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Geometric branch-and-bound methods for constrained global optimization problems

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Abstract Geometric branch-and-bound methods are popular solution algorithms in deterministic global optimization to solve problems in small dimensions. The aim of this paper is to formulate a geometric branch-and-bound method for constrained global optimization problems which allows the use of arbitrary bounding operations. In particular, our main goal is to prove the convergence of the suggested method using the concept of the rate of convergence in geometric branch-and-bound methods as introduced in some recent publications. Furthermore, some efficient further discarding tests using necessary conditions for optimality are derived and illustrated numerically on an obnoxious facility location problem.

Keywords Global optimization · Geometric branch-and-bound · Approximation algorithms · Continuous location

1 Introduction

Geometric branch-and-bound methods are a general class of solution algorithms in deterministic global optimization. These methods can be used to solve non-convex global optimization problems in small dimensions, say for problems with up to 6 or 10 variables. The main task throughout geometric branch-and-bound methods is the calculation of the required lower bounds which has been studied for example in Hansen et al. (1985), Ratschek and Voller (1991), Plastria (1992), Horst et al. (2000), Drezner and Suzuki (2004), and Blanquero and Carrizosa (2009). For a survey, see any classical textbook on global optimization such as Horst and Tuy (1996) or Floudas (1999). Recently, a general convergence theory which allows theoretical results about all these bounding procedures was introduced in Schöbel and Scholz (2010) and Scholz (2011). A detailed overview on geometric branch-and-bound methods, its bounding operations, and applications can be found in the textbook Scholz (2012).

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The aim of this paper is to analyze geometric branch-and-bound methods for constrained global optimization problems. Two of the first works on this subject are Ratschek and Rokne (1988) and Hansen (1992), both using interval analysis. Androulakis et al. (1995) presented a technique to calculate the required lower bounds making use of a convex relaxation of the original problem. An improvement of these methods for twice continuously differentiable functions can be found in Adjiman et al. (1998). Furthermore, Sun and Johnson (2005) suggested a local sampling technique such that unwanted subboxes could be discarded throughout the algorithm. Some further speed-up methods using Fritz John conditions for optimality were also discussed in Kearfott (1992) and Hansen (1992). Therein, interval Newton and Gauß-Seidel methods were employed to check subboxes for their feasibility.

The present paper differs from all these publications in several points. Our key issue is to formulate a geometric branch-and-bound method for constrained global optimization problems (see Sect. 3), in such a way that the convergence of the method can be proven using the concept of the rate of convergence as introduced in Schöbel and Scholz (2010) and Scholz (2011), see Sect. 4. To improve the efficiency of the suggested method, some general further discarding tests using necessary conditions for optimality are introduced in Sect. 5. Numerical results on an obnoxious facility location problem in Sect. 6 illustrate the huge improvement using these further discarding tests.

2 Definitions and notations

Let $f, g_1, \ldots, g_m, h_1, \ldots, h_r : \mathbb{R}^n \to \mathbb{R}$. Then we consider the *constrained global optimization problem*

$$\begin{aligned} \min\ & f(x) \text{ s.t.} \\ g_i(x) &\leq 0 \quad \text{for} \quad i = 1, \ldots, m, \\ h_j(x) &= 0 \quad \text{for} \quad j = 1, \ldots, r, \\ x &\in \mathbb{R}^n. \end{aligned}$$

Note that this problem is equivalent to

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ x \in \mathbb{R}^n \tag{1}$$

defining the function $g : \mathbb{R}^n \to \mathbb{R}$ with

$$g(x) = \max\{g_1(x), \ldots, g_m(x), h_1(x), -h_1(x), \ldots, h_r(x), -h_r(x)\}.$$

Therefore, we will only analyze problems of form (1) although all following results are also valid for more general problems. One of our main goals is to present an algorithm which finds an ε-optimal solution for problem (1) such that the constraint $g(x) \leq 0$ is violated only within a small accuracy of $\alpha \geq 0$.

Definition 1 Let $\varepsilon, \alpha \geq 0$ and assume that problem (1) has at least one feasible solution. Then we say an $x^* \in \mathbb{R}^n$ is an (ε, α)-*optimal solution* for (1) if x^* is within an absolute accuracy of ε from the global minimum and $g(x^*) \leq \alpha$.

Note that a $(0, 0)$-optimal solution is an optimal solution for problem (1). Furthermore, we will use the following notation throughout the paper.

Notation 2 A compact *box* or *hyperrectangle* with sides parallel to the axes is denoted by

$$X = [x_1^L, x_1^R] \times \cdots \times [x_n^L, x_n^R] \subset \mathbb{R}^n.$$

The *diameter* of a box $X \subset \mathbb{R}^n$ is

$$\delta(X) = \max\{\|x - \tilde{x}\|_2 : x, \tilde{x} \in X\} = \sqrt{\left(x_1^R - x_1^L\right)^2 + \cdots + \left(x_n^R - x_n^L\right)^2}$$

and the *center* of a box $X \subset \mathbb{R}^n$ is defined by

$$c(X) = \left(\frac{1}{2}(x_1^L + x_1^R), \ldots, \frac{1}{2}(x_n^L + x_n^R)\right).$$

In order to apply the algorithm presented in the next section, we further need the following definition [see Schöbel and Scholz (2010)].

Definition 3 Let $X \subset \mathbb{R}^n$ be a box and consider $f : X \to \mathbb{R}$. A *bounding operation* is a procedure to calculate for any subbox $Y \subset X$ a *lower bound* $LB(Y) \in \mathbb{R}$ with

$$LB(Y) \leq f(x) \quad \text{for all} \quad x \in Y$$

and to specify a point $r(Y) \in Y$. Formally, we obtain the bounding operation

$$(LB(Y), r(Y))$$

for all subboxes $Y \subset X$.

3 The branch-and-bound algorithm

In this section we present a general branch-and-bound method for constrained optimization problems. We remark that some related algorithms can also be found in classical textbooks on global optimization as mentioned in the introduction, e.g., Horst and Tuy (1996). However, the following algorithm differs at several points in order to prove the convergence without the knowlegde of explicit bounding operations but only using the concept of the rate of convergence as recently introduced in Schöbel and Scholz (2010), see Definition 4 and Theorem 2 in Sect. 4.

For our approach, consider a feasible problem

$$\min \ f(x) \quad \text{s.t.} \quad g(x) \leq 0, \ x \in \mathbb{R}^n$$

and assume that $\{x \in \mathbb{R}^n : g(x) \leq 0\} \subset X$, where X is a box with sides parallel to the axes, say

$$X = [x_1^L, x_1^R] \times \cdots \times [x_n^L, x_n^R] \subset \mathbb{R}^n.$$

Furthermore, we assume that bounding operations

$$(LB_f(Y), r(Y)) \qquad \text{and} \qquad (LB_g(Y), r(Y))$$

for f and g, respectively, are known. Then the following algorithm finds an (ε, α)-optimal solution for any accuracies $\varepsilon, \alpha > 0$.

(1) Let $\mathcal{X}$ be a list of boxes, initialize $\mathcal{X} := \{X\}$, and set $UB := \infty$.
(2) Apply both bounding operations to X and set $LB_{min} := LB_f(X)$. If $g(r(X)) \leq \alpha$, set $UB = f(r(X))$ and $x^* := r(X)$.
(3) If $UB - LB_{min} \leq \varepsilon$, the algorithm stops. Else set

$$\delta_{max} = \max\{\delta(Y) : Y \in \mathcal{X}\}.$$

(4) Select a box $Y \in \mathcal{X}$ with $\delta(Y) = \delta_{max}$ and split it into s subboxes Y_1 to Y_s such that $Y = Y_1 \cup \ldots \cup Y_s$.
(5) Set $\mathcal{X} = (\mathcal{X} \setminus Y) \cup \{Y_1, \ldots, Y_s\}$, i.e., delete Y from $\mathcal{X}$ and add $Y_1, \ldots, Y_s$.
(6) Apply both bounding operations to Y_1 to Y_s, let

$$I = \{k \in \{1, \ldots, s\} : g(r(Y_k)) \leq \alpha\},$$

and set

$$UB = \min\{UB, \min\{f(r(Y_k)) : k \in I\}\}.$$

If $UB = f(r(Y_k))$ for a $k \in I$, set $x^* = r(Y_k)$.
(7) For all $Z \in \{Y_1, \ldots, Y_s\}$, if $LB_g(Z) > 0$ set $\mathcal{X} = \mathcal{X} \setminus Z$.
(8) For all $Z \in \mathcal{X}$, if $LB_f(Z) > UB$ set $\mathcal{X} = \mathcal{X} \setminus Z$. If UB has not changed it is sufficient to check only the subboxes Y_1 to Y_s which were not deleted in the previous step.
(9) Whenever possible, apply some further discarding test, i.e., delete boxes $Z \in \mathcal{X}$ which do not contain any optimal solution (see Sect. 5).
(10) Set $LB_{min} = \min\{LB_f(Y) : Y \in \mathcal{X}\}$.
(11) Return to Step **(3)**.

For boxes in small dimensions, say $n \leq 3$, we suggest a split into $s = 2^n$ congruent subboxes. In higher dimensions, boxes can be bisected perpendicular to the direction of the maximum width component in two subboxes.

Lemma 1 *If the algorithm terminates, then x^* is an (ε, α)-optimal solution for*

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ x \in \mathbb{R}^n. \tag{2}$$

Proof First of all note that if the algorithm terminates we have $UB < \infty$ and the algorithm ensures $g(x^*) \leq \alpha$, see Step **(6)** of the algorithm.

Furthermore, note that in Step **(7)** to Step **(9)** only boxes are deleted which do not contain any optimal solution for the original problem (2). Hence, since throughout the algorithm we have

$$LB_{min} \leq \min\{f(x) : g(x) \leq 0,\ x \in \mathbb{R}^n\} \leq f(x^*) = UB,$$

the termination rule in Step **(3)** ensures that x^* is within an absolute accuracy of ε from the global minimum of problem (2). Thus, x^* is an (ε, α)-optimal solution. □

In the following section, we show that the algorithm also terminates indeed under weak assumptions.

4 Convergence theory

In this section, we discuss the convergence of the algorithm. To this end, let us recall the following definition from Schöbel and Scholz (2010).

Definition 4 Let $X \subset \mathbb{R}^n$ be a box and consider $f : X \to \mathbb{R}$. We say a bounding operation $(LB(Y), r(Y))$ has the *rate of convergence* $p \in \mathbb{N}$ if there exists a fixed constant $C > 0$ such that

$$f(r(Y)) - LB(Y) \leq C \cdot \delta(Y)^p$$

for all boxes $Y \subset X$.

The next theorem shows that the algorithm terminates after a finite number of iterations if we assume bounding operations with a rate of convergence of at least one.

We remark again that although the convergence of related algorithms can be found for example in Horst and Tuy (1996), the following result is more general since we only use the general concept of the rate of convergence and the knowledge on explicit bounding operations is not necessary.

Theorem 2 *Consider the geometric branch-and-bound algorithm for constrained optimization problems with bounding operations*

$$(LB_f(Y), r(Y)) \quad \text{and} \quad (LB_g(Y), r(Y))$$

for f and g, respectively, which have a rate of convergence of $p \geq 1$. Furthermore, assume that each selected box throughout the algorithm is split into $s = 2^n$ congruent smaller boxes.

Then the algorithm terminates after a finite number of iterations for every $\varepsilon, \alpha > 0$.

Proof Since in Step **(4)** of the algorithm a box with largest diameter is selected for a split into $s = 2^n$ smaller subboxes, we find

$$g(r(Y)) - LB_g(Y) \leq \alpha$$

for all boxes $Y \in \mathcal{X}$ after a finite number of iterations. In other words, we know that $g(r(Y)) \leq \alpha$ or that $LB_g(Y) > 0$ for all boxes Y which occur in the remainder of the algorithm.

Moreover, we also have

$$f(r(Y)) - LB_f(Y) \leq \varepsilon$$

after a finite number of iterations. Thus, after some time we obtain for all $Y \in \mathcal{X}$ in Step **(10)**

$$UB - LB_f(Y) \leq f(r(Y)) - LB_f(Y) \leq \varepsilon,$$

where $UB \leq f(r(Y))$ holds due to Step **(6)** since $g(r(Y)) \leq \alpha$ for all $Y \in \mathcal{X}$, see above.

To sum up, we find $UB - LB_{min} \leq \varepsilon$ after a finite number of iterations and the algorithm terminates. □

To sum up, Theorem 2 presents the termination of the algorithm and Lemma 1 says that if the algorithm terminates, then we found an (ε, α)-optimal solution x^*. In other words, if bounding operations for f and g with a rate of convergence of at least one are employed, then the algorithm terminates with an (ε, α)-optimal solution.

Furthermore, note that the algorithm analogously terminates for a bisecting splitting rule with $s = 2$. But in this case the algorithm might take much more iterations compared to the $s = 2^n$ splitting rule.

5 Discarding tests for constrained optimization problems

Making use of the Fritz John conditions for optimality, our aim in this section is to derive some futher discarding tests, see Step **(9)**, in order to speed-up the branch-and-bound algorithm.

Theorem 3 (Fritz John conditions for optimality) *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and assume that f and g are continuously differentiable at $\hat{x} \in \mathbb{R}^n$. A necessary condition for $\hat{x}$ to be optimal for*

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \le 0,\ x \in \mathbb{R}^n$$

is that there exist multipliers $\lambda, \mu \ge 0$ such that

$$\lambda \cdot \nabla f(\hat{x}) + \mu \cdot \nabla g(\hat{x}) = 0, \qquad \mu \cdot g(\hat{x}) = 0, \qquad \text{and} \qquad \lambda + \mu > 0.$$

Proof See any standard textbook on nonlinear programming such as Bazaraa et al. (1993). □

We remark that some similar necessary conditions for the case that the objective function and the constrains are nondifferentiable can be found for example in Craven and Mond (1976). Thus, it might be possible to formulate our further results also for nondifferentiable functions in a similar way.

The next corollary is crucial for our following considerations.

Corollary 4 *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and assume that f and g are continuously differentiable at $\hat{x} \in \mathbb{R}^n$. If*

$$\frac{\partial f}{\partial x_i}(\hat{x}) \cdot \frac{\partial g}{\partial x_j}(\hat{x}) - \frac{\partial f}{\partial x_j}(\hat{x}) \cdot \frac{\partial g}{\partial x_i}(\hat{x}) \neq 0 \tag{3}$$

for some $1 \le i < j \le n$ then $\hat{x}$ is not optimal for the optimization problem

$$\min\ f(x) \quad \text{s.t.} \quad g(x) \le 0,\ x \in \mathbb{R}^n.$$

Proof From the Fritz John conditions for optimality it follows that a necessary condition for $\hat{x}$ to be an optimal solution for the constrained optimization problem is that the gradients $\nabla f(\hat{x})$ and $\nabla g(\hat{x})$ are linear dependent.

On the other hand, if $\hat{x}$ satisfies the condition given in Eq. (3) for some $1 \le i < j \le n$ then $\nabla f(\hat{x})$ and $\nabla g(\hat{x})$ are linear independent and, hence, $\hat{x}$ cannot be an optimal solution. □

Now we can formulate some further discarding tests to delete boxes throughout the branch-and-bound algorithm which do not contain any optimal solution, see Step **(9)** in Sect. 3. The idea of using the Fritz John conditions to detect subboxes which do not contain any optimal solution is not new and can be found for example in Hansen (1992) and references therein. Some of these discarding tests are based on Newton or Gauß-Seidel methods [see Hansen (1992)]. The use of the following discarding tests accelerates the convergence of the algorithm enormously (see Sect. 6), although the tests are easy to implement and their runtimes seem to be small compared to the previous mentioned techniques.

We assume that we are in a position to calculate lower bounds $H(Y)^L$ and upper bounds $H(Y)^R$ for some functions $h : \mathbb{R}^n \to \mathbb{R}$ on subboxes $Y \subset X$. In other words, we have to calculate real values $H(Y)^L, H(Y)^R \in \mathbb{R}$ such that

$$H(Y)^L \le h(x) \le H(Y)^R \qquad \text{for all} \quad x \in Y.$$

One of the easiest ways to do so is interval analysis [see e.g., Ratschek and Rokne (1988), Neumaier (1990), or Hansen (1992)] for details on this subject. For example, the natural interval extension directly yields the required bounds such that

$$h(Y) \; = \; \{h(x) \; : \; x \in Y\} \; \subset \; [H(Y)^L, \; H(Y)^R].$$

Hence, we obtain the following discarding tests.

Lemma 5 *Let $f, g : \mathbb{R}^n \to \mathbb{R}$, consider a box $Y \subset X$, and assume that f is continuously differentiable for all $x \in Y$. Furthermore, let*

$$\frac{\partial f}{\partial x_k}(Y) \; \subset \; [F_k(Y)^L, F_k(Y)^R] \quad \text{and} \quad g(Y) \; \subset \; [G(Y)^L, G(Y)^R]$$

for all $k = 1, \ldots, n$. If $G(Y)^R < 0$ and if there is an $s \in \{1, \ldots, n\}$ with

$$F_s(Y)^L \; > \; 0 \quad \text{or} \quad F_s(Y)^R \; < \; 0$$

then Y does not contain any optimal solutions for the optimization problem

$$\min \; f(x) \quad \text{s.t.} \quad g(x) \; \leq \; 0, \; x \in \mathbb{R}^n.$$

Proof If $G(Y)^R < 0$, we know that $g(x) < 0$ for all $x \in Y$. Hence, $\nabla f(x) = 0$ is a necessary condition for optimality, see Theorem 3. But if

$$F_s(Y)^L \; > \; 0 \quad \text{or} \quad F_s(Y)^R \; < \; 0$$

for an $s \in \{1, \ldots, n\}$, there is no $x \in Y$ such that $\nabla f(x) = 0$. □

Lemma 6 *Let $f, g : \mathbb{R}^n \to \mathbb{R}$, consider a box $Y \subset X$, and assume that f and g are continuously differentiable for all $x \in Y$. Moreover, for all $1 \leq i < j \leq n$ define*

$$h_{ij}(x) \; := \; \frac{\partial f}{\partial x_i}(x) \cdot \frac{\partial g}{\partial x_j}(x) \; - \; \frac{\partial f}{\partial x_j}(x) \cdot \frac{\partial g}{\partial x_i}(x)$$

and assume

$$h_{ij}(Y) \; \subset \; [H_{ij}(Y)^L, H_{ij}(Y)^R].$$

If there are $1 \leq k < s \leq n$ with

$$H_{ks}(Y)^L \; > \; 0 \quad \text{or} \quad H_{ks}(Y)^R \; < \; 0$$

then Y does not contain any optimal solutions for the optimization problem

$$\min \; f(x) \quad \text{s.t.} \quad g(x) \; \leq \; 0, \; x \in \mathbb{R}^n.$$

Proof Follows directly from Corollary 4. □

6 Numerical results

For some numerical results, the algorithm was implemented in JAVA, compiled by JAVA 2 SDK 1.4, using double precision arithmetic. All tests were run on a 3.0 GHz computer with 4 GB of memory.

In our following studies, the algorithm was run twice. In the first run, we made use of both further discarding tests as presented in Sect. 5, see Lemmas 5 and 6, while in the second

Table 1 Input data for the constrained function g

k	1	2	3
b_k	(2, 3)	(8, 4)	(4, 7)
v_k	−8	−8	−8
s_k	−0.2	−0.2	−0.2
r_k	$(\sqrt{2}, 1)$	$(\sqrt{3}, \sqrt{2})$	(1, 1)

Table 2 Input data for the example problem

k	1	2	3	4	5	6	7	8	9	10
a_k	(2, 3)	(7, 1)	(8, 9)	(2, 5)	(6, 6)	(4, 9)	(9, 3)	(4, 3)	(3, 1)	(1, 8)
w_k	30	96	85	92	84	28	4	31	83	74

run no further discarding test was used. All boxes throughout the algorithm were bisected perpendicular to the direction of the maximum width component in two subboxes.

As example problem, we consider the following constrained obnoxious facility location problem [see Plastria (1995) or Drezner and Suzuki (2004)]. We want to minimize the sum of the reciprocal squared distance from m given demand points $a_1, \ldots, a_m \in \mathbb{R}^2$ to a new facility $x \in \mathbb{R}^2$. Hence, for the Euclidean norm the objective function reads as

$$f(x) = \sum_{k=1}^{m} \frac{w_k}{\max\{\|x - a_k\|_2^2, \epsilon\}}, \tag{4}$$

where $\epsilon > 0$ is a small number and $w_1, \ldots, w_m$ are non-negative weights. Obviously, the corresponding unconstrained minimization problem has no optimal solution $x \in \mathbb{R}^2$. Therefore, we consider the constraint

$$g(x) = 2 + \sum_{k=1}^{3} v_k \cdot \exp\left(s_k \cdot \left\|r_k^T \cdot (x - b_k)\right\|_2^2\right) \leq 0, \tag{5}$$

where the input data is given in Table 1. Note that f and g are continuously differentiable for all boxes $Y \subset X$ with

$$\min\{\|x - a_k\|_2^2 : x \in Y,\ k = 1, \ldots, m\} \geq \epsilon.$$

6.1 Example problem

As a first example problem, consider the objective function (4) with $m = 10$ and the input data collected in Table 2.

As inital box we chose $X = [0, 10] \times [0, 10]$ such that

$$\{x \in \mathbb{R}^2 : g(x) \leq 0\} \subset X.$$

Furthermore, we set $\varepsilon = 10^{-6}$ and $\alpha = 10^{-10}$. For all required bounds throughout the algorithm we employed the natural interval bounding operation derived from interval analysis with a rate of convergence of one [see Scholz (2011)].

In the first run, i.e., using both further discarding tests for all boxes $Y \subset X$ such that $f(x)$ is differentiable for all $x \in Y$, the algorithm stopped after only 255 iterations and a runtime

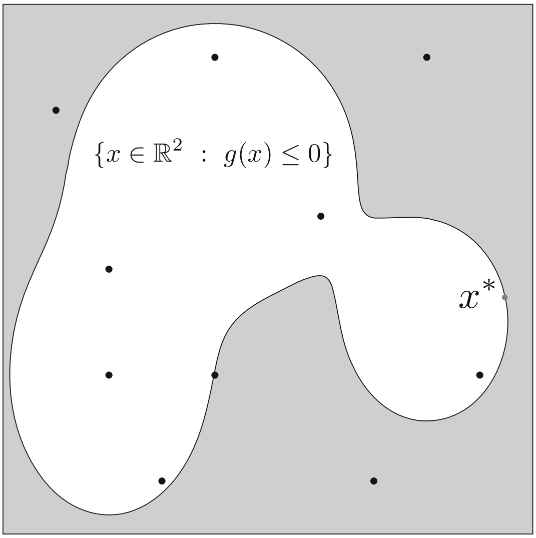

Fig. 1 Demand points (*black dots*) and feasible area (*white area*) for the example problem

Table 3 Numerical results for the obnoxious facility location problem problem

m	Iterations with discarding tests			Iterations without discarding tests		
	Min	Max	Ave.	Min	Max	Ave.
10	208	1,037	459.2	17,288	53,555	28,334.5
20	164	985	390.2	28,472	85,733	44,098.8
50	269	1,142	448.1	60,614	253,992	88,059.3
100	210	1,102	465.5	64,441	220,859	131,303.2

of 0.06 s. In the second run, i.e., without the further discarding tests, the algorithm needed 68,040 iterations and a runtime of 61.55 s. In both cases we found the optimal solution (see Fig. 1)

$$x^* = (9.472471, \ 4.469520).$$

6.2 Further computational experiences

For some further computational experiences, for the objective function f we generated $10 \leq m \leq 100$ demand points $a_1, \ldots, a_m$ uniformly distributed in $X = [0, 10] \times [0, 10]$ and weights $w_k \in [2, 10]$. The constraint was the same as before [see Eq. (5)].

Ten problems were run for different values of m twice, one time with the discarding tests and one time without the discarding tests. We again chose $\varepsilon = 10^{-6}$ and $\alpha = 10^{-10}$ and all required bounds were calculated using the natural interval bounding operation. Our results can be found in Tables 3 and 4 where we reported the number of iterations and the runtimes, respectively.

As can be seen, the discarding tests for constrained optimization problems derived from the Fritz John conditions for optimality improve the algorithm enormously. Furthermore, the number of iterations increases much faster without using any discarding test compared to an

Table 4 Numerical results for the obnoxious facility location problem

m	Runtime (s) with discarding tests			Runtime (s) without discarding tests		
	Min	Max	Ave.	Min	Max	Ave.
10	0.01	0.06	0.03	2.61	37.05	10.45
20	0.01	0.07	0.03	8.96	91.48	26.97
50	0.04	0.14	0.06	47.66	847.08	141.74
100	0.05	0.23	0.11	59.13	610.12	255.75

Table 5 Numerical results for the obnoxious facility location problem: comparison of the two discarding tests

Discarding 1	Discarding 2	Runtime (s)		Iterations	
		Inner	Boundary	Inner	Boundary
Yes	No	0.15	18.07	1,550.0	40,674.5
No	Yes	974.27	0.06	251,057.8	554.2
Yes	Yes	0.14	0.06	1,536.6	479.9

almost constant number of iterations. We remark that we obtained similar results for different constraint functions g even if the feasible set was not connected.

6.3 Comparison of the two discarding tests

In a third study, we compared the discarding test derived from Lemma 5 (discarding 1) and the one derived from Lemma 6 (discarding 2) separately. To this end, we generated 20 problem instances, each with $m = 20$ demand points again uniformly distributed in $X = [0, 10] \times [0, 10]$, weights $w_k \in [2, 10]$, and the same constraint as before.

In 10 out of these 20 problem instances, the optimal solution x^* was found to be in the interior of the feasible area, i.e., $g(x^*) < 0$. To be more accurate, for these problem instances (inner) the algorithm terminated with an x^* such that $g(x^*) < -0.1$. In the other 10 problem instances, the optimal solution was found on the boundary of the feasible area. These problem instances (boundary) were identified when the algorithm terminated with an x^* such that $g(x^*) \in [-\alpha, \alpha] = [-10^{-10}, 10^{-10}]$.

The average runtimes in seconds and the average number of iterations to solve these problems with (yes) and without (no) the discarding tests are given in Table 5.

This study shows that the discarding test derived from Lemma 5 (discarding 1) is efficient especially if the optimal solution is in the interior of the feasible area. On the other hand, the discarding test derived from Lemma 6 (discarding 2) accelerates the algorithm enormously for problem instances with an optimal solution on the boundary of the feasible area. To sum up, both discarding tests together are a very good choice for general problem instances.

7 Conclusions

Summarizing, in this paper we suggested a geometric branch-and-bound method for constrained global optimization problems. The convergence of the algorithm was shown and some further discarding tests using necessary conditions for optimality were derived.

Although some numerical results on the constrained obnoxious facility location problem illustrated the runtime of the method as well as the efficiency of the further discarding tests, let us point out some limitations of our approach.

First of all, note that the convergence was only shown for feasible and bounded problems, i.e., for problems with

$$\emptyset \neq \{x \in \mathbb{R}^n : g(x) \leq 0\} \subset X.$$

Hence, knowledge of the initial box X and the feasible area was assumed which might be difficult in particular problems. On the other hand, under these assumptions we presented the convergence of the algorithm using the general concept of the rate of convergence for bounding operations.

Finally, we want to mention that in our numerical results two cases appeared. In the first case, the algorithm terminated with x^* on the boundary of the feasible area, i.e., $g(x^*) = 0$. In the second case, we found $g(x^*) < 0$. In both cases the further discarding tests lead to a much more efficient algorithm. However, we only presented some brief numerical results and extensive studies are left for a further publication.

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