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ERRATUM

Erratum to: On the Stability of the Flux Reconstruction Schemes on Quadrilateral Elements for the Linear Advection Equation

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The authors would like to report a mistake in (3.32) in the original paper. In the paper cited as citation number 12, i.e., Castonguay et al. [1], the summation over the elements is performed after transforming Θ_{adv} to the physical domain. When this transformation is carried out in [1], there is a J_k term that appears in Θ_{adv} which cancels out with the J_k in the LHS. This cancellation happens in our case as well through all the extra terms, which was missed. This can be seen by looking at (3.34), where J_k appears in all the terms of Θ_{extra}. In the cited paper, the cancellation is important to obtain the result $\Theta_{adv} \leq 0$ which is used by us. Thus, the correct approach would be to transform the quantities to physical domain, perform the cancellation and then sum over all the elements. This mistake affects the final result obtained in the paper and thereby some of the statements made in the Abstract and the Conclusions sections as well. All the derivations up until Lemma 3.4 are correct. Here, we give the corrected version of Theorem 3.5 and its proof. We apologize for any inconvenience caused.

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Theorem 3.5 *If the FR scheme for a 2D conservation law with periodic boundary conditions is used in conjunction with the Lax–Friedrichs formulation for the common interface flux*

$$\boldsymbol{f}^* = \{\{\boldsymbol{f}^D\}\} + \frac{\lambda}{2}\left(\max_{u \in [u_-^D, u_+^D]} \left|\frac{\partial \boldsymbol{f}}{\partial u} \cdot n\right|\right)[[u^D]] \tag{3.25}$$

with $0 \leq \lambda \leq 1$*, and if a non-negative value of the VCJH parameter c is used, then it can be shown that for a linear advective flux and any Cartesian mesh, the following holds*

$$\frac{d}{dt}\|u^D\|^2_{W^{2p,2}_\delta} \leq 0 \tag{3.26}$$

for a modified Sobolev norm defined as follows

$$\|u^D\|^2_{W^{2p,2}_\delta} = \sum_{k=1}^{N}\left(\int_{\Omega_k}\left[\left(u_k^D\right)^2 + \frac{c}{2}\left(\left(\frac{\partial^p u_k^D}{\partial \xi^p}\right)^2 + \left(\frac{\partial^p u_k^D}{\partial \eta^p}\right)^2\right) + \frac{c^2}{4}\left(\frac{\partial^{2p} u_k^D}{\partial \xi^p \partial \eta^p}\right)^2\right] d\Omega_k\right) \tag{3.27}$$

Note: For brevity of proof, we discuss the properties of the above norm in Appendix A.

Proof Equations (3.28)–(3.31) are correct and remain as it is. However, instead of (3.32), let us just substitute (3.29), (3.30) and the other results similar to (3.31) into (3.28) without summing over all the elements to get

$$\begin{aligned}
&\frac{1}{2}J_k\frac{d}{dt}\left(\int_{\Omega_k}(u_k^D)^2 d\Omega_k + \frac{c}{2}\int_{\Omega_k}\left(\left(\frac{\partial^p u_k^D}{\partial \xi^p}\right)^2 + \left(\frac{\partial^p u_k^D}{\partial \eta^p}\right)^2\right)d\Omega_k \right.\\
&\left. + \frac{c^2}{4}\int_{\Omega_k}\left(\frac{\partial^{2p} u_k^D}{\partial \xi^p \partial \eta^p}\right)^2 d\Omega_k\right)\\
&= -\int_{\Omega_S}\hat{u}^D(\hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{f}}^D)d\Omega_S - \int_{\Gamma_S}\hat{u}^D(\hat{\boldsymbol{f}}^C \cdot \hat{\boldsymbol{n}})d\Gamma_S\\
&+ c\left(\left[\frac{1}{2}\frac{\partial^p \hat{u}_R^D}{\partial \eta^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D \cdot \hat{\boldsymbol{n}})_R}{\partial \eta^p} - \frac{\partial^p \hat{u}_R^D}{\partial \eta^p}\frac{\partial^p(\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{n}})_R^*}{\partial \eta^p}\right]\right.\\
&+ \left[\frac{1}{2}\frac{\partial^p \hat{u}_L^D}{\partial \eta^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D \cdot \hat{\boldsymbol{n}})_L}{\partial \eta^p} - \frac{\partial^p \hat{u}_L^D}{\partial \eta^p}\frac{\partial^p(\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{n}})_L^*}{\partial \eta^p}\right]\\
&+ \left[\frac{1}{2}\frac{\partial^p \hat{u}_T^D}{\partial \xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D \cdot \hat{\boldsymbol{n}})_T}{\partial \xi^p} - \frac{\partial^p \hat{u}_T^D}{\partial \xi^p}\frac{\partial^p(\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{n}})_T^*}{\partial \xi^p}\right]\\
&\left.+ \left[\frac{1}{2}\frac{\partial^p \hat{u}_B^D}{\partial \xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D \cdot \hat{\boldsymbol{n}})_B}{\partial \xi^p} - \frac{\partial^p \hat{u}_B^D}{\partial \xi^p}\frac{\partial^p(\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{n}})_B^*}{\partial \xi^p}\right]\right)
\end{aligned}$$

Transforming the RHS of the above equation to the physical domain, we get,

$$\frac{1}{2}J_k\frac{d}{dt}\left(\int\limits_{\Omega_k}(u_k^D)^2 d\Omega_k+\frac{c}{2}\int\limits_{\Omega_k}\left(\left(\frac{\partial^p u_k^D}{\partial\xi^p}\right)^2+\left(\frac{\partial^p u_k^D}{\partial\eta^p}\right)^2\right)d\Omega_k+\frac{c^2}{4}\int\limits_{\Omega_k}\left(\frac{\partial^{2p}u_k^D}{\partial\xi^p\partial\eta^p}\right)^2 d\Omega_k\right)$$

$$\begin{aligned}
&= -J_k\int\limits_{\Omega_k}u^D(\nabla\cdot\boldsymbol{f}^D)d\Omega_k - J_k\int\limits_{\Gamma_k}u^D(\boldsymbol{f}^C\cdot\boldsymbol{n})d\Gamma_k \\
&\quad + c\Bigg(J_{y_k}^{2p+1}J_k\left[\frac{1}{2}\frac{\partial^p u_R^D}{\partial y^p}\frac{\partial^p F_R^D}{\partial y^p}-\frac{\partial^p u_R^D}{\partial y^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_R^*}{\partial y^p}\right]_k \\
&\quad\quad + J_{y_k}^{2p+1}J_k\left[-\frac{1}{2}\frac{\partial^p u_L^D}{\partial y^p}\frac{\partial^p F_L^D}{\partial y^p}-\frac{\partial^p u_L^D}{\partial y^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_L^*}{\partial y^p}\right] \\
&\quad\quad + J_{x_k}^{2p+1}J_k\left[\frac{1}{2}\frac{\partial^p u_T^D}{\partial x^p}\frac{\partial^p G_T^D}{\partial x^p}-\frac{\partial^p u_T^D}{\partial x^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_T^*}{\partial x^p}\right] \\
&\quad\quad + J_{x_k}^{2p+1}J_k\left[-\frac{1}{2}\frac{\partial^p u_B^D}{\partial x^p}\frac{\partial^p G_B^D}{\partial x^p}-\frac{\partial^p u_B^D}{\partial x^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_B^*}{\partial x^p}\right]\Bigg)
\end{aligned}$$

It is clear from the above equation that J_k cancels across all the terms. After cancellation of J_k, we sum over all the elements to get

$$\frac{d}{dt}\|u^D\|^2 = \Theta_{adv} + c\Theta_{extra}$$

where

$$\|u^D\|^2=\sum_{k=1}^{N}\left(\int\limits_{\Omega_k}\left[(u_k^D)^2+\frac{c}{2}\left(\left(\frac{\partial^p u_k^D}{\partial\xi^p}\right)^2+\left(\frac{\partial^p u_k^D}{\partial\eta^p}\right)^2\right)+\frac{c^2}{4}\left(\frac{\partial^{2p}u_k^D}{\partial\xi^p\partial\eta^p}\right)^2\right]d\Omega_k\right)$$

is a broken Sobolev norm of the solution in the entire domain,

$$\Theta_{adv}=\sum_{k=1}^{N}\left(-\int\limits_{\Omega_k}u^D(\nabla\cdot\boldsymbol{f}^D)d\Omega_k-\int\limits_{\Gamma_k}u^D(\boldsymbol{f}^C\cdot\boldsymbol{n})d\Gamma_k\right)$$

and

$$\begin{aligned}
\Theta_{extra} = \sum_{k=1}^{N}\Bigg(&J_{y_k}^{2p+1}\left[\frac{1}{2}\frac{\partial^p u_R^D}{\partial y^p}\frac{\partial^p F_R^D}{\partial y^p}-\frac{\partial^p u_R^D}{\partial y^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_R^*}{\partial y^p}\right]_k \\
&+ J_{y_k}^{2p+1}\left[-\frac{1}{2}\frac{\partial^p u_L^D}{\partial y^p}\frac{\partial^p F_L^D}{\partial y^p}-\frac{\partial^p u_L^D}{\partial y^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_L^*}{\partial y^p}\right] \\
&+ J_{x_k}^{2p+1}\left[\frac{1}{2}\frac{\partial^p u_T^D}{\partial x^p}\frac{\partial^p G_T^D}{\partial x^p}-\frac{\partial^p u_T^D}{\partial x^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_T^*}{\partial x^p}\right] \\
&+ J_{x_k}^{2p+1}\left[-\frac{1}{2}\frac{\partial^p u_B^D}{\partial x^p}\frac{\partial^p G_B^D}{\partial x^p}-\frac{\partial^p u_B^D}{\partial x^p}\frac{\partial^p(\boldsymbol{f}\cdot\boldsymbol{n})_B^*}{\partial x^p}\right]\Bigg)
\end{aligned}$$

Note that the difference is that, now there is no J_k appearing in any of the above terms. Note the change in the definition of the broken Sobolev norm as well where the term J_k no longer appears.

This expression can be rewritten as a sum over all the edges instead of all the elements. Consider one such summation along an interior vertical edge. Let − and + subscripts denote the element on the left and right. For the element on the left, this edge is its right boundary and for the right element, it is the left boundary. Also, note that for a Cartesian mesh with no mortar elements, the J_y for these left and right elements are the same, since it is the edge length of their common boundary. Adding the 2 terms coming from each element from this edge, we get

$$J_y^{2p+1}\left[\frac{1}{2}\frac{\partial^p u_-^D}{\partial y^p}\frac{\partial^p F_-^D}{\partial y^p}-\frac{\partial^p u_-^D}{\partial y^p}\frac{\partial^p (\boldsymbol{f}\cdot\boldsymbol{n})_-^*}{\partial y^p}-\frac{1}{2}\frac{\partial^p u_+^D}{\partial y^p}\frac{\partial^p F_+^D}{\partial y^p}-\frac{\partial^p u_+^D}{\partial y^p}\frac{\partial^p (\boldsymbol{f}\cdot\boldsymbol{n})_+^*}{\partial y^p}\right] \quad (3.35)$$

The above equation is the corrected version of Eq. (3.35). Since the J_k terms got cancelled out, J_+ and J_- do not appear in the equation above.

Now, we use the fact that f is a linear advective flux, i.e., $F^D = au^D$ and $G^D = bu^D$. Also, from the definition of the Lax–Friedrichs flux, we have,

$$(\boldsymbol{f}\cdot\boldsymbol{n})_-^* = \frac{1}{2}a\left(u_-^D + u_+^D\right) + \frac{\lambda}{2}|a|\left(u_-^D - u_+^D\right) \quad (3.36)$$

$$(\boldsymbol{f}\cdot\boldsymbol{n})_+^* = -\frac{1}{2}a\left(u_-^D + u_+^D\right) + \frac{\lambda}{2}|a|\left(u_+^D - u_-^D\right) \quad (3.37)$$

Substituting these results in (3.35), we get

$$J_y^{2p+1}\left[-\frac{\lambda}{2}|a|\left(\frac{\partial^p u_-^D}{\partial y^p}\frac{\partial^p u_-^D}{\partial y^p}-\frac{\partial^p u_-^D}{\partial y^p}\frac{\partial^p u_+^D}{\partial y^p}\right)-\frac{\lambda}{2}|a|\left(\frac{\partial^p u_+^D}{\partial y^p}\frac{\partial^p u_+^D}{\partial y^p}-\frac{\partial^p u_-^D}{\partial y^p}\frac{\partial^p u_+^D}{\partial y^p}\right)\right] \quad (3.38)$$

which can be further simplified as

$$J_y^{2p+1}\left[-\frac{\lambda}{2}|a|\left(\frac{\partial^p u_-^D}{\partial y^p}-\frac{\partial^p u_+^D}{\partial y^p}\right)^2\right] \quad (3.39)$$

From this equation, we can see that the terms contributing to Θ_{extra} are non-positive. The same is true for Θ_{adv} and therefore, for any $c \geq 0$, we have

$$\frac{d}{dt}\|u^D\|^2 = \Theta_{adv} + c\Theta_{extra} \leq 0$$

which proves Theorem 3.5.

Conclusions

The corrected version of Theorem 3.5 shows that the FR approach for the linear advection equation is stable on Cartesian meshes when using the Lax Friedrichs-type flux and when the VCJH parameter $c \geq 0$. This result is in agreement with the previous stability results obtained in 1D and on triangles.

Reference

1. Castonguay, P., Vincent, P.E., Jameson, A.: A new class of high-order energy stable flux reconstruction schemes for triangular elements. J. Sci. Comput. (2011)