\geq 0}} \mathcal{J}[\varphi] \mathrm{d}^2 F_t$, $t \mapsto \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \mathrm{d}^3 F_t$ are continuous on $[s, \infty)$ ($\forall\, s > 0$). Also we note that the measure $t \mapsto F_{t+s}$ is a distributional solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_s and thus it satisfies the integral equation (1.9). Next in order to use our test functions in $C_b^{1,1}(\mathbb{R}_{\geq 0})$ we consider smooth truncations $\varphi_n(x) = x^2 e^{-x/n}$. We compute $\sup_{n \geq 1, x \geq 0} |D^2 \varphi_n(x)| \leq 4$ and, using (3.6),

$$\sup_{n \geq 1} |\mathcal{K}[\varphi_n](x, y, z)| \leq 4(1 + y)(1 + z), \quad \sup_{n \geq 1} |\mathcal{J}[\varphi_n](y, z)| \leq 2(1 + y)^{5/2}(1 + z)^{5/2},$$

$$\lim_{n \to \infty} \mathcal{K}[\varphi_n](x, y, z) = \mathcal{K}[\varphi](x, y, z), \qquad \lim_{n \to \infty} \mathcal{J}[\varphi_n](y, z) = \mathcal{J}[\varphi](y, z)$$

for all $x, y, z \geq 0$. Since $\sup_{t \geq s} \|F_t\|_{5/2} < \infty$, it follows from dominated convergence theorem that the integral equation (1.9) of the solution $t \mapsto F_{t+s}$ holds also for the function $\varphi(x) = x^2$. Since $s > 0$ is arbitrary, this proves that the function $t \mapsto M_2(F_t)$ belongs to $C^1((0, \infty))$ and satisfies the differential equation (3.40).

Now let us compute $\mathcal{J}[\varphi](y, z)$. By definition of $\mathcal{J}[\varphi]$ we have $\mathcal{J}[\varphi](y, z) = \mathcal{J}[\varphi](z, y)$ and $\mathcal{J}[\varphi](0, 0) = 0$. Suppose for instance $0 \leq y \leq z$ and $z > 0$. By considering $\mathcal{J}^+[\varphi](y, z) = \int_0^y + \int_y^z + \int_z^{y+z}$ we compute

$$\mathcal{J}[\varphi](y, z) = \left(\frac{1}{2} + \frac{8}{15} \cdot \frac{y}{z} - \frac{13}{210} \left(\frac{y}{z} \right)^2 \right) y z^{3/2} + \frac{z^{5/2}}{3} - \frac{1}{2}(y^2 + z^2)\sqrt{z}.$$

Since $\frac{1}{2} + \frac{8}{15}\theta - \frac{13}{210}\theta^2 \leq \frac{102}{105} < 1$ for all $0 \leq \theta \leq 1$, it follows that

$$\mathcal{J}[\varphi](y, z) \leq y z^{3/2} \vee y^{3/2} z + \frac{1}{3}(y^{5/2} \vee z^{5/2}) - \frac{1}{2}(y^2 + z^2)\sqrt{y \vee z} \qquad \forall\, y, z \geq 0.$$

From this we obtain

$$\int_{\mathbb{R}^2_{\geq 0}} \mathcal{J}[\varphi] \mathrm{d}^2 F_t \leq 2E M_{3/2}(F_t) - \frac{1}{3} N M_{5/2}(F_t).$$

Using the Hölder inequality (3.37) and the conservation of mass and energy $M_0(F_t) = N$, $M_1(F_t) = E$ we have $E \leq \sqrt{N}\sqrt{M_2(F_t)}$, $M_{3/2}(F_t) \leq \sqrt{E}\sqrt{M_2(F_t)}$ and $M_{5/2}(F_t) \geq [M_2(F_t)]^{3/2}/\sqrt{E}$. Thus

$$\int_{\mathbb{R}^2_{\geq 0}} \mathcal{J}[\varphi] \mathrm{d}^2 F_t \leq 2\sqrt{NE} M_2(F_t) - \frac{N}{3\sqrt{E}}[M_2(F_t)]^{3/2}. \qquad (3.41)$$

Next we estimate the cubic term. Recalling that $W(x, y, z) > 0$ implies $x_* = y + z - x > 0$ and so $\Delta\varphi(x, y, z) = x^2 + x_*^2 - y^2 - z^2 = 2(yz - xx_*)$, it follows that

$$\mathcal{K}[\varphi](x, y, z) = 2W(x, y, z)(yz - xx_*) \leq 2W(x, y, z)yz \leq 2\sqrt{yz}$$

for all $x, y, z \geq 0$ hence

$$\int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \mathrm{d}^3 F_t \leq 2N[M_{1/2}(F_t)]^2 \leq 2N^2 E \leq 2\frac{N^3}{E} M_2(F_t). \tag{3.42}$$

Here we used $M_{1/2}(F_t)^2 \leq NE$ and $E^2 \leq NM_2(F_t)$. Combining (3.41) and (3.42) with (3.40) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2(F_t) \leq AM_2(F_t) - B[M_2(F_t)]^{3/2}, \quad t > 0 \tag{3.43}$$

where

$$A = 2\left(\sqrt{NE} + \frac{N^3}{E}\right), \quad B = \frac{N}{3\sqrt{E}}. \tag{3.44}$$

By solving the differential inequality (3.43) we conclude

$$M_2(F_t) \leq (A/B)^2(1 - e^{-\frac{1}{2}At})^{-2}, \quad t > 0. \tag{3.45}$$

Therefore the inequality (3.38) follows from (3.39), (3.45) and (3.44). $\qquad\square$

Theorem 3.2 *Let $F_0 \in \mathcal{B}^+_{-1/2,1}(\mathbb{R}_{\geq 0})$ and suppose*

$$M_{-1/2}(F_0) \leq \frac{1}{80}[N(F_0)E(F_0)]^{1/4}. \tag{3.46}$$

Then there exists a unique conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0. Moreover F_t is also a strong solution on $[0, \infty)$ and satisfies

$$\sup_{t \geq 0} M_{-1/2}(F_t) \leq \frac{1}{4}[N(F_0)E(F_0)]^{1/4}. \tag{3.47}$$

In particular F_t has no condensation for all $t \in [0, \infty)$.

Proof Denote $N = N(F_0)$, $E = E(F_0)$ and let F_t be a conservative distributional solution of Eq. (1.5) on $[0, \infty)$ obtained by Theorem 1.1 with the initial datum F_0, in particular F_t has the moment production (1.15). Then, by Lemma 3.6, F_t satisfies (3.38). To prove the theorem, we need only to prove that $T_{F_.,\max} = \infty$ and F_t satisfies (3.47). In fact if $T_{F_.,\max} = \infty$ holds true, then we conclude from Proposition 3.1 and Theorem 3.1 that this F_t is a strong solution on $[0, \infty)$ and F_t is the unique one in the class of conservative distributional solutions of Eq. (1.5) on $[0, \infty)$ having the same initial datum F_0.

Let us introduce two numbers which will play important roles:

$$\beta = \frac{NE}{[M_{-1/2}(F_0)]^4}, \quad \gamma = \frac{E}{N^{5/3}}. \tag{3.48}$$

Using Hölder inequality (3.37) we have $N \leq (M_{-1/2}(F_0))^{2/3}E^{1/3}$ i.e.

$$M_{-1/2}(F_0) \geq \sqrt{N^3/E}. \tag{3.49}$$

From (3.48), (3.49) we deduce

$$(\gamma/\beta)^{1/4} = N^{-2/3}M_{-1/2}(F_0), \quad \gamma \geq \beta^{1/3}.$$

Then, by Lemma 3.4,

$$M_{-1/2}(F_t) \leq N^{2/3}\left(\frac{1 + (\gamma/\beta)^{1/4}}{\sqrt{1 - t/T_{F_0}}} - 1\right) \qquad \forall t \in [0, T_{F_0}) \tag{3.50}$$

where T_{F_0} is given in (3.12) which can be also rewritten in terms of $(\gamma/\beta)^{1/4}$ as

$$T_{F_0} = \frac{1}{2N^{4/3}(1 + (\gamma/\beta)^{1/4})^2}.$$

According to the differential inequality (3.9), in order to obtain an upper bound of $M_{-1/2}(F_t)$ on $[t_0, \infty)$ for some $0 < t_0 < T_{F_0}$, we need to have a lower bound for $M_{1/2}(F_t)$ on $[t_0, \infty)$. The condition (3.46) which now reads $\beta \geq (80)^4$ is just designed for this popups. Numerical computation suggests that a good choice for t_0 is given by

$$1 - \exp(-(\sqrt{NE} + N^3/E)t_0) = \frac{36}{37} \quad \text{i.e.} \quad t_0 = \frac{\log(37)}{\sqrt{NE} + N^3/E}. \tag{3.51}$$

We need to prove that this t_0 satisfies

$$t_0 \leq \frac{1}{2}T_{F_0} \quad \text{i.e.} \quad 4\log(37) \leq \frac{\sqrt{NE} + N^3/E}{N^{4/3}(1 + (\gamma/\beta)^{1/4})^2}. \tag{3.52}$$

To prove (3.52) we use $\gamma = E/N^{5/3}$ to write

$$\sqrt{NE} + N^3/E = N^{4/3}(\sqrt{\gamma} + \gamma^{-1}) \tag{3.53}$$

so that the inequality (3.52) is equivalent to

$$4\log(37) \leq \frac{\sqrt{\gamma} + \gamma^{-1}}{(1 + (\gamma/\beta)^{1/4})^2}. \tag{3.54}$$

Using $\gamma \geq \beta^{1/3}$ and omitting γ^{-1} we see that the right hand side of (3.54) is larger than $\left(\gamma^{1/4}/(1 + (\gamma/\beta)^{1/4})\right)^2 \geq \beta^{1/6}\left(1 + \beta^{-1/6}\right)^{-2} > 4\log(37)$ where the last inequality is because $\beta^{1/6} \geq (80)^{2/3}$. Thus (3.54) holds true. By the way, from $\gamma \geq \beta^{1/3}$ we also have

$$\frac{E}{N^{5/3}} = \gamma \geq (80)^{4/3}. \tag{3.55}$$

Now applying (3.38) with (3.51), (3.53) and $E = N^{5/3}\gamma$ we obtain

$$M_{1/2}(F_t) \geq \frac{6}{37} \cdot \frac{NE}{\sqrt{NE} + N^3/E} = 3M_*^2 \qquad \forall t \geq t_0 \tag{3.56}$$

where

$$M_* = N^{2/3}\left(\frac{2\sqrt{\gamma}}{37(1 + \gamma^{-3/2})}\right)^{1/2}. \tag{3.57}$$

Inserting (3.56) into the differential inequality (3.9) leads to

$$\frac{d}{dt}M_{-1/2}^{(\varepsilon)}(F_t) \leq 2N^2 - 3M_*^2 M_{-1/2}^{(\varepsilon)}(F_t) + [M_{-1/2}^{(\varepsilon)}(F_t)]^3 \qquad \forall t \geq t_0 \tag{3.58}$$

from which we see that in order to get a global upper bound of $(t, \varepsilon) \mapsto M_{-1/2}^{(\varepsilon)}(F_t)$, it needs only to prove the following inequalities (the reason will be clear later):

$$N^2 \leq M_*^3, \quad M_{-1/2}(F_t) \leq M_* \quad \forall t \in [0, t_0]. \tag{3.59}$$

The first inequality in (3.59) is equivalent to $\frac{\sqrt{\gamma}}{1+\gamma^{-3/2}} \geq 37/2$ which is exactly satisfied since $\gamma \geq (80)^{4/3}$. To prove the second one in (3.59) we first use (3.50) and $t_0 \leq \frac{1}{2}T_{F_0}$ to get

$$M_{-1/2}(F_t) \leq N^{2/3}\left(\sqrt{2}\,(1+(\gamma/\beta)^{1/4}) - 1\right) \qquad \forall\, t \in [0, t_0].$$

Then, by definition of M_* and $\gamma^{-3/2} \leq (80)^{-2}$, we see that a sufficient condition for the second inequality in (3.59) to hold is

$$\sqrt{2}\,(1+(\gamma/\beta)^{1/4}) - 1 \leq \left(\frac{2\sqrt{\gamma}}{37(1+(80)^{-2})}\right)^{1/2}.$$

But this inequality does hold true because $\beta^{1/4} \geq 80$ and $\gamma^{1/4} \geq (80)^{1/3}$.

Now from (3.58), $N^2 \leq M_*^3, 2M_*^3 - 3M_*^2 M + M^3 = (M - M_*)^2(M + 2M_*)$, and denoting $M_\varepsilon(t) = M_{-1/2}^{(\varepsilon)}(F_t)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}M_\varepsilon(t) \leq (M_\varepsilon(t) - M_*)^2(M_\varepsilon(t) + 2M_*) \qquad \forall\, t \geq t_0.$$

Since $M_\varepsilon(t_0) \leq M_{-1/2}(F_{t_0}) \leq M_*$, it follows that

$$(M_\varepsilon(t) - M_*)_+ \leq \int_{t_0}^{t} [(M_\varepsilon(\tau) - M_*)_+]^2(M_\varepsilon(\tau) + 2M_*)\mathrm{d}\tau, \quad t \in [t_0, \infty).$$

Thus we conclude from Gronwall Lemma that $(M_\varepsilon(t) - M_*)_+ \equiv 0$, i.e. $M_{-1/2}^{(\varepsilon)}(F_t) = M_\varepsilon(t) \leq M_*$ for all $t \geq t_0$ and all $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ leads to $M_{-1/2}(F_t) \leq M_*$ for all $t \geq t_0$. Combining this with the second inequality in (3.59) we conclude that

$$M_{-1/2}(F_t) \leq M_* \qquad \forall\, t \in [0, \infty). \tag{3.60}$$

This implies that $T_{F,\max} = \infty$ by definition of $T_{F,\max}$. Finally since $N^{2/3}\gamma^{1/4} = (NE)^{1/4}$, it follows from (3.57) that $M_* \leq (NE)^{1/4}\sqrt{2/37} \leq \frac{1}{4}(NE)^{1/4}$ and thus we obtain (3.47) from (3.60). This completes the proof. □

There are many F_0 that satisfy the condition (3.46). For instance for any $G_0 \in \mathcal{B}_{-1/2,1}^+(\mathbb{R}_{\geq 0})$ with $M_{-1/2}(G_0) > 0$, define $F_0 = \rho G_0$ with a constant $\rho > 0$. Then

$$\frac{M_{-1/2}(F_0)}{[N(F_0)E(F_0)]^{1/4}} = \sqrt{\rho}\,\frac{M_{-1/2}(G_0)}{[N(G_0)E(G_0)]^{1/4}} \tag{3.61}$$

and so F_0 satisfies (3.46) when ρ is small enough.

We have proved that the condition (3.46) implies (3.55) i.e. $\frac{E}{N^{5/3}} \geq (80)^{4/3}$, from which and (1.14) one sees that the condition (3.46) belongs to the case of high temperature: $\overline{T}/\overline{T}_c > 783$.

4 Regularity and Mild Solutions

In this section we use the above results to study regularity of distributional solutions and prove the existence and stability of mild solutions. Without risk of confusion we use short notations for the norms of $L^1(\mathbb{R})_+$ and $L^\infty(\mathbb{R})_+$:

$$\|f\|_{L^1} \equiv \|f\|_{L^1(\mathbb{R}_+)}, \quad \|f\|_{L^\infty} \equiv \|f\|_{L^\infty(\mathbb{R}_+)}.$$

As usual we denote $f(t) = f(\cdot, t)$ when the x-variable has been taken certain integration or norm, for instance $N(f(t)) = N(f(\cdot, t))$, $\|f(t)\|_{L^1} = \|f(\cdot, t)\|_{L^1}$, etc. From Lemma 2.2 and the following propositions one will see that just as $M_{-1/2}(F_t)$ plays the important role in the existence of strong solutions, $\|f(t)\|_{L^1} = M_{-1/2}(f(t))$ also controls everything for the existence and stability of local and global (bounded or unbounded) mild solutions.

Proposition 4.1 *Let F_t be a distributional solution of Eq. (1.5) on $[0, \infty)$ whose initial datum F_0 is regular and satisfies $M_{-1/2}(F_0) < \infty$, and let $T_{F,\max}$ be defined in (3.11). Then F_t is regular for all $t \in [0, T_{F,\max})$ and its density $f(\cdot, t)$ is a mild solution of Eq. (1.5) on $[0, T_{F,\max})$ satisfying $f \in C([0, T_{F,\max}); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$, where f_0 is the density of F_0. In particular if F_t is conservative, so is $f(\cdot, t)$ on $[0, T_{F,\max})$.*

Proof Denote $T_{\max} = T_{F,\max}$. By Proposition 3.1, F_t is a strong distributional solution on $[0, T_{\max})$, and from the relation (1.11) we have $F_t(\{0\}) = 0$ for all $t \in [0, T_{\max})$, which means that the origin $x = 0$ has no contribution with respect to the measure F_t and thus the integration domain $\mathbb{R}_{\geq 0}$ can be replaced by $\mathbb{R}_+ = \mathbb{R}_{>0}$. Let

$$\nu_t(E) = \int_E \frac{1}{\sqrt{x}} dF_t(x), \quad V_t(\delta) = \sup_{\mathrm{mes}(U) < \delta} \nu_t(U), \quad 0 \leq t < T_{\max}$$

where $E \subset \mathbb{R}_+$ is any Borel set, U is chosen from all open sets in $\mathbb{R}_+$, and $\mathrm{mes}(\cdot)$ denotes the Lebeague measure. We are going to establish Gronwall inequality for $V_t(\delta)$ on $t \in [0, T_{\max})$.

Given any $T \in (0, T_{\max})$ and take any open set $U \subset \mathbb{R}_+$ satisfying $\mathrm{mes}(U) < \delta$. Applying the integral equation (3.3) to a monotone sequence $0 \leq \varphi_n \in C_b(\mathbb{R}_{\geq 0})$ satisfying

$$\varphi_n(x) \nearrow \psi_U(x) := \frac{1}{\sqrt{x}} \mathbf{1}_U(x) \quad (n \to \infty) \quad \forall x \in \mathbb{R}_+$$

(for instance one may take $\varphi_n(x) = (x + \frac{1}{n})^{-1/2}(1 - \exp(-n\mathrm{dist}(x, U^c)))$ where $U^c = \mathbb{R} \setminus U$) and then omitting negative parts, we deduce from monotone convergence that

$$\nu_t(U) \leq \nu_0(U) + \int_0^t d\tau \int_{\mathbb{R}_+^2} \mathcal{J}^+[\psi_U] d^2 F_\tau + \int_0^t d\tau \int_{\mathbb{R}_+^3} \mathcal{K}^+[\psi_U] d^3 F_\tau.$$

Next we compute for all $x, y, z > 0$

$$\mathcal{J}^+[\psi_U](y, z) \leq \frac{1}{\sqrt{yz}} \mathrm{mes}(U) \leq \frac{1}{\sqrt{yz}} \delta,$$

$$\mathcal{K}^+[\psi_U](x, y, z) \leq \frac{1}{\sqrt{xyz}} [\mathbf{1}_U(x) + \mathbf{1}_U(y + z - x)].$$

Since $\mathbb{R}_+ \cap (U + x - z)$ is open and $\mathrm{mes}(\mathbb{R}_+ \cap (U + x - z)) \leq \mathrm{mes}(U) < \delta$, this gives

$$\int_{\mathbb{R}_+} \mathbf{1}_U(y + z - x) \frac{1}{\sqrt{y}} dF_\tau(y) = \int_{\mathbb{R}_+ \cap (U + x - z)} \frac{1}{\sqrt{y}} dF_\tau(y) \leq V_\tau(\delta).$$

It follows that

$$\nu_t(U) \leq V_0(\delta) + \int_0^t [M_{-1/2}(F_\tau)]^2 d\tau \, \delta + 2 \int_0^t [M_{-1/2}(F_\tau)]^2 V_\tau(\delta) d\tau.$$

Taking $\sup_{\mathrm{mes}(U)<\delta}$ leads to

$$V_t(\delta) \leq V_0(\delta) + 2 \int_0^t [M_{-1/2}(F_\tau)]^2 \Big(\frac{\delta}{2} + V_\tau(\delta)\Big) \mathrm{d}\tau, \quad 0 \leq t \leq T$$

and so, by Gronwall Lemma,

$$\frac{\delta}{2} + V_t(\delta) \leq \Big(\frac{\delta}{2} + V_0(\delta)\Big) \exp\Big(2 \int_0^t [M_{-1/2}(F_\tau)]^2 \mathrm{d}\tau\Big), \quad 0 \leq t \leq T.$$

By assumption on F_0 we have $\lim_{\delta\to 0^+} V_0(\delta) = 0$ and thus $\lim_{\delta\to 0^+} V_t(\delta) = 0$ for all $t \in [0, T]$. Since $T \in (0, T_{\max})$ is arbitrary, this proves that ν_t is absolutely continuous with respect to the Lebesgue measure for every $t \in [0, T_{\max})$, and thus there is a unique $0 \leq f(\cdot, t) \in L^1(\mathbb{R}_{\geq 0})$ such that $\frac{1}{\sqrt{x}}\mathrm{d}F_t(x) = \mathrm{d}\nu_t(x) = f(x, t)\mathrm{d}x$. That is, we have proved that F_t is regular for all $t \in [0, T_{\max})$ and its density $f(\cdot, t)$ belongs to $L^1(\mathbb{R}_+)$ for all $t \in [0, T_{\max})$.

Since $\|f(t)\|_{L^1} = M_{-1/2}(F_t) < \infty$ for all $t \in [0, T_{\max})$, it follows from (2.26) and (3.13) that

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}_+^3} w(x, y, z)[f'f'_*(1 + f + f_*)) + ff_*(1 + f' + f'_*)]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\leq 4 \sup_{0\leq t\leq T} \Big(M_{1/2}(f(t))\|f(t)\|_{L^1} + \|f(t)\|_{L^1}^3\Big) < \infty \quad \forall T \in (0, T_{\max}). \quad (4.1)$$

From this and that F_t is a strong solution of Eq. (1.5) on $[0, T_{\max})$ we conclude that the equation

$$\int_{\mathbb{R}_{\geq 0}} \varphi(x)\Big(f(x, t) - f_0(x) - \int_0^t Q(f)(x, \tau)\mathrm{d}\tau\Big)\sqrt{x}\,\mathrm{d}x$$

$$= \int_{\mathbb{R}_{\geq 0}} \varphi(x)\mathrm{d}\Big(F_t - F_0 - \int_0^t Q(F_\tau)\mathrm{d}\tau\Big)\mathrm{d}x = 0$$

holds for all $t \in [0, T_{\max})$ and all $\varphi \in L_0^\infty(\mathbb{R}_{\geq 0})$. Thus for any $t \in [0, T_{\max})$, there is a null set $Z_t \subset \mathbb{R}_{\geq 0}$ such that

$$f(x, t) = f_0(x) + \int_0^t Q(f)(x, \tau)\mathrm{d}\tau \quad \forall x \in \mathbb{R}_{\geq 0}\backslash Z_t.$$

Now we consider the nonnegative measurable function $\widetilde{f}(x, t) := |f_0(x) + \int_0^t Q(f)(x, \tau)\mathrm{d}\tau|$ on $\mathbb{R}_+ \times [0, T_{\max})$. We have $\widetilde{f}(x, 0) \equiv f_0(x)$ and, by nonnegativity of f, $\widetilde{f}(x, t) = f(x, t)$ for all $t \in [0, T_{\max})$ and all $x \in \mathbb{R}_+\backslash Z_t$. But the advantage of $\widetilde{f}$ is that there is a null set Z which is independent of t such that for every $x \in \mathbb{R}_+\backslash Z$ the function $t \mapsto \widetilde{f}(x, t)$ is continuous on $[0, T_{\max})$. Thus it follows from Fubini theorem that $\widetilde{f}(\cdot, t)$ is a mild solution of Eq. (1.5) on $[0, T_{\max})$. Again, since $\widetilde{f}(x, t) = f(x, t)$ for all $t \in [0, T_{\max})$ and all

$x \in \mathbb{R}_+ \setminus Z_t$, it follows that $\widetilde{f}(\cdot, t)$ is also the same density of F_t for $t \in [0, T_{\max})$. Thus by rewriting $\widetilde{f}(\cdot, t)$ as $f(\cdot, t)$ we conclude that the density $f(\cdot, t)$ of F_t is a mild solution of Eq. (1.5) on $[0, T_{F, \max})$.

Finally for any $T \in (0, T_{\max})$, let C_T be the left hand side of (4.1). Then we deduce from (4.1) and the definition of mild solutions that $\|f(\tau) - f(t)\|_{L^1} \le 2C_T |s - t|$ for all $s, t \in [0, T]$. Therefore $f \in C([0, T_{F, \max}); L^1(\mathbb{R}_+))$.   □

Proposition 4.2 *Let $0 \le f_0 \in L^1(\mathbb{R}_+)$ have finite mass and energy and let $\mathcal{T}(f_0) = \{0 < T \le \infty \mid Eq.$ (1.5) has a conservative mild solution $f(\cdot, t)$ on $[0, T)$ satisfying $f \in C([0, T); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0\}$, $T_{f_0, \max} = \sup \mathcal{T}(f_0)$. Then*

(a) *$\mathcal{T}(f_0)$ is non-empty, $T_{f_0, \max} \in \mathcal{T}(f_0)$, and there exists a unique conservative mild solution $f(\cdot, t)$ of Eq.(1.5) on $[0, T_{f_0, \max})$ satisfying $f \in C([0, T_{f_0, \max}); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$. Besides, if $T_{f_0, \max} < \infty$ then $\|f(t)\|_{L^1} \to \infty$ as $t \nearrow T_{f_0, \max}$.*

(b) *Let $F_0 \in \mathcal{B}^+_{-1/2, 1}(\mathbb{R}_{\ge 0})$ be regular with the density f_0. Then there exists a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ such that $T_{F, \max} = T_{f_0, \max}$ and F_t is regular on $[0, T_{f_0, \max})$ with the density $f(\cdot, t)$ obtained in part (a). And such an F_t is unique on $[0, T_{f_0, \max})$.*

(c) *Let $0 \le g_0 \in L^1(\mathbb{R}_+)$ have finite mass and energy and let $g(\cdot, t)$ be conservative mild solutions of Eq. (1.5) on $[0, T_{g_0, \max})$ satisfying $g \in C([0, T_{g_0, \max}); L^1(\mathbb{R}_+))$ and $g(0, \cdot) = g_0$. Then for any $T \in (0, T_{f_0, \max} \wedge T_{g_0, \max})$*

$$\|f(t) - g(t)\|_1 \le C \Psi_{f_0}(\|f_0 - g_0\|_1) e^{ct}, \quad \forall t \in [0, T] \tag{4.2}$$

where

$$\|h\|_1 := \int_{\mathbb{R}_+} (1 + x)|h(x)|\sqrt{x}\,dx,$$

$\Psi_{f_0}(\varepsilon) := \Psi_{F_0}(\varepsilon)$ is defined by (3.24) with F_0 defined by $dF_0(x) = f_0(x)\sqrt{x}\,dx$, and $C = C_T, c = c_T$ are finite positive constants depending only on $N(f_0)$, $E(f_0)$, $\sup_{0 \le t \le T} \|f(t)\|_{L^1}$, and $\sup_{0 \le t \le T} \|g(t)\|_{L^1}$.

(d) *Suppose $T_{f_0, \max} = \infty$ and let $f(\cdot, t)$ be the unique conservative mild solution of Eq. (1.5) on $[0, \infty)$ satisfying $f \in C([0, \infty); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$ obtained in part (a). Let $F_0, F_t \in \mathcal{B}^+_1(\mathbb{R}_{\ge 0})$ be regular measures with densities $f_0, f(\cdot, t)$ respectively. Then F_t is the unique conservative strong solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0.*

Proof (a)–(b): Let $F_0 \in \mathcal{B}^+_{-1/2, 1}(\mathbb{R}_{\ge 0})$ be regular with the density f_0, let F_t be a conservative distributional solution of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0, and let $T_{F, \max}$ be defined in (3.11). By Proposition 4.1, F_t is regular on $[0, T_{F, \max})$ and its density $f(\cdot, t)$ is a conservative mild solution of Eq. (1.5) on $[0, T_{F, \max})$ satisfying $f \in C([0, T_{F, \max}); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$. This implies $T_{F, \max} \in \mathcal{T}(f_0)$. We claim $T_{F, \max} = T_{f_0, \max}$. Otherwise, $T_{F, \max} < T_{f_0, \max}$, then there exists $T^* \in \mathcal{T}(f_0)$ such that $T^* > T_{F, \max}$ and Eq. (1.5) has a conservative mild solution $f^*(\cdot, t)$ of Eq. (1.5) on $[0, T^*)$ satisfying $f^*(\cdot, 0) = f_0$ and $f^* \in C([0, T^*); L^1(\mathbb{R}_+))$. Take $t_0 \in (T_{F, \max}, T^*)$ and let F_t^* be defined by $dF_t^*(x) = f^*(x, t)\sqrt{x}\,dx$, $t \in [0, t_0]$. Applying Theorem 1.1 to the initial datum $F_{t_0}^*$, there exists a conservative distributional solution F_t^{**} of Eq. (1.5) on $[t_0, \infty)$ with the initial datum $F_{t_0}^{**} = F_{t_0}^*$. Let $\widetilde{F}_t = F_t^*$ for $t \in [0, t_0]$, $\widetilde{F}_t = F_t^{**}$ for $t \in [t_0, \infty)$. Then $\widetilde{F}_t$ satisfies (i), (ii)′, (iii)′ in the equivalent definition of distributional solutions Eq. (1.5) proved in Remark 1.1. So $\widetilde{F}_t$ is a conservative distributional solution of Eq. (1.5) on $[0, \infty)$ with the

initial datum $\widetilde{F}_0 = F_0$. By Theorem 3.1 we conclude $T_{\widetilde{F},\max} = T_{F,\max}$. On the other hand, from $M_{-1/2}(\widetilde{F}_{t_0}) = M_{-1/2}(F_{t_0}^*) = \|f^*(t_0)\|_{L^1} < \infty$ we have $T_{\widetilde{F},\max} \geq t_0 > T_{F,\max}$. This contradiction proves $T_{F,\max} = T_{f_0,\max}$.

Next let $\tilde{f}(\cdot, t)$ be another conservative mild solution of Eq. (1.5) on $[0, T_{f_0,\max})$ satisfying $\tilde{f}(\cdot, 0) = f_0$ and $\tilde{f} \in C([0, T_{f_0,\max}); L^1(\mathbb{R}_+))$. Take any $T \in (0, T_{f_0,\max})$ and let $F_t^{(T)}$ be defined by $dF_t^{(T)}(x) = \tilde{f}(x, t)\sqrt{x}dx$, $t \in [0, T]$. As shown above, $F_t^{(T)}$ can be extended as a conservative distributional solution $\widetilde{F}_t$ of Eq. (1.5) on $[0, \infty)$ with the initial datum $\widetilde{F}_0 = F_0$. By Theorem 3.1 we conclude $T_{\widetilde{F},\max} = T_{F,\max} = T_{f_0,\max}$ and $\widetilde{F}_t = F_t$ for all $t \in [0, T_{f_0,\max})$. In particular, $F_t^{(T)} = \widetilde{F}_t = F_t$ for all $t \in [0, T]$ and so $\tilde{f}(\cdot, t) = f(\cdot, t)$ for all $t \in [0, T]$. Since T is arbitrary in $(0, T_{f_0,\max})$, we conclude $\tilde{f}(\cdot, t) = f(\cdot, t)$ for all $t \in [0, T_{f_0,\max})$. Thus $f(\cdot, t)$ is the unique conservative mild solution $f(\cdot, t)$ of Eq. (1.5) on $[0, T_{f_0,\max})$ satisfying $f(\cdot, 0) = f_0$ and $f \in C([0, T_{f_0,\max}); L^1(\mathbb{R}_+))$. The uniqueness of F_t on $[0, T_{f_0,\max})$ follows from Theorem 3.1.

Now assume $T_{f_0,\max} < \infty$. Then using Proposition 3.1 and $T_{F,\max} = T_{f_0,\max} < \infty$ we have $\|f(t)\|_{L^1} = M_{-1/2}(F_t) \to \infty$ as $t \nearrow T_{f_0,\max}$.

(c) Let G_t be the conservative distributional solutions of Eq. (1.5) on $[0, \infty)$ obtained in part (b) corresponding to $g(\cdot, t)$, i.e. G_t is regular on $[0, T_{g_0,\max})$ with the density $g(\cdot, t)$ for all $t \in [0, T_{g_0,\max})$. Since $T_{F,\max} = T_{f_0,\max}$, $T_{G,\max} = T_{g_0,\max}$, $\|f(t)\|_{L^1} = M_{-1/2}(F_t)$ for all $t \in [0, T_{f_0,\max})$, $\|g(t)\|_{L^1} = M_{-1/2}(G_t)$ for all $t \in [0, T_{g_0,\max})$, and $\|f(t) - g(t)\|_1 = \|F_t - G_t\|_1$ for all $t \in [0, T_{f_0,\max} \wedge T_{g_0,\max})$, the stability estimate (4.2) follows from Theorem 3.1.

(d) From $dF_t(x) = f(x, t)\sqrt{x}dx$ and $f \in C([0, \infty); L^1(\mathbb{R}_+))$, it is easily seen that $t \mapsto \mathcal{Q}_2^{\pm}(F_t)$, $t \mapsto \mathcal{Q}_3^{\pm}(F_t)$ belong to $C([0, \infty); \mathcal{B}_0(\mathbb{R}_{\geq 0}))$ and F_t satisfies the integral equation (3.3) for all $t \geq 0$ and thus F_t is a conservative distributional solution with the initial datum F_0. The conclusion of part (d) then follows from the equivalent definition of strong solutions (see Remark 3.1) and the uniqueness theorem (Theorem 3.1). □

As did for the classical Boltzmann equation, the collision integral $Q(f)$ can be decomposed as positive and negative parts:

$$Q(f)(x) = Q^+(f)(x) - Q^-(f)(x), \tag{4.3}$$

$$Q^+(f)(x) = \int_{\mathbb{R}_+^2} w(x, y, z)f(y)f(z)(1 + f(x_*))dydz, \tag{4.4}$$

$$Q^-(f)(x) = f(x)L(f)(x), \tag{4.5}$$

$$L(f)(x) = \int_{\mathbb{R}_+^2} w(x, y, z)[f(x_*)(1 + f(y) + f(z)) - f(y)f(z)]dydz. \tag{4.6}$$

Notice that, according to Lemma 2.2 and Lemma 2.3, for any $0 \leq f \in L^1(\mathbb{R}_+, \sqrt{x}dx)$, the function $x \mapsto L(f)(x)$ is well-defined, nonnegative on $\mathbb{R}_+$, and satisfies

$$M_{1/2}(f) \leq L(f)(x) \leq \sqrt{x}N(f) + M_{1/2}(f) + 2[M_{-1/2}(f)]^2. \tag{4.7}$$

The following proposition gives an exponential-positive representation (i.e. Duhamel's formula) for a class of mild solutions.

Proposition 4.3 *Let $0 \leq f_0 \in L^1(\mathbb{R}_+)$ have finite mass and energy and let $f \in C([0, T_{f_0,\max}); L^1(\mathbb{R}_+))$ be the unique conservative mild solution of Eq. (1.5) on $[0, T_{f_0,\max})$*

satisfying $f(\cdot, 0) = f_0$. *Then there is a null set* $Z \subset \mathbb{R}_+$ *such that for all* $x \in \mathbb{R}_+ \backslash Z$ *and all* $t \in [0, T_{f_0,\max})$

$$f(x, t) = f_0(x) e^{-\int_0^t L(f)(x, \tau) d\tau} + \int_0^t Q^+(f)(x, \tau) e^{-\int_\tau^t L(f)(x, s) ds} d\tau \qquad (4.8)$$

where $Q^+(f)$, $L(f)$ *are defined in* (4.4)–(4.6).

Proof By definition of mild solutions and $Q(f) = Q^+(f) - f L(f)$ there is a null set $Z \subset \mathbb{R}_+$ which is independent of t such that for every $x \in \mathbb{R}_+ \backslash Z$

$$\frac{\partial}{\partial t} f(x, t) = Q^+(f)(x, t) - f(x, t) L(f)(x, t) \quad \text{a.e.} \quad t \in [0, T_{f_0,\max}). \qquad (4.9)$$

Notice that for every $x > 0$ the nonnegative function $t \mapsto L(f)(x, t)$ is locally integrable on $[0, T_{f_0,\max})$. In fact applying (4.7) and $f \in C([0, T_{f_0,\max}); L^1(\mathbb{R}_+))$ we have

$$\sup_{t \in [0,T]} L(f)(x, t) \leq \sup_{t \in [0,T]} (\sqrt{x} N(f(t)) + M_{1/2}(f(t)) + 2\|f(t)\|_{L^1}^2) < \infty \qquad (4.10)$$

for all $T \in (0, T_{f_0,\max})$ and all $x > 0$. Therefore, for every $x \in \mathbb{R}_+ \backslash Z$, the function $t \mapsto f(x, t) e^{\int_0^t L(f)(x, \tau) d\tau}$ is also absolutely continuous on $[0, T]$ for all $T \in (0, T_{f_0,\max})$ and thus the Duhamel's formula (4.8) follows from the differential equation (4.9). $\qquad \square$

For bounded mild solutions we have the following

Proposition 4.4 *Let* $0 \leq f_0 \in L^1(\mathbb{R}_+)$ *have finite mass and energy and let* $f \in C([0, T_{f_0,\max}); L^1(\mathbb{R}_+))$ *be the unique conservative mild solution of Eq.* (1.5) *on* $[0, T_{f_0,\max})$ *satisfying* $f(\cdot, 0) = f_0$. *Suppose in addition* $f_0 \in L^\infty(\mathbb{R}_+)$. *Then* $f(\cdot, t) \in L^\infty(\mathbb{R}_+)$ *for all* $t \in [0, T_{f_0,\max})$ *and there hold two estimates: for all* $t \in [0, T_{f_0,\max})$,

$$\|f(t)\|_{L^\infty} \leq \max\{1, \|f_0\|_{L^\infty}\} \exp\left(2 \int_0^t \|f(\tau)\|_{L^1}^2 d\tau\right), \qquad (4.11)$$

$$\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty} \exp\left(\int_0^t [\|f(\tau)\|_{L^1}^2 - M_{1/2}(f(\tau))] d\tau\right)$$

$$+ \int_0^t \|f(\tau)\|_{L^1}^2 \exp\left(\int_\tau^t [\|f(s)\|_{L^1}^2 - M_{1/2}(f(s))] ds\right) d\tau. \qquad (4.12)$$

Besides, if $T_{f_0,\max} < \infty$ *then* $\|f(t)\|_{L^\infty} \to \infty$ *as* $t \nearrow T_{f_0,\max}$. $\qquad \square$

Proof Let $A(t)$ be the right hand side of (4.11) i.e.

$$A(t) := \max\{1, \|f_0\|_{L^\infty}\} e^{2 \int_0^t a(\tau) d\tau}, \quad a(t) := \|f(t)\|_{L^1}^2, \quad t \in [0, T_{f_0,\max}).$$

By definition of mild solutions and $f(x, 0) = f_0(x) \leq A(0)$ for all $x \in \mathbb{R}_+ \backslash Z$ (here and below $Z \subset \mathbb{R}_+$ denotes any null set which is independent of time variable) we have for all

$t \in [0, T_{f_0,\max})$

$$(f(x,t) - A(t))_+ = \int_0^t (Q(f)(x,\tau) - 2A(\tau)a(\tau))\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}d\tau.$$

Taking integration with respect to $x \in \mathbb{R}_+$ and omitting the negative part $Q^-(f) \geq 0$ gives

$$\int_{\mathbb{R}_+} (f(x,t) - A(t))_+ dx \leq \int_0^t d\tau \int_{\mathbb{R}_+} Q^+(f)(x,\tau)\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}dx$$

$$- \int_0^t d\tau \int_{\mathbb{R}_+} 2A(\tau)a(\tau)\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}dx.$$

Next for the integrand of $Q^+(f)(x,\tau)$, we have

$$f(y,\tau)f(z,\tau)(1 + f(x_*,\tau)) \leq f(y,\tau)f(z,\tau)(f(x_*,\tau) - A(\tau))_+ + 2A(\tau)f(y,\tau)f(z,\tau)$$

where we used $1 \leq A(\tau)$. Then applying the first inequality in (2.24) to $(f(x_*,\tau) - A(\tau))_+$ and recalling $a(\tau) = \|f(\tau)\|_{L^1}^2$ we deduce

$$\int_0^t d\tau \int_{\mathbb{R}_+} Q^+(f)(x,\tau)\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}dx \leq \int_0^t a(\tau)d\tau \int_{\mathbb{R}_+} (f(x,\tau) - A(\tau))_+ dx$$

$$+ \int_0^t d\tau \int_{\mathbb{R}_+} 2A(\tau)a(\tau)\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}dx.$$

Notice that the common terms $\int_0^t d\tau \int_{\mathbb{R}_+} 2A(\tau)a(\tau)\mathbf{1}_{\{f(x,\tau)>A(\tau)\}}dx$ in the right hand sides of the above successive inequalities cancel each other. It follows that for all $t \in [0, T_{\max})$

$$\int_{\mathbb{R}_+} (f(x,t) - A(t))_+ dx \leq \int_0^t a(\tau)d\tau \int_{\mathbb{R}_+} (f(x,\tau) - A(\tau))_+ dx.$$

By Gronwall Lemma we conclude $\int_{\mathbb{R}_+} (f(x,t) - A(t))_+ dx = 0$ for all $t \in [0, T_{\max})$. This implies $f(\cdot,t) \in L^\infty(\mathbb{R}_+)$ and $\|f(t)\|_{L^\infty} \leq A(t)$ for all $t \in [0, T_{\max})$, i.e. (4.11) holds true.

To prove (4.12) we denote $b(t) = M_{1/2}(f(t))$ and use the first inequality in (2.25) to get $L(f)(x,t) \geq b(t)$ for all $x > 0$. Then we deduce from Duhamel's formula (4.8) that

$$f(x,t) \leq f_0(x)e^{-\int_0^t b(\tau)d\tau} + \int_0^t a(\tau)(1 + \|f(\tau)\|_{L^\infty})e^{-\int_\tau^t b(s)ds}d\tau$$

for all $x \in \mathbb{R}_+ \backslash Z$ and all $t \in [0, T_{f_0,\max})$. This gives

$$\|f(t)\|_{L^\infty}e^{\int_0^t b(\tau)d\tau} \leq \|f_0\|_{L^\infty} + \int_0^t a(\tau)(1 + \|f(\tau)\|_{L^\infty})e^{\int_0^\tau b(s)ds}d\tau$$

for all $t \in [0, T_{f_0,\max})$ and thus (4.12) follows from Gronwall Lemma.

Finally by the conservation of mass we have $\|f(t)\|_{L^1} \leq 2[N(f_0)]^{2/3}[\|f(t)\|_{L^\infty}]^{1/3}$ for all $t \in [0, T_{f_0, \max})$ and so the L^∞-blow up follows from the L^1-blow up in part (a) of Proposition 4.2. $\qquad\square$

Now we turn to the global mild solutions. As explained at the end of Sect. 3, the following result belongs to the case of high temperature.

Theorem 4.1 *Let $0 \leq f_0 \in L^1(\mathbb{R}_+)$ have finite mass and energy and suppose*

$$\|f_0\|_{L^1} \leq \frac{1}{80}[N(f_0)E(f_0)]^{1/4}. \tag{4.13}$$

Then there exists a unique conservative mild solution $f(\cdot, t)$ of Eq. (1.5) on $[0, \infty)$ satisfying $f \in C([0, \infty); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$. Moreover we have

$$\sup_{t \geq 0} \|f(t)\|_{L^1} \leq \frac{1}{4}[N(f_0)E(f_0)]^{1/4}. \tag{4.14}$$

Besides, if in addition $f_0 \in L^\infty(\mathbb{R}_+)$, then $f(\cdot, t) \in L^\infty(\mathbb{R}_+)$ for all $t \geq 0$ and

$$\sup_{t \geq 0} \|f(t)\|_{L^\infty} \leq 2\max\{1, \|f_0\|_{L^\infty}\}. \tag{4.15}$$

Proof Let F_0 be defined by $dF_0(x) = f_0(x)\sqrt{x}dx$. Then (4.13) says that F_0 satisfies the condition (3.46) in Theorem 3.2 which insures that there exists a unique conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0. Since F_t is also a strong solution on $[0, \infty)$, and $T_{F., \max} = \infty$, by Proposition 4.1 and Proposition 4.2 we conclude that F_t is regular on $[0, \infty)$ and its density $f(\cdot, t)$ is the unique conservative mild solution of Eq. (1.5) on $[0, \infty)$ satisfying $f(\cdot, 0) = f_0$ and $f \in C([0, \infty); L^1(\mathbb{R}_+))$. Since $\|f(t)\|_{L^1} = M_{-1/2}(F_t)$, $N(F_0) = N(f_0)$, $E(F_0) = E(f_0)$, the estimate (4.14) follows from (3.46).

Let us denote $N = N(f_0)$, $E = E(f_0)$. By Theorem 1.1 and the uniqueness of F_t, F_t has the moment production (1.15). Thus from Lemma 3.6 and $M_{1/2}(f(t)) = M_{1/2}(F_t)$ we have for all $t > 0$

$$M_{1/2}(f(t)) \geq \frac{1}{6}\left[1 - \exp\left(-(\sqrt{NE} + N^3/E)t\right)\right]\frac{NE}{\sqrt{NE} + N^3/E}. \tag{4.16}$$

Now assume further that $f_0 \in L^\infty(\mathbb{R}_+)$. Then, from inequalities (4.14), (4.11) and $T_{f_0, \max} = \infty$ we have for all $t \geq 0$

$$\|f(t)\|_{L^1}^2 \leq \frac{1}{16}\sqrt{NE} =: a, \quad \|f(t)\|_{L^\infty} \leq \max\{1, \|f_0\|_{L^\infty}\}e^{2at}. \tag{4.17}$$

Let us choose t_0 satisfying

$$1 - \exp\left(-(\sqrt{NE} + N^3/E)t_0\right) = \frac{6}{7} \quad \text{i.e.} \quad t_0 = \frac{\log 7}{\sqrt{NE} + N^3/E}.$$

Then we deduce from (4.16) that

$$M_{1/2}(f(t)) \geq \frac{1}{7} \cdot \frac{NE}{\sqrt{NE} + N^3/E} =: b, \quad \forall t \geq t_0. \tag{4.18}$$

Applying (4.12) to the mild solution $t \mapsto f(\cdot, t + t_0)$ on $[0, \infty)$ with the initial datum $f(\cdot, t_0)$ and using the first inequality in (4.17) and (4.18) we have

$$\|f(t)\|_{L^\infty} \leq \|f(t_0)\|_{L^\infty} e^{(a-b)(t-t_0)} + a \int_{t_0}^{t} e^{(a-b)(t-\tau)} \mathrm{d}\tau, \quad t \geq t_0. \tag{4.19}$$

This gives

$$\|f(t)\|_{L^\infty} \leq \max \left\{ \|f(t_0)\|_{L^\infty}, \frac{a}{b-a} \right\}, \quad t \geq t_0 \tag{4.20}$$

provided $b > a$. But this is true because $b > 2a$. In fact by definition of a, b in (4.17), (4.18), it is easily checked that $b > 2a$ is equivalent to $\frac{E^{3/2}}{N^{5/2}} > 7$, and the latter is obvious since the assumption $M_{-1/2}(F_0) = \|f_0\|_{L^1} \leq \frac{1}{80}(NE)^{1/4}$ implies (3.55) and so $\frac{E^{3/2}}{N^{5/2}} \geq (80)^2 > 7$. Thus (4.20) holds true. From (4.20), the second inequality in (4.17) and $a/(b-a) < 1$ we obtain

$$\|f(t)\|_{L^\infty} \leq e^{2at_0} \max\{1, \|f_0\|_{L^\infty}\}, \quad t \geq 0.$$

Finally by definition of t_0 and a we have $e^{2at_0} \leq 7^{\frac{1}{8}} < 2$. This proves (4.15) and completes the proof of the theorem. $\qquad\square$

5 Regularity for Solutions with $f_0(x) \leq Kx^{-1}$

In this section we study regularity of such a distributional solution F_t whose initial datum F_0 is regular with density f_0 satisfying $f_0(x) \leq Kx^{-1}$ ($\forall x > 0$). It seems very difficult to prove any expected regularity without using suitable approximate solutions. This is why we introduced the second cutoff (2.31)–(2.32) and constructed approximate solutions. Theorem 5.1 below is based on the new positivity in Lemma 2.3 and the following two relevant lemmas.

Lemma 5.1 *Let f be a nonnegative measurable function on $\mathbb{R}_+$ satisfying $N(f) < \infty$. Then*

$$\int_{\mathbb{R}_+^2} w(x, y, z) f(y) f(z) \mathrm{d}y \mathrm{d}z \leq 2\sqrt{2}[N(f)]^2 x^{-1} \quad \forall x > 0. \tag{5.1}$$

Proof Since $w(x, y, z)$ is symmetric with respect to (y, z) and that $w(x, y, z) > 0$ implies $y + z > x$, it follows that

$$\int_{\mathbb{R}_+^2} w(x, y, z) f(y) f(z) \mathrm{d}y \mathrm{d}z \leq 2 \int_{0 < y \leq z, \, z \geq x/2} \frac{\sqrt{y}}{\sqrt{x}} f(y) f(z) \mathrm{d}y \mathrm{d}z$$

$$\leq \frac{2\sqrt{2}}{x} \int_{0 < y \leq z, \, z \geq x/2} \sqrt{y}\sqrt{z} f(y) f(z) \mathrm{d}y \mathrm{d}z \leq \frac{2\sqrt{2}}{x}[N(f)]^2.$$

$\qquad\square$

Lemma 5.2 *Let $5/6 < \alpha < 5/4$. Then, for all $x > 0$ we have*

$$I_\alpha(x) := \int_{\mathbb{R}_+^2} w(x, y, z) x_*^{-\alpha} y^{-\alpha} z^{-\alpha} \mathrm{d}y \mathrm{d}z = I_\alpha(1) x^{2-3\alpha} < \infty. \tag{5.2}$$

In particular for $\alpha = 1$ we have $I_1(1) < 19$ and

$$\int_{\mathbb{R}_+^2} w(x, y, z) x_*^{-1} y^{-1} z^{-1} \mathrm{d}y\mathrm{d}z = I_1(1)x^{-1}. \qquad (5.3)$$

Proof Fix $x > 0$. The equality (5.2) follows by changing variables $y = xu, z = xv$ which gives the constant

$$I_\alpha(1) = 2 \int_{u \geq v > 0, u+v > 1} \frac{\min\{1, \sqrt{u+v-1}, \sqrt{u}, \sqrt{v}\}}{(u+v-1)^\alpha u^\alpha v^\alpha} \mathrm{d}u\mathrm{d}v.$$

To prove $I_\alpha(1) < \infty$ we use $\min\{1, \sqrt{u+v-1}, \sqrt{u}, \sqrt{v}\} \leq (u+v-1)^{1/4} v^{1/4}$ and change variables $(u, v) = (r^2 \cos^2\theta, r^2 \sin^2\theta)$ $(r > 1, 0 < \theta < \pi/4)$ to get

$$I_\alpha(1) \leq 8 \left(\int_1^\infty \frac{\mathrm{d}r}{(r^2 - 1)^{\alpha-1/4} r^{4\alpha-7/2}} \right) \left(\int_0^{\pi/4} \frac{\mathrm{d}\theta}{\cos^{2\alpha-1}(\theta) \sin^{2\alpha-3/2}(\theta)} \right).$$

The two integrals are finite when $5/6 < \alpha < 5/4$. For $\alpha = 1$, by direct and careful calculation we have $I_1(1) = 8 \int_0^{\pi/2} \theta \sin(\theta)(1 + \cos^2(\theta))^{-1/2} \mathrm{d}\theta + 7\pi^2/6 < 19$. $\qquad \square$

The following simple lemma is also useful for proving Theorem 5.1.

Lemma 5.3 *Let $f_i, g_i \in [0, \infty), i = 1, 2, \ldots, n$ $(n \geq 2)$. Then*

$$\prod_{i=1}^n f_i \leq \prod_{i=1}^n f_i \wedge g_i + \sum_{i=1}^n \left(\prod_{1 \leq j \leq n, j \neq i} f_j \right)(f_i - g_i)_+. \qquad (5.4)$$

Proof Consider $\prod_{i=1}^n f_i - \prod_{i=1}^n f_i \wedge g_i$ and use $0 \leq f_i \wedge g_i \leq f_i$ and $f_i - f_i \wedge g_i = (f_i - g_i)_+$. The inequality follows easily by induction on the number n. $\qquad \square$

Theorem 5.1 *Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ be a regular measure with density f_0 satisfying*

$$f_0(x) \leq Kx^{-1} \quad \forall x \in \mathbb{R}_+$$

for some $0 < K < \infty$. Let $T_K = (6[N(F_0)]^2 K^{-1} + 40K^2)^{-1}$. Then there exists a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 such that $F_t|_{[0,T_K]}$ is a regular strong solution on $[0, T_K]$ and its density $f(x, t)$ satisfies

$$f(x, t) \leq 5Kx^{-1} \quad \forall (x, t) \in \mathbb{R}_+ \times [0, T_K]. \qquad (5.5)$$

In particular F_t has no condensation for all $t \in [0, T_K]$.

Proof Step 1: Let f^n with $f^n|_{t=0} = f_0$ be conservative mild solutions of the approximate equations constructed in Sect. 2 (see (2.27) (2.28)) with $w_n(x, y, z) = \frac{1}{\sqrt{x}} S_n(x, x_*; y, z)$, where $S_n(x, x_*; y, z)$ are taken as the second cutoff (2.31). Let $C_1 = I_1(1)$ be given in Lemma 5.2 and let $C_0 = 2\sqrt{2}[N(F_0)]^2, \lambda = C_0 K^{-1} + C_1 K^2$, and

$$\Phi(x, t) = (1 - 2\lambda t)^{-1/2} Kx^{-1}, \quad t \in [0, T_K].$$

In this step we prove that $f^n(x, t) \leq \Phi(x, t)$ for all $(x, t) \in \mathbb{R}_+ \times [0, T_K]$ and all $n \geq 1$.

First of all by computing differences of coefficients $6 - 4\sqrt{2}$, $40 - 2I_1(1)(> 2)$ etc., we have $2\lambda T_K < 1$ and $(1 - 2\lambda T_K)^{-1/2} < 5$. This implies $\Phi(x, t) \leq 5Kx^{-1}$ for all $x > 0, t \in [0, T_K]$. Next using the absolute continuity of $t \mapsto f^n(x, t)$ and $f^n(x, 0) = f_0(x) \leq \Phi(x, 0)$ we have, for almost every $x \in \mathbb{R}_+$ and for all $t \in [0, T_K]$,

$$\left(f^n(x, t) - \Phi(x, t)\right)_+ = \int_0^t (Q_n(f^n)(x, \tau) - \partial_\tau \Phi(x, \tau))\mathbf{1}_{\{f^n(x,\tau) > \Phi(x,\tau)\}}d\tau. \quad (5.6)$$

Now consider decomposition $Q_n(f) = Q_n^+(f) - fL_n(f)$ as given in (4.3)–(4.6) by replacing $w(x, y, z)$ with $w_n(x, y, z)$. Thanks to Lemma 2.3 we have $L_n(f^n)(x, \tau) \geq 0$ hence $Q_n(f^n)(x, \tau) \leq Q_n^+(f^n)(x, \tau)$. Multiplying $\sqrt{x}$ to both sides of (5.6) and taking integration over $\mathbb{R}_+$ we deduce

$$N((f^n(t) - \Phi(t))_+) \leq \int_0^t d\tau \int_{\mathbb{R}_+} \sqrt{x}(Q_n^+(f^n)(x, \tau) - \partial_\tau \Phi(x, \tau))\mathbf{1}_{\{f^n(x,\tau) > \Phi(x,\tau)\}}dx$$

for all $t \in [0, T_K]$. Notice that there is no problem of integrability because, for all $t \in [0, T_K]$,

$$\partial_t \Phi(x, t) = \lambda(1 - 2\lambda t)^{-3/2}Kx^{-1},$$
$$Kx^{-1}\mathbf{1}_{\{f^n(x,t) > \Phi(x,t)\}} \leq \Phi(x, t)\mathbf{1}_{\{f^n(x,t) > \Phi(x,t)\}} \leq f^n(x, t),$$

and $(1 - 2\lambda t)^{-3/2} \leq (1 - 2\lambda T_K)^{-3/2} < 5^3$. For convenience of derivation, let us use the notation (1.7), i.e. $f^n = f^n(x, \tau)$, $f_*^n = f^n(x_*, \tau)$, $f^{n'} = f^n(y, \tau)$, $f_*^{n'} = f^n(z, \tau)$, etc. For the quadratic term of $Q_n^+(f^n)(x, \tau)$ we use Lemma 5.1 and $w_n(x, y, z) \leq w(x, y, z)$ to get

$$\int_{\mathbb{R}_+^2} w_n(x, y, z)f^{n'}f_*^{n'}dydz \leq C_0 x^{-1}$$

where we have used the conservation of mass and $N(f_0) = N(F_0)$. For the cubic term of $Q_n^+(f^n)(x, \tau)$ we use Lemma 5.3 to get

$$\int_{\mathbb{R}_+^2} w_n(x, y, z)f^{n'}f_*^{n'}f_*^n dydz \leq \int_{\mathbb{R}_+^2} w_n(x, y, z)\Phi'\Phi_*'\Phi_* dydz$$
$$+ \int_{\mathbb{R}_+^2} w_n(x, y, z)\big[f_*^{n'}f_*^n(f^{n'} - \Phi')_+$$
$$+ f^{n'}f_*^n(f_*^{n'} - \Phi_*')_+ + f^{n'}f_*^{n'}(f_*^n - \Phi_*)_+\big]dydz$$

and using Lemma 5.2 with $\alpha = 1$ we have (since $w_n(x, y, z) \leq w(x, y, z)$)

$$\int_{\mathbb{R}_+^2} w_n(x, y, z)\Phi'\Phi_*'\Phi_* dydz \leq (1 - 2\lambda \tau)^{-3/2}K^3C_1x^{-1}, \quad x > 0.$$

Combining these with the inequalities (2.37) for cubic terms we deduce

$$\int_{\mathbb{R}_+} \sqrt{x}(Q_n^+(f^n)(x,\tau) - \partial_\tau \Phi(x,\tau))\mathbf{1}_{\{f^n(x,\tau)>\Phi(x,\tau)\}}dx$$

$$\leq \int_{\mathbb{R}_+^3} S_n(x,x_*;y,z)[f_*^{n'}f_*^n(f^{n'}-\Phi')_+ + f^{n'}f_*^n(f_*^{n'}-\Phi_*')_+ + f^{n'}f_*^{n'}(f_*^n-\Phi_*)_+]dxdydz$$

$$+ \int_{\mathbb{R}_+} \sqrt{x}[C_0 x^{-1} + C_1(1-2\lambda\tau)^{-3/2}K^3 x^{-1} - \lambda(1-2\lambda\tau)^{-3/2}Kx^{-1}]\mathbf{1}_{\{f^n(x,\tau)>\Phi(x,\tau)\}}dx$$

$$\leq a_n N((f^n(\tau)-\Phi(\tau))_+) + b(\tau)$$

where $a_n = 3n[N(F_0)]^2$,

$$b(\tau) = \int_{\mathbb{R}_+} \sqrt{x}[C_0 K^{-1} - (\lambda - C_1 K^2)(1-2\lambda\tau)^{-3/2}]Kx^{-1}\mathbf{1}_{\{f^n(x,\tau)>\Phi(x,\tau)\}}dx.$$

Now by our choice for λ we have

$$C_0 K^{-1} - (\lambda - C_1 K^2)(1-2\lambda\tau)^{-3/2} = C_0 K^{-1} - C_0 K^{-1}(1-2\lambda\tau)^{-3/2} \leq 0$$

and so $b(\tau) \leq 0$ for all $\tau \in [0, T_K]$. Thus

$$N((f^n(t) - \Phi(t))_+) \leq a_n \int_0^t N((f^n(\tau)-\Phi(\tau))_+)d\tau \qquad \forall t \in [0, T_K].$$

By Gronwall Lemma we conclude $N((f^n(t) - \Phi(t))_+) = 0$ for all $t \in [0, T_K]$. Thus $f^n(x,t) \leq \Phi(x,t)$ for all $t \in [0, T_K]$ and for a.e. $x \in \mathbb{R}_+$ and so the function $f^n \wedge \Phi$ is the same (up to a null set in $\mathbb{R}_+$) mild solution restricted on $\mathbb{R}_+ \times [0, T_K]$ with the initial datum f_0. If we rewrite $f^n \wedge \Phi$ as f^n, then the mild solution f^n satisfies $f^n \leq \Phi$ on $\mathbb{R}_+ \times [0, T_K]$. In particular we have

$$f^n(x,t) \leq 5Kx^{-1} \quad \forall(x,t) \in \mathbb{R}_+ \times [0, T_K], \quad \forall n \geq 1. \tag{5.7}$$

Step 2: In this step we prove that a subsequence of $\{f^n\}_{n=1}^\infty$ (restricted on $\mathbb{R}_+ \times [0, T_K]$) converges in L^1-weak topology to a density of a strong solution F_t on $[0, T_K]$ with the initial datum F_0. To shorten notations we define $F_t^n \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ by

$$dF_t^n(x) = f^n(x,t)\sqrt{x}dx, \quad t \in [0, T_K]$$

and recall that $dF_0(x) = f_0(x)\sqrt{x}dx$. By conservation of mass and energy we have

$$\int_{\mathbb{R}_+} (1+x)dF_t^n(x) \equiv N(F_0) + E(F_0), \qquad t \in [0, T_K], \ n \geq 1. \tag{5.8}$$

From (5.7) and $\int_E x^{-1/2}dx \leq \int_0^{\text{mes}(E)} x^{-1/2}dx = 2\sqrt{\text{mes}(E)}$ for every measurable set $E \subset \mathbb{R}_+$, we have

$$\sup_{n\geq 1, t\in[0,T_K]} \int_E dF_t^n(x) \leq 10K\sqrt{\text{mes}(E)} \to 0 \quad \text{as} \quad \text{mes}(E) \to 0. \tag{5.9}$$

Also we have for all $0 \leq \alpha < 1/2$

$$\sup_{n \geq 1, t \in [0, T_K]} \int_{\mathbb{R}_+} x^{-\alpha} dF_t^n(x) \leq 5K \int_0^1 x^{-1/2-\alpha} dx + N(F_0) < \infty.$$

Taking $\alpha = 1/3$ and using $W_n(x, y, z) \leq W(x, y, z)$ and

$$\int_{\mathbb{R}_{\geq 0}} \sqrt{x} W(x, y, z) dx \leq \sqrt{y} + \sqrt{z}, \quad W(x, y, z) \leq (xyz)^{-1/3} \tag{5.10}$$

we obtain for all $\psi \in L^\infty(\mathbb{R}_{\geq 0})$

$$\sup_{n \geq 1, t \in [0, T_K]} \left(\int_{\mathbb{R}_+^2} |\mathcal{J}_n^\pm[\psi]| d^2 F_t^n + \int_{\mathbb{R}_+^3} |\mathcal{K}_n^\pm[\psi]| d^3 F_t^n \right) \leq C \|\psi\|_{L^\infty} \tag{5.11}$$

here and below C denotes any constant that depends only on $N(F_0)$, $E(F_0)$ and K. Next using the weak formula (2.40) we have, for all $0 \leq s < t \leq T_K$ and all $\psi \in L^\infty(\mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \psi d(F_t^n - F_s^n) = \int_s^t d\tau \int_{\mathbb{R}_+^2} \mathcal{J}_n[\psi] d^2 F_\tau^n + \int_s^t d\tau \int_{\mathbb{R}_+^3} \mathcal{K}_n[\psi] d^3 F_\tau^n \tag{5.12}$$

which together with (5.11) gives the uniform strong continuity on $[0, T_K]$:

$$\sup_{n \geq 1} \|F_t^n - F_s^n\|_0 \leq C|t - s| \quad \forall t, s \in [0, T_K]. \tag{5.13}$$

From (5.8), (5.9), (5.13) and the criterion of L^1-weakly relative compactness we conclude that there exist a subsequence of $\{F_t^n\}_{n=1}^\infty$, still denote it as $\{F_t^n\}_{n=1}^\infty$, and a positive regular Borel measure $F_t \in C([0, T_K]; \mathcal{B}_0(\mathbb{R}_+))$ such that for all $t \in [0, T_K]$

$$\lim_{n \to \infty} \int_{\mathbb{R}_+} \psi dF_t^n = \int_{\mathbb{R}_+} \psi dF_t \quad \forall \psi \in L^\infty(\mathbb{R}_+). \tag{5.14}$$

Let $f(\cdot, t)$ be the density of F_t. From (5.14), (5.7) we have $f(x, t) \leq \Phi(x, t)$ for all $t \in [0, T_K]$ and for a.e. $x \in \mathbb{R}_+$. Let $f(x, t)$ be replaced by $f(x, t) \wedge \Phi(x, t)$ which we still denote as $f(x, t)$. Then $f(\cdot, t)$ is the same density of F_t for all $t \in [0, T_K]$ and satisfies (5.5). From (5.14) and the conservation of mass and energy for F_t^n we also have

$$N(F_t) = N(F_0), \quad E(F_t) \leq E(F_0), \quad t \in [0, T_K]. \tag{5.15}$$

From these and $f^n(x, t)\sqrt{x} \leq 5Kx^{-1/2}$, $f(x, t)\sqrt{x} \leq 5Kx^{-1/2}$ on $\mathbb{R}_+ \times [0, T_K]$ and (5.14) we see that the convergence (5.14) can be extended as follows: for all $t \in [0, T_K]$ and all $\psi \in L^\infty(\mathbb{R}_+)$

$$\lim_{n \to \infty} \int_{\mathbb{R}_+} \psi(x)(1 + \sqrt{x}) dF_t^n(x) = \int_{\mathbb{R}_+} \psi(x)(1 + \sqrt{x}) dF_t(x), \tag{5.16}$$

$$\lim_{n \to \infty} \int_{\mathbb{R}_+} \psi(x) x^{-1/3} dF_t^n(x) = \int_{\mathbb{R}_+} \psi(x) x^{-1/3} dF_t(x). \tag{5.17}$$

Also from the pointwise convergence (2.34) and (5.10) we have for all $(x, y, z) \in \mathbb{R}_{>0}^3$

$$\lim_{n \to \infty} \mathcal{J}_n^\pm[\psi](y, z) = \mathcal{J}^\pm[\psi](y, z), \quad \lim_{n \to \infty} \mathcal{K}_n^\pm[\psi](x, y, z) = \mathcal{K}^\pm[\psi](x, y, z);$$

$$\sup_{n \geq 1} |\mathcal{J}_n^\pm[\psi](y, z)|, |\mathcal{J}^\pm[\psi](y, z)| \leq \|\psi\|_{L^\infty}(\sqrt{y} + \sqrt{z}),$$

$$\sup_{n \geq 1} |\mathcal{K}_n^\pm[\psi](x, y, z)|, |\mathcal{K}^\pm[\psi](x, y, z)| \leq 2\|\psi\|_{L^\infty}(xyz)^{-1/3}.$$

Thus we deduce from elementary convergence properties of integrals with product measures (see e.g. Lemma 4 in [12] and its application in the same paper) that

$$\lim_{n \to \infty} \int_{\mathbb{R}_+^2} \mathcal{J}_n^\pm[\psi] \mathrm{d}^2 F_t^n = \int_{\mathbb{R}_+^2} \mathcal{J}^\pm[\psi] \mathrm{d}^2 F_t, \quad \lim_{n \to \infty} \int_{\mathbb{R}_+^3} \mathcal{K}_n^\pm[\psi] \mathrm{d}^3 F_t^n = \int_{\mathbb{R}_+^3} \mathcal{K}^\pm[\psi] \mathrm{d}^3 F_t$$

for all $t \in [0, T_K]$. From these and the bound (5.11) which also holds for $\int_{\mathbb{R}_{\geq 0}^2} |\mathcal{J}^\pm[\psi]| \mathrm{d}^2 F_t + \int_{\mathbb{R}_{\geq 0}^3} |\mathcal{K}^\pm[\psi]| \mathrm{d}^3 F_t$ we conclude that for all $t \in [0, T_K]$ and all $\psi \in L^\infty(\mathbb{R}_+)$

$$\int_{\mathbb{R}_{\geq 0}} \psi \mathrm{d} F_t = \int_{\mathbb{R}_{\geq 0}} \psi \mathrm{d} F_0 + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}_{\geq 0}^2} \mathcal{J}[\psi] \mathrm{d}^2 F_\tau + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\psi] \mathrm{d}^3 F_\tau. \quad (5.18)$$

Thus as shown above we have $\|F_t - F_s\|_0 \leq C|t - s|$ for all $t, s \in [0, T_K]$. And since $x^{-1/3} f(x, t)\sqrt{x} \leq 5Kx^{-5/6}$, it follows that $\sup_{t \in [0, T_K]} M_{-1/3}(F_t) < \infty$ and

$$M_{-1/3}(|F_t - F_s|) \leq C\|F_t - F_s\|_0^{1/3} \leq C|t - s|^{1/3}, \quad s, t \in [0, T_K]$$

from which and the basic estimates of collision integrals (2.19) (with $k = 0$) and (2.20) we see that $t \mapsto \mathcal{Q}_2^\pm(F_t), t \mapsto \mathcal{Q}_3^\pm(F_t)$ belong to $C([0, T_K]; \mathcal{B}_0(\mathbb{R}_{\geq 0}))$. These together with (5.18) imply that the dual form (3.3) holds true. Thus we conclude from the equivalent definition of strong solutions showed in Remark 3.1 that F_t is a strong solution of Eq. (1.5) on $[0, T_K]$.

Step 3 (extension): Taking F_{T_K} as an initial datum, according to Theorem 1.1, there exists a conservative distributional F_t of Eq. (1.5) on $[T_K, \infty)$ such that $F_t|_{t=T_K} = F_{T_K}$. As before, it is easily seen that the measure F_t defined for all $t \in [0, \infty)$ in that way is a distributional solution of Eq. (1.5) on $[0, \infty)$. And from (5.15) we have $E(F_t) \leq E(F_0)$ for all $t \in [0, \infty)$ and so it follows from Theorem 1.1(b) that F_t conserves also the energy. Thus F_t is a desired solution claimed in the theorem. □

6 Condensation in Finite Time

As mentioned in the Introduction, our strategy for investigating the problem of condensation in finite time is to assume to the contrary that the distributional solution under consideration has no condensation at a finite time, then derive some necessary condition on the initial datum.

Let $\underline{D}_\alpha(F), \overline{D}_\alpha(F)$ be defined in (1.19) for $F \in \mathcal{B}^+(\mathbb{R}_{\geq 0})$. As did in [15], in order to connect Eq. (1.8), we use a smooth version of $\underline{D}_\alpha(F), \overline{D}_\alpha(F)$. For $F \in \mathcal{B}^+(\mathbb{R}_{\geq 0}), \alpha \geq 0$,

and $\varepsilon > 0$, let

$$N_\alpha(F, \varepsilon) = \frac{1}{\varepsilon^\alpha} N_0(F, \varepsilon), \qquad N_0(F, \varepsilon) = \int\limits_{[0,\varepsilon]} \left(1 - \frac{x}{\varepsilon}\right)^2 dF(x),$$

$$\underline{N}_\alpha(F, \varepsilon) = \inf_{0<\delta\leq\varepsilon} N_\alpha(F, \delta), \quad \overline{N}_\alpha(F, \varepsilon) = \sup_{0<\delta\leq\varepsilon} N_\alpha(F, \delta),$$

$$\underline{N}_\alpha(F) = \lim_{\varepsilon\to 0^+} \underline{N}_\alpha(F, \varepsilon), \qquad \overline{N}_\alpha(F) = \lim_{\varepsilon\to 0^+} \overline{N}_\alpha(F, \varepsilon),$$

$$A_\alpha(F, \varepsilon) = \frac{1}{\varepsilon^\alpha} \int\limits_{[0,\varepsilon]} \left(\frac{x}{\varepsilon}\right)^2 dF(x).$$

It is easily seen that $\underline{N}_\alpha(F), \overline{N}_\alpha(F)$ are equivalent to $\underline{D}_\alpha(F), \overline{D}_\alpha(F)$ respectively. For instance for the most interesting case $\alpha = 1/2$ we have

$$\underline{D}_{1/2}(F) \leq (5/4)^2\sqrt{5}\,\underline{N}_{1/2}(F), \quad \overline{D}_{1/2}(F) \leq (5/4)^2\sqrt{5}\,\overline{N}_{1/2}(F). \tag{6.1}$$

In fact for any $r > 1$ and any $\varepsilon > 0$

$$N_{1/2}(F, r\varepsilon) \geq \frac{1}{\sqrt{r\varepsilon}} \int\limits_{[0,\varepsilon]} \left(1 - \frac{x}{r\varepsilon}\right)^2 dF(x) \geq \frac{1}{\sqrt{r}}\left(1 - \frac{1}{r}\right)^2 \frac{F([0,\varepsilon])}{\sqrt{\varepsilon}}.$$

Choosing $r = 5$ gives the best constant in (6.1).

Notice that the integrand $\varphi_\varepsilon(x) = [(1 - x/\varepsilon)_+]^2 = (1 - x/\varepsilon)^2 \mathbf{1}_{[0,\varepsilon]}(x)$ defining $N_\alpha(F, \varepsilon)$ is convex and belongs to $C^{1,1}(\mathbb{R}_{\geq 0})$. In the following, φ_ε always stands for this special function.

Lemma 6.1 ([15]) *Let* $F \in \mathcal{B}^+(\mathbb{R}_{\geq 0})$, $\alpha \geq 0$, $\varepsilon > 0$. *Then*

$$\underline{N}_\alpha(F, \varepsilon)[A_{\frac{1-\alpha}{2}}(F, \varepsilon)]^2 \leq \int\limits_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon]d^3 F \quad for \quad 0 \leq \alpha \leq 1/2 \tag{6.2}$$

Also we have

$$F(\{0\}) = 0 \implies \sqrt{N_\alpha(F, \varepsilon)} \leq \int\limits_0^1 \theta^{\alpha/2-1}\sqrt{A_\alpha(F, \varepsilon\theta)}\,d\theta. \tag{6.3}$$

Note To check (6.3) with [15] one may replace the closed interval $[0, \varepsilon]$ by $(0, \varepsilon]$ and define $N_\alpha^\circ(F, \varepsilon) = \frac{1}{\varepsilon^\alpha} \int\limits_{(0,\varepsilon]} (1 - x/\varepsilon)^2 dF(x)$ as used in [15]. Then $F(\{0\}) = 0 \implies N_\alpha(F, \varepsilon) \equiv N_\alpha^\circ(F, \varepsilon)$.

The following proposition provides a necessary condition on the initial data for the absence of condensation at a finite time.

Proposition 6.1 *Let* F_t *be a distributional solution of Eq.* (1.5) *with the initial datum* F_0 *and suppose there is* $T \in (0, \infty)$ *such that* $F_T(\{0\}) = 0$. *Then*

$$\sqrt{\underline{D}_{1/2}(F_0)\overline{D}_{1/2}(F_0)} \leq \frac{50\sqrt{5}}{\sqrt{T}} \exp\left(\frac{2}{3}\sqrt{N(F_0)E(F_T)}\,T\right). \tag{6.4}$$

Proof Applying (6.1) to $F = F_0$ we see that to prove (6.4) it suffices to prove

$$\sqrt{\underline{N}_{1/2}(F_0)\overline{N}_{1/2}(F_0)} \leq \frac{2^5}{\sqrt{T}} \exp\left(\frac{2}{3}\sqrt{N(F_0)E(F_T)}\,T\right). \tag{6.5}$$

Define for $t \geq 0$, $\alpha \geq 0$ and $\varepsilon > 0$

$$m(t) = \exp\left(\int_0^t M_{1/2}(F_\tau)d\tau\right), \qquad N_\alpha^*(t, \varepsilon) = m(t)N_\alpha(F_t, \varepsilon),$$

$$\underline{N}_\alpha^*(t, \varepsilon) = m(t)\underline{N}_\alpha(F_t, \varepsilon) = \inf_{0<\delta\leq\varepsilon} N_\alpha^*(t, \delta), \quad \overline{N}_\alpha^*(t, \varepsilon) = m(t)\overline{N}_\alpha(F_t, \varepsilon) = \sup_{0<\delta\leq\varepsilon} N_\alpha^*(t, \delta).$$

Notice that since all $\overline{N}_\alpha^*(t, \varepsilon)$ are nonnegative, the following estimates involving these terms make sense even if possibly $\overline{N}_\alpha^*(t, \varepsilon) = \infty$ for large α.

Step 1: We prove that for any $\varepsilon > 0$ and any $\alpha \geq 0$

$$N_\alpha^*(t, \varepsilon) \geq N_\alpha^*(s, \varepsilon) + \frac{1}{\varepsilon^\alpha}\int_s^t m(\tau)d\tau \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon]d^3 F_\tau, \quad 0 \leq s < t. \tag{6.6}$$

Using Eq. (1.8), Lemma 3.1 and recalling $\int_{\mathbb{R}_{\geq 0}^3} \varphi_\varepsilon dF_t = N_0(F_t, \varepsilon)$ we have

$$\frac{\partial}{\partial t} N_0(F_t, \varepsilon) \geq -M_{1/2}(F_t)N_0(F_t, \varepsilon) + \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon]d^3 F_t, \quad \forall t \geq 0$$

and so by definition of $m(t)$ and $N_0^*(t, \varepsilon) = m(t)N_0(F_t, \varepsilon)$,

$$\frac{\partial}{\partial t} N_0^*(t, \varepsilon) \geq m(t) \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon]d^3 F_t, \quad \forall t \geq 0.$$

Since $N_\alpha^*(t, \varepsilon) = \frac{1}{\varepsilon^\alpha}N_0^*(t, \varepsilon)$, this gives (6.6).

By the way, applying (6.6) to $\alpha = 0$ and letting $\varepsilon \to 0^+$ leads to a weak version of (1.18): $m(t)F_t(\{0\}) \geq m(s)F_s(\{0\})$ for $0 \leq s < t$. In particular we see that the assumption $F_T(\{0\}) = 0$ implies $F_t(\{0\}) = 0$ for all $t \in [0, T]$.

Step 2: To prove (6.5) we can assume $\underline{N}_{1/2}(F_0) > 0$. Let us define for any $\lambda \in (1/2, 1)$

$$D_\lambda = \lambda\underline{N}_{1/2}(F_0) \quad \text{if} \quad \underline{N}_{1/2}(F_0) < \infty; \quad D_\lambda = 1 \quad \text{if} \quad \underline{N}_{1/2}(F_0) = \infty.$$

Here the case "$\underline{N}_{1/2}(F_0) = \infty$" is just considered in logic; if (6.5) is true, we must have $\underline{N}_{1/2}(F_0) < \infty$. By definitions of $\underline{N}_{1/2}(F_0, \varepsilon)$ and D_λ, there is $\varepsilon_\lambda > 0$ such that

$$\underline{N}_{1/2}(F_0, \varepsilon) > D_\lambda \qquad \forall \varepsilon \in (0, \varepsilon_\lambda].$$

We now prove that for any $\alpha \geq 0$ and any $\varepsilon \in (0, \varepsilon_\lambda]$

$$\overline{N}_{\frac{1}{4}+\frac{\alpha}{2}}^*(s, \varepsilon) \leq \frac{2^4}{\sqrt{D_\lambda}} \cdot \frac{e^{ct}}{\sqrt{t-s}}\left(\frac{1}{2}+\alpha\right)^{-2}\sqrt{N_\alpha^*(t, \varepsilon)} \quad \forall 0 \leq s < t \leq T. \tag{6.7}$$

where $c = \sqrt{N(F_0)E(F_T)}$.

Applying (6.6) to $\alpha = 1/2$ and $s = 0$ we have $N_{1/2}^*(t, \varepsilon) \geq N_{1/2}^*(0, \varepsilon)$ for all $t \geq 0$ and so

$$\underline{N}_{1/2}^*(t, \varepsilon) \geq \underline{N}_{1/2}^*(0, \varepsilon) = \underline{N}_{1/2}(F_0, \varepsilon) > D_\lambda \qquad \forall t \geq 0, \ \forall \varepsilon \in (0, \varepsilon_\lambda]. \tag{6.8}$$

Given any $\alpha \geq 0$ and $0 \leq s < t \leq T$. Using (6.6) we have (omitting $N^*_\alpha(s, \varepsilon)$)

$$\frac{1}{\varepsilon^\alpha} \int_s^t m(\tau)d\tau \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi_\varepsilon]d^3 F_\tau \leq N^*_\alpha(t, \varepsilon) \qquad \forall \varepsilon \in (0, \varepsilon_\lambda]. \tag{6.9}$$

While by Lemma 6.1 we have

$$\underline{N}_{1/2}(F_\tau, \varepsilon)[A_{\frac{1}{4}}(F_\tau, \varepsilon)]^2 \leq \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi_\varepsilon]d^3 F_\tau. \tag{6.10}$$

Multiplying $\frac{1}{\varepsilon^\alpha} m(\tau)$ to both sides of (6.10) and recalling definition of $\underline{N}^*_\alpha(t, \varepsilon)$ gives

$$\underline{N}^*_{1/2}(\tau, \varepsilon)\left[A_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon)\right]^2 \leq \frac{1}{\varepsilon^\alpha} m(\tau) \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi_\varepsilon]d^3 F_\tau, \quad \tau \in [s, t].$$

Then taking integration $\int_s^t$ and using (6.9) and the lower bound (6.8) of $\underline{N}^*_{1/2}(\tau, \varepsilon)$ we obtain

$$\int_s^t [A_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon)]^2 d\tau \leq \frac{1}{D_\lambda} N^*_\alpha(t, \varepsilon) \qquad \forall \varepsilon \in (0, \varepsilon_\lambda]. \tag{6.11}$$

On the other hand, using $F_t(\{0\}) \equiv 0$ on $[0, T]$ and Lemma 6.1 we have

$$\sqrt{N_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon)} \leq \int_0^1 \theta^{\frac{1}{8}+\frac{\alpha}{4}-1}\sqrt{A_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon\theta)}\, d\theta \qquad \forall \tau \in [s, t].$$

Multiplying $\sqrt{m(\tau)}$ to both sides and noticing that $m(\tau) \leq m(t)$ we have

$$\sqrt{N^*_{\frac{1}{4}+\frac{\alpha}{2}}(\tau, \varepsilon)} \leq \sqrt{m(t)} \int_0^1 \theta^{\frac{1}{8}+\frac{\alpha}{4}-1}\sqrt{A_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon\theta)}\, d\theta \qquad \forall \tau \in [s, t]. \tag{6.12}$$

By conservation of mass $N(F_\tau) \equiv N(F_0)$ and the non-decrease of the energy $\tau \mapsto E(F_\tau)$ on $[0, \infty)$ we have $M_{1/2}(F_\tau) \leq \sqrt{N(F_0)E(F_T)} =: c$ for all $\tau \in [0, T]$ and so $m(t) \leq e^{ct}$. By taking square of both sides of (6.12) and using Cauchy–Schwarz inequality twice together with (6.11) we deduce

$$\int_s^t N^*_{\frac{1}{4}+\frac{\alpha}{2}}(\tau, \varepsilon)d\tau \leq \frac{e^{ct}}{\frac{1}{8}+\frac{\alpha}{4}} \int_0^1 \theta^{\frac{1}{8}+\frac{\alpha}{4}-1}\sqrt{t-s}\sqrt{\int_s^t \left[A_{\frac{1}{4}+\frac{\alpha}{2}}(F_\tau, \varepsilon\theta)\right]^2 d\tau}\, d\theta$$

$$\leq \frac{e^{ct}}{\sqrt{D_\lambda}}\sqrt{t-s}\left(\frac{1}{8}+\frac{\alpha}{4}\right)^{-2}\sqrt{\overline{N}^*_\alpha(t, \varepsilon)} \tag{6.13}$$

where we used inequalities $N^*_\alpha(t, \varepsilon\theta) \leq \overline{N}^*_\alpha(t, \varepsilon\theta) \leq \overline{N}^*_\alpha(t, \varepsilon)$ for all $\theta \in (0, 1)$. On the other hand using (6.6) we have $N^*_{\frac{1}{4}+\frac{\alpha}{2}}(s, \varepsilon) \leq N^*_{\frac{1}{4}+\frac{\alpha}{2}}(\tau, \varepsilon)$ for all $\tau \in [s, t]$. This gives

$$N^*_{\frac{1}{4}+\frac{\alpha}{2}}(s, \varepsilon) \leq \frac{2^4}{\sqrt{D_\lambda}}\frac{e^{ct}}{\sqrt{t-s}}\left(\frac{1}{2}+\alpha\right)^{-2}\sqrt{\overline{N}^*_\alpha(t, \varepsilon)} \qquad \forall \varepsilon \in (0, \varepsilon_\lambda]. \tag{6.14}$$

Since $\varepsilon \mapsto \overline{N}^*_\alpha(t, \varepsilon)$ is non-decreasing, (6.7) follows from (6.14).

Step 3: We now use an iteration process supplied by the inequality (6.7) to deduce (6.5). Let us fix any $\varepsilon \in (0, \varepsilon_\lambda]$ and consider

$$\alpha_n = \frac{1}{2} - \frac{1}{2^n}, \quad t_k = \left(1 - \frac{1}{2^k}\right)T, \quad n = 1, 2, 3, \ldots; \quad k = 0, 1, 2, \ldots.$$

Applying (6.7) to $\alpha = \alpha_{n-k}, t = t_k, s = t_{k-1}$ for $k = 1, 2, \ldots, n - 1; n \geq 2$, and noticing that $\alpha_{n-(k-1)} = \frac{1}{4} + \frac{\alpha_{n-k}}{2}, \frac{1}{2} + \alpha_{n-k} = 1 - \frac{1}{2^{n-k}}$ and $t_k - t_{k-1} = \frac{T}{2^k}$, we compute

$$\overline{N}^*_{\alpha_{n-(k-1)}}(t_{k-1}, \varepsilon) \leq C 2^{k/2} e^{-\frac{cT}{2^k}} \left(1 - \frac{1}{2^{n-k}}\right)^{-2} \sqrt{\overline{N}^*_{\alpha_{n-k}}(t_k, \varepsilon)}. \tag{6.15}$$

where $C = \frac{2^4}{\sqrt{D_\lambda T}} e^{cT}$. For any fixed $n \geq 2$ let us denote

$$A_k := \overline{N}^*_{\alpha_{n-k}}(t_k, \varepsilon), \quad B_k := C 2^{k/2} e^{-\frac{cT}{2^k}} \left(1 - \frac{1}{2^{n-k}}\right)^{-2}, \quad k = 0, 1, \ldots, n - 1.$$

Then (6.15) reads

$$A_{k-1} \leq B_k \sqrt{A_k}, \quad k = 1, 2, \ldots, n - 1. \tag{6.16}$$

Since $\alpha_1 = 0$ and the conservation of mass imply that $A_{n-1} = \overline{N}^*_0(t_{n-1}, \varepsilon) \leq m(T)N(F_0)$, it follows from (6.16) that A_k are finite ($k = 1, 2, \ldots, n - 1$) and iterating (6.16) gives

$$A_0 \leq \left(\prod_{k=1}^{n-1} (B_k)^{\frac{1}{2^{k-1}}}\right) (A_{n-1})^{\frac{1}{2^{n-1}}}. \tag{6.17}$$

Also since $t_0 = 0$ hence $A_0 = \overline{N}^*_{\alpha_n}(0, \varepsilon)$, it follows from (6.17) and $A_{n-1} \leq m(T)N(F_0)$ that

$$\overline{N}^*_{\alpha_n}(0, \varepsilon) \leq P_n [m(T)N(F_0)]^{\frac{1}{2^{n-1}}}, \quad n = 2, 3, 4, \ldots$$

where

$$P_n = \prod_{k=1}^{n-1} \left(C 2^{k/2} e^{-\frac{cT}{2^k}} \left(1 - \frac{1}{2^{n-k}}\right)^{-2}\right)^{\frac{1}{2^{k-1}}}.$$

On the other hand by definition of $\overline{N}^*_\alpha(t, \varepsilon)$ and $\alpha_n = \frac{1}{2} - \frac{1}{2^n}$ we have

$$\overline{N}^*_{\alpha_n}(0, \varepsilon) = \overline{N}_{\alpha_n}(F_0, \varepsilon) \geq N_{\alpha_n}(F_0, \varepsilon) = \varepsilon^{\frac{1}{2^n}} N_{1/2}(F_0, \varepsilon)$$

and so

$$N_{1/2}(F_0, \varepsilon) \leq (1/\varepsilon)^{\frac{1}{2^n}} P_n [m(T)N(F_0)]^{\frac{1}{2^{n-1}}}, \quad n = 2, 3, 4, \ldots.$$

Letting $n \to \infty$ and computing

$$\lim_{n \to \infty} P_n = \prod_{k=1}^{\infty} C^{\frac{2}{2^k}} \prod_{k=1}^{\infty} 2^{\frac{k}{2^k}} \prod_{k=1}^{\infty} e^{-\frac{2cT}{4^k}} \lim_{n \to \infty} \prod_{k=1}^{n-1} \left(1 - \frac{1}{2^{n-k}}\right)^{\frac{-4}{2^k}} = C^2 \cdot 2^2 \cdot e^{-\frac{2cT}{3}} \cdot 1$$

we obtain

$$N_{1/2}(F_0, \varepsilon) \leq (2C e^{-\frac{cT}{3}})^2 = \frac{2^{10}}{D_\lambda T} e^{\frac{4}{3}cT}.$$

Since $\varepsilon \in (0, \varepsilon_\lambda]$ is arbitrary, taking upper limit $\limsup_{\varepsilon \to 0^+}$ leads to

$$\overline{N}_{1/2}(F_0) \le \frac{2^{10}}{D_\lambda T} e^{\frac{4}{3} cT}.$$

This implies $\overline{N}_{1/2}(F_0) < \infty$ hence $\underline{N}_{1/2}(F_0) < \infty$ and so $D_\lambda = \lambda \underline{N}_{1/2}(F_0)$. Thus

$$\underline{N}_{1/2}(F_0) \overline{N}_{1/2}(F_0) \le \frac{2^{10}}{\lambda T} e^{\frac{4}{3} cT}.$$

Since $\lambda \in (1/2, 1)$ is arbitrary, letting $\lambda \to 1^-$ gives (6.5) and finishes the proof. $\square$

As a consequence of Proposition 6.1 we obtain the following

Theorem 6.1 *Let F_t be a distributional solution of Eq. (1.5) with the initial datum F_0. Then*
(a)

$$\underline{D}_{1/2}(F_0) \overline{D}_{1/2}(F_0) = \infty \implies F_t(\{0\}) > 0 \quad for \ all \ \ t > 0.$$

In general, if

$$\sqrt{\underline{D}_{1/2}(F_0) \overline{D}_{1/2}(F_0)} > \frac{50\sqrt{5}}{\sqrt{T}} \exp\left(\frac{2}{3}\sqrt{N(F_0)E(F_T)}\, T\right) \quad for \ some \ \ T \in (0, \infty)$$

then

$$F_t(\{0\}) > 0 \quad for \ all \ \ t \ge T.$$

(b) Suppose F_t also conserves the energy and

$$\sqrt{\underline{D}_{1/2}(F_0) \overline{D}_{1/2}(F_0)} > 213 [N(F_0)E(F_0)]^{1/4}.$$

Then

$$F_t(\{0\}) > 0 \quad for \ all \ \ t \ge \frac{3}{4}(N(F_0)E(F_0))^{-1/2}.$$

Proof Part (a) follows directly from Proposition 6.1 and the propagation of condensation (1.18). Part (b) is a special case of part (a) with $E(F_T) = E(F_0)$ and $T = \frac{3}{4}(N(F_0)E(F_0))^{-1/2}$ which is the minimizer of $\frac{50\sqrt{5}}{\sqrt{T}} \exp(\frac{2}{3}\sqrt{N(F_0)E(F_0)}\, T)$ with respect to $T \in (0, \infty)$. The minimum is $50\sqrt{20e/3}\,[N(F_0)E(F_0)]^{1/4}$ and $213 > 50\sqrt{20e/3}$. $\square$

As a corollary of Theorem 5.1 and Theorem 6.1 we have the following result concerning the strict positivity of the critical time of condensation.

Theorem 6.2 *Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\ge 0})$ be a regular measure with density f_0 satisfying*

$$\sqrt{\underline{D}_{1/2}(F_0) \overline{D}_{1/2}(F_0)} > 213 [N(F_0)E(F_0)]^{1/4}, \quad f_0(x) \le K x^{-1} \ \ \forall x \in \mathbb{R}_+$$

for some $0 < K < \infty$. Then for the positive number $T_K = (6[N(F_0)]^2 K^{-1} + 40K^2)^{-1}$, there exists a conservative distributional solution F_t of Eq. (1.5) with the initial datum F_0, such that F_t is regular for all $t \in [0, T_K]$ and thus the critical time t_c of condensation of F_t is strictly positive and $t_c \in [T_K, \frac{3}{4}(N(F_0)E(F_0))^{-1/2}]$.

Proof By assumption we have $2K \ge \overline{D}_{1/2}(F_0) \ge \sqrt{\underline{D}_{1/2}(F_0) \overline{D}_{1/2}(F_0)}$ and so $4K^2 > (213)^2 [N(F_0)E(F_0)]^{1/2}$ which apparently implies that $T_K < \frac{3}{4}[N(F_0)E(F_0)]^{-1/2}$. The conclusion of the theorem then follows from Theorem 5.1, Theorem 6.1, and the definition of the critical time t_c (see (1.21)). $\square$

7 Some Examples

As applications of main results of the paper we present here several examples mentioned in the Introduction.

Example 7.1 Here we show that if $\alpha > 1/2$, then there exist extensively conservative distributional solutions F_t of Eq. (1.5) on $[0, \infty)$ satisfying $D_\alpha(F_t) = \infty$ for all $t \in [0, \infty)$, but F_t have no condensation for all $t \in [0, \infty)$. Given any regular measure $G_0 \in \mathcal{B}^+_{-1/2,1}(\mathbb{R}_{\geq 0})$ satisfying $M_{-1/2}(G_0) < \frac{1}{85}[N(G_0)E(G_0)]^{1/4}$. For any $\alpha > 1/2$, take $1/2 < \beta < \alpha \wedge 1$ and consider a local surgery F_0 to G_0 near the origin: $\mathrm{d}F_0(x) = f_0(x)\sqrt{x}\mathrm{d}x$, $f_0(x) = x^{\beta-3/2}\mathbf{1}_{\{0<x\leq\eta\}} + g_0(x)\mathbf{1}_{\{x>\eta\}}$ with $\eta > 0$, where g_0 is the density of G_0. We have $\lim_{\eta\to 0^+}(M_{-1/2}(F_0), N(F_0), E(F_0)) = (M_{-1/2}(G_0), N(G_0), E(G_0))$. So we can fix a small $\eta > 0$ such that $M_{-1/2}(F_0) < \frac{1}{80}[N(F_0)E(F_0)]^{1/4}$, i.e. $\|f_0\|_{L^1} < \frac{1}{80}[N(f_0)E(f_0)]^{1/4}$. According to Theorem 3.2, Theorem 4.1 and part (d) of Proposition 4.2, there is a unique conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0, and F_t is regular with density $f(\cdot, t)$ which is a unique conservative mild solution of Eq. (1.5) on $[0, \infty)$ satisfying $f \in C([0, \infty); L^1(\mathbb{R}_+))$ and $f(\cdot, 0) = f_0$. Of course F_t has no condensation for all $t \geq 0$. On the other hand, using the Duhamel's formula (4.8) and (4.10) we deduce

$$f(x, t) \geq f_0(x)e^{-c(x)t} \qquad \text{a.e. } x \in \mathbb{R}_+ \text{ for all } t \geq 0$$

where $c(x) = \sqrt{x}N(f_0) + \frac{9}{8}\sqrt{N(f_0)E(f_0)}$ (by (4.7), (4.14)). This implies $\underline{D}_\beta(F_t) > 0$ and so $D_\alpha(F_t) = \infty$ for all $t \geq 0$ because $\alpha > \beta$. This shows also that the unboundedness of a mild solution near the origin does not generally imply the condensation in finite time. $\square$

Example 7.2 In this example we show that the condensation can happen for arbitrary temperature. Given any $G_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$. As did above we make a surgery to G_0 near the origin: for any $K > 1$ we define $F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$ by $\mathrm{d}F_0(x) = K^{1/4}x^{-1}\mathbf{1}_{\{0<x\leq 1/K\}}\sqrt{x}\mathrm{d}x + \mathbf{1}_{\{x>1/K\}}\mathrm{d}G_0(x)$. We have $\|F_0 - G_0\|_1 \to 0$ as $K \to \infty$ and

$$D_{1/2}(F_0) = 2K^{1/4}, \quad N(F_0) \leq 2K^{-1/4} + N(G_0), \quad E(F_0) \leq \frac{2}{3}K^{-5/4} + E(G_0). \quad (7.1)$$

So for any $T \in (0, \infty)$ we can choose K large enough such that

$$D_{1/2}(F_0) > \frac{50\sqrt{5}}{\sqrt{T}} \exp\left(\frac{2}{3}\sqrt{N(F_0)E(F_0)}\, T\right).$$

Let F_t, G_t be conservative distributional solutions of Eq. (1.5) on $[0, \infty)$ with the initial data F_0, G_0 respectively. By Theorem 6.1, $F_t(\{0\}) > 0$ for all $t \geq T$. Notice that $D_{1/2}(F_0) > 0$ implies $M_{-1/2}(F_0) = \infty$. So there is no contradiction to the stability theorem (Theorem 3.1) even if we can assume that the original initial datum G_0 satisfies $M_{-1/2}(G_0) \leq \frac{1}{80}[N(G_0)E(G_0)]^{1/4}$ so that G_t has no condensation for all $t \geq 0$. Notice also that F_0 satisfies $D_\alpha(F_0) = 0$ for all $0 < \alpha < 1/2$ and so our previous result Theorem 2.1(b) of [15] cannot apply to F_t. Finally we note that if K is large enough, then F_0 has almost the same temperature as G_0 so that it can be arbitrarily high/low as that of G_0. $\square$

Example 7.3 We show that the initial data F_0 satisfying the conditions in Theorem 6.2 exist extensively. Take any regular measure $G_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$ and let g_0 be its density. As in Example 7.2, for any $K > 1$ we define $F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0})$ by $\mathrm{d}F_0(x) = f_0(x)\sqrt{x}\mathrm{d}x$ with $f_0(x) = K^{1/4}x^{-1}\mathbf{1}_{\{0<x\leq 1/K\}} + \mathbf{1}_{\{x>1/K\}}\min\{g_0(x), Kx^{-1}\}$, $x \in \mathbb{R}_+$. It is easily seen that

$f_0(x) \leq Kx^{-1}$ for all $x \in \mathbb{R}_+$ and (7.1) holds for all $K > 1$. Thus for $K > 1$ large enough, the condition $D_{1/2}(F_0) > 213[N(F_0)E(F_0)]^{1/4}$ is also satisfied. By Theorem 6.2, this example shows also that the critical time of condensation can be strictly positive even for a large class of initial data which are unbounded near the origin. □

Example 7.4 Consider a family of regular initial data $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$, $\mathrm{d}F_0(x) = f_0(x)\sqrt{x}\mathrm{d}x$, given by $f_0(x) = a[x\log(1/x)]^{-1}\mathbf{1}_{\{0<x\leq 1/2\}} + bx^{-\gamma}\mathbf{1}_{\{x>1/2\}}$ with constants $a > 0, b > 0, \gamma > 5/2$. For such an F_0 we have $D_{1/2}(F_0) = 0$ and $M_{-1/2}(F_0) = \infty$ so that neither the condensation results in Sect. 6 nor the regularity results in Sect. 4 can be used, but f_0 satisfies $f_0(x) \leq Kx^{-1}$ ($\forall x > 0$) with $K = \max\{a(\log 2)^{-1}, b2^{\gamma-1}\}$ and so by Theorem 5.1 there exists a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum F_0 such that F_t is regular on $[0, T_K]$. To see whether F_t has condensation in finite time, we now have no other method but to check the low temperature condition. Let $\overline{T}, \overline{T}_c$ be the kinetic temperature and the critical temperature defined in (1.14) with $N = N(F_0)$, $E = E(F_0)$. Then, for some explicit constants $C_2 \geq C_1 > 0$ depending only on γ, we have $C_1(a \vee b)^{-2/3} \leq \overline{T}/\overline{T}_c \leq C_2(a \vee b)^{-2/3}$. If $a \vee b$ is large enough such that $\overline{T}/\overline{T}_c < 1$, then according to part (III.2) of Theorem 1.2, F_t has condensation in finite time with the critical time $t_c \in [T_K, \infty)$; while if $a \vee b$ is so small that $\overline{T}/\overline{T}_c \geq 1$, we do not know whether F_t has condensation in finite time since as mentioned above the results obtained so far do not work for this case of such F_0. □

Example 7.5 As the last example we discuss the size of the universal constant θ_* and its influence on the working area of the condensation condition (1.24).

We first show that $\theta_* \leq 1/2$. Assume $\theta_* > 1/2$. Let $1/2 < \beta < \theta_* \wedge 1$ and let F_t be the conservative regular distributional solution of Eq. (1.5) on $[0, \infty)$ constructed in Example 7.1. In the condition (1.24) let us choose $N = N(F_0)$, $E = E(F_0)$. From $D_\beta(F_0) > 0$ and $3/2 > \beta, \theta_* > \beta$ we have

$$\lim_{\rho\to 0^+} \inf_{0<\varepsilon\leq\rho} \frac{F_0([0, \varepsilon])}{\varepsilon^{3/2}} = D_{3/2}(F_0) = \infty, \quad \lim_{\rho\to 0^+} \frac{F_0([0, \rho])}{\rho^{\theta_*}} = D_{\theta_*}(F_0) = \infty$$

and so F_0 satisfies (1.24) sufficiently: the left hand side of (1.24) is ∞. But F_t has no condensation for all $t \geq 0$. This proves $\theta_* \leq 1/2$. Another example for proving $\theta_* \leq 1/2$ is the regular equilibrium $F_t \equiv F_{\mathrm{be}}$ with density $f_{\mathrm{be}}(x) = (e^{x/\kappa} - 1)^{-1}$.

Next we show that if $\theta_* < 1/2$, then for any $N > 0$, $E > 0$, the condition (1.20) does not imply (1.24), and the same holds also true for $\theta_* = 1/2$ if $K^* > 214(NE)^{1/4}$.

Fix $\gamma \geq 3$ and let $a = 107(NE)^{1/4}$. Consider a conservative distributional solution F_t of Eq. (1.5) on $[0, \infty)$ with the initial datum $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ given by

$$\mathrm{d}F_0(x) = (ax^{-1}\mathbf{1}_{\{0<x\leq\eta\}} + bx^{-\gamma}\mathbf{1}_{\{x\geq R\}})\sqrt{x}\mathrm{d}x$$

where

$$0 < \eta < \eta_0(N, E) := \min\left\{\left(\frac{N}{4a}\right)^2, \left(\frac{3E}{4a}\right)^{2/3}, \frac{E}{6N}\right\}$$

and $b > 0$, $R > 0$ are determined by the given mass and energy: $N(F_0) = N$, $E(F_0) = E$, i.e.,

$$b = (\gamma - 3/2)R^{\gamma-3/2}N_\eta, \quad R = \frac{\gamma - 5/2}{\gamma - 3/2} \cdot \frac{E_\eta}{N_\eta}$$

where $N_\eta = N - 2a\eta^{1/2}$, $E_\eta = E - \frac{2}{3}a\eta^{3/2}$. Notice that our restriction on η and $\gamma \geq 3$ imply $N_\eta > N/2$, $E_\eta > E/2$, and $R > E/(6N) > \eta$. By construction, F_0 satisfies (1.20): $D_{1/2}(F_0) = 2a > 213(NE)^{1/4}$ and so F_t has condensation in finite time. On the other hand, let $\nu > 0$, $\rho_0 = \rho_0(N, E, \nu) > 0$, $K^* = K^*(N, E, \nu) > 0$ be given in part (II) of Theorem 1.2, and suppose first that $\theta_* < 1/2$. In this case we take any $0 < \delta < 1$ and let η be further small:

$$0 < \eta < \min\left\{\eta_0(N, E), \left(\frac{\delta\nu}{2a}\right)^2 \left(\frac{E}{6N}\right)^3, \left(\frac{\delta K^*}{2a}\right)^{\frac{1}{\frac{1}{2}-\theta_*}}\right\}.$$

We show that F_0 does not satisfy (1.24):

$$\sup_{0<\rho\leq\rho_0} \min\left\{\inf_{0<\varepsilon\leq\rho} \frac{F_0([0,\varepsilon])}{\nu\varepsilon^{3/2}}, \frac{F_0([0,\rho])}{K^*\rho^{\theta_*}}\right\} \leq \delta. \tag{7.2}$$

In fact, for any $0 < \rho \leq \rho_0$, if $\rho \geq R$, then $\rho > E/(6N) := \varepsilon_0$ and so (since $\eta < \varepsilon_0 < R$)

$$\inf_{0<\varepsilon\leq\rho} \frac{F_0([0,\varepsilon])}{\nu\varepsilon^{3/2}} \leq \frac{F_0([0,\varepsilon_0])}{\nu\varepsilon_0^{3/2}} = \frac{2a}{\nu\varepsilon_0^{3/2}}\eta^{1/2} < \delta;$$

while if $\rho < R$, then

$$\frac{F_0([0,\rho])}{K^*\rho^{\theta_*}} = \frac{2a}{K^*\rho^{\theta_*}}(\eta \wedge \rho)^{1/2} \leq \frac{2a}{K^*}\eta^{\frac{1}{2}-\theta_*} < \delta.$$

Thus for any $0 < \rho \leq \rho_0$, one of the two is less than δ. This proves (7.2).

Finally suppose $\theta_* = 1/2$. In this case we assume that there is $0 < \delta < 1$ such that $K^* \geq \frac{214}{\delta}(NE)^{1/4} = \frac{2a}{\delta}$. Then for η satisfying

$$0 < \eta < \min\left\{\eta_0(N, E), \left(\frac{\delta\nu}{2a}\right)^2 \left(\frac{E}{6N}\right)^3\right\}$$

the same argument applies and so (7.2) with $\theta_* = 1/2$ holds also for such δ. $\qquad\square$

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