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On efficiency and mixed duality for a new class of nonconvex multiobjective variational control problems

Tadeusz Antczak

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Abstract In this paper, we extend the notions of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity to the continuous case and we use these concepts to establish sufficient optimality conditions for the considered class of nonconvex multiobjective variational control problems. Further, multiobjective variational control mixed dual problem is given for the considered multiobjective variational control problem and several mixed duality results are established under (Φ, ρ)-invexity.

Keywords Multiobjective variational problems · Efficient solution · (Φ, ρ)-invexity · (Φ, ρ)-pseudo-invexity · (Φ, ρ)-quasi-invexity · Mixed duality

Mathematics Subject Classification 65K10 · 90C29 · 26B25

1 Introduction

During the last two decades, multiobjective control problems have been considered in flight control design, in the control of space structures, in industrial process control, in impulsive control problems, in the control of production and inventory, and other diverse fields. The multiobjective variational programming problem with equality and inequality restrictions was considered by many authors (see, for example, [7,13,20,21], and references here)

Chandra et al. [4] gave the Fritz-John necessary optimality conditions for the existence of an optimal solution for the single objective control problem. In [14], Mond and Smart gave duality results and sufficiency conditions for control problems under invexity assumptions. Bhatia and Kumar [2] extended the work of Mond and Smart to the content of multiobjective control problems and established duality results for Wolfe as well as Mond–Weir-type duals under ρ-invexity assumptions and their generalizations. In [15], Mukherejee and Rao

T. Antczak (✉)
Faculty of Mathematics and Computer Science, University of Łódź,
Banacha 22, 90-238 Lodz, Poland
e-mail: antczak@math.uni.lodz.pl

extended mixed-type duality to the class of multiobjective variational problems and proved duality results under generalized ρ-invexity. Mishra and Mukherejee [11] obtained duality results for multiobjective control problems under V-invexity assumptions and their generalizations. They extended the results of Bhatia and Kumar [2] to a wider class of multiobjective control problems. Bhatia and Mehra [3] extended the concepts of B-type I and generalized B-type I functions to the continuous case and they used these concepts to establish sufficient optimality conditions and duality results for multiobjective variational programming problems. Nahak and Nanda [17] discussed duality theorems and related efficient solutions of the primal and dual problems for multiobjective variational control problems with (F, ρ)-convexity. Reddy and Mukherjee [18] studied duality theorems and related efficient solutions of the primal and dual problems for multiobjective fractional control problems under (F, ρ)-convexity. Ahmad and Gulati [1] studied mixed type duality for multiobjective variational problems also under (F, ρ)-convexity, obtaining new optimality results. Using the relationship between the efficient solution of the multiobjective control problem and the optimal solution of the associated scalar control problem, Gramatovici [6] derived the necessary optimality conditions for the multiobjective control problems with invex functions. Kim and Kim [9] introduced new classes of generalized V-type I functions for variational problems and they proved a number of sufficiency results and duality theorems using Lagrange multiplier conditions under various types of generalized V-type I invexity requirements. Further, under the generalized V-type I invexity assumptions and their generalizations, they obtained duality results for Mond–Weir type duals. Also Hachimi and Aghezzaf [7] obtained several mixed type duality results for multiobjective variational programming problems, but under a new introduced concept of generalized type I functions. In [10], Khazafi et al. introduced the classes of (B, ρ)-type I functions and generalized (B, ρ)-type I functions and derived a series of sufficient optimality conditions and mixed type duality results for multiobjective control problems.

Our aim in this paper is to provide several sufficient optimality conditions and mixed duality results for a multiobjective variational control problem under generalized convexity restrictions on the components of functions describing the constraints and the objective functions. In our approach, the usual convexity requirement for functions is relaxed. In this paper, therefore, we introduce the concepts of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity for a multiobjective variational control problem, as a new condition on functions of this kind of problem. Then, we use these mentioned concepts of generalized invexity to establish several sufficient optimality conditions for a new class of nonconvex multiobjective variational control problems.

Further, for the considered multiobjective variational control problem, its vector variational control mixed dual problem is given and several duality theorems are established between these vector optimization problems under (Φ, ρ)-invexity. Since (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity notions unify several classes of generalized convex functions, therefore, the results established in this paper for multiobjective variational control problems are more general than those in a fairly large number of works.

2 Preliminaries and notations

The following convention for equalities and inequalities will be used in the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, where the symbol $()^T$ stands for the transpose, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

All vectors will be taken as column vectors.

Let $I = [a, b]$ be a real interval and let $A = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, q\}$ and $K = \{1, \ldots, s\}$.

In this paper, we assume that $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a, b]$.

Denote by X the space of piecewise smooth state functions $x : I \to R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by $z = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t z(s)\,ds$, where $x(a)$ is a given boundary value. Therefore, $\frac{d}{dt} \equiv D$ except at discontinuities. For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively.

Let $f : I \times R^n \times R^n \to R^p$ be a p-dimensional function and each its component is a continuously differentiable real scalar function, $g : I \times R^n \times R^n \to R^q$ and $h : I \times R^n \times R^n \to R^s$ be continuously differentiable q-dimensional and s-dimensional functions, respectively. Here t is the independent variable and $x(t)$ is the state variable. In order to consider $f^1(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f^1 with respect to t, x and $\dot{x}$, respectively, by f_t^1, f_x^1, $f_{\dot{x}}^1$ such that $f_x^1 = \left(\frac{\partial f^1}{\partial x_1}, \ldots, \frac{\partial f^1}{\partial x_n}\right)$ and $f_{\dot{x}}^1 = \left(\frac{\partial f^1}{\partial \dot{x}_1}, \ldots, \frac{\partial f^1}{\partial \dot{x}_n}\right)$. Similarly the partial derivatives of the vector function g and the vector function h can be written, using matrices with q rows and s rows instead of one, respectively.

In [5], Caristi et al. introduced the concept of (Φ, ρ)-invexity as a generalization of invexity notion, previously defined in the literature by Hanson [8].

In this section, we extend the definitions of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity notions to the continuous case. Thus, we generalize the definitions of generalized convexity introduced by Caristi et al. [5] for scalar optimization problems to the case of multiobjective variational control ones.

Before we introduce the definitions mentioned above, we give a definition of convexity of a functional $\Phi : I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$.

Definition 1 Let $\Phi : I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$. A functional $\Phi(t, x, \dot{x}, z, \dot{z}, (\cdot, \cdot))$ is convex on R^{n+1} if, for any $x, z \in R^n, \dot{x}, \dot{z} \in R^n$, the following inequality

$$\Phi\left(t, x, \dot{x}, z, \dot{z}; (\lambda(\xi_1, \rho_1) + (1-\lambda)(\xi_2, \rho_2))\right)$$
$$\leqq \lambda\Phi\left(t, x, \dot{x}, z, \dot{z}; (\xi_1, \rho_1)\right) + (1-\lambda)\,\Phi\left(t, x, \dot{x}, z, \dot{z}; (\xi_2, \rho_2)\right)$$

holds for all $\xi_1, \xi_2 \in R^n$, $\rho_1, \rho_2 \in R$ and for any $\lambda \in [0, 1]$.

Let $\Psi : X \to R$ defined by $\Psi(x) = \int_a^b \varphi(t, x, \dot{x})dt$, where $\varphi : I \times R^n \times R^n \to R$, be differentiable. For notational convenience, we use $\varphi\left(t, x, \dot{x}\right)$ for $\varphi\left(t, x(t), \dot{x}(t)\right)$. The following definitions introduce the concepts of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity for the functional Ψ.

Definition 2 Let $\overline{x} \in X$ be given. If there exist a real number ρ and a functional Φ: $I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$, where $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (\cdot, \cdot))$ is convex on R^{n+1}, $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for every $x \in X$ and any $a \in R_+$, such that the inequality

$$\int_a^b \varphi\left(t, x, \dot{x}\right) dt - \int_a^b \varphi\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\geqq \int_a^b \Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; \left(\varphi_x\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[\varphi_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho\right)\right) dt (>)$$

holds for all $x \in X$, $(x \neq \overline{x})$, then the functional Ψ is said to be (strictly) (Φ, ρ)-invex at $\overline{x}$ on X. If the inequality above is satisfied for every $\overline{x} \in X$, then Ψ is said to be (strictly) (Φ, ρ)-invex on X.

Definition 3 Let $\overline{x} \in X$ be given. If there exist a real number ρ and a functional Φ: $I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$, where $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (\cdot, \cdot))$ is convex on R^{n+1}, $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for every $x \in X$ and any $a \in R_+$ such that the inequality

$$\int_a^b \varphi\left(t, x, \dot{x}\right) dt - \int_a^b \varphi\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\leqq \int_a^b \Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; \left(\varphi_x\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[\varphi_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho\right)\right) dt\ (<)$$

holds for all $x \in X$, $(x \neq \overline{x})$, then the functional Ψ is said to be (strictly) (Φ, ρ)-incave at $\overline{x}$ on X. If the inequality above is satisfied for every $\overline{x} \in X$, then Ψ is said to be (strictly) (Φ, ρ)-incave on X.

Definition 4 Let $\overline{x} \in X$ be given. If there exist a real number ρ and a functional Φ: $I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$, where $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (\cdot, \cdot))$ is convex on R^{n+1}, $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for every $x \in X$ and any $a \in R_+$ such that the relation

$$\int_a^b \varphi\left(t, x, \dot{x}\right) dt < \int_a^b \varphi\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\Longrightarrow \int_a^b \Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; \left(\varphi_x\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[\varphi_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho\right)\right) dt < 0$$

holds for all $x \in X$, then the functional Ψ is said to be (Φ, ρ)-pseudo-invex at $\overline{x}$ on X. If the relation above is satisfied for every $\overline{x} \in X$, then Ψ is said to be (Φ, ρ)-pseudo-invex on X.

Definition 5 Let $\overline{x} \in X$ be given. If there exist a real number ρ and a functional Φ: $I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$, where $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (\cdot, \cdot))$ is convex on R^{n+1}, $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for every $x \in X$ and any $a \in R_+$ such that the relation

$$\int_a^b \varphi\left(t, x, \dot{x}\right) dt \leqq \int_a^b \varphi\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\Longrightarrow \int_a^b \Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; \left(\varphi_x\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[\varphi_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho\right)\right) dt < 0$$

holds for all $x \in X$, $x \neq \overline{x}$, then the functional Ψ is said to be strictly (Φ, ρ)-pseudo-invex at $\overline{x} \in X$ on X. If the relation above is satisfied for every $\overline{x} \in X$, then Ψ is said to be strictly (Φ, ρ)-pseudo-invex on X.

Definition 6 Let $\overline{x} \in X$ be given. If there exist a real number ρ and a functional Φ: $I \times R^n \times R^n \times R^n \times R^n \times R^n \times R \to R$, where $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (\cdot, \cdot))$ is convex on R^{n+1}, $\Phi(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for every $x \in X$ and any $a \in R_+$ such that the relation

$$\int_a^b \varphi\left(t, x, \dot{x}\right) dt \leqq \int_a^b \varphi\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\Longrightarrow \int_a^b \Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; \left(\varphi_x\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[\varphi_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho\right)\right) dt \leqq 0$$

holds for all $x \in X$, then the functional Ψ is said to be (Φ, ρ)-quasi-invex at $\overline{x} \in X$ on X. If the relation above is satisfied for every $\overline{x} \in X$, then Ψ is said to be (Φ, ρ)-quasi-invex on X.

The concept of (Φ, ρ)-invexity generalizes and extends a lot of generalized convexity notions previously defined in the literature. In order to illustrate this fact, we give an example of a functional Ψ which is (Φ, ρ)-invex, but it is not invex.

Example 7 Define the function $\varphi : I \times R^2 \times R^2 \to R$ by $\varphi(t, x, \dot{x}) = x_1(t)x_2(t)$. Consider the functional Ψ defined by $\Psi(x) = \int_0^1 \varphi(t, x, \dot{x})dt$. We set $\rho = -1$ and

$$\Phi\left(t, x, \dot{x}, \overline{x}, \dot{\overline{x}}; (\beta, \rho)\right) = -\frac{1}{2}\left(\overline{x}_1\beta_1 + \overline{x}_2\beta_2\right) + 2\left(2^\rho - 1\right)|x_1(t)x_2(t)|.$$

Then, by Definition 2, it can be proved that the functional Ψ is (Φ, ρ)-invex on $R^2 \times R^2$. Note, moreover, that the functional Ψ is not invex on $R^2 \times R^2$ with respect to any function $\eta : I \times R^n \times R^n \times R^n \times R^n \to R^n$ (see, Definition 4 [16]).

In the multiobjective variational control problem, under given conditions, the state vector $x(t)$ is brought from specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way to minimize a given functional. A more precise mathematical formulation is given in the following vector optimization problem:

$$\text{Minimize} \int_a^b f\left(t, x(t), \dot{x}(t)\right) dt = \left(\int_a^b f^1\left(t, x(t), \dot{x}(t)\right), \ldots, \int_a^b f^p\left(t, x(t), \dot{x}(t)\right)\right) \tag{MVCP}$$

$$\begin{aligned} \text{subject to } & g\left(t, x(t), \dot{x}(t)\right) \leqq 0, \ t \in I \\ & h\left(t, x(t), \dot{x}(t)\right) = 0, \ t \in I, \\ & x(a) = \alpha, \ x(b) = \beta, \end{aligned}$$

where $f = (f_1, \ldots, f_p) : I \times R^n \times R^n \to R^p$, is a p-dimensional function and each of its component is a continuously differentiable real scalar function, $g : I \times R^n \times R^n \to R^q$ and $h : I \times R^n \times R^n \to R^s$ are assumed to be continuously differentiable q-dimensional and s-dimensional functions, respectively.

Let S denote the set of all feasible solutions of (MVCP), i.e.:

$$S = \{x : x \in X \text{ verifying the constraints of (MVCP)}\}.$$

Definition 8 A feasible solution $\overline{x}$ of the considered multiobjective variational control problem (MVCP) is said to be efficient of (MVCP) if there exists no other $x \in S$ such that

$$\int_a^b f\left(t, x, \dot{x}\right) dt \leq \int_a^b f\left(t, \overline{x}, \dot{\overline{x}}\right) dt,$$

that is, there exists no other $x \in S$ such that

$$\int_a^b f^i\left(t, x, \dot{x}\right) dt \leqq \int_a^b f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt, \qquad \forall i \in A,$$

$$\int_a^b f^r\left(t, x, \dot{x}\right) dt < \int_a^b f^r\left(t, \overline{x}, \dot{\overline{x}}\right) dt \qquad \text{for some } r \in A.$$

3 Optimality conditions

In order to prove sufficient optimality conditions for the considered multiobjective variational programming problem (MVCP), we give the Karush-Kuhn-Tucker necessary optimality conditions for such a vector optimization problem. This theorem is the continuous version of Theorem 2.2 [19] (see also [3,12,13]).

Theorem 9 *Let $\overline{x}$ be a normal efficient solution in problem (MVCP) at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist $\overline{\lambda} \in R^p$ and the piecewise smooth functions $\overline{\xi}(\cdot) : I \to R^m$ and $\overline{\zeta}(\cdot) : I \to R^s$ such that*

$$\begin{aligned} &\overline{\lambda}^T f_x\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\xi}(t)^T g_x\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\zeta}(t)^T h_x\left(t, \overline{x}, \dot{\overline{x}}\right) \\ &= \frac{d}{dt}\left[\overline{\lambda}^T f_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\xi}(t)^T g_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\zeta}(t)^T h_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \quad t \in I, \end{aligned} \tag{1}$$

$$\int_a^b \overline{\xi}(t)^T g\left(t, \overline{x}, \dot{\overline{x}}\right) dt = 0, \tag{2}$$

$$\overline{\lambda} \geq 0, \ \overline{\lambda}^T e = 1, \quad \overline{\xi}(t) \geqq 0. \tag{3}$$

For notational convenience, we use ξ for $\xi(t)$ and ζ for $\zeta(t)$.

Theorem 10 *Let $\overline{x}$ be a feasible solution in the considered multiobjective variational programming problem (MVCP) and the Karush-Kuhn-Tucker conditions (1)–(3) be satisfied at this point with $\overline{\lambda} \in R^p$ and the piecewise smooth functions $\overline{\xi}(\cdot) : I \to R^m$ and $\overline{\zeta}(\cdot) : I \to R^s$. Further, assume that the following hypotheses are fulfilled:*

(a) $f^i(t, \cdot, \cdot)$, $i = 1, \ldots, p$, *is strictly (Φ, ρ_{f_i})-invex at $\overline{x}$ on S,*
(b) $g^j(t, \cdot, \cdot)$, $j = 1, .., q$, *is (Φ, ρ_{g_j})-invex at $\overline{x}$ on S,*
(c) $h^k(t, \cdot, \cdot)$, $k \in K^+(t) = \left\{k \in K : \overline{\zeta}_k(t) > 0\right\}$, *is (Φ, ρ_{h_k})-invex at $\overline{x}$ on S,*
(d) $-h^k(t, \cdot, \cdot)$, $k \in K^-(t) = \left\{k \in K : \overline{\zeta}_k(t) < 0\right\}$, *is (Φ, ρ_{h_k})-invex at $\overline{x}$ on S,*
(e) $\sum_{i=1}^p \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \overline{\xi}_i \rho_{g_j} + \sum_{k \in K^+(t)} \overline{\zeta}_k \rho_{h_k} - \sum_{k \in K^-(t)} \overline{\zeta}_k \rho_{h_k} \geqq 0.$

Then $\overline{x}$ is an efficient solution in problem (MVCP).

Proof Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution in problem (MVCP). Then, there exists $\widetilde{x} \in S$ such that

$$\int_a^b f^i\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt \leqq \int_a^b f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt, \quad \forall i \in A, \tag{4}$$

and

$$\int_a^b f^r\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt < \int_a^b f^r\left(t, \overline{x}, \dot{\overline{x}}\right) dt \quad \text{for some } r \in A. \tag{5}$$

Since the hypotheses (a)–(d) are fulfilled, therefore, by Definition 2, the following inequalities

$$\int_a^b f^i\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt - \int_a^b f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$
$$> \int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(f_x^i\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[f_{\dot{x}}^i\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{f_i}\right)\right) dt, \quad i \in A, \tag{6}$$

$$\int_a^b g^j\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt - \int_a^b g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$
$$\geqq \int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(g_x^j\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[g_{\dot{x}}^j\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{g_j}\right)\right) dt, \quad j \in J, \tag{7}$$

$$\int_a^b h^k\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt - \int_a^b h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\geqq \int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt, \quad k \in K^+(t). \quad (8)$$

$$-\int_a^b h^k\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt + \int_a^b h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\geqq \int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(-h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[-h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt, \quad k \in K^-(t). \quad (9)$$

Combining (4)–(6) and taking into account that $\overline{\lambda} \geq 0$, we get

$$\int_a^b \overline{\lambda}_i \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(f_x^i\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[f_{\dot{x}}^i\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{f_i}\right)\right) dt \leqq 0, \quad i \in A \quad (10)$$

and

$$\int_a^b \overline{\lambda}_r \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(f_x^r\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[f_{\dot{x}}^r\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{f_r}\right)\right) dt < 0$$

$$\text{for at least one } r \in A. \quad (11)$$

Adding both sides of (10) and (11), we obtain

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(f_x^i\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[f_{\dot{x}}^i\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{f_i}\right)\right) dt < 0. \quad (12)$$

Since $\overline{\xi}_j(t) \geqq 0$, $j \in J$, then (7) gives

$$\int_a^b \sum_{j=1}^q \overline{\xi}_j g^j\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt - \int_a^b \sum_{j=1}^q \overline{\xi}_j g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt$$

$$\geqq \int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(g_x^j\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[g_{\dot{x}}^j\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{g_j}\right)\right) dt. \quad (13)$$

Using the feasibility of $\widetilde{x}$ in problem (MVCP) together with the Karush-Kuhn-Tucker necessary optimality condition (2), we get

$$\int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(g_x^j\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[g_{\dot{x}}^j\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{g_j}\right)\right) dt \leqq 0. \quad (14)$$

The inequalities (8) and (9) yield, respectively,

$$\int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k h^k\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt - \int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k h^k\left(t,\overline{x},\dot{\overline{x}}\right)dt$$
$$\geqq \int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt, \tag{15}$$

$$\int_a^b \sum_{k\in K^-(t)} \overline{\zeta}_k h^k\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt - \int_a^b \sum_{k\in K^-(t)} \overline{\zeta}_k h^k\left(t,\overline{x},\dot{\overline{x}}\right)dt$$
$$\geqq \int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(-h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[-h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt. \tag{16}$$

Adding both sides of the inequalities (15) and (16), we obtain

$$\int_a^b \sum_{k\in K} \overline{\zeta}_k h^k\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt - \int_a^b \sum_{k\in K} \overline{\zeta}_k h^k\left(t,\overline{x},\dot{\overline{x}}\right)dt$$
$$\geqq \int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt$$
$$+\int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(-h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[-h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt.$$

Using the feasibility of $\widetilde{x}$ in problem (MVCP) together with (2) and (3), we have

$$\int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt$$
$$+\int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(-h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[-h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt \leqq 0. \tag{17}$$

Combining (12), (14) and (17), we get

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{f_i}\right)\right)dt$$
$$+\int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{g_j}\right)\right)dt$$

$$+\int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt$$

$$+\int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(-h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[-h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt<0. \tag{18}$$

We denote

$$\widehat{\lambda}_i=\frac{\overline{\lambda}_i}{\sum_{i=1}^p\overline{\lambda}_i+\sum_{j=1}^q\overline{\xi}_j(t)+\sum_{k\in K^+(t)}\overline{\zeta}_k(t)-\sum_{k\in K^-(t)}\overline{\zeta}_k(t)},\quad i\in A, \tag{19}$$

$$\widehat{\xi}_j(t)=\frac{\overline{\xi}_j(t)}{\sum_{i=1}^p\overline{\lambda}_i+\sum_{j=1}^q\overline{\xi}_j(t)+\sum_{k\in K^+(t)}\overline{\zeta}_k(t)-\sum_{k\in K^-(t)}\overline{\zeta}_k(t)},\quad j\in J, \tag{20}$$

$$\widehat{\zeta}_k(t)=\frac{\overline{\zeta}_k(t)}{\sum_{i=1}^p\overline{\lambda}_i+\sum_{j=1}^q\overline{\xi}_j(t)+\sum_{k\in K^+(t)}\overline{\zeta}_k(t)-\sum_{k\in K^-(t)}\overline{\zeta}_k(t)},\quad k\in K^+(t), \tag{21}$$

$$\widehat{\zeta}_k(t)=\frac{-\overline{\zeta}_k(t)}{\sum_{i=1}^p\overline{\lambda}_i+\sum_{j=1}^q\overline{\xi}_j(t)+\sum_{k\in K^+(t)}\overline{\zeta}_k(t)-\sum_{k\in K^-(t)}\overline{\zeta}_k(t)},\quad k\in K^-(t). \tag{22}$$

By (19)–(22), it follows that $0\leqq\widehat{\lambda}_i\leqq 1, i\in A$, but $\widehat{\lambda}_i>0$ for at least one $i\in A, 0\leqq \widehat{\xi}_j(t)\leqq 1, j\in J, 0\leqq\widehat{\zeta}_k(t)\leqq 1, k\in K$, and, moreover,

$$\sum_{i=1}^p\widehat{\lambda}_i+\sum_{j=1}^q\widehat{\xi}_j(t)+\sum_{k\in K^+(t)}\widehat{\zeta}_k(t)+\sum_{k\in K^-(t)}\widehat{\zeta}_k(t)=1. \tag{23}$$

Combining (18)–(22), we get

$$\int_a^b\sum_{i=1}^p\widehat{\lambda}_i\Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{f_i}\right)\right)dt$$

$$+\int_a^b\sum_{j=1}^q\widehat{\xi}_j\Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{g_j}\right)\right)dt$$

$$+\int_a^b\sum_{k\in K^+(t)}\widehat{\zeta}_k\Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt$$

$$+\int_a^b\sum_{k\in K^-(t)}\widehat{\zeta}_k\Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(-h_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[-h_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{h_k}\right)\right)dt<0. \tag{24}$$

By Definition 2, it follows that the functional $\Phi(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}, \cdot)$ is convex on R^{n+1}. Since (23) holds, then, by (24), Definition 1 implies

$$\int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(\left[\sum_{i=1}^p \widehat{\lambda}_i f_x^i\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{j=1}^q \widehat{\xi}_j g_x^j\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{k\in K^+(t)} \widetilde{\zeta}_k h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{k\in K^-(t)} \left(-\widehat{\zeta}_k\right) h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right] - \frac{d}{dt}\left[\sum_{i=1}^p \widehat{\lambda}_i f_{\dot{x}}^i\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{j=1}^q \widehat{\xi}_j g_{\dot{x}}^j\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{k\in K^+(t)} \widehat{\zeta}_k h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right) + \sum_{k\in K^-(t)} \left(-\widehat{\zeta}_k\right) h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \sum_{i=1}^p \widehat{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \widehat{\xi}_j \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widehat{\zeta}_k \rho_{h_k}\right)\right) dt < 0.$$

Hence, the Karush-Kuhn-Tucker necessary optimality condition (2) yields

$$\int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(0, \sum_{i=1}^p \widehat{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \widehat{\xi}_j \rho_{g_j} + \sum\nolimits_{k\in K^+(t)\cup K^-(t)} \widehat{\zeta}_k \rho_{h_k}\right)\right) dt < 0. \tag{25}$$

From the hypothesis (e), we have

$$\sum_{i=1}^p \widehat{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \widehat{\xi}_j \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widehat{\zeta}_k \rho_{h_k} \geqq 0. \tag{26}$$

By Definition 2, it follows that $\Phi(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}, (0, a)) \geqq 0$ for any $a \in R_+$. Thus, (26) implies that the following inequality

$$\int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(0, \sum_{i=1}^p \widehat{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \widehat{\xi}_j \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widehat{\zeta}_k \rho_{h_k}\right)\right) dt \geqq 0$$

holds, contradicting (25). Thus, the proof of theorem is completed. □

Theorem 11 *Let $\overline{x}$ be a feasible solution in the considered multiobjective variational programming problem (MVCP) and the Karush-Kuhn-Tucker conditions (1)–(3) be satisfied at this point with $\overline{\lambda} \in R^p$ and the piecewise smooth functions $\overline{\xi}(\cdot) : I \to R^m$ and $\overline{\zeta}(\cdot) : I \to R^s$. Further, assume that the following hypotheses are fulfilled:*

(a) $f^i(t, \cdot, \cdot)$, $i = 1, \ldots, p$, *is strictly* $\left(\Phi, \rho_{f_i}\right)$*-pseudo-invex at* $\overline{x}$ *on* S,
(b) $g^j(t, \cdot, \cdot)$, $j = 1, \ldots, q$, *is* $\left(\Phi, \rho_{g_j}\right)$*-quasi-invex at* $\overline{x}$ *on* S,
(c) $h^k(t, \cdot, \cdot)$, $k \in K^+(t) = \left\{k \in K : \overline{\zeta}_k(t) > 0\right\}$, *is* $\left(\Phi, \rho_{h_k}\right)$*-quasi-invex at* $\overline{x}$ *on* S,
(d) $-h^k(t, \cdot, \cdot)$, $k \in K^-(t) = \left\{k \in K : \overline{\zeta}_k(t) < 0\right\}$, *is* $\left(\Phi, \rho_{h_k}\right)$*-quasi-invex at* $\overline{x}$ *on* S,
(e) $\sum_{i=1}^p \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \overline{\xi}_i \rho_{g_j} + \sum_{k\in K^+(t)} \overline{\zeta}_k \rho_{h_k} - \sum_{k\in K^-(t)} \overline{\zeta}_k \rho_{h_k} \geqq 0.$

Then $\overline{x}$ is an efficient solution in problem (MVCP).

Proof Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution in problem (MVCP). Then, there exists $\widetilde{x}$ feasible in problem (MVCP) such that

$$\int_a^b f^i\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt \leqq \int_a^b f^i\left(t,\overline{x},\dot{\overline{x}}\right)dt, \quad i\in A, \tag{27}$$

and

$$\int_a^b f^r\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt < \int_a^b f^r\left(t,\overline{x},\dot{\overline{x}}\right)dt \quad \text{for some } r\in A. \tag{28}$$

By Definition 5, (27) and (28) yield

$$\int_a^b \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{f_i}\right)\right)dt \leqq 0, \quad i\in A.$$

and

$$\int_a^b \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\, f_x^r\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[f_{\dot{x}}^r\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{f_r}\right)dt < 0, \text{ for some } r\in A.$$

Since $\overline{\lambda} \geq 0$, then the inequality above gives

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{f_i}\right)\right)dt < 0. \tag{29}$$

Using the feasibility of $\overline{x}$ and $\widetilde{x}$ in problem (MVCP) together with the Karush-Kuhn-Tucker necessary optimality conditions (2) and (3), we obtain

$$\int_a^b \overline{\xi}_j g^j\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt \leqq \int_a^b \overline{\xi}_j g^j\left(t,\overline{x},\dot{\overline{x}}\right)dt, \quad j=1,\ldots,q.$$

Thus, by Definition 6, the hypothesis (b) yields

$$\int_a^b \overline{\xi}_j \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{g_j}\right)\right)dt \leqq 0, \quad j\in J.$$

Adding both sides of the inequalities above, we obtain

$$\int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t,\widetilde{x},\dot{\widetilde{x}},\overline{x},\dot{\overline{x}};\left(g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}\left[g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right],\rho_{g_j}\right)\right)dt \leqq 0. \tag{30}$$

Using the feasibility of $\overline{x}$ and $\widetilde{x}$ in problem (MVCP), we have

$$\int_a^b h^k\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt = \int_a^b h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt,\ k \in K^+(t), \tag{31}$$

$$\int_a^b -h^k\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt = \int_a^b -h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt,\ k \in K^-(t). \tag{32}$$

Hence, by hypotheses (c) and (d), the inequalities (31) and (32) imply, respectively,

$$\int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt \leqq 0, \quad k \in K^+(t), \tag{33}$$

$$\int_a^b \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(-h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[-h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt \leqq 0, \quad k \in K^-(t). \tag{34}$$

Thus,

$$\int_a^b \sum_{k \in K^+(t)} \overline{\zeta}_k \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt \leqq 0, \tag{35}$$

$$\int_a^b \sum_{k \in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(-h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[-h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt \leqq 0. \tag{36}$$

Combining (29), (30), (35) and (36), we get

$$\begin{aligned}
&\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(f_x^i\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[f_{\dot{x}}^i\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{f_i}\right)\right) dt \\
&+ \int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(g_x^j\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[g_{\dot{x}}^j\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{g_j}\right)\right) dt \\
&+ \int_a^b \sum_{k \in K^+(t)} \overline{\zeta}_k \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt \\
&+ \int_a^b \sum_{k \in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t, \widetilde{x}, \dot{\widetilde{x}}, \overline{x}, \dot{\overline{x}}; \left(-h_x^k\left(t, \overline{x}, \dot{\overline{x}}\right) - \frac{d}{dt}\left[-h_{\dot{x}}^k\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \rho_{h_k}\right)\right) dt < 0.
\end{aligned}$$

The rest of proof is the same as in proof of Theorem 10. □

4 Mixed duality

Let M be a subset of J and $L = J/M$ such that $M \cup L = J$, and let

$$\xi_M(t)^T g^M\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = \sum_{j \in M} \xi_j(t) g^j\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)$$

and

$$\xi_L(t)^T g^L\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = \sum_{j \in L} \xi_j(t) g^j\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right).$$

In this section, we prove duality results between the considered multiobjective variational control programming problem (MVCP) and its mixed type multiobjective variational dual problem (VMD) defined as follows

$$\text{Maximize} \int_a^b \left(f\left(t, y(t), \dot{y}(t)\right) + \xi_M(t)^T g^M\left(t, y(t), \dot{y}(t)\right) e\right) dt \tag{VMD}$$

$$\begin{aligned}
\text{subject to} \quad & \Big[\lambda^T f_y\left(t, y(t), \dot{y}(t)\right) + \xi(t)^T g_y\left(t, y(t), \dot{y}(t)\right) \\
& \quad + \zeta(t)^T h_y\left(t, y(t), \dot{y}(t)\right)\Big] = \frac{d}{dt}\Big[\lambda^T f_{\dot{y}}\left(t, y(t), \dot{y}(t)\right) \\
& \quad + \xi(t)^T g_{\dot{y}}\left(t, y(t), \dot{y}(t)\right) + \zeta(t)^T h_{\dot{y}}\left(t, y(t), \dot{y}(t)\right)\Big], \quad t \in I, \\
& \int_a^b \xi_L(t)^T g^L\left(t, y(t), \dot{y}(t)\right) dt \geqq 0, \\
& \int_a^b \zeta(t)^T h\left(t, y(t), \dot{y}(t)\right) dt = 0, \\
& y(a) = \alpha, \quad y(b) = \beta, \\
& \lambda \geq 0, \ \lambda^T e = 1, \ \xi(t) \geqq 0,
\end{aligned}$$

where $e = (1, \ldots, 1) \in R^p$ is a p-dimensional vector. It may be noted here that the above dual constraints are written using the Karush-Kuhn-Tucker necessary optimality conditions for the problem (MVCP) (see Theorem 9).

Remark 12 Let $L = \varnothing$. Then, the dual (VMD) reduces to the well-known Wolfe dual. If $M = \varnothing$, then (VMD) becomes Mond–Weir type dual.

Let Ω_D be the set of all feasible solutions (y, λ, ξ, ζ) in mixed type multiobjective variational dual problem (VMD). We denote by Y the set $Y = \{y \in X : (y, \lambda, \xi, \zeta) \in \Omega_D\}$.

Theorem 13 (Weak duality)*: Let x and (y, λ, ξ, ζ) be any feasible solutions in problems (MVCP) and (VMD), respectively. Further, assume that the following hypotheses are fulfilled:*

(a) $f^i(t, \cdot, \cdot)$, $i = 1, \ldots, p$, *is strictly* $\left(\Phi, \rho_{f_i}\right)$*-invex at* y *on* $S \cup Y$,
(b) $g^j(t, \cdot, \cdot)$, $j = 1, .., q$, *is* $\left(\Phi, \rho_{g_j}\right)$*-invex at* y *on* $S \cup Y$,
(c) $h^k(t, \cdot, \cdot)$, $k \in K^+(t) = \{k \in K : \zeta_k(t) > 0\}$, *is* $\left(\Phi, \rho_{h_k}\right)$*-invex at* y *on* $S \cup Y$,

(d) $-h^k(t,\cdot,\cdot), k \in K^-(t) = \{k \in K : \zeta_k(t) < 0\}$, *is* (Φ, ρ_{h_k})*-invex at* y *on* $S \cup Y$,

(e) $\sum_{i=1}^{p} \lambda_i \rho_{f_i} + \sum_{j=1}^{q} \xi_i \rho_{g_j} + \sum_{k \in K^+(t)} \zeta_k \rho_{h_k} - \sum_{k \in K^-(t)} \zeta_k \rho_{h_k} \geqq 0.$

Then, the following cannot hold

$$\int_a^b f^i\left(t,x,\dot{x}\right)dt \leqq \int_a^b \left(f^i\left(t,y,\dot{y}\right) + \xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt, \quad i \in A, \tag{37}$$

and

$$\int_a^b f^r\left(t,x,\dot{x}\right)dt < \int_a^b \left(f^r\left(t,y,\dot{y}\right) + \xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt \quad \textit{for some } r \in A. \tag{38}$$

Proof We proceed by contradiction. Suppose, contrary to the result, that the inequalities (37) and (38) are satisfied. Since the hypotheses (a)–(d) are fulfilled, therefore, by Definition 2, the following inequalities

$$\int_a^b f^i\left(t,x,\dot{x}\right)dt - \int_a^b f^i\left(t,y,\dot{y}\right)dt > \int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(f_y^i\left(t,y,\dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t,y,\dot{y}\right)\right],\rho_{f_i}\right)\right)dt, \quad i \in A, \tag{39}$$

$$\int_a^b g^j\left(t,x,\dot{x}\right)dt - \int_a^b g^j\left(t,y,\dot{y}\right)dt \geqq \int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(g_y^j\left(t,y,\dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right],\rho_{g_j}\right)\right)dt, \quad j \in J, \tag{40}$$

$$\int_a^b h^k\left(t,x,\dot{x}\right)dt - \int_a^b h^k\left(t,y,\dot{y}\right)dt \geqq \int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(h_y^k\left(t,y,\dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t,y,\dot{y}\right)\right],\rho_{h_k}\right)\right)dt, \quad k \in K^+(t), \tag{41}$$

$$-\int_a^b h^k\left(t,x,\dot{x}\right)dt + \int_a^b h^k\left(t,y,\dot{y}\right)dt \geqq \int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(-h_y^k\left(t,y,\dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t,y,\dot{y}\right)\right],\rho_{h_k}\right)\right)dt, \quad k \in K^-(t) \tag{42}$$

hold. Since $\xi(t) \geqq 0$, then (40) gives

$$\int_a^b \xi_j g^j\left(t, x, \dot{x}\right) dt - \int_a^b \xi^j g^j\left(t, y, \dot{y}\right) dt$$
$$\geqq \int_a^b \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt, \quad j \in J. \tag{43}$$

Using the feasibility of x and (y, λ, ξ, ζ) in problems (MVCP) and (VMD), respectively, we get

$$-\int_a^b \sum_{j \in M} \xi_j g^j\left(t, y, \dot{y}\right) dt$$
$$\geqq \int_a^b \sum_{j \in M} \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt \tag{44}$$

and

$$\int_a^b \sum_{j \in L} \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt \leqq 0. \tag{45}$$

By (39), (44) and (45), we get

$$\int_a^b f^i\left(t, x, \dot{x}\right) dt - \int_a^b f^i\left(t, y, \dot{y}\right) dt - \int_a^b \sum_{j \in M} \xi_j g^j\left(t, y, \dot{y}\right) dt$$
$$> \int_a^b \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j \in M} \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt$$
$$+ \int_a^b \sum_{j \in L} \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt, \quad i \in A.$$

Thus, by $M \cup L = J$, the inequality above yields

$$\int_a^b f^i\left(t, x, \dot{x}\right) dt - \int_a^b f^i\left(t, y, \dot{y}\right) dt - \int_a^b \sum_{j \in M} \xi_j g^j\left(t, y, \dot{y}\right) dt$$
$$> \int_a^b \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt, \quad i \in A. \tag{46}$$

Combining (37), (38) and (46), we obtain

$$\int_a^b \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt < 0, \quad i \in A. \tag{47}$$

Since $\lambda \geq 0$ and $\lambda^T e = 1$, then (47) gives

$$\int_a^b \sum_{i=1}^p \lambda_i \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt < 0. \tag{48}$$

Thus, (41) and (42) yield, respectively,

$$\int_a^b \sum_{k \in K^+(t)} \zeta_k h^k\left(t, x, \dot{x}\right) dt - \int_a^b \sum_{k \in K^+(t)} \zeta_k h^k\left(t, y, \dot{y}\right) dt$$
$$\geqq \int_a^b \sum_{k \in K^+(t)} \zeta_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt, \tag{49}$$

and

$$\int_a^b \sum_{k \in K^-(t)} \zeta_k h^k\left(t, x, \dot{x}\right) dt - \int_a^b \sum_{k \in K^-(t)} \zeta_k h^k\left(t, y, \dot{y}\right) dt$$
$$\geqq \int_a^b \sum_{k \in K^-(t)} (-\zeta_k) \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt. \tag{50}$$

Adding both sides of (49) and (50), we get

$$\int_a^b \sum_{k\in K} \zeta_k h^k\left(t, x, \dot{x}\right) dt - \int_a^b \sum_{k\in K} \zeta_k h^k\left(t, y, \dot{y}\right) dt$$
$$\geqq \int_a^b \sum_{k\in K^+(t)} \zeta_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^-(t)} (-\zeta_k)\, \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt.$$

Hence, by the feasibility of x and of (y, λ, ξ, ζ) in problems (MVCP) and (VMD), respectively, it follows that

$$\int_a^b \sum_{k\in K^+(t)} \zeta_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^-(t)} (-\zeta_k)\, \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt \leqq 0. \tag{51}$$

Hence, (48) and (51) yield

$$\int_a^b \sum_{i=1}^p \lambda_i \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \xi_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^+(t)} \zeta_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^-(t)} (-\zeta_k)\, \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt < 0. \tag{52}$$

We denote

$$\widetilde{\lambda}_i = \frac{\lambda_i}{\sum_{i=1}^p \lambda_i + \sum_{j=1}^q \xi_j(t) + \sum_{k\in K^+(t)} \zeta_k(t) - \sum_{k\in K^-(t)} \zeta_k(t)}, \quad i \in A, \tag{53}$$

$$\widetilde{\xi}_j(t) = \frac{\xi_j(t)}{\sum_{i=1}^p \lambda_i + \sum_{j=1}^q \xi_j(t) + \sum_{k\in K^+(t)} \zeta_k(t) - \sum_{k\in K^-(t)} \zeta_k(t)}, \quad j \in J, \tag{54}$$

$$\widetilde{\zeta}_k(t) = \frac{\zeta_k(t)}{\sum_{i=1}^p \lambda_i + \sum_{j=1}^q \xi_j(t) + \sum_{k\in K^+(t)} \zeta_k(t) - \sum_{k\in K^-(t)} \zeta_k(t)}, \quad k \in K^+(t), \tag{55}$$

$$\widetilde{\zeta}_k(t) = \frac{-\zeta_k(t)}{\sum_{i=1}^p \lambda_i + \sum_{j=1}^q \xi_j(t) + \sum_{k\in K^+(t)} \zeta_k(t) - \sum_{k\in K^-(t)} \zeta_k(t)}, \quad k \in K^-(t). \quad (56)$$

By (53)–(56), it follows that $0 \leqq \widetilde{\lambda}_i \leqq 1, i \in A$, but $\widetilde{\lambda}_i > 0$ for at least one $i \in A$, $0 \leqq \widetilde{\xi}_j(t) \leqq 1, j \in J$, $0 \leqq \widetilde{\zeta}_k(t) \leqq 1, k \in K$, and, moreover,

$$\sum_{i=1}^p \widetilde{\lambda}_i + \sum_{j=1}^q \widetilde{\xi}_j(t) + \sum_{k\in K^+(t)} \widetilde{\zeta}_k(t) + \sum_{k\in K^-(t)} \widetilde{\zeta}_k(t) = 1. \quad (57)$$

Combining (52)–(56), we get

$$\begin{aligned}
&\int_a^b \sum_{i=1}^p \widetilde{\lambda}_i \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt \\
&+ \int_a^b \sum_{j=1}^q \widetilde{\xi}_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt \\
&+ \int_a^b \sum_{k\in K^+(t)} \widetilde{\zeta}_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt \\
&+ \int_a^b \sum_{k\in K^-(t)} \widetilde{\zeta}_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt < 0.
\end{aligned} \quad (58)$$

By Definition 2, it follows that the functional $\Phi(t, x, \dot{x}, y, \dot{y}, \cdot)$ is convex on R^{n+1}. Thus, since (57) holds, then Definition 1 implies

$$\begin{aligned}
&\int_a^b \sum_{i=1}^p \widetilde{\lambda}_i \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(f_y^i\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, y, \dot{y}\right)\right], \rho_{f_i}\right)\right) dt \\
&\quad + \int_a^b \sum_{j=1}^q \widetilde{\xi}_j \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(g_y^j\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right], \rho_{g_j}\right)\right) dt \\
&\quad + \int_a^b \sum_{k\in K^+(t)} \widetilde{\zeta}_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt \\
&\quad + \int_a^b \sum_{k\in K^-(t)} \widetilde{\zeta}_k \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(-h_y^k\left(t, y, \dot{y}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, y, \dot{y}\right)\right], \rho_{h_k}\right)\right) dt \\
&\geqq \int_a^b \Phi\left(t, x, \dot{x}, y, \dot{y}; \left(\left[\sum_{i=1}^p \widetilde{\lambda}_i f_y^i\left(t, y, \dot{y}\right) + \sum_{j=1}^q \widetilde{\xi}_j g_y^j\left(t, y, \dot{y}\right) + \sum_{k\in K^+(t)} \widetilde{\zeta}_k h_y^k\left(t, y, \dot{y}\right)\right.\right.\right. \\
&\qquad \left. + \sum_{k\in K^-(t)} \left(-\widetilde{\zeta}_k\right) h_y^k\left(t, y, \dot{y}\right)\right] - \frac{d}{dt}\left[\sum_{i=1}^p \widetilde{\lambda}_i f_{\dot{y}}^i\left(t, y, \dot{y}\right) + \sum_{j=1}^q \widetilde{\xi}_j g_{\dot{y}}^j\left(t, y, \dot{y}\right)\right.
\end{aligned}$$

$$+\sum_{k\in K^+(t)}\widetilde{\zeta}_k h^k_{\dot{y}}\left(t,y,\dot{y}\right)+\sum_{k\in K^-(t)}\left(-\widetilde{\zeta}_k\right)h^k_{\dot{y}}\left(t,y,\dot{y}\right)\Bigg],$$

$$\sum_{i=1}^{p}\widetilde{\lambda}_i\rho_{f_i}+\sum_{j=1}^{q}\widetilde{\xi}_j\rho_{g_j}+\sum_{k\in K^+(t)\cup K^-(t)}\widetilde{\zeta}_k\rho_{h_k}\Bigg)\Bigg)dt. \tag{59}$$

Combining (58) and (59), we have

$$\int_a^b\Phi\Bigg(t,x,\dot{x},y,\dot{y};\Bigg(\Bigg[\sum_{i=1}^{p}\widetilde{\lambda}_i f^i_y\left(t,y,\dot{y}\right)+\sum_{j=1}^{q}\widetilde{\xi}_j g^j_y\left(t,y,\dot{y}\right)$$

$$+\sum_{k\in K^+(t)}\widetilde{\zeta}_k h^k_y\left(t,y,\dot{y}\right)+\sum_{k\in K^-(t)}\left(-\widetilde{\zeta}_k\right)h^k_y\left(t,y,\dot{y}\right)\Bigg]-\frac{d}{dt}\Bigg[\sum_{i=1}^{p}\widetilde{\lambda}_i f^i_{\dot{y}}\left(t,y,\dot{y}\right)$$

$$+\sum_{j=1}^{q}\widetilde{\xi}_j g^j_{\dot{y}}\left(t,y,\dot{y}\right)+\sum_{k\in K^+(t)}\widetilde{\zeta}_k h^k_{\dot{y}}\left(t,y,\dot{y}\right)+\sum_{k\in K^-(t)}\left(-\widetilde{\zeta}_k\right)h^k_{\dot{y}}\left(t,y,\dot{y}\right),$$

$$\sum_{i=1}^{p}\widetilde{\lambda}_i\rho_{f_i}+\sum_{j=1}^{q}\widetilde{\xi}_j\rho_{g_j}+\sum_{k\in K^+(t)\cup K^-(t)}\widetilde{\zeta}_k\rho_{h_k}\Bigg)\Bigg)dt<0.$$

Hence, the first constraint of (VMD) yields

$$\int_a^b\Phi\Bigg(t,x,\dot{x},y,\dot{y};\Bigg(0,\sum_{i=1}^{p}\widetilde{\lambda}_i\rho_{f_i}+\sum_{j=1}^{q}\widetilde{\xi}_j\rho_{g_j}+\sum_{k\in K^+(t)\cup K^-(t)}\widetilde{\zeta}_k\rho_{h_k}\Bigg)\Bigg)dt<0. \tag{60}$$

From the hypothesis (e), it follows that

$$\sum_{i=1}^{p}\widetilde{\lambda}_i\rho_{f_i}+\sum_{j=1}^{q}\widetilde{\xi}_j\rho_{g_j}+\sum_{k\in K^+(t)\cup K^-(t)}\widetilde{\zeta}_k\rho_{h_k}\geqq 0. \tag{61}$$

By Definition 2, we have that $\Phi(t,x,\dot{x},y,\dot{y};(0,a))\geqq 0$ for any $a\in R_+$. Thus, (61) implies that the following inequality

$$\int_a^b\Phi\Bigg(t,x,\dot{x},y,\dot{y};\Bigg(0,\sum_{i=1}^{p}\widetilde{\lambda}_i\rho_{f_i}+\sum_{j=1}^{q}\widetilde{\xi}_j\rho_{g_j}+\sum_{k\in K^+(t)\cup K^-(t)}\widetilde{\zeta}_k\rho_{h_k}\Bigg)\Bigg)dt\geqq 0$$

holds, contradicting (60). This completes the proof of theorem. □

Theorem 14 (Weak duality): *Let x and (y,λ,ξ,ζ) be any feasible solutions in problems (MVCP) and (VMD), respectively. Further, assume that the following hypotheses are fulfilled:*

(a) $f^i(t,\cdot,\cdot)+\xi_M(t)^T g^M(t,\cdot,\cdot)$, $i=1,\ldots,p$, *is strictly (Φ,ρ_{f_i})-pseudo-invex at y on $S\cup Y$,*
(b) $\xi_L(t)^T g^L(t,\cdot,\cdot)$ *is (Φ,ρ_{g_L})-quasi-invex at y on $S\cup Y$,*
(c) $\zeta(t)^T h(t,\cdot,\cdot)$ *is (Φ,ρ_h)-quasi-invex at y on $S\cup Y$,*
(d) $\sum_{i=1}^{p}\lambda_i\rho_{f_i}+\rho_{g_L}+\rho_h\geqq 0$.

Then, the following cannot hold

$$\int_a^b f^i\left(t,x,\dot{x}\right)dt \leqq \int_a^b \left(f^i\left(t,y,\dot{y}\right)+\xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt, \quad i\in A, \tag{62}$$

and

$$\int_a^b f^r\left(t,x,\dot{x}\right)dt < \int_a^b \left(f^r\left(t,y,\dot{y}\right)+\xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt \text{ for some } r\in A. \tag{63}$$

Proof We proceed by contradiction. Suppose, contrary to the result, that the inequalities (62) and (63) are satisfied. Hence, by $x\in D$ and $(y,\lambda,\xi,\zeta)\in\Omega_D$, (62) and (63) yield, respectively,

$$\int_a^b \left(f^i\left(t,x,\dot{x}\right)+\xi_M(t)^T g^M\left(t,x,\dot{x}\right)\right)dt \leqq \int_a^b \left(f^i\left(t,y,\dot{y}\right)+\xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt, \quad i\in A, \tag{64}$$

$$\int_a^b \left(f^r\left(t,x,\dot{x}\right)+\xi_M(t)^T g^M\left(t,x,\dot{x}\right)\right)dt < \int_a^b \left(f^r\left(t,y,\dot{y}\right)+\xi_M(t)^T g^M\left(t,y,\dot{y}\right)\right)dt \quad \text{for some } r\in A. \tag{65}$$

Thus, by (64) and (65), Definition 5 implies

$$\int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(f_y^i\left(t,y,\dot{y}\right)+\sum_{j\in M}\xi_j g_y^j\left(t,y,\dot{y}\right)-\frac{d}{dt}\left[f_{\dot{y}}^i\left(t,y,\dot{y}\right)+\sum_{j\in M}\xi_j g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right],\rho_{f_i}\right)\right)dt<0, \quad i\in A. \tag{66}$$

By $x\in D$ and $(y,\lambda,\xi,\zeta)\in\Omega_D$, it follows that

$$\int_a^b \xi_L(t)^T g^L\left(t,x,\dot{x}\right)dt \leqq \int_a^b \xi_L(t)^T g^L\left(t,y,\dot{y}\right)dt, \tag{67}$$

$$\int_a^b \zeta(t)^T h\left(t,x,\dot{x}\right)dt = \int_a^b \zeta(t)^T h\left(t,y,\dot{y}\right)dt. \tag{68}$$

By hypotheses (b)-(c), (67) and (68), Definition 6 implies, respectively,

$$\int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{j\in L}\xi_j g_y^j\left(t,y,\dot{y}\right)-\frac{d}{dt}\left[\sum_{j\in L}\xi_j g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right],\rho_{g_L}\right)\right)dt \leqq 0, \quad (69)$$

$$\int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{k=1}^s\zeta_k h_y^k\left(t,y,\dot{y}\right)-\frac{d}{dt}\left[\sum_{k=1}^s\zeta_k h_{\dot{y}}^k\left(t,y,\dot{y}\right)\right],\rho_h\right)\right)dt \leqq 0. \quad (70)$$

Since $\lambda \geq 0$, then, combining (66), (69) and (70), we have

$$\begin{aligned}\int_a^b \sum_{i=1}^p \lambda_i \Bigg\{\Phi\Bigg(t,x,\dot{x},y,\dot{y};\Bigg(f_y^i\left(t,y,\dot{y}\right)+\sum_{j\in M}\xi_j g_y^j\left(t,y,\dot{y}\right)\\ -\frac{d}{dt}\left[f_{\dot{y}}^i\left(t,y,\dot{y}\right)+\sum_{j\in M}\xi_j g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right],\rho_{f_i}\Bigg)\Bigg)\\ +\Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{j\in L}\xi_j g_y^j\left(t,y,\dot{y}\right)-\frac{d}{dt}\left[\sum_{j\in L}\xi_j g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right],\rho_{g_L}\right)\right)\\ +\Phi\left(t,x,\dot{x},y,\dot{y};\left(\sum_{k=1}^s\zeta_k h_y^k\left(t,y,\dot{y}\right)-\frac{d}{dt}\left[\sum_{k=1}^s\zeta_k h_{\dot{y}}^k\left(t,y,\dot{y}\right)\right],\rho_h\right)\right)\Bigg\} dt < 0.\end{aligned} \quad (71)$$

Since the functional $\Phi(t,x,\dot{x},y,\dot{y},\cdot)$ is convex on R^{n+1}, $\lambda^T e = 1$ and $M \cup L = J$, then Definition 1 implies

$$\begin{aligned}\int_a^b \Phi\Bigg(t,x,\dot{x},y,\dot{y};\Bigg(\sum_{i=1}^p\lambda_i f_y^i\left(t,y,\dot{y}\right)+\sum_{j=1}^q\xi_j g_y^j\left(t,y,\dot{y}\right)+\sum_{k=1}^s\zeta_k h_y^k\left(t,y,\dot{y}\right)\\ -\frac{d}{dt}\left[\sum_{i=1}^p\lambda_i f_{\dot{y}}^i\left(t,y,\dot{y}\right)+\sum_{j=1}^q\xi_j g_{\dot{y}}^j\left(t,y,\dot{y}\right)+\sum_{k=1}^q\zeta_k h_{\dot{y}}^k\left(t,y,\dot{y}\right)\right],\\ \sum_{i=1}^p\lambda_i\rho_{f_i}+\rho_{g_L}+\rho_h\Bigg)\Bigg)dt < 0.\end{aligned}$$

Hence, the first constraint of (VMD) yields

$$\int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(0,\sum_{i=1}^p\lambda_i\rho_{f_i}+\rho_{g_L}+\rho_h\right)\right)dt < 0. \quad (72)$$

Since $\Phi(t,x,\dot{x},y,\dot{y};(0,a)) \geqq 0$ for any $a \in R_+$, then hypothesis (d) implies that the following inequality

$$\int_a^b \Phi\left(t,x,\dot{x},y,\dot{y};\left(0,\sum_{i=1}^p\lambda_i\rho_{f_i}+\rho_{g_L}+\rho_h\right)\right)dt \geqq 0$$

holds, contradicting (72). Thus, the proof of theorem completes. □

Theorem 15 (Strong duality)*: Let $\overline{x}$ be an efficient solution in the considered multiobjective variational programming problem (MVCP). Further assume that the Kuhn–Tucker constraint qualification is satisfied for (MVCP). Then there exist $\overline{\lambda} \in R^p_+$ and the piecewise smooth functions $\overline{\xi}(\cdot) : I \to R^m$ and $\overline{\zeta}(\cdot) : I \to R^s$ such that $\overline{x}$ is a feasible solution for problem (VMD). If also the weak duality theorem holds between (MVCP) and (VMD), then $\overline{x}$ is an efficient solution for mixed type dual problem (VMD) and the objective function values are equal.*

Proof By assumption, $\overline{x}$ is an efficient solution in the considered multiobjective variational programming problem (MVCP). Hence, by Theorem 9, there exist $\overline{\lambda} \in R^p$ and piecewise smooth functions $\overline{\xi}(\cdot) : I \to R^m$ and $\overline{\zeta}(\cdot) : I \to R^s$ such that the Karush-Kuhn-Tucker optimality conditions (1)–(3) are satisfied. Thus, $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ is a feasible solution in mixed dual problem (VMD) and the two objective functionals have same values. Efficiency of $\overline{x}$ in problem (VMD) follows directly from the weak duality theorem (Theorem 13).

Proposition 16 *Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ be a feasible solution in mixed type dual problem (VMD) such that $\overline{y} \in S$. Further, assume that the following hypotheses are fulfilled:*

(a) $f^i(t, \cdot, \cdot), i = 1, \ldots, p$, *is strictly* (Φ, ρ_{f_i})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(b) $g^j(t, \cdot, \cdot), j = 1, \ldots, q$, *is* (Φ, ρ_{g_j})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(c) $h^k(t, \cdot, \cdot), k \in K^+(t) = \left\{k \in K : \overline{\zeta}_k(t) > 0\right\}$, *is* (Φ, ρ_{h_k})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(d) $-h^k(t, \cdot, \cdot), k \in K^-(t) = \left\{k \in K : \overline{\zeta}_k(t) < 0\right\}$, *is* (Φ, ρ_{h_k})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(e) $\sum_{i=1}^p \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \overline{\xi}_j \rho_{g_j} + \sum_{k \in K^+(t)} \overline{\zeta}_k \rho_{h_k} - \sum_{k \in K^-(t)} \overline{\zeta}_k \rho_{h_k} \geqq 0$.

Then $\overline{y}$ is efficient in the considered multiobjective variational control problem (MVCP).

Theorem 17 (Converse duality)*: Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ be an efficient solution in mixed type multiobjective variational dual problem (VMD) and $\overline{y} \in S$. Further, assume that the hypotheses (a)-(e) of* Proposition 16 *are fulfilled. Then $\overline{y}$ is efficient for the considered multiobjective variational control problem (MVCP).*

Proof Proof follows directly from Proposition 16. □

Theorem 18 (Strict converse duality)*: Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ be feasible solutions in problems (MVCP) and (VMD), respectively, such that*

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt \leqq \int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{y}, \dot{\overline{y}}\right) dt + \int_a^b \sum_{j \in M} \overline{\xi}_j g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt. \quad (73)$$

Further, assume that the following hypotheses are fulfilled:

(a) $f^i(t, \cdot, \cdot), i = 1, \ldots, p$, *is strictly* (Φ, ρ_{f_i})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(b) $g^j(t, \cdot, \cdot), j = 1, \ldots, q$, *is* (Φ, ρ_{g_j})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(c) $h^k(t, \cdot, \cdot), k \in K^+(t) = \left\{k \in K : \overline{\zeta}_k(t) > 0\right\}$, *is* (Φ, ρ_{h_k})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(d) $-h^k(t, \cdot, \cdot), k \in K^-(t) = \left\{k \in K : \overline{\zeta}_k(t) < 0\right\}$, *is* (Φ, ρ_{h_k})*-invex at* $\overline{y}$ *on* $S \cup Y$,
(e) $\sum_{i=1}^p \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^q \overline{\xi}_j \rho_{g_j} + \sum_{k \in K^+(t)} \overline{\zeta}_k \rho_{h_k} - \sum_{k \in K^-(t)} \overline{\zeta}_k \rho_{h_k} \geqq 0$.

Then $\overline{x} = \overline{y}$ and $\overline{y}$ is efficient for the considered multiobjective variational control problem (MVCP).

Proof Suppose, contrary to the result, that $\overline{x} \neq \overline{y}$. Since $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ are feasible solutions in problems (MVCP) and (VMD), respectively, then

$$\int_a^b \sum_{j\in L} \overline{\xi}_j g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt \leqq \int_a^b \sum_{j\in L} \overline{\xi}_j g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt, \tag{74}$$

$$\int_a^b \sum_{k\in K} \overline{\zeta}_k h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt = \int_a^b \sum_{k\in K} \overline{\zeta}_k h^k\left(t, \overline{y}, \dot{\overline{y}}\right) dt. \tag{75}$$

The hypotheses (b)–(d), by Definition 2, yield

$$\begin{aligned}
&\int_a^b g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt \\
&\quad \geqq \int_a^b \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(g_y^j\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{g_j}\right)\right) dt, \quad j \in J,
\end{aligned} \tag{76}$$

$$\begin{aligned}
&\int_a^b h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b h^k\left(t, \overline{y}, \dot{\overline{y}}\right) dt \\
&\quad \geqq \int_a^b \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right)\right) dt, \quad k \in K^+(t),
\end{aligned} \tag{77}$$

$$\begin{aligned}
&-\int_a^b h^k\left(t, \overline{x}, \dot{\overline{x}}\right) dt + \int_a^b h^k\left(t, \overline{y}, \dot{\overline{y}}\right) dt \\
&\quad \geqq \int_a^b \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(\left[-h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right] - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right)\right) dt, \quad k \in K^-(t).
\end{aligned} \tag{78}$$

By $\overline{\xi}_j \geqq 0$, $j \in J$, (76) gives

$$\begin{aligned}
&\int_a^b \sum_{j=1}^q \overline{\xi}_j g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b \sum_{j=1}^q \overline{\xi}_j g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt \\
&\quad \geqq \int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(g_y^j\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{g_j}\right)\right) dt.
\end{aligned} \tag{79}$$

Since $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\zeta}\right)$ are feasible solutions in problems (MVCP) and (VMD), respectively, then (79) implies

$$\int_a^b \sum_{j\in M} \overline{\xi}_j g^j\left(t,\overline{x},\dot{\overline{x}}\right)dt + \int_a^b \sum_{j\in M} \overline{\xi}_j \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(g_y^j\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{g_j}\right)\right)dt \leqq 0 \tag{80}$$

and, by (74),

$$\int_a^b \sum_{j\in L} \overline{\xi}_j \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(g_y^j\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{g_j}\right)\right)dt \leqq 0. \tag{81}$$

Adding both sides of (80) and (81), we obtain

$$\int_a^b \sum_{j\in M} \overline{\xi}_j g^j\left(t,\overline{x},\dot{\overline{x}}\right)dt + \int_a^b \sum_{j=1}^{q} \overline{\xi}_j \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(g_y^j\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{g_j}\right)\right)dt \leqq 0 \tag{82}$$

Since $\overline{\zeta}_k > 0, k \in K^+(t)$, and $-\overline{\zeta}_k > 0, k \in K^-(t)$, then (77) and (78) yield

$$\begin{aligned}
&\int_a^b \sum_{k\in K^+(t)\cup K^-(t)} \overline{\zeta}_k h^k\left(t,\overline{x},\dot{\overline{x}}\right)dt - \int_a^b \sum_{k\in K^+(t)\cup K^-(t)} \overline{\zeta}_k h^k\left(t,\overline{y},\dot{\overline{y}}\right)dt \\
&\geqq \int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(h_y^k\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{h_k}\right)\right)dt \\
&+ \int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(-h_y^k\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{h_k}\right)\right)dt.
\end{aligned} \tag{83}$$

By the feasibility of $\overline{x}$ and of $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\zeta}\right)$ in problems (MVCP) and (VMD), respectively, it follows that

$$\begin{aligned}
&\int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(h_y^k\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{h_k}\right)\right)dt \\
&+ \int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t,\overline{x},\dot{\overline{x}},\overline{y},\dot{\overline{y}};\left(-h_y^k\left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t,\overline{y},\dot{\overline{y}}\right)\right],\rho_{h_k}\right)\right)dt \leqq 0.
\end{aligned} \tag{84}$$

We denote

$$\widetilde{\lambda}_i = \frac{\overline{\lambda}_i}{\sum_{i=1}^{p} \overline{\lambda}_i + \sum_{j=1}^{q} \overline{\xi}_j(t) + \sum_{k\in K^+(t)} \overline{\zeta}_j(t) - \sum_{k\in K^-(t)} \overline{\zeta}_j(t)}, \quad i \in A, \tag{85}$$

$$\widetilde{\xi}_j(t) = \frac{\overline{\xi}_j(t)}{\sum_{i=1}^{p} \overline{\lambda}_i + \sum_{j=1}^{q} \overline{\xi}_j(t) + \sum_{k\in K^+(t)} \overline{\zeta}_j(t) - \sum_{k\in K^-(t)} \overline{\zeta}_j(t)}, \quad j \in J, \tag{86}$$

$$\widetilde{\zeta}_k(t) = \frac{\overline{\zeta}_k(t)}{\sum_{i=1}^{p} \overline{\lambda}_i + \sum_{j=1}^{q} \overline{\xi}_j(t) + \sum_{k\in K^+(t)} \overline{\zeta}_j(t) - \sum_{k\in K^-(t)} \overline{\zeta}_j(t)}, \quad k \in K^+(t), \tag{87}$$

$$\widetilde{\zeta}_k(t) = \frac{-\overline{\zeta}_k(t)}{\sum_{i=1}^{p} \overline{\lambda}_i + \sum_{j=1}^{q} \overline{\xi}_j(t) + \sum_{k\in K^+(t)} \overline{\zeta}_j(t) - \sum_{k\in K^-(t)} \overline{\zeta}_j(t)}, \quad k \in K^-(t). \tag{88}$$

By (85)–(88), it follows that $0 \leqq \widetilde{\lambda}_i \leqq 1, i \in A$, but $\widetilde{\lambda}_i > 0$ for at least one $i \in A$, $0 \leqq \widetilde{\xi}_j(t) \leqq 1$, $j \in J$, $0 \leqq \widetilde{\zeta}_k(t) \leqq 1$, $k \in K^+(t) \cup K^-(t)$, and, moreover,

$$\sum_{i=1}^{p} \widetilde{\lambda}_i + \sum_{j=1}^{q} \widetilde{\xi}_j(t) + \sum_{k\in K^+(t)} \widetilde{\zeta}_k(t) + \sum_{k\in K^-(t)} \widetilde{\zeta}_k(t) = 1. \tag{89}$$

Then, by (85)–(88), the hypothesis (e) yields

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \rho_{f_i} + \sum_{j=1}^{q} \widetilde{\xi}_j(t) \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widetilde{\zeta}_k(t) \rho_{h_k} \geqq 0. \tag{90}$$

By Definition 2, it follows that $\Phi(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; (0, a)) \geqq 0$ for any $a \in R_+$. Hence, (90) gives

$$\int_a^b \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(0, \sum_{i=1}^{p} \widetilde{\lambda}_i \rho_{f_i} + \sum_{j=1}^{q} \widetilde{\xi}_j(t) \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widetilde{\zeta}_k(t) \rho_{h_k}\right)\right) dt \geqq 0. \tag{91}$$

Thus, the first constraint of vector variational control mixed type dual problem (VMD) implies

$$\begin{aligned}
\int_a^b \Phi\Bigg(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \Bigg(\Bigg[\sum_{i=1}^{p} \widetilde{\lambda}_i f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) + \sum_{j=1}^{q} \widetilde{\xi}_j g_y^j\left(t, \overline{y}, \dot{\overline{y}}\right) \\
+ \sum_{k\in K^+(t)} \widetilde{\zeta}_k h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) + \sum_{k\in K^-(t)} \left(-\widetilde{\zeta}_k\right) h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right)\Bigg] \\
-\frac{d}{dt}\Bigg[\sum_{i=1}^{p} \widetilde{\lambda}_i f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right) + \sum_{j=1}^{q} \widetilde{\xi}_j g_{\dot{y}}^j\left(t, \overline{y}, \dot{\overline{y}}\right) \\
+ \sum_{k\in K^+(t)} \widetilde{\zeta}_k h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right) + \sum_{k\in K^-(t)} \left(-\widetilde{\zeta}_k\right) h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\Bigg], \\
\sum_{i=1}^{p} \widetilde{\lambda}_i \rho_{f_i} + \sum_{j=1}^{q} \widetilde{\xi}_j \rho_{g_j} + \sum_{k\in K^+(t)\cup K^-(t)} \widetilde{\zeta}_k \rho_{h_k}\Bigg)\Bigg) dt \geqq 0.
\end{aligned} \tag{92}$$

By Definition 2, it follows that the functional $\Phi(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}, \cdot)$ is convex on R^{n+1}. Thus, by Definition 1 and (85)–(90), (92) yields

$$\int_a^b \sum_{i=1}^p \widetilde{\lambda}_i \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \widetilde{\xi}_j \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(g_y^j\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{g_j}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^+(t)} \widetilde{\zeta}_k \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^-(t)} \left(-\widetilde{\zeta}_k\right) \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(-h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right) dt\right) \geqq 0.$$

Again by (85)–(90), it follows that

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt$$
$$+ \int_a^b \sum_{j=1}^q \overline{\xi}_j \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(g_y^j\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[g_{\dot{y}}^j\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{g_j}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^+(t)} \overline{\zeta}_k \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right)\right) dt$$
$$+ \int_a^b \sum_{k\in K^-(t)} \left(-\overline{\zeta}_k\right) \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(-h_y^k\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[-h_{\dot{y}}^k\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{h_k}\right)\right) dt$$
$$\geqq 0. \tag{93}$$

Combining (82), (84) and (93), we get

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt$$
$$\geqq \int_a^b \sum_{j\in M} \overline{\xi}_j g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt. \tag{94}$$

Hence, the hypothesis (a) and Definition 2 yield

$$\int_a^b f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b f^i\left(t, \overline{y}, \dot{\overline{y}}\right) dt$$
$$> \int_a^b \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt, \quad i \in A.$$

Multiplying both sides of the above inequalities by $\overline{\lambda}_i$, $i = 1, \ldots, p$, where $\overline{\lambda} \geq 0$, $\overline{\lambda}^T e = 1$, we obtain

$$\int_a^b \overline{\lambda}_i f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b \overline{\lambda}_i f^i\left(t, \overline{y}, \dot{\overline{y}}\right) dt$$

$$\geqq \int_a^b \overline{\lambda}_i \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt, \quad i \in A, \quad (95)$$

$$\int_a^b \overline{\lambda}_r f^r\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b \overline{\lambda}_r f^r\left(t, \overline{y}, \dot{\overline{y}}\right) dt > \int_a^b \overline{\lambda}_r \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^r\left(t, \overline{y}, \dot{\overline{y}}\right)\right.\right.$$

$$\left.\left. - \frac{d}{dt}\left[f_{\dot{y}}^r\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_r}\right)\right) dt \quad \text{for at least one } r \in A. \quad (96)$$

By (95) and (96), we get

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt - \int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{y}, \dot{\overline{y}}\right) dt$$

$$> \int_a^b \sum_{i=1}^p \overline{\lambda}_i \Phi\left(t, \overline{x}, \dot{\overline{x}}, \overline{y}, \dot{\overline{y}}; \left(f_y^i\left(t, \overline{y}, \dot{\overline{y}}\right) - \frac{d}{dt}\left[f_{\dot{y}}^i\left(t, \overline{y}, \dot{\overline{y}}\right)\right], \rho_{f_i}\right)\right) dt. \quad (97)$$

By (94) and (97), it follows that the following inequality

$$\int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt > \int_a^b \sum_{i=1}^p \overline{\lambda}_i f^i\left(t, \overline{y}, \dot{\overline{y}}\right) dt + \int_a^b \sum_{j \in M} \overline{\xi}_j g^j\left(t, \overline{y}, \dot{\overline{y}}\right) dt$$

holds, contradicting the assumption (73). Hence, $\overline{x} = \overline{y}$ and efficiency of $\overline{y}$ in the multiobjective variational control problem (MVCP) follows by the weak duality theorem (Theorem 13). This completes the proof of theorem. □

5 Conclusion

In this paper, we have generalized (Φ, ρ)-invexity notion and its generalizations to the continuous case. Then we have used these classes of generalized convex functions to derive several sufficient optimality conditions and mixed type duality results for a new class of nonconvex multiobjective variational control problems. Our results apparently generalize a fairly large number of sufficient optimality conditions and duality results previously obtained in the literature for multiobjective variational control problems under other generalized convexity notions.

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