J Stat Phys (2014) 156:156–176
DOI 10.1007/s10955-014-0996-9

A Phase Transition in a Quenched Amorphous Ferromagnet

Alexei Daletskii · Yuri Kondratiev · Yuri Kozitsky · Tanja Pasurek

Received: 4 March 2013 / Accepted: 4 April 2014 / Published online: 22 April 2014

Abstract Quenched thermodynamic states of an amorphous ferromagnet are studied. The magnet is a countable collection of point particles chaotically distributed over $\mathbb{R}^d$, $d \geq 2$. Each particle bears a real-valued spin with symmetric a priori distribution; the spin-spin interaction is pair-wise and attractive. Two spins are supposed to interact if they are neighbors in the graph defined by a homogeneous Poisson point process. For this model, we prove that with probability one: (a) quenched thermodynamic states exist; (b) they are multiple if the intensity of the underlying point process and the inverse temperature are big enough; (c) there exist multiple quenched thermodynamic states which depend on the realizations of the underlying point process in a measurable way.

Keywords Random Gibbs measure · Geometric random graph · Poisson point process · Percolation · Unbounded spin model · Wells inequality

Mathematics Subject Classification 82B44 · 82B21 · 82A57

A. Daletskii
Department of Mathematics, University of York, York YO1 5DD, UK
e-mail: alex.daletskii@york.ac.uk

Yu. Kondratiev · T. Pasurek
Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
e-mail: kondrat@math.uni-bielefeld.de

T. Pasurek
e-mail: tpasurek@math.uni-bielefeld.de

Yu. Kozitsky (✉)
Instytut Matematyki, Uniwersytet Marii Curie-Sklodowskiej, 20-031 Lublin, Poland
e-mail: jkozi@hektor.umcs.lublin.pl

1 Introduction

1.1 Setup

In this paper, we study thermodynamic states of the following system. A countable collection of 'particles' is distributed over $\mathbb{R}^d$, $d \geq 2$, in such a way that every bounded $\Lambda \subset \mathbb{R}^d$ contains only finite number of them. Each 'particle' represents a cluster of negligible size of magnetically active physical particles. In our model, this amounts to assuming that a 'particle' is characterized by its location $x \in \mathbb{R}^d$ and spin $\sigma_x \in \mathbb{R}$. The locations of 'particles' constitute a locally finite set (configuration) $\gamma \subset \mathbb{R}^d$, and the spins take real values. We also assume that each σ_x is characterized by one and the same symmetric a priori distribution χ on $\mathbb{R}$. The interaction is supposed to be pair-wise and attractive. For the 'particles' located at x and y, the interaction energy is $\phi(|x-y|)\sigma_x\sigma_y$, with $\phi(r) \geq \phi_* > 0$ for $r \in [0, r_*]$, and $\phi(r) = 0$ for $r > r_*$. If γ were a crystalline lattice, then the model would be a standard lattice system of 'unbounded spins'. The study of Gibbs states of such spin systems goes back to the seminal paper [29], further continued in [6,37,38]. In [28], a similar model living on a more general discrete metric space was studied. The next step was made in [26] where the underlying set was a countable graph with globally unbounded vertex degrees. In that paper, a class of graphs was introduced in which vertices of large degree are sparse—a property formulated in [26] as a weighted summability of the vertex degrees. For such graphs, tempered Gibbs states of unbounded spin systems were constructed and studied. A natural continuation of those works would be to pass to random graphs, in which this kind of summability holds for almost all realizations. In the present paper we do this step by letting the underlying set $\gamma \subset \mathbb{R}^d$ be random, obeying the Poisson law with homogeneous density $\lambda > 0$. The graph structure on γ is then defined by the spin-spin interaction: $x, y \in \gamma$ are adjacent (neighbors) if $\phi(|x-y|) > 0$. We call this model the *amorphous ferromagnet*, cf. [34, Sect. 11].

In view of the mentioned randomness, there can be two types of thermodynamic state of our model. In the first case, the randomness is taken into account already at the level of local states defined on the space of (marked) configurations $\hat{\gamma} = \{(x, \sigma_x) : x \in \gamma\}$. The global thermodynamic states constructed in this way are then *annealed states*; they describe the thermal equilibrium of the whole system. The second approach, which we follow in this paper, consists in constructing thermodynamic states of the spin system alone for fixed *typical* configurations γ. These are *quenched states*, cf. [8]. The global observables characterizing such states ought to be *self-averaging* – taking the same value for all typical (i.e., for almost all) configurations γ. Note that studying quenched states is a more difficult problem, as compared to that of annealed ones, since the present spatial irregularities do not allow for applying here most of the methods effective for regular systems. In what follows, we aim at proving:

- Existence, for almost all configurations γ, of thermodynamic states with properties suitable for physical applications.
- Measurability of thermodynamic states with respect to γ.
- Multiplicity of such states, for almost γ, for temperatures $T < T_*$, where the self-averaging parameter $T_* > 0$ may depend on the model parameters λ, ϕ_*, r_*, d.

There are only few publications on the mathematically rigorous theory of phase transitions in spin systems of general type living on non-crystalline (amorphous) substances, see [12, 20–22,36] where annealed states were considered. The reason for this is presumably that the corresponding methods, e.g., infrared estimates, are essentially based on the translation invariance (and other symmetries) of the underlying crystals. At the same time, for Ising spins $\sigma_x = \pm 1$, there exist methods applicable to the corresponding models on graphs, see [23,30]. For such models, see also [11], the main idea of proving the existence of phase transitions is to relate the appearance of multiple phases of the spin system to the Bernoulli bond percolation on the underlying graph. In our model, we deal with a random graph with vertex set γ and the adjacency relation $x \sim y$ defined by the property $|x - y| \leq r_*$. That is, the set of edges of the graph is $\varepsilon_\gamma = \{\{x, y\} \subset \gamma : x \sim y\}$, and the graph itself—known as the *Gilbert graph* – is then the pair $(\gamma, \varepsilon_\gamma)$. It has various applications and is intensively studied, see, e.g., [5,18,31,33]. The probability distribution of $(\gamma, \varepsilon_\gamma)$ is described in Sect. 2.1 below. If this graph has an infinite connected component, which is a random event with probability dependent on λ, under certain conditions one can observe the Bernoulli bond percolation with a nontrivial percolation threshold $q_* \in (0, 1)$. We combine the mentioned methods and prove that the mean magnetization in our model can be positive for almost all configurations γ, and hence the quenched Gibbs states can be multiple, if the particle density λ and the inverse temperature $\beta = 1/T$ are large enough[1]. Finally, let us mention that, for our model with $\sigma_x \in \mathbb{R}$, the problem of uniqueness of Gibbs states remains open, see Sect. 2.3 below. We also note that the method developed in this article can be used to study annealed states of amorphous ferromagnetic substances where the spin and the particle configurations are in thermal equilibrium.

1.2 The Overview of the Results

In the sequel, by π_λ we denote the homogeneous Poisson measure with density $\lambda > 0$ – the probability distribution of the configurations γ. In Proposition 4 and Theorem 1 below, we show that there exists a set of configurations A_1 such that: (a) $\pi_\lambda(A_1) = 1$ for all $\lambda > 0$; (b) $\mathcal{G}_t(\beta|\gamma) \neq \emptyset$ for all $\beta > 0$ and all $\gamma \in A_1$. Here $\mathcal{G}_t(\beta|\gamma)$ is the set of *tempered* Gibbs measures of our spin system on γ at a given β. Tempered Gibbs measures are Gibbs measures supported on the configurations with tempered growth of $|\sigma_x|$ as $|x| \to +\infty$. Note that, for spin models on graphs with unbounded vertex degrees and with single-spin distributions with noncompact support, there may exist states supported on configurations of spins with rapidly increasing $|\sigma_x|$, whereas for typical ferromagnetic configurations in physical substances, most of the spins take values close to same $s > 0$. The proof of Theorem 1 is based on [26, Theorem 3.1] and on the property of π_λ obtained in Proposition 4. Here A_1 appears as the set of all those configurations for which the quantities in (11) are finite, and hence the graph $(\gamma, \varepsilon_\gamma)$ belongs to the class of 'sparse' graphs introduced in [26].

The Ising model on γ is a particular case of our model, which corresponds to the choice $\chi(d\sigma) = \delta(\sigma^2 - 1)d\sigma$. As is well-known, the set of Gibbs states of this model, $\mathcal{G}_t^{\mathrm{Ising}}(\beta|\gamma)$, is nonempty for all γ. Next, for

$$A_2(\beta) := \{\gamma : |\mathcal{G}_t(\beta|\gamma)| > 1\},$$
$$A_2^{\mathrm{Ising}}(\beta) := \{\gamma : |\mathcal{G}_t^{\mathrm{Ising}}(\beta|\gamma)| > 1\},$$

by the Wells correlation inequality [42], it follows that, cf. Proposition 5 below,

$$A_2(\beta) \supseteq A_2^{\mathrm{Ising}}(a^2\beta) \tag{1}$$

where $a > 0$ is determined by the measure χ, see (27). For the reader convenience, we present here a complete proof of the Wells inequality, which is a refinement of that in [10, Appendix]. By standard results on the continuum percolation driven by the Poisson random

[1] From now on, the parameters r_* and ϕ_* are fixed and mostly suppressed from the notations.

point process, see [31,33] and also [17, Corollary 3.7] and [18, Theorem 3.1], it follows that, for π_λ-almost all γ, the corresponding graph $(\gamma, \varepsilon_\gamma)$ has an infinite connected component whenever $\lambda > \lambda_*$, where a non-random parameter $\lambda_* > 0$ is determined by the parameter r_* and the dimension of the space d. Suppose now that each edge of this infinite connected component is removed independently with probability $1 - q$ and kept with probability q. If the graph obtained in this way still possesses an infinite connected component, then one says that the Bernoulli bond percolation with bond probability q occurs on the infinite connected component of $(\gamma, \varepsilon_\gamma)$. As in the previous result, it is possible to show, see Propositions 2 and 3 below, that for $\lambda > \lambda_*$ there exists $q_* \in (0, 1)$ such that

$$\pi_\lambda(A_3(q)) = 1, \qquad \text{for } q > q_* \text{ and } \lambda > \lambda_*, \tag{2}$$

where $A_3(q)$ is the set of all those γ such that the Bernoulli bond percolation with bond probability q occurs on the infinite connected component of $(\gamma, \varepsilon_\gamma)$. By [23, Theorem 2.1], we know that

$$A_2^{\text{Ising}}(\beta) \supseteq A_3(q), \quad \text{for } \beta > [\log(1+q) - \log(1-q)]/2,$$

Then combining (1) and (2), we conclude that, for $\lambda > \lambda_*$, there exists $\beta_* \geq [\log(1+q_*) - \log(1-q_*)]/2$ such that, for all $\beta > a^2\beta_*$ and π_λ-almost all γ, the set $\mathcal{G}_\mathrm{t}(\beta|\gamma)$ contains at least two elements, see Theorem 2 below. In principle, we could stop here. However, in that case one important aspect of the theory would have been omitted. This is the dependence of our tempered Gibbs measures on γ. In mathematical theories of random systems [9], Gibbs measures are supposed to depend on the random parameters in a measurable way. Then they are called *random Gibbs measures*. In Sect. 4, we study the measurability issue by employing marked configurations $\hat{\gamma}$ consisting of pairs (x, σ_x). In this setting, random Gibbs measures are obtained as conditional measures on the space of marked configurations. In Theorem 3, we show that the random Gibbs measures of our model are multiple if the conditions of Theorem 2 are satisfied.

For the sake of clarity, in this paper we restricted ourselves to the simplest model of amorphous substances—the Gilbert graph model based on the Poisson point process. In a similar way, one can prove the statements mentioned above if the underlying graph is as in the random connection model, see [18,31,33] or a tempered Gibbs random field, see [17, Corollary 3.7]. The only condition is that the graph almost surely has the summability property as in Proposition 4, see [14] for more detail.

2 Quenched Gibbs Measures

2.1 The Underlying Graph

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ stand for the set of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. The space of all configurations is defined as

$$\Gamma = \left\{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \quad \text{for any } \Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d) \right\}, \tag{3}$$

where $|A|$ stands for the cardinality of A. The space Γ is equipped with the vague topology being the weakest one in which the maps $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x)$ are continuous for all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, see, e.g., [3]. By $\mathcal{B}(\Gamma)$ we denote the corresponding Borel σ-field. The vague topology is metrizable in such a way that the corresponding metric space is complete and separable. By π_λ we denote the homogeneous

Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity (density) $\lambda > 0$. Note that the set of all finite configurations Γ_0 is a Borel subset of Γ such that $\pi_\lambda(\Gamma_0) = 0$. That is, π_λ-almost all configurations $\gamma \in \Gamma$ are infinite.

Each γ can be considered as a graph with vertex set γ and adjacency relation $x \sim y$ defined by the condition $|x - y| \le r_*$. Then $\varepsilon_\gamma = \{\{x, y\} \subset \gamma : x \sim y\}$ is its edge set. The probability distribution of the random graph $(\gamma, \varepsilon_\gamma)$ is constructed in the following way, cf. [17]. Let ε denote a set of pairs of distinct points, i.e., of $e = \{x, y\}$, $x, y \in \mathbb{R}^d$, $x \neq y$. We say that ε is locally finite if ε_Λ is finite for each $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$. Here $\varepsilon_\Lambda := \{\{x, y\} \in \varepsilon : \{x, y\} \subset \Lambda\}$. Let E be the set of all locally finite ε, and $\mathcal{F}(E)$ be the σ-field of subsets of E generated by the counting maps $\varepsilon \mapsto |\varepsilon_\Lambda|$ with all possible $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$. Note that E is, in fact, the set of locally finite configurations, cf. (3), over the symmetrization of the set $\mathbb{R}^d \times \mathbb{R}^d \backslash \{(x, x) : x \in \mathbb{R}^d\}$. Let $\mathcal{P}(E)$ be the set of all probability measures on $(E, \mathcal{F}(E))$. Each $\varsigma \in \mathcal{P}(E)$ can uniquely be determined by its Laplace transform, which we introduce in the following way. Let $\theta : \mathbb{R}^d \times \mathbb{R}^d \to (-1, 0]$ be measurable, symmetric, and local, i.e., $\theta(x, y) = 0$ whenever x or y is in $\Lambda^c := \mathbb{R} \backslash \Lambda$ for some $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$. By Θ we denote the set of all such functions. Then for a $\varsigma \in \mathcal{P}(E)$, its Laplace transform is defined as

$$L_\varsigma(\theta) = \int_E \exp\left[\sum_{\{x,y\}\in\varepsilon} \log(1 + \theta(x, y))\right] \varsigma(d\varepsilon), \qquad \theta \in \Theta.$$

Now let $g : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ be measurable and symmetric. For each $\theta \in \Theta$, the pointwise product $g\theta$ is also in Θ. An *independent g-thinning* of a given $\varsigma \in \mathcal{P}(E)$, cf. [15, Sect. 11.2], is the measure ς^g defined by the relation

$$L_{\varsigma^g}(\theta) = L_\varsigma(g\theta). \tag{4}$$

The g-thinning of ς means that each configuration ε distributed according to ς is ‘thinned’ in the sense that each $\{x, y\} \in \varepsilon$ is removed from the edge configuration with probability $1 - g(x, y)$ and kept with probability $g(x, y)$. The probability distribution of such ‘thinned’ configurations is then ς^g.

Given $\gamma \in \Gamma$, we define $\varsigma(\cdot|\gamma) \in \mathcal{P}(E)$ by its Laplace transform

$$L(\theta|\gamma) = \exp\left[\sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} \log(1 + j(x, y)\theta(x, y))\right] \tag{5}$$

where $j(x, y) = 1$ for $|x - y| \le r^*$ and $j(x, y) = 0$ otherwise. For a measurable $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ := [0, +\infty)$, the function

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} \sum_{y\in\gamma\backslash x} \Psi(x, y) \in \mathbb{R}$$

is measurable. Therefore, the map $\Gamma \ni \gamma \mapsto L(\theta|\gamma) \in \mathbb{R}$ is also measurable for each θ. Thus, ς defined in (5) is a probability kernel from $(\Gamma, \mathcal{B}(\Gamma))$ to $(E, \mathcal{F}(E))$, cf. [17, Lemma 2.4]. The probability distribution of the graph $(\gamma, \varepsilon_\gamma)$ is now defined by the measure

$$\zeta_\lambda(d\gamma, d\varepsilon) = \varsigma(d\varepsilon|\gamma)\pi_\lambda(d\gamma). \tag{6}$$

It may happen that ζ_λ-typical graphs $(\gamma, \varepsilon_\gamma)$ have only finite connected components. Given g as in (4), by $\varsigma^g(\cdot|\gamma)$ we denote the independent g-thinning of $\varsigma(\cdot|\gamma)$, and set

$$\zeta^g_\lambda(d\gamma, d\varepsilon) := \varsigma^g(d\varepsilon|\gamma)\pi_\lambda(d\gamma).$$

The following fact is known, see [17, Lemma 2.4 and Corollary 3.7] and especially [18, Theorem 3.1].

Proposition 1 *Let* $g : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ *be measurable, symmetric, and translation invariant. Then there exists* $c(d) > 0$ *such that, for* λ *satisfying the condidion*

$$\lambda \int_{\mathbb{R}^d} g(x, 0)dx > c(d), \tag{7}$$

ζ_λ^g*-almost all graphs* (γ, ε) *have infinite connected components.*

Then by Proposition 1 we obtain the following fact.

Proposition 2 *There exists* λ_* *such that, for* $\lambda > \lambda_*$*, the graph* $(\gamma, \varepsilon_\gamma)$ *has an infinite connected component for* π_λ*-almost all* γ*.*

We prove Proposition 2 by taking in (7) $g(x, y) = j(x, y)$ and

$$\lambda_* = c(d)/V(r^*), \tag{8}$$

where $V(r^*)$ is the volume of the ball $\{x \in \mathbb{R}^d : |x| \leq r^*\}$. Note that there can only be a single infinite connected component, see [31, Theorem 6.3, page 172].

For a constant function $q(x, y) \equiv q \in [0, 1]$, let us consider the independent q-thinning of $\varsigma(\cdot|\gamma)$. If the corresponding random graph has an infinite connected component, then the Bernoulli bond percolation with bond probability q occurs on the infinite connected component of $(\gamma, \varepsilon_\gamma)$. The next fact can also be deduced from Proposition 1.

Proposition 3 *Let* λ_* *be as in Proposition 2 and inequality* $\lambda > \lambda_*$ *hold. Then, there exists* $q_* \in (0, 1)$ *such that for* π_λ*-almost all* γ*, the Bernoulli bond percolation with bond probability* $q > q_*$ *occurs on the infinite connected component of* $(\gamma, \varepsilon_\gamma)$*.*

Indeed, by (7) and (8) one can take

$$q_* = \frac{\lambda_*}{\lambda} = \frac{c(d)}{\lambda V(r^*)}.$$

Our next step is to prove that, for π_λ-almost all γ, the graph $(\gamma, \varepsilon_\gamma)$ belongs to the class of 'sparse' graphs studied in [26]. For $x \in \gamma$, let $n_\gamma(x)$ be the number of neighbors of x in γ, i.e., $n_\gamma(x) := |\{y \in \gamma : y \sim x\}|$. Clearly, $n_\gamma(x)$ is finite for all $\gamma \in \Gamma$. Note, however, that

$$\sup_{x\in\gamma} n_\gamma(x) = +\infty, \tag{9}$$

also for π_λ-almost all γ. Set

$$w_\alpha(x) = \exp(-\alpha|x|), \qquad \alpha > 0, \quad x \in \mathbb{R}^d. \tag{10}$$

For $x \in \gamma$ and $\theta > 0$, we then consider, cf. Eqs. (4) and (5) in [26],

$$\begin{aligned} a_\gamma(\alpha, \theta) &:= \sum_{\{x,y\}\in\varepsilon_\gamma} [w_\alpha(x) + w_\alpha(y)][n_\gamma(x)n_\gamma(y)]^\theta, \\ b_\gamma(\alpha) &:= \sum_{x\in\gamma} w_\alpha(x), \end{aligned} \tag{11}$$

and

$$\begin{aligned} A_{1,a} &= \{\gamma \in \Gamma : \forall \alpha > 0\ \forall \theta > 0\ \ a_\gamma(\alpha, \theta) < +\infty\} \\ A_{1,b} &= \{\gamma \in \Gamma : \forall \alpha > 0\ \ b_\gamma(\alpha) < +\infty\} \\ A_1 &= A_{1,a} \cap A_{1,b}. \end{aligned}$$

Proposition 4 *For all $\lambda > 0$, it follows that $A_1 \in \mathcal{B}(\Gamma)$ and $\pi_\lambda(A_1) = 1$.*

Proof For each γ, by (11) we have that $b_\gamma(\alpha) \leq b_\gamma(\alpha')$ whenever $\alpha > \alpha'$. Likewise, $a_\gamma(\alpha, \theta)$ is decreasing in α and increasing in θ. This yields that

$$\begin{aligned} A_{1,a} &= \bigcap_{\alpha,\theta \in \mathbb{Q}_+} \{\gamma \in \Gamma : a_\gamma(\alpha, \theta) < +\infty\}, \\ A_{1,b} &= \bigcap_{\alpha \in \mathbb{Q}_+} \{\gamma \in \Gamma : b_\gamma(\alpha) < +\infty\}, \end{aligned}$$

where $\mathbb{Q}_+$ is the set of all positive rational numbers. Hence, it is enough to obtain the π_λ-a.s. finiteness of $a_\gamma(\alpha, \theta)$ and $b_\gamma(\alpha)$ for fixed $\alpha \in \mathbb{Q}_+$ and $\theta \in \mathbb{Q}_+$.

By the definition of the Poisson measure π_λ, for each $n \in \mathbb{N}$ and any measurable and symmetric function $f : \mathbb{R}^d \times \Gamma \to \mathbb{R}_+$, we have that

$$\int_\Gamma \left(\sum_{x \in \gamma} f(x, \gamma \backslash x) \right) \pi_\lambda(d\gamma) = \lambda \int_\Gamma \left(\int_{\mathbb{R}^d} f(x, \gamma) dx \right) \pi_\lambda(d\gamma) \tag{12}$$

- the Mecke identity. Then

$$\int_\Gamma b_\gamma(\alpha) \pi_\lambda(d\gamma) = \int_\Gamma \left(\sum_{x \in \gamma} w_\alpha(x) \right) \pi_\lambda(d\gamma) = \lambda \int_{\mathbb{R}^d} w_\alpha(x) dx < \infty,$$

which yields

$$\pi_\lambda(\{\gamma : b_\gamma(\alpha) + \infty\}) = 0.$$

Next, we rewrite (11) in the form

$$a_\gamma(\alpha, \theta) = \sum_{x \in \gamma} w_\alpha(x) m_\gamma(\theta, x), \qquad m_\gamma(\theta, x) := \sum_{y : y \sim x} [n_\gamma(x) n_\gamma(y)]^\theta. \tag{13}$$

Set $B_r(x) = \{y \in \mathbb{R}^d : |x - y| \leq r\}$, $r > 0$, and let $\mathcal{I} : \mathbb{R}^d \to \{0, 1\}$ be the indicator of the ball $B_{2r_*}(0)$. Clearly,

$$\max_{y \in B_{r_*}(x) \cap \gamma} n_\gamma(y) \leq \sum_{z \in \gamma} \mathcal{I}(z - x).$$

Applying this in (13) we get

$$m_\gamma(\theta, x) \leq [n_\gamma(x)]^{\theta + 1} \max_{y : y \sim x} [n_\gamma(y)]^\theta \leq \left(\sum_{y \in \gamma \backslash x} \mathcal{I}(y - x) \right)^{2\theta + 1}.$$

By the latter and (12), as well as by the translation invariance of π_λ, we then obtain from (13)

$$\begin{aligned}\int_\Gamma a_\gamma(\alpha,\theta)\pi_\lambda(d\gamma) &\le \int_\Gamma \sum_{x\in\gamma} w_\alpha(x)\left(\sum_{y\in\gamma\setminus x}\mathcal{I}(y-x)\right)^{2\theta+1}\pi_\lambda(d\gamma)\\ &= \int_{\mathbb{R}^d} w_\alpha(x)\left\{\int_\Gamma\left(\sum_{y\in\gamma}\mathcal{I}(y-x)\right)^{2\theta+1}\pi_\lambda(d\gamma)\right\}dx\\ &= \left(\int_{\mathbb{R}^d} w_\alpha(x)dx\right)\cdot\int_\Gamma\left(\sum_{y\in\gamma}\mathcal{I}(y)\right)^{2\theta+1}\pi_\lambda(d\gamma)\\ &= \ell_{2\theta+1}\left(\lambda V(2r^*)\right)\cdot\int_{\mathbb{R}^d} w_\alpha(x)dx < \infty.\end{aligned}$$

Here

$$\ell_\vartheta(\varkappa) := e^{-\varkappa}\sum_{k=1}^\infty k^\vartheta \varkappa^k/k!, \qquad \vartheta,\varkappa > 0,$$

and $V(2r^*) = \int_{\mathbb{R}^d}\mathcal{I}(x)dx$ is the volume of the ball $B_{2r_*}(0)$. The latter estimate leads to the conclusion

$$\pi_\lambda(\{\gamma : a_\gamma(\alpha,\theta) = +\infty\}) = 0,$$

which completes the proof.

2.2 The Gibbs Specification

As mentioned in the Introduction, the a priori distribution of the spin of a 'particle' is determined by a finite symmetric measure χ on $\mathbb{R}$. We assume that, for some $u > 2$ and $\varkappa > 0$, the following holds

$$\int_{\mathbb{R}} \exp\left(\varkappa|t|^u\right)\chi(dt) < \infty. \tag{14}$$

For a fixed γ, let $\mathbb{R}^\gamma$ stand for the space of all maps $\sigma : \gamma \to \mathbb{R}$. We equip it with the topology of point-wise convergence and the corresponding Borel σ-field $\mathcal{B}(\mathbb{R}^\gamma)$. Let also $\mathcal{P}(\mathbb{R}^\gamma)$ denote the set of all probability measures on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and $\gamma \in \Gamma$, we set $\gamma_\Lambda = \gamma\cap\Lambda$ and denote by σ_Λ the restriction of σ to γ_Λ, i.e., $\sigma_\Lambda = (\sigma_x)_{x\in\gamma_\Lambda}$. For $\sigma,\bar\sigma \in \mathbb{R}^\gamma$ and $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, by $\sigma_\Lambda \times \bar\sigma_{\Lambda^c}$ we denote the element $\sigma' \in \mathbb{R}^\gamma$ such that $\sigma'_\Lambda = \sigma_\Lambda$ and $\sigma'_{\Lambda^c} = \bar\sigma_{\Lambda^c}$.

The magnet that we study is characterized by a ferromagnetic spin-spin interaction, which for fixed $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ and $\gamma \in \Gamma$ is described by the following relative energy functional

$$-E^\gamma_\Lambda(\sigma_\Lambda|\bar\sigma_{\Lambda^c}) = \sum_{\{x,y\}\subset\gamma_\Lambda}\phi(|x-y|)\sigma_x\sigma_y + \sum_{x\in\gamma_\Lambda,\, y\in\gamma_{\Lambda^c}}\phi(|x-y|)\sigma_x\bar\sigma_y. \tag{15}$$

Here $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable and bounded function such that, $\phi(r) \ge \phi_* > 0$ for $r \in [0, r_*]$ and $\phi(r) = 0$ for all $r > r_*$.

For $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, by $\mathcal{B}_\Lambda(\mathbb{R}^\gamma)$ we denote the smallest σ-subfield of $\mathcal{B}(\mathbb{R}^\gamma)$ which contains all sets $A = \{\sigma \in \mathbb{R}^\gamma : \sigma_\Lambda \in A^0\}$, $A^0 \in \mathcal{B}(\mathbb{R}^{\gamma_\Lambda})$, where $\mathcal{B}(\mathbb{R}^{\gamma_\Lambda})$ is the corresponding Borel σ-field. The algebra of *local sets* is

$$\mathcal{B}_{\mathrm{loc}}(\mathbb{R}^\gamma) := \bigcup_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)} \mathcal{B}_\Lambda(\mathbb{R}^\gamma). \tag{16}$$

We will use the following topology on $\mathcal{P}(\mathbb{R}^\gamma)$, see [16, Definition 4.2, p. 59].

Definition 1 The topology of local set convergence ($\mathfrak{L}$-topology for short) is the weakest topology on $\mathcal{P}(\mathbb{R}^\gamma)$ that makes the evaluation maps $\mu \mapsto \mu(A)$ continuous for all $A \in \mathcal{B}_{\mathrm{loc}}(\mathbb{R}^\gamma)$.

For $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $\bar{\sigma} \in \mathbb{R}^\gamma$, we define

$$\Pi^\gamma_\Lambda(A|\bar{\sigma}) = \frac{1}{Z^\gamma_\Lambda(\bar{\sigma})} \int\limits_{\mathbb{R}^{\gamma_\Lambda}} \mathbb{I}_A(\sigma_\Lambda \times \bar{\sigma}_{\Lambda^c}) \exp\left(-\beta E^\gamma_\Lambda(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c})\right) \chi_\Lambda(d\sigma_\Lambda), \tag{17}$$

where $\mathbb{I}_A$ is the indicator of $A \in \mathcal{B}(\mathbb{R}^\gamma)$, E^γ_Λ is as in (15), and

$$\chi_\Lambda(d\sigma_\Lambda) := \bigotimes_{x \in \gamma_\Lambda} \chi(d\sigma_x),$$

$$Z^\gamma_\Lambda(\bar{\sigma}) := \int\limits_{\mathbb{R}^{|\Lambda|}} \exp\left(-\beta E^\gamma_\Lambda(\sigma_\Lambda|\bar{\sigma}_{\Lambda^c})\right) \chi_\Lambda(d\sigma_\Lambda). \tag{18}$$

Thus, for each $A \in \mathcal{B}(\mathbb{R}^\gamma)$, $\Pi^\gamma_\Lambda(A|\cdot)$ is $\mathcal{B}(\mathbb{R}^\gamma)$-measurable, and, for each $\bar{\sigma} \in \mathbb{R}^\gamma$, $\Pi^\gamma_\Lambda(\cdot|\bar{\sigma})$ is a probability measure on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$. The collection of probability kernels $\{\Pi^\gamma_\Lambda : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$ is called the *Gibbs specification* of the model we consider, see [16, Chap. 2]. It enjoys the consistency property

$$\int\limits_{\mathbb{R}^\gamma} \Pi^\gamma_{\Lambda_1}(A|\sigma)\Pi^\gamma_{\Lambda_2}(d\sigma|\bar{\sigma}) = \Pi^\gamma_{\Lambda_2}(A|\bar{\sigma}),$$

which holds for all $A \in \mathcal{B}(\mathbb{R}^\gamma)$, $\bar{\sigma} \in \mathbb{R}^\gamma$, and all $\Lambda_1, \Lambda_2 \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $\Lambda_1 \subset \Lambda_2$.

Definition 2 A probability measure μ on $(\mathbb{R}^\gamma, \mathcal{B}(\mathbb{R}^\gamma))$ is said to be a *quenched Gibbs measure* (for a fixed $\gamma \in \Gamma$) if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\mu(A) = \int\limits_{\mathbb{R}^\gamma} \Pi^\gamma_\Lambda(A|\sigma)\mu(d\sigma), \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}^\gamma).$$

The set of all such measures is denoted by $\mathcal{G}(\beta|\gamma)$.

In modern equilibrium statistical mechanics, the notion of thermodynamic phase of a system of bounded spins living on a fixed graph like $\mathbb{Z}^d$ is attributed to the extreme elements of the set of corresponding Gibbs measures, see, e.g., [16, Chap. 7] or [40, Chap. III]. However, for unbounded spins, not all extreme Gibbs measures may have physical meaning. It is believed that the measures corresponding to observed thermodynamic states should be supported on spin configurations with ‘tempered growth’ see [6,29,37,38] or a more recent development in [2,28] and [1, Chap. 3]. In this approach, only tempered Gibbs measures are taken into account, and hence a phase transition is related to the existence of multiple tempered Gibbs measures[2]. We take this approach and study quenched Gibbs measures introduced in Definition 2 with a priori prescribed support properties. We call them *tempered Gibbs states*. Thus, for an $\alpha > 0$, we define

$$\Sigma(\alpha) := \left\{ \sigma \in \mathbb{R}^\gamma : \sum_{x \in \gamma} |\sigma_x|^2 w_\alpha(x) < \infty \right\}, \tag{19}$$

where the weights w_α are as in (10). For each fixed $\gamma \in \Gamma$, $\Sigma(\alpha)$ is a Borel subset of $\mathbb{R}^\gamma$ and its elements are called tempered configurations. Then

$$\mathcal{G}_{\mathrm{t}}(\beta|\gamma) := \{\mu \in \mathcal{G}(\beta|\gamma) : \mu(\Sigma(\alpha)) = 1\} \tag{20}$$

is the set of tempered Gibbs states.

Theorem 1 *Let the single-spin measure χ be such that (14) holds and A_1 be as in Proposition 4. Then, for all $\gamma \in A_1$ and all $\beta > 0$, the set of Gibbs states $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$ is nonempty. Moreover, for each $\gamma \in A_1$ and each positive ϑ and α, there exists a finite $C_\gamma(\vartheta, \alpha) > 0$ such that the estimate*

$$\int_{\mathbb{R}^\gamma} \exp\left(\vartheta \sum_{x \in \gamma} |\sigma_x|^2 w_\alpha(x) \right) \mu(d\sigma) \le C_\gamma(\vartheta, \alpha) \tag{21}$$

holds uniformly for all $\mu \in \mathcal{G}_{\mathrm{t}}(\beta|\gamma)$.

Proof The proof of all the statements of the theorem follows by Theorem 1 of [26] since all the conditions of that theorem are satisfied in view of Proposition 4 and the assumed properties of χ.

Let us make some comments. The existence of Gibbs measures follows from the relative compactness of the family $\{\Pi^\gamma_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$ in the $\mathfrak{L}$-topology, see Definition 1, for at least some $\bar{\sigma} \in \Sigma(\alpha)$. A typical choice of $\bar{\sigma}$, for which the compactness is proven, is $\bar{\sigma}_x = s \in \mathbb{R}$ for all $x \in \gamma$. Note that such $\bar{\sigma}$ is tempered, see (19). Then the accumulation points of the family $\{\Pi^\gamma_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$ are shown to obey the Dobrushin–Lanford–Ruelle equation and to satisfy the estimate in (21), in which the constant $C_\gamma(\vartheta, \alpha)$ can be expressed explicitly in terms of the weights as in (10) and the parameters defined in (11), cf. Proposition 4 above. Thus, the accumulation points are tempered measures, and hence belong to $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$. Let us stress again that $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$ is nonempty for all those γ, for which both $a_\gamma(\alpha, \theta)$ and $b_\gamma(\alpha)$ are finite. By similar arguments, one can show that $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$ is compact in the $\mathfrak{L}$-topology. Finally, let us mention that $u > 2$ in (14) and θ in Proposition 4 should be such that $\theta(u-2) > 2$. Under this condition the sufficiently fast decay of the tail of χ compensates destabilizing effect of the property (9) of the underlying graph, see [26] for more detail. Since Proposition 4 holds for all $\theta > 0$, then we just assume that $u > 2$.

A sequence $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ is called *cofinal* if $\Lambda_n \subset \Lambda_{n+1}$, $n \in \mathbb{N}$, and each $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ is contained in a certain Λ_n. Given $\bar{\sigma} \in \Sigma(\alpha)$, the relative compactness of the family $\{\Pi^\gamma_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$ yields that there exists a cofinal sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ such that the sequence $\{\Pi^\gamma_{\Lambda_n}(\cdot|\bar{\sigma})\}_{n\in\mathbb{N}}$ converges in the $\mathfrak{L}$-topology to a certain element of $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$. For $a > 0$, by

$$\mu^{\pm a} \in \mathcal{G}_{\mathrm{t}}(\beta|\gamma) \tag{22}$$

[2] Note that the theory of quantum stabilization and phase transitions in quantum anharmonic crystals developed in [2,25,28] and [1, Chap. 6] with the use of tempered Gibbs measures is consistent with the corresponding phenomena observed experimentally.

we denote limiting elements of $\mathcal{G}_t(\beta|\gamma)$ which correspond to $\bar{\sigma}_x = \pm a$ for all $x \in \gamma$. Each such $\mu^{\pm a}$ depends on the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ along which it has been attained. Note that only limiting Gibbs measures can approximate thermodynamic states of large finite systems, see [16, Sect. 7.1].

Now we turn to the single-spin measure χ. If it has compact support, as was the case in [36], then (14) clearly holds for any u and $\varkappa$. The most known example of such χ is

$$\chi(dt) = [\delta_{-1}(dt) + \delta_{+1}(dt)]/2, \tag{23}$$

which corresponds to an Ising magnet. Here δ_s is the Dirac measure concentrated at $s \in \mathbb{R}$. We reserve a special notation $\mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$ for the set of all corresponding Gibbs measures. By $\nu^{\pm} \in \mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$, we denote the limiting Gibbs measures as in (22) with $a = 1$. In this case, however, $\nu^{\pm}$ are independent of the sequences $\{\Lambda_n\}_{n\in\mathbb{N}}$ along which they were attained. This holds because, for each x and $\nu \in \mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$,

$$\int_{\mathbb{R}^\gamma} \sigma_x \nu^-(d\sigma) \le \int_{\mathbb{R}^\gamma} \sigma_x \nu(d\sigma) \le \int_{\mathbb{R}^\gamma} \sigma_x \nu^+(d\sigma).$$

That is, ν^+ and ν^- are the maximum and minimum elements of $\mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$, respectively, cf. [28, Theorem 3.8].

In the case of 'unbounded' spins, a natural choice of the single-spin measure is

$$\chi(dt) = \exp(-V(t))\, dt,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a measurable even function such that: (a) the set $\{t \in \mathbb{R} : V(t) < +\infty\}$ is of positive Lebesgue measure; (b) $V(t)$ increases at infinity as $|t|^{u+\epsilon}$ with some $\epsilon > 0$ and u as in (14). This includes the case where V is a polynomial of even degree at least 4 with positive leading coefficient, cf. [26–29].

2.3 The Question of Uniqueness

Once the existence of Gibbs states has been established, the problem of their uniqueness/nonuniqueness arises. Thus, prior to proving non-uniqueness of $\mu \in \mathcal{G}_t(\beta|\gamma)$, which holds for π_λ-almost all γ whenever β and λ are large enough, see Theorem 2 below, we address the question of whether the same uniqueness does actually hold for some values of these parameters. For small enough λ, all the connected components of the graph $(\gamma, \varepsilon_\gamma)$ are finite, and hence $\mathcal{G}_t(\beta|\gamma)$ is a singleton for all β. On the other hand, $\mathcal{G}_t(\beta|\gamma)$ is a singleton if and only if, for each $x \in \gamma$, arbitrary $\bar{\sigma} \in \Sigma(\alpha)$, and any cofinal sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$, one has

$$\lim_{n\to+\infty} \int_{\mathbb{R}^\gamma} \sigma_x \Pi^\gamma_{\Lambda_n}(d\sigma|\bar{\sigma}) = 0. \tag{24}$$

This equivalence holds for any symmetric ferromagnet satisfying the bound in (21), that can be proven by standard arguments based on the Strassen theorem, see [28] for more detail. Actually, for models with 'unbounded' spins living on an infinite connected graph with globally unbounded degree, which by (9) is the case in our situation, there are no tools[3] for proving (24). For the Ising ferromagnet, the uniqueness in question can be obtained by percolation arguments, see [19, Theorem 7.2].

[3] The celebrated Dobrushin uniqueness technique is not applicable here.

3 The Phase Transition

3.1 The Statement

Recall that by a phase transition in the considered ferromagnet we mean the fact that the set of tempered Gibbs states $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$ for π_λ-almost all γ contains at least two elements (if β and λ are big enough). It is equivalent to the appearance of a nonzero magnetization in states $\mu^{\pm a} \in \mathcal{G}_{\mathrm{t}}(\beta|\gamma)$, cf. [16, Chap. 19] and (24).

Let us note that there is no interaction between spins in different connected components of the underlying graph. Then for a phase transition to occur it is necessary that the graph $(\gamma, \varepsilon_\gamma)$ possess an infinite connected component, which holds for π_λ-almost all γ whenever $\lambda > \lambda_*$, see [31,33] and also [17, Corollary 3.7] and [18, Theorem 3.1]. For $\lambda < \lambda_*$, we have no infinite connected component of $(\gamma, \varepsilon_\gamma)$ and thus $|\mathcal{G}(\beta|\gamma)| = 1$ for all β and π_λ-almost all γ. In order to obtain a sufficient condition for a phase transition to occur, we will explore the well-known relationship between the Bernoulli bond percolation on the fixed sample graph $(\gamma, \varepsilon_\gamma)$, established in Propositions 2 and 3, and the existence of multiple Gibbs states in the corresponding Ising model, established in [23]. Our goal is to prove the following statement.

Theorem 2 *Let the measure χ be as in Theorem 1 and such that $\chi(\{0\}) < \chi(\mathbb{R})$. Assume also that the intensity λ of the underlying Poisson point process satisfies the condition $\lambda > \lambda_*$, and thus the typical graph $(\gamma, \varepsilon_\gamma)$ has an infinite connected component. Then there exists a constant $\beta^* > 0$ such that, for $\beta > \beta^*$ and π_λ-almost all γ, the sets $\mathcal{G}_{\mathrm{t}}(\beta|\gamma)$ contain at least two elements.*

The proof of this statement is based on the following result, cf. (22).

Lemma 1 *Let the conditions of Theorem 2 be satisfied. Then there exist $a > 0$, and $\beta_* > 0$ such that, for π_λ-almost all γ, and for all $\beta > a^2\beta_*$, all $\mu^{+a} \in \mathcal{G}(\beta|\gamma)$, and some $o \in \gamma$, the following estimate holds:*

$$\int_{\mathbb{R}^\gamma} \sigma_o \mu^{+a}(d\sigma) > 0. \tag{25}$$

The proof of this lemma is given in the next subsection.

Proof of Theorem 2: Since the integral in (25) is the limit of those in (24) with $\bar{\sigma}_x = a$, then (25) contradicts (24) and hence implies non-uniqueness, which ought to hold for $\beta > \beta^* := a^2\beta_*$. On the other hand, by the invariance of χ and of the interaction in (15) with respect to the transformation $\sigma \to -\sigma$ and $\bar{\sigma} \to -\bar{\sigma}$, we have

$$\int_{\mathbb{R}^\gamma} \sigma_o \mu^{+a}(d\sigma) = -\int_{\mathbb{R}^\gamma} \sigma_o \mu^{-a}(d\sigma).$$

Then (25) yields $\mu^{+a} \neq \mu^{-a}$ and hence the multiplicity in question. Note that o in (25) belongs to the infinite connected component of $(\gamma, \varepsilon_\gamma)$, and the integral in (25) is the mean value of the spin at this vertex in state μ^{+a}.

3.2 Proof of Lemma 1

First, by means of the percolation arguments of [23], we prove the lemma for the Ising model. Then we extend the proof to the general case by comparison inequalities.

Recall that the single-spin measure of the Ising model is given in (23), $\mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$ denotes the set of all corresponding Gibbs measures, and $\nu^+ \in \mathcal{G}^{\mathrm{Ising}}(\beta|\gamma)$ is the maximum Gibbs measure as in (22) with $a = 1$. We are going to use the key fact proven in [23]: the Ising model with constant intensities $\phi(|x-y|) = \phi_* > 0$ on the edges of an infinite graph has at least two phases if and only if the graph admits the Bernoulli bond percolation with critical probability $q_* \in (0,1)$ if $\beta > [\log(1+q_*) - \log(1-q_*)]/2\phi_*$. In our case, for π_λ-almost all γ, the graph $(\gamma, \varepsilon_\gamma)$ admits this percolation and the threshold probability satisfies $q_* \geq \lambda_*/\lambda$, see Proposition 3. Then, for some $o \in \gamma$, it follows that

$$\int\limits_{\mathbb{R}^\gamma} \sigma_o \tilde{\nu}^+(d\sigma) > 0, \tag{26}$$

see [23, Theorem 2.1] and also the proof of Lemma 4.2 therein. Here $\tilde{\nu}^+$ is the corresponding Gibbs measure of the Ising model with $\phi(|x-y|) = \phi_* > 0$. By the standard GKS inequality, see, e.g., [23, Sect. 3.4], we have

$$\int\limits_{\mathbb{R}^\gamma} \sigma_o \nu^+(d\sigma) \geq \int\limits_{\mathbb{R}^\gamma} \sigma_o \tilde{\nu}^+(d\sigma),$$

which together with (26) yields the proof in this case.

Now we turn to the general case and estimate the integral in (25) from below by the corresponding integral with respect to the maximum Gibbs measure ν^+ of the Ising model with a rescaled interaction intensity. The proof of the lemma immediately follows from the Wells inequality used, e.g., in [35].

Proposition 5 (Wells inequality) *Let $a > 0$ be such that*

$$\chi\left([a\sqrt{2}, +\infty)\right) \geq \chi([0,a]). \tag{27}$$

Then, for each $x \in \gamma$ and each $\mu^{+a} \in \mathcal{G}(\beta|\gamma)$, as well as for $\nu^+ \in \mathcal{G}^{\mathrm{Ising}}(a^2\beta|\gamma)$, we have that

$$\int\limits_{\mathbb{R}^\gamma} \sigma_x \mu^{+a}(d\sigma) \geq a \int\limits_{\mathbb{R}^\gamma} \sigma_x \nu^+(d\sigma). \tag{28}$$

As the original publication [42] is hardly attainable and the proof in [10] contains numerical inaccuracies, for the reader convenience in Appendix we give a short proof of this inequality in the form suitable for our purposes.

4 Random Gibbs Measures

Random Gibbs measures of spin systems are supposed to depend on the random parameters (γ in ours case) in a measurable way, see, e.g., [9, Definition 6.2.5] and [27, Definition 2.3]. At the same time, the measures introduced in Definition 2 and (20) are defined on spaces which themselves depend on γ, and thus one cannot speak of the corresponding measurability in this setting. In order to settle this problem, we use some of the results of [14] where random Gibbs measures are defined as conditional measures on spaces of marked configurations. Here we outline the main points of this construction and prove the non-uniqueness of such measures, see Theorem 3 below.

For our model, the space of marked configurations is

$$\widehat{\Gamma} = \{\hat{\gamma} = (\gamma, \sigma) : \gamma \in \Gamma,\ \sigma \in \mathbb{R}^{\gamma}\},$$

where Γ is as in (3). Thus, the elements of $\widehat{\Gamma}$ consist of pairs (x, σ_x), and $\hat{\gamma} = \hat{\gamma}'$ implies $\gamma = \gamma'$. To relate $\hat{\gamma}$ with γ we use the canonical projection

$$\widehat{\Gamma} \ni \hat{\gamma} \mapsto \hat{p}(\hat{\gamma}) = \gamma \in \Gamma, \tag{29}$$

and equip $\widehat{\Gamma}$ with the following topology. Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a bounded continuous function with support contained in $\Lambda \times \mathbb{R}$ for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. The topology in question is the weakest one which makes the maps

$$\widehat{\Gamma} \ni \hat{\gamma} \mapsto \sum_{x \in \hat{p}(\hat{\gamma})} f(x, \sigma_x) \in \mathbb{R} \tag{30}$$

continuous for all possible $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ as above. This topology is completely and separably metrizable, see [13, Sect. 2], and thus $\widehat{\Gamma}$ is a Polish space. Let $\mathcal{B}(\widehat{\Gamma})$ denote the corresponding Borel σ-field. From the definition of the topologies of Γ and $\widehat{\Gamma}$, it follows that the projection defined in (29) is continuous, and hence, for each $\gamma \in \Gamma$, we have that

$$\hat{p}^{-1}(\{\gamma\}) =: \mathbb{R}^{\gamma} \in \mathcal{B}(\widehat{\Gamma}).$$

For each fixed $\gamma \in \Gamma$, $\mathbb{R}^{\gamma}$ is a Polish space embedded into $\widehat{\Gamma}$, which is a Polish space as well. By the Kuratowski theorem [39, page 21], the latter implies that the Borel σ-fields $\mathcal{B}(\mathbb{R}^{\gamma})$ and

$$\mathcal{A}(\mathbb{R}^{\gamma}) := \{A \in \mathcal{B}(\widehat{\Gamma}) : A \subset \mathbb{R}^{\gamma}\}$$

are measurably isomorphic. Thus, any probability measure μ on $\mathcal{B}(\widehat{\Gamma})$ with the property $\mu(\mathbb{R}^{\gamma}) = 1$ can be redefined as a measure on $\mathcal{B}(\mathbb{R}^{\gamma})$.

Let $\mathcal{P}(\widehat{\Gamma})$ be the set of all probability measures on $(\widehat{\Gamma}, \mathcal{B}(\widehat{\Gamma}))$. We equip it with the topology defined as follows. For a fixed $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, let $\mathcal{B}_{\Lambda}(\widehat{\Gamma})$ be the smallest σ-subfield of $\mathcal{B}(\widehat{\Gamma})$ such that the maps as in (30) are $\mathcal{B}_{\Lambda}(\widehat{\Gamma})$-measurable for all bounded measurable functions $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ with support contained in $\Lambda \times \mathbb{R}$. Then we set, cf. (16),

$$\mathcal{B}_{\mathrm{loc}}(\widehat{\Gamma}) = \bigcup_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)} \mathcal{B}_{\Lambda}(\widehat{\Gamma}).$$

Now the $\mathfrak{L}$-topology on $\mathcal{P}(\widehat{\Gamma})$ is defined as in Definition 1 by using $\mathcal{B}_{\mathrm{loc}}(\widehat{\Gamma})$ as the algebra of local sets. A map $\widehat{\Gamma} \ni \hat{\gamma} \mapsto \varphi(\hat{\gamma}) \in \mathbb{R}$ is called *local* if it is $\mathcal{B}_{\Lambda}(\widehat{\Gamma})$-measurable for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Local maps $\Gamma \ni \gamma \mapsto \varphi(\gamma) \in \mathbb{R}$ are defined in the same way.

Let $\bar{\sigma}$ in (17) be fixed in such a way that $\sigma_x = s$ for some $s \in \mathbb{R}$ and all $x \in \gamma$. Then, for each $\gamma \in \Gamma$ and $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and for such $\bar{\sigma}$, we can define a probability measure on $(\widehat{\Gamma}, \mathcal{B}(\widehat{\Gamma}))$ by setting

$$\widehat{\Pi}^{s}_{\Lambda}(A|\gamma) = \Pi^{\gamma}_{\Lambda}(A \cap \mathbb{R}^{\gamma}|\bar{\sigma}), \tag{31}$$

where $\Pi^{\gamma}_{\Lambda}(\cdot|\bar{\sigma})$ is given in (17). By the very construction, the map $\Gamma \ni \gamma \mapsto \widehat{\Pi}^{s}_{\Lambda}(A|\gamma) \in \mathbb{R}$ is measurable for each $A \in \mathcal{B}(\widehat{\Gamma})$. Thus, we can set

$$\hat{\eta}^{s}_{\Lambda}(\cdot) := \int_{\Gamma} \widehat{\Pi}^{s}_{\Lambda}(\cdot|\gamma)\pi_{\lambda}(d\gamma), \tag{32}$$

which is an element of $\mathcal{P}(\widehat{\Gamma})$, equipped with the $\mathfrak{L}$-topology defined above. It can be shown, see [14, Corollary 4.2], that, for each $s \in \mathbb{R}$, the family $\{\hat{\eta}^s_\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ is relatively compact in $\mathcal{P}(\widehat{\Gamma})$. Let $\hat{\eta}^s$ be its accumulation point and $\{\Lambda_n\}_{n\in\mathbb{N}}$ be the cofinal sequence such that $\hat{\eta}^s_{\Lambda_n} \to \hat{\eta}^s$ as $n \to +\infty$. By $\hat{\eta}^s_\Gamma$ we denote the projection of $\hat{\eta}^s$ on Γ. For a bounded local function $f : \Gamma \to \mathbb{R}$, $\hat{f} := f \circ \hat{p} : \widehat{\Gamma} \to \mathbb{R}$ is then also local, and hence

$$\int_\Gamma f d\pi_\lambda = \int_{\widehat{\Gamma}} \hat{f} d\hat{\eta}^s_{\Lambda_n} \to \int_{\widehat{\Gamma}} \hat{f} d\hat{\eta}^s = \int_\Gamma f d\hat{\eta}^s_\Gamma, \quad n \to +\infty.$$

Thus, $\hat{\eta}^s_\Gamma = \pi_\lambda$, and we can disintegrate, cf. (32), and obtain

$$\hat{\eta}^s(A) = \int_\Gamma \eta^s(A|\gamma)\pi_\lambda(d\gamma), \quad A \in \mathcal{B}(\widehat{\Gamma}), \tag{33}$$

where η^s is a regular conditional measure such that $\eta^s(A|\gamma) = \eta^s(A \cap \mathbb{R}^\gamma|\gamma)$ for almost all γ. As in (31), we then redefine $\eta^s(\cdot|\gamma)$ as a measure on $\mathbb{R}^\gamma$, for which we keep the same notation. One can prove [14] that, for almost all γ, $\eta^s(\cdot|\gamma) \in \mathcal{G}_t(\beta|\gamma)$. Thus, $\eta^s(\cdot|\gamma)$ is a *random Gibbs measure*.

In principle, we could construct such Gibbs measures without the study performed in Sect. 2, just by showing that the family $\{\hat{\eta}^s_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ is relatively compact in $\mathcal{P}(\widehat{\Gamma})$. However, this way has the following drawbacks: (a) in contrast to those in (22), the measures $\eta^s(\cdot|\gamma)$ need not be limiting and hence cannot approximate thermodynamic states of large finite systems; (b) there is no control on the sets of γ, as well as on their dependence on λ and s, for which $\eta^s(\cdot|\gamma)$ exist, cf. Proposition 4; (c) nothing can be said of the integrability properties of $\eta^s(\cdot|\gamma)$, cf. (21); (d) it is unclear whether we can have $\eta^{+a} \neq \eta^{-a}$. These problems are partly resolved in the statement below. Recall that each μ^{+a} is a measure on $\mathbb{R}^\gamma$, $\gamma \in A_1$, see Proposition 4, and is attained along a cofinal sequence $\mathcal{L} := \{\Lambda_n\}_{n\in\mathbb{N}}$, that will be indicated as $\mu^{+a}_{\mathcal{L}}$. For each $\gamma \in A_1$, such measures μ^{+a} constitute the set of accumulation points of the family $\{\Pi^\gamma_\Lambda(\cdot|\bar{\sigma}) : \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}$. The meaning of the theorem below is that, for a full π_λ-measure subset $A'_1 \subset A_1$, there exists a (measurable) selection $\{\mu^{+a}_{\mathcal{L}(\gamma)}\}_{\gamma\in A'_1}$ such that $\mu^{+a}_{\mathcal{L}(\gamma)} = \eta^{+a}(\cdot|\gamma)$ for all $\gamma \in A'_1$.

Theorem 3 *For arbitrary positive a and λ, there exists $A'_1 \subset A_1$ such that: (i) $\pi_\lambda(A'_1) = 1$; (ii) for each $\gamma \in A'_1$, there exists a cofinal sequence $\mathcal{L}(\gamma)$ such that $\mu^{+a}_{\mathcal{L}(\gamma)} = \eta^{+a}(\cdot|\gamma)$. Therefore, $\eta^{+a}(\cdot|\gamma) \neq \eta^{-a}(\cdot|\gamma)$, and hence quenched random Gibbs measures are multiple whenever the conditions of Theorem 2 are satisfied.*

Proof From now on we fix $s = +a$ and $\{\Lambda_n\}_{n\in\mathbb{N}}$, and hence $\hat{\eta}^s$. Note that we cannot expect that the assumed convergence $\hat{\eta}^s_{\Lambda_n} \to \hat{\eta}^s$ does imply that $\Pi^\gamma_{\Lambda_n}(\cdot|\bar{\sigma}) \to \eta^s(\cdot|\gamma)$, which would yield the proof.

Let $\{\Xi_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_b(\mathbb{R}^d)$ be a partition of $\mathbb{R}^d$. Fix dense subsets $\{y^n_k\}_{k\in\mathbb{N}} \subset \Xi_n$. As in the proof of Lemma 2.3 in [24, p. 20], by means of this partition we introduce a linear order of the elements of each $\gamma \in \Gamma$. If $x, y \in \gamma$ belong to distinct Ξ_n, we set $x < y$ whenever $x \in \Xi_n$, $y \in \Xi_m$, and $n < m$. If both x and y lie in the same Ξ_n, let k be the smallest integer such that $|x - y^n_k| \neq |y - y^n_k|$. Then we set $x < y$ if $|x - y^n_k| < |y - y^n_k|$. Next, we enumerate the elements of γ in accordance with the order in such a way that $x_1 < x_2 < \cdots < x_k < \cdots$. This defines an *enumeration* on $\widehat{\Gamma}$, that is, the map

$$\hat{\gamma} \mapsto \epsilon(\hat{\gamma}) = \{(x_1, \sigma_{x_1}), \ldots, (x_k, \sigma_{x_k}), \ldots\} \tag{34}$$

such that all (x_k, σ_{x_k}) are distinct and $(x_k, \sigma_{x_k}) \in \hat{\gamma}$. This map is *measurable* in the sense that $\{\hat{\gamma} : x_k \in \Delta,\ \sigma_{x_k} \in \Sigma\} \in \mathcal{B}(\widehat{\Gamma})$ for each $k \in \mathbb{N}$, $\Delta \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $\Sigma \in \mathcal{B}(\mathbb{R})$. The latter fact can be proven by a slight generalization of the proof of Lemma 2.3 in [24, p. 20]. Then, for a fixed $\Delta \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and any $k \in \mathbb{N}$, $p \in (\mathbb{Q}^d)^{\mathbb{N}}$, and $q \in \mathbb{Q}^{\mathbb{N}}$, the real and imaginary parts of the function

$$\hat{\gamma} \mapsto \exp[i(p_k \cdot x_k + q_k \sigma_{x_k})] \in \mathbb{C}$$

are as in (30), and hence are $\mathcal{B}_\Delta(\widehat{\Gamma})$- measurable. Here $i = \sqrt{-1}$ and $p \cdot x$ stands for the scalar product in $\mathbb{R}^d$. Let $\mathcal{D} = \{\Delta_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ be a cofinal sequence. For ϵ as in (34), $\Delta \in \mathcal{D}$, $p \in (\mathbb{Q}^d)^{\mathbb{N}}$, and $q \in \mathbb{Q}^{\mathbb{N}}$, we set

$$f_{\Delta,p,q}(\hat{\gamma}) = \exp\left(i \sum_{j=1}^{|\gamma_\Delta|} \left[p_{k_j} \cdot x_{k_j} + q_{k_j} \sigma_{x_{k_j}} \right] \right), \tag{35}$$

where $\gamma = \hat{p}(\hat{\gamma})$, (x_k, σ_{x_k}) is the k-th element of the sequence $\epsilon(\hat{\gamma})$, and the sum runs over the set of all those j for which $x_{k_j} \in \Delta$. By construction, each such f is $\mathcal{B}_\Delta(\widehat{\Gamma})$-measurable. Let $\mathcal{F}$ be the (countable) family of all such functions. It has the following properties: (a) is closed under point-wise multiplication; (b) separates points of $\widehat{\Gamma}$. The latter means that, for any two distinct $\hat{\gamma}, \hat{\gamma}' \in \widehat{\Gamma}$, one finds $f \in \mathcal{F}$ such that $f(\hat{\gamma}) \neq f(\hat{\gamma}')$. By Fernique's theorem [41, p. 6], property (b) implies that $\sigma(\mathcal{F}) = \mathcal{B}(\widehat{\Gamma})$. That is, the smallest σ-field of subsets of $\widehat{\Gamma}$ with respect to which each $f \in \mathcal{F}$ is measurable, is $\mathcal{B}(\widehat{\Gamma})$. Combining this with property (a) we then obtain, see [7, . 149] or the proof of Theorem 1.3.26 in [1, p. 113], that $\mathcal{F}$ is a separating class for $\mathcal{P}(\widehat{\Gamma})$. That is, $\mu, \nu \in \mathcal{P}(\widehat{\Gamma})$ coincide if and only if

$$\int f d\mu = \int f d\nu, \qquad \text{for all } f \in \mathcal{F}.$$

For a fixed triple Δ, p, q, by the assumed convergence $\hat{\eta}^s_{\Lambda_n} \to \hat{\eta}^s$ we have that

$$\int_{\widehat{\Gamma}} f_{\Delta,p,q}(\hat{\gamma}) \hat{\eta}^s_{\Lambda_n}(d\hat{\gamma}) = \int_{\Gamma} g^{(n)}_{\Delta,p,q}(\gamma) \pi_\lambda(d\gamma) \to \int_{\widehat{\Gamma}} f_{\Delta,p,q}(\hat{\gamma}) \hat{\eta}^s(d\hat{\gamma}), \tag{36}$$

as $n \to +\infty$. Here, cf. (33) and (35),

$$\begin{aligned} g^{(n)}_{\Delta,p,q}(\gamma) &:= \int_{\widehat{\Gamma}} f_{\Delta,p,q}(\hat{\gamma}) \widehat{\Pi}^s_{\Lambda_n}(d\hat{\gamma}|\gamma) = \exp\left(i \sum_{j=1}^{|\gamma_\Delta|} p_{k_j} \cdot x_{k_j} \right) h^{(n)}_{\Delta,q}(\gamma), \\ h^{(n)}_{\Delta,q}(\gamma) &:= \int_{\mathbb{R}^\gamma} \exp\left(i \sum_{x\in\gamma_\Delta} q_{k(x)} \sigma_x \right) \Pi^\gamma_{\Lambda_n}(d\sigma|\bar{\sigma}), \end{aligned} \tag{37}$$

where $k(x)$ is the number of $x \in \gamma = \hat{p}(\hat{\gamma})$ defined by the enumeration (34). Obviously, $|h^{(n)}_{\Delta,q}(\gamma)| \leq 1$ for all $n \in \mathbb{N}$, and $\{h^{(n)}_{\Delta,q}\}_{n\in\mathbb{N}} \subset L^1(\Gamma, d\pi_\lambda)$. By Komlós' theorem (see, e.g., [4,27]), there exists a subsequence $\{h^{(n_l)}_{\Delta,q}\}_{l\in\mathbb{N}} \subset \{h^{(n)}_{\Delta,q}\}_{n\in\mathbb{N}}$ such that, for each further subsequence $\{h^{(n_{l_m})}_{\Delta,q}\}_{m\in\mathbb{N}} \subset \{h^{(n_l)}_{\Delta,q}\}_{l\in\mathbb{N}}$, one has

$$\frac{1}{M} \sum_{m=1}^{M} h^{(n_{l_m})}_{\Delta,q}(\gamma) \to h_{\Delta,q}(\gamma), \qquad \text{for } \pi_\lambda - \text{a.a. } \gamma \in \Gamma, \tag{38}$$

where $h_{\Delta,q}$ is a certain element of $L^1(\Gamma, d\pi_\lambda)$. Note that the subsequence $\{n_l\}_{l\in\mathbb{N}}$ depends on the choice of Δ, q. However, by the diagonal procedure as in [27] one can pick $\{n_{l_m}\}_{m\in\mathbb{N}} \subset \{n_l\}_{l\in\mathbb{N}}$ such that (38) holds for all $\Delta \in \mathcal{D}$, and $q \in \mathbb{Q}^{\mathbb{N}}$. Then by (37) and (38) we get

$$\int_{\mathbb{R}^\gamma} \exp\left(i \sum_{x\in\gamma_\Delta} q_{k(x)}\sigma_x\right) P_M^\gamma(d\sigma|\bar{\sigma}) \to h_{\Delta,q}(\gamma), \quad \text{for } \pi_\lambda - \text{a.a. } \gamma \in \Gamma, \tag{39}$$

where $\Lambda^m := \Lambda_{n_{l_m}}$ and

$$P_M^\gamma(d\sigma|\bar{\sigma}) := \frac{1}{M}\sum_{m=1}^{M} \Pi^\gamma_{\Lambda^m}(d\sigma|\bar{\sigma}), \qquad \bar{\sigma}_x = s = +a, \quad x \in \gamma. \tag{40}$$

Note that by Komlós' theorem the convergence in (39) holds also for the Cesàro means of each subsequence of $\{\Pi^\gamma_{\Lambda^m}(\cdot|\bar{\sigma})\}_{m\in\mathbb{N}}$. For any $\gamma \in A_1$, by Proposition 4 $\{\Pi^\gamma_{\Lambda^m}(\cdot|\bar{\sigma})\}_{m\in\mathbb{N}}$ is relatively compact in the $\mathfrak{L}$-topology. Thus, one can pick $\mathcal{L}(\gamma) = \{\Lambda^{m_l}(\gamma)\}_{l\in\mathbb{N}} \subset \{\Lambda^m\}_{m\in\mathbb{N}}$, for which, cf. (22),

$$\Pi^\gamma_{\Lambda^{m_l}(\gamma)}(\cdot|\bar{\sigma}) \to \mu^{+a}_{\mathcal{L}(\gamma)} \in \mathcal{G}_{\mathrm{t}}(\beta|\gamma), \quad l \to +\infty. \tag{41}$$

Note that the dependence of the sequence $\mathcal{L}(\gamma)$ on γ can be very irregular in view of the so called *chaotic size dependence*, see [32] and the discussion in [27]. Now let $A_1' \subset A_1$ be such that also (39) holds for all Δ and q. Then, for a fixed $\gamma \in A_1'$, by (40) and (41) there exists the sequence $\{P_M^\gamma\}$ such that $P_M^\gamma \to \mu^{+a}_{\mathcal{L}(\tau)}$, which yields

$$h_{\Delta,q}(\gamma) = \int_{\mathbb{R}^\gamma} \exp\left(i \sum_{x\in\gamma_\Delta} q_{k(x)}\sigma_x\right) \mu^{+a}_{\mathcal{L}(\gamma)}(d\sigma), \quad \gamma \in A_1'.$$

Since $h_{\Delta,q} \in L^1(\Gamma, d\pi_\lambda)$, we can integrate and by (33), (36), (37), and (38) obtain

$$\begin{aligned}
&\int_\Gamma \exp\left(i\sum_{j=1}^{|\gamma_\Delta|} p_{k_j}\cdot x_{k_j}\right)\left[\int_{\mathbb{R}^\gamma} \exp\left(i\sum_{x\in\gamma_\Delta} q_{k(x)}\sigma_x\right)\eta^{+a}(d\sigma|\gamma)\right]\pi_\lambda(d\gamma)\\
&= \int_\Gamma \exp\left(i\sum_{j=1}^{|\gamma_\Delta|} p_{k_j}\cdot x_{k_j}\right)\left[\int_{\mathbb{R}^\gamma} \exp\left(i\sum_{x\in\gamma_\Delta} q_{k(x)}\sigma_x\right)\mu^{+a}_{\mathcal{L}(\gamma)}(d\sigma)\right]\pi_\lambda(d\gamma),
\end{aligned}$$

which holds for all $\Delta \in \mathcal{D}$, $p \in (\mathbb{Q}^d)^{\mathbb{N}}$, and $q \in \mathbb{Q}^{\mathbb{N}}$. This yields the proof since the integrand functions constitute separating classes.

Acknowledgments This work was financially supported by the DFG through SFB 701: "Spektrale Strukturen und Topologische Methoden in der Mathematik" and through the research project 436 POL 125/0-1. The support from the ZiF Research Group "Stochastic Dynamics: Mathematical Theory and Applications" (Universität Bielefeld) was also very helpful. The authors benefited from discussions with Philippe Blanchard and Stas Molchanov, for which they are cordially indebted. The authors thank also both referees for constructive criticism and valuable suggestions.

Appendix: Proof of Proposition 5

For the general choice of χ, let $\Pi^{\gamma,+a}_{\Lambda}$ be defined as in (17) with $\bar{\sigma}_x = +a$ for all $x \in \gamma$. Each μ^{+a} is the limit of $\{\Pi^{\gamma,+a}_{\Lambda_n}\}_{n\in\mathbb{N}}$ for some cofinal sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$. In the case of unbounded spins, this convergence alone does not yet imply the convergence of the moments of $\Pi^{\gamma,+a}_{\Lambda_n}$ to that on the left-hand side of (28). Then we use the uniform in n bound as in (21), which can also be proven for all $\Pi^{\gamma,+a}_{\Lambda}$, and obtain

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,+a}_{\Lambda_n}(d\sigma) \to \int_{\mathbb{R}^\gamma} \sigma_x \mu^{+a}(d\sigma), \qquad n \to +\infty.$$

Since the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is exhausting, it contains a cofinal subsequence, $\{\Lambda_{n_k}\}_{k\in\mathbb{N}}$, such that also

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,\mathrm{Ising}}_{\Lambda_{n_k}}(d\sigma) \to \int_{\mathbb{R}^\gamma} \sigma_x \nu^{+}(d\sigma), \qquad n \to +\infty,$$

where $\Pi^{\gamma,\mathrm{Ising}}_{\Lambda_{n_k}}$ is the kernel (17) corresponding to the Ising single-spin measure (23), interaction intensities $a^2\phi(|x-y|)$, and the choice $\bar{\sigma}_x = +1$ for all $x \in \gamma$. Thus, the validity of (28) will follow if we prove that, for each Λ which contains x, the following holds

$$\int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,+a}_{\Lambda}(d\sigma) \geq a \int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,\mathrm{Ising}}_{\Lambda}(d\sigma). \tag{42}$$

Let $Z^{\gamma}_{\Lambda}(a)$ and $Z^{\gamma,\mathrm{Ising}}_{\Lambda}(1)$ be the corresponding normalizing factors defined in (18). Then by (17) we have, cf. (15),

$$\begin{aligned}
&\int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,+a}_{\Lambda}(d\sigma) - a\int_{\mathbb{R}^\gamma} \sigma_x \Pi^{\gamma,\mathrm{Ising}}_{\Lambda}(d\sigma) = \left(Z_{\Lambda}(a) Z^{\gamma,\mathrm{Ising}}_{\Lambda}(1)\right)^{-1} \\
&\quad\times \int_{\mathbb{R}^\gamma}\int_{\mathbb{R}^\gamma} (\sigma_x - a\tilde{\sigma}_x) \exp\Bigg\{\beta \sum_{\{x,y\}\subset\gamma_\Lambda} \phi(|y-z|)[\sigma_y\sigma_z + a^2\tilde{\sigma}_y\tilde{\sigma}_z] \\
&\quad + \sum_{y\in\gamma_\Lambda} [\sigma_y + a\tilde{\sigma}_y]K_y\Bigg\} \bigotimes_{x\in\gamma_\Lambda} \left(\chi(d\sigma_x)\otimes\chi^{\mathrm{Ising}}(d\tilde{\sigma}_x)\right),
\end{aligned} \tag{43}$$

where χ^{Ising} is given in (23) and $K_y = \beta a \sum_{z\in\gamma_{\Lambda^c}:z\sim y} \phi(|y-z|) \geq 0$. Then (42) will follow from the positivity of the integral on the right-hand side of (43). Now we rewrite the integrand in (43) in the variables $u^{\pm}_x := (\sigma_x \pm a\tilde{\sigma}_x)/\sqrt{2}$, and then expand the exponent and write the integral as the sum of the products over $x \in \gamma_\Lambda$ of 'one-site' integrals having the form

$$\begin{aligned}
&C_x \int_{\mathbb{R}^2} (u^+_x)^{m_x}(u^-_x)^{n_x}\chi(d\sigma_x)\otimes\chi^{\mathrm{Ising}}(d\tilde{\sigma}_x) \\
&= C_x \int_{\mathbb{R}} \left[(\sigma_x + a)^{m_x}(\sigma_x - a)^{n_x} + (\sigma_x - a)^{m_x}(\sigma_x + a)^{n_x}\right]\chi(d\sigma_x), \quad C_x \geq 0.
\end{aligned} \tag{44}$$

Thus, to prove the statement we have to show that the integral on the right-hand side of (44) is nonnegative for all values of $m_x, n_x \in \mathbb{N}_0$. By the assumed symmetry of χ, this integral vanishes if m_x and n_x are of different parity. If both m_x and n_x are even, then the positivity is immediate. Thus, it is left to consider the case where $m_x = 2k+1$ and $n_x = 2l+1$. By the symmetry of χ, it is enough to take $k \geq l$. Thus, we have to prove the positivity of the following integral

$$\int_{\mathbb{R}} \left[(\sigma+a)^{2k+1}(\sigma-a)^{2l+1} + (\sigma-a)^{2k+1}(\sigma+a)^{2l+1}\right]\chi(d\sigma)$$

$$= 2\int_0^{+\infty} (\sigma^2-a^2)^{2l+1}\left[(\sigma+a)^{k-l} + (\sigma+a)^{k-l}\right]\chi(d\sigma).$$

The function $\varphi(\sigma) := (\sigma+a)^{k-l} + (\sigma+a)^{k-l}$ is increasing on $[0,+\infty)$. The integral on the right-hand side of the latter equality can be written in the form

$$\int_0^{+\infty} (\sigma^2-a^2)^{2l+1}\varphi(\sigma)\chi(d\sigma) = I_1(a) + I_2(a) + I_3(a),$$

$$I_1(a) := \int_0^{a} (\sigma^2-a^2)^{2l+1}\varphi(\sigma)\chi(d\sigma) \geq -a^{4l+2}\varphi(a)\chi([0,a]),$$

$$I_2(a) := \int_a^{a\sqrt{2}} (\sigma^2-a^2)^{2l+1}\varphi(\sigma)\chi(d\sigma) \geq 0,$$

$$I_3(a) := \int_{a\sqrt{2}}^{+\infty} (\sigma^2-a^2)^{2l+1}\varphi(\sigma)\chi(d\sigma) \geq a^{4l+2}\varphi(a\sqrt{2})\chi([a\sqrt{2},+\infty)) \tag{45}$$

In view of (27), the sum on the right-hand side of (45) is nonnegative, which completes the proof.

References

1. Albeverio, S., Kondratiev, Yu., Kozitsky, Yu., Röckner, M., The Statistical Mechanics of Quantum Lattice Systems. A Path Integral Approach. EMS Tracts in Mathematics, vol. 8. European Mathematical Society, Zürich (2009)
2. Albeverio, S., Kondratiev, Y., Kozitsky, Y., Röckner, M.: Phase transitions and quantum effects in anharmonic crystals. Int. J. Modern Phys. B **26**, 1250063 (2012)
3. Albeverio, S., Kondratiev, Yu.G., Röckner, M.: Analysis and geometry on configuration spaces. J. Func. Anal. **154**, 444–500 (1998)
4. Balder, E.J.: Infinite-dimensional extensions of a theorem of Komlós. Probab. Theor. Relat. Fields **81**, 185–188 (1989)
5. Balister, P., Bollobás, B., Sarkar, A., Percolation, connectivity, coverage and colouring of random geometric graphs. In: Handbook of Large-Scale Random Networks, Bolyai Society Mathematical Studies, vol. 18, pp. 117–142, Springer, Berlin (2009)
6. Bellissard, J., Høegh-Krohn, R.: Compactness and the maximal Gibbs state for random Gibbs fields on a lattice. Commun. Math. Phys. **84**, 297–327 (1982)
7. Bogachev, V.I.: Measure Theory, vol. 1. Springer, Berlin (2007)
8. Ben Arous, G., Molchanov, S., Ramírez, A.F.: Transition from the annealed to the quenched asymptotics for a random walk on random obstacles. Ann. Probab. **33**, 2149–2187 (2005)
9. Bovier, A., Statistical Mechanics of Disordered Systems: A Mathematical Perspective. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge (2006)

10. Bricmont, J., Lebowitz, J.L., Pfister, Ch.E.: Periodic Gibbs states of ferromagnetic spin systems. J. Stat. Phys. **24**, 269–277 (1981)
11. Chayes, J.T., Chayes, L., Kotecký, R.: The analysis of the Widom–Rowlonson model by stochastic geometric methods. Commun. Math. Phys. **172**, 551–569 (1995)
12. Chayes, L., Shlosman, S.B., Zagrebnov, V.A.: Discontinuity of the magnetization in diluted $O(n)$-models. J. Stat. Phys. **98**, 537–549 (2000)
13. Conrad, F., Grothaus, M.: N/V-limit for Langevin dynamics in continuum. Rev. Math. Phys. **23**, 1–51 (2011)
14. Daletskii, A., Kondratiev, Yu., Kozitsky, Yu., Pasurek, T., Gibbs states on random configurations, preprint SFB-701, 13026, 2013, Bielefeld; arXiv: 1307.4718vl [math-ph]
15. Daley, D.J., Vere-Jones, D.: An Introduction to the Theory of Point Processes, vol. 2, 2nd edn. Springer, New York (2003)
16. Georgii, H.-O.: Gibbs Measures and Phase Transitions (de Gruyter Studies in Mathematics, vol. 9. de Gruyter, Berlin (1988)
17. Georgii, H.-O., Häggström, O.: Phase transition in continuum Potts models. Commun. Math. Phys. **181**, 507–528 (1996)
18. Georgii, H.-O., Phase transition and percolation in Gibbsian particle models. In: Statistical physics and spatial statistics (Wuppertal, 1999). Lecture Notes in Physics, vol. 554, pp. 267–294,Springer, Berlin (2000)
19. Georgii, H.-O., Häggström, O., Maes, C., The random geometry of equilibrium phases. In: Phase transitions and critical phenomena, vol. 18, pp. 1–142, Academic Press, San Diego (2001)
20. Georgii, H.-O., Zagrebnov, V.A.: On the interplay of magnetic and molecular forces in Curie-Weiss ferrofluid models. J. Stat. Phys. **93**, 79–107 (1998)
21. Gruber, Ch., Griffiths, R.B.: Phase transition in a ferromagnetic fluid. Physica A **138**, 220–230 (1986)
22. Gruber, Ch., Tamura, H., Zagrebnov, V.A.: Berezinskii–Kosterlitz–Thouless order in two-dimensional O(2)-ferrofluid. J. Stat. Phys. **106**, 875–893 (2002)
23. Häggström, O.: Markov random fields and percolation on general graphs. Adv. Appl. Probab. **32**, 39–66 (2000)
24. Kallenberg, O.: Random Measures, 3rd edn. Akademie-Verlag, Berlin (1983)
25. Kargol, A., Kondratiev, Yu., Kozitsky, Yu.: Phase transitions and quantum stabilization in quantum anharmonic crystals. Rev. Math. Phys. **20**, 529–595 (2008)
26. Kondratiev, Yu., Kozitsky, Yu., Pasurek, T.: Gibbs random fields with unbounded spins on unbounded degree graphs. J. Appl. Probab. **47**, 856–875 (2010)
27. Kondratiev, Yu., Kozitsky, Yu., Pasurek, T.: Gibbs measures of disordered lattice systems with unbounded spins. Markov Process. Relat. Fields **18**, 553–582 (2012)
28. Kozitsky, Yu., Pasurek, T.: Euclidean Gibbs measures of interacting quantum anharmonic oscillators. J. Stat. Phys. **127**, 985–1047 (2007)
29. Lebowitz, J.L., Presutti, E.: Statistical mechanics of systems of unbounded spins. Commun. Math. Phys. **50**, 195–218 (1976)
30. Lyons, R.: The Ising model and percolation on trees and tree-like graphs. Commun. Math. Phys. **125**, 337–353 (1989)
31. Meester, R., Roy, R.: Continuum Percolation. Cambridge Tracts in Mathematics, vol. 119. Cambridge University Press, Cambridge (1996)
32. Newman, C.M., Stein, D.L.: Thermodynamic chaos and the structure of short-range spin glasses. In: Bovier, A., Picco, P. (eds.) Mathematical Aspects of Spin Glasses and Neural Networks. Progress in Probability, vol. 41, pp. 243–287. Birkhäuser Boston, Boston (1998)
33. Penrose, M.D.: Random Geometric Graphs, (Oxford Studies in Probability, 5. Oxford University Press, Oxford (2003)
34. O'Handley, R.C.: Modern Magnetic Materials: Principles and Applications. Wiley, New York (2000)
35. Osada, H., Spohn, H.: Gibbs measures relative to Brownian motion. Ann. Probab. **27**, 1183–1207 (1999)
36. Romano, S., Zagrebnov, V.A.: Orientational ordering transition in a continuous-spin ferrofluid. Phys. A **253**, 483–497 (1998)
37. Park, Y.M., Yoo, H.J.: A characterization of Gibbs states of lattice boson systems. J. Stat. Phys. **75**, 215–239 (1994)
38. Park, Y.M., Yoo, H.J.: Uniqueness and clustering properties of Gibbs states for classical and quantum unbounded spins. J. Stat. Phys. **80**, 223–271 (1995)
39. Parthasarathy, K.R.: Probability Measures on Metric Spaces (Probability and Mathematical Statistics 3. Academic Press Inc, New York (1967)
40. Simon, B.: The Statistical Mechanics of Lattice Gases. Princeton Series in Physics, vol. I. Princeton University Press, Princeton (1993)

41. Vakhania, N.N., Tarieladze, V.I., Chobanian, S.A.: Probability Distributions on Banach Spaces. D. Reidel Publishing Company, Dordrecht (1987)
42. Wells, D., Some Moment Inequalities and a Result on Multivariable Unimodality Thesis, Indiana University,Bloomington (1977)