J Dyn Diff Equat (2018) 30:637–665
https://doi.org/10.1007/s10884-016-9565-z

A Widom–Rowlinson Jump Dynamics in the Continuum

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Received: 26 April 2016 / Revised: 13 November 2016 / Published online: 21 November 2016

Abstract We study the dynamics of an infinite system of point particles of two types. They perform random jumps in $\mathbb{R}^d$ in the course of which particles of different types repel each other whereas those of the same type do not interact. The states of the system are probability measures on the corresponding configuration space. The main result is the construction of the global (in time) Markov evolution of such states by means of correlation functions. It is proved that for each initial sub-Poissonian state μ_0, the states evolve $\mu_0 \mapsto \mu_t$ in such a way that μ_t is sub-Poissonian for all $t > 0$. The mesoscopic (approximate) description of the evolution of states is also given. The stability of translation invariant stationary states is studied. In particular, we show that some of such states can be unstable with respect to space-dependent perturbations.

Keywords Configuration space · Stochastic semigroup · Correlation function · Scale of Banach spaces · Denjoy–Carleman theorem · Mesoscopic limit · Kinetic equation

Mathematics Subject Classification 82C22 · 70F45 · 60K35

1 Introduction

1.1 Posing

In this paper, we study the dynamics of an infinite system of point particles of two types placed in $\mathbb{R}^d$. The particles perform random jumps in the course of which particles of different types repel each other whereas those of the same type do not interact. We do not require that the

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repulsion is of hard-core type. This model can be viewed as a dynamical version of the Widom–Rowlinson model [15] of equilibrium statistical mechanics – one of the few models of phase transitions in continuum particle systems, see the corresponding discussion in [7] where a similar birth-and-death model was introduced and studied. Note that the latter paper and this our work are the only publications where the dynamics of two-component infinite systems of interacting particles have been studied so far.

The phase space of our model is defined as follows. Let Γ denote the set of all $\gamma \subset \mathbb{R}^d$ that are *locally finite*, i.e., such that $\gamma \cap \Lambda$ is a finite set whenever $\Lambda \subset \mathbb{R}^d$ is compact. Thus, Γ is a configuration space as defined in [1,3,8,11]. In order to take into account the particle's type we use the Cartesian product $\Gamma^2 = \Gamma \times \Gamma$, see [5,7,9], the elements of which are denoted by $\gamma = (\gamma_0, \gamma_1)$. In a standard way, Γ^2 is equipped with a σ-field of measurable subsets which allows one to deal with probability measures as states of the system. Among them one may distinguish Poissonian states in which the particles are independently distributed over $\mathbb{R}^d$. *Sub-Poissonian* states are characterized by a rather weak dependence between the particle's positions, see Definition 2.1 below. As was shown in [10], for infinite particle systems with birth-and-death dynamics the evolution of states exists and is such that they remain sub-Poissonian globally in time if the birth of the particles is in a sense controlled by their state-dependent death. For conservative dynamics, in which the particles do not appear or disappear and only change their positions, the interaction may in general change the sub-Poissonian character of the state in finite time (even cause an explosion), e.g., due to an infinite number of simultaneous correlated jumps. Thus, the conceptual outcome of the present study is that this is not the case for the considered model. Note that we do not impose any kind of restrictions on the model parameters, and the existence of the global in time evolution of states is proved to hold even if there may exist multiple equilibrium states (phase transitions), and hence no ergodicity can be expected. Our another aim in this paper is to study the dynamics of the considered model in the mesoscopic limit, which yields its though an approximate (mean-field like) but more detailed picture. We do this and show how this approximate picture and the description of the evolution of states are related to each other.

1.2 Presenting the Results

The Markov evolution of states of the system which we consider is described by the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \qquad F_t|_{t=0} = F_0, \tag{1.1}$$

where $F_t : \Gamma^2 \to \mathbb{R}$ is an *observable* and the operator L specifies the model. It has the following form

$$\begin{aligned}(LF)(\gamma_0, \gamma_1) = &\sum_{x\in\gamma_0} \int_{\mathbb{R}^d} c_0(x, y, \gamma_1)[F(\gamma_0\backslash x \cup y, \gamma_1) - F(\gamma_0, \gamma_1)]dy \\ &+ \sum_{x\in\gamma_1} \int_{\mathbb{R}^d} c_1(x, y, \gamma_0)[F(\gamma_0, \gamma_1\backslash x \cup y) - F(\gamma_0, \gamma_1)]dy.\end{aligned} \tag{1.2}$$

The evolution of states is supposed to be obtained by solving the Fokker-Planck equation

$$\frac{d}{dt}\mu_t = L^*\mu_t, \qquad \mu_t|_{t=0} = \mu_0, \tag{1.3}$$

related to that in (1.1) by the duality

$$\int_{\Gamma^2} F_t(\gamma)\mu_0(d\gamma) = \int_{\Gamma^2} F_0(\gamma)\mu_t(d\gamma). \tag{1.4}$$

As is usual for models of this kind, the direct meaning of (1.1) or (1.3) can only be given for states of finite systems, cf. [12]. In this case, the Banach space where the Cauchy problem in (1.3) is defined can be the space of signed measures with finite variation. For infinite systems, the evolution of states is constructed indirectly, by employing correlation functions and or the related Bogoliubov functionals, see [3,6–10] and the references quoted therein.

In this paper, in describing the evolution of states, see Theorem 3.5 below, we mostly follow the scheme elaborated in [10]. It consists in: (a) constructing the evolution of correlation functions $k_0 \mapsto k_t$, $t < T < +\infty$, based on the Cauchy problem in (3.1); (b) proving that each k_t is the correlation function of a unique sub-Poissonian state μ_t; (c) constructing the continuation of thus obtained evolution $k_{\mu_0} = k_0 \mapsto k_t = k_{\mu_t}$ to all $t > 0$. Step (a) is performed by means of Ovcyannikov-like arguments similar to those used, e.g., in [3,6,7]. Step (b) is based on the use of the Denjoy–Carleman theorem [4]. In realizing step (c), we crucially use the result of (b). Our description of the mesoscopic limit is based on the scaling procedure described in Sect. 4. It is equivalent to the Lebowitz-Penrose scaling used in [7], and also to the Vlasov scaling used in [3,6]. In this procedure, passing to the mesoscopic level amounts to considering the system at different spatial scales parameterized by $\varepsilon \in (0; 1]$ in such a way that $\varepsilon = 1$ corresponds to the micro-level, whereas the limit $\varepsilon \to 0$ yields the meso-level description in which the corpuscular structure disappears and the system turns into a (two-component) medium characterized by a density function. The evolution of the latter is supposed to be found from the kinetic equation (3.15). In Theorem 3.8, we show that the kinetic equation has a unique global (in time) solution in the corresponding Banach space. In Theorem 3.9, we demonstrate that the micro- and meso-scopic descriptions are indeed connected by the scaling procedure in the sense of Definition 3.6. In Theorems 3.10 and 3.11, we describe the stability of translation invariant stationary solutions of the kinetic equation. In particular, we show that some of such solutions can be unstable with respect to space-dependent perturbations.

The rest of the paper has the following structure. In Sect. 2, we give necessary information on the analysis on configuration spaces and on the description of sub-Poissonian states on such spaces with the help of Bogoliubov functionals and correlation functions. We also describe in detail the model which we consider. In Sect. 3, we formulate the results mentioned above and prove Theorems 3.10 and 3.11. We also provide some comments; in particular, we relate our results with those of [7] describing a birth-and-death version of the Widom–Rowlinson dynamics in the continuum. Section 4 is dedicated to developing our main technical tool—Proposition 4.2. By means of it we realize step (a) in proving Theorem 3.5, see above. Steps (b) and (c) are based on Lemmas 5.1, 5.2, 5.4 and 5.5 proved in Sect. 5. Section 6 is dedicated to the proof of Theorems 3.8 and 3.10.

2 Preliminaries and the Model

2.1 Two-Component Configuration Spaces

Here we present necessary information on the subject. A more detailed description can be found in, e.g., [5,7,9].

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denote the sets of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. The configuration space Γ mentioned above is equipped with the vague topology and thus with the corresponding Borel σ-field $\mathcal{B}(\Gamma)$. It is known, see [11, Sect. 2.2], that $\mathcal{B}(\Gamma) = \sigma\{N_\Delta : \Delta \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\}$, that is, $\mathcal{B}(\Gamma)$ is generated by the counting maps $\Gamma \ni \gamma \mapsto N_\Delta(\gamma) := |\gamma \cap \Delta|$, where $|\cdot|$ denotes cardinality. The elements of $\Gamma^2 := \Gamma \times \Gamma$ are $\gamma = (\gamma_0, \gamma_1)$, i.e., the one-component configurations are always written with the subscript $i = 0, 1$. By $\mathcal{B}(\Gamma^2)$ we denote the corresponding product σ-field. For $\Lambda_i \in \mathcal{B}(\mathbb{R}^d)$, $i = 0, 1$, we denote $\Lambda = \Lambda_0 \times \Lambda_1$ and set

$$\Gamma^2_\Lambda = \left\{\gamma = (\gamma_0, \gamma_1) \in \Gamma^2 : \gamma_i \subset \Lambda_i, \ i = 0, 1\right\}.$$

Clearly $\Gamma^2_\Lambda \in \mathcal{B}(\Gamma^2)$ and hence

$$\mathcal{B}(\Gamma^2_\Lambda) := \left\{A \cap \Gamma^2_\Lambda : A \in \mathcal{B}(\Gamma^2)\right\}$$

is a sub-field of $\mathcal{B}(\Gamma^2)$. Let $p_\Lambda : \Gamma^2 \to \Gamma^2_\Lambda$ be the projection

$$p_\Lambda(\gamma) = \gamma_\Lambda := (\gamma_0 \cap \Lambda_0, \gamma_1 \cap \Lambda_1).$$

It is clearly measurable, and thus the sets

$$p_\Lambda^{-1}(A_\Lambda) := \{\gamma \in \Gamma^2 : p_\Lambda(\gamma) \in A_\Lambda\}, \quad A_\Lambda \in \mathcal{B}(\Gamma^2_\Lambda),$$

belong to $\mathcal{B}(\Gamma^2)$ for each Borel Λ_i, $i = 0, 1$.

Let $\mathcal{P}(\Gamma^2)$ denote the set of all probability measures on $(\Gamma^2, \mathcal{B}(\Gamma^2))$. For a given $\mu \in \mathcal{P}(\Gamma^2)$, its projection on $(\Gamma^2_\Lambda, \mathcal{B}(\Gamma^2_\Lambda))$ is

$$\mu^\Lambda(A_\Lambda) := \mu\left(p_\Lambda^{-1}(A_\Lambda)\right), \qquad A_\Lambda \in \mathcal{B}(\Gamma^2_\Lambda). \tag{2.1}$$

Let π be the standard homogeneous Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with density (intensity) $\varkappa = 1$. Then the product measure $\pi^2 := \pi \otimes \pi$ is a probability measure on $(\Gamma^2, \mathcal{B}(\Gamma^2))$. By $\mathcal{P}_\pi(\Gamma^2)$ we denote the set of all $\mu \in \mathcal{P}(\Gamma^2)$, for each of which the projections μ^Λ, with all possible $\Lambda = \Lambda_0 \times \Lambda_1$, $\Lambda_i \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $i = 0, 1$, are absolutely continuous with respect to the corresponding projections of π^2. It is known, see [5, Proposition 3.1], that for each $\mu \in \mathcal{P}_\pi(\Gamma^2)$ the following holds

$$\mu\left(\{\gamma = (\gamma_0, \gamma_1) \in \Gamma^2 : \gamma_0 \cap \gamma_1 = \emptyset\}\right) = 1.$$

Since we are going to deal with elements of $\mathcal{P}_\pi(\Gamma^2)$ only, from now on we assume that the configurations γ_0 and γ_1 are subsets of one and the same space $\mathbb{R}^d$.

Let $\Gamma_0 \subset \Gamma$ be the set of all finite configurations. It is known that $\Gamma_0 \in \mathcal{B}(\Gamma)$, see [11, Sect. 2.2]. Hence $\widetilde{\mathcal{B}}(\Gamma_0) := \{A \subset \Gamma_0 : A \in \mathcal{B}(\Gamma)\}$ ia a sub-σ-field of $\mathcal{B}(\Gamma)$. At the same time, Γ_0 can be equipped with the topology related to the Euclidean topology of $\mathbb{R}^d$, see [11, Sect. 2.1]. Let $\mathcal{B}(\Gamma_0)$ be the corresponding Borel σ-field of subsets of Γ_0. Clearly, a function, $g : \Gamma_0 \to \mathbb{R}$ is $\mathcal{B}(\Gamma_0)/\mathcal{B}(\mathbb{R})$-measurable if and only if there exist symmetric Borel functions $g^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ such that $g(\{x_1, \ldots, x_n\}) = g^{(n)}(x_1, \ldots, x_n)$, $n \in \mathbb{N}$. The relationship between this measurability and the corresponding property of $g : \Gamma_0 \subset \Gamma \to \mathbb{R}$ is clarified by Obata's result, see Lemma 1.1 and Proposition 1.3 in [14], by which $\widetilde{\mathcal{B}}(\Gamma_0) = \mathcal{B}(\Gamma_0)$. Thus, such a function g is $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$-measurable if and only if there exist $\{g^{(n)} : n \in \mathbb{N}\}$ with the above mentioned properties. For completeness, one adds to this family also $g^{(0)} = g(\varnothing)$.

By the very definition of $\mathcal{B}(\Gamma^2)$ we have that $\Gamma_0 \times \Gamma_0 =: \Gamma^2_0 \in \mathcal{B}(\Gamma^2)$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and also $\mathbb{N}^2_0 = \{n = (n_0, n_1) : n_i \in \mathbb{N}_0, i = 0, 1\}$. Then a function $G : \Gamma^2_0 \to \mathbb{R}$ is $\mathcal{B}(\Gamma^2)/\mathcal{B}(\mathbb{R})$-measurable if and only if for each $n \in \mathbb{N}^2_0$, there exists a Borel function

$G^{(n)} : (\mathbb{R}^d)^{n_0} \times (\mathbb{R}^d)^{n_1} \to \mathbb{R}$, symmetric with respect to the permutations of the components of each of η_i, $i = 0, 1$, such that

$$G(\eta) = G(\eta_0, \eta_1) = G^{(n)}(x_1, \ldots, x_{n_0}; y_1, \ldots, y_{n_1}),$$

for $\eta_0 = \{x_1, \ldots, x_{n_0}\}$ and $\eta_1 = \{y_1, \ldots, y_{n_1}\}$.

By $B_{\mathrm{bs}}(\Gamma_0^2)$ we denote the set of all measurable functions $G : \Gamma_0^2 \to \mathbb{R}$ that have the following properties: (a) there exists $C_G > 0$ such that $|G(\eta)| \leq C_G$ for all $\eta \in \Gamma_0^2$; (b) there exists $\Lambda = \Lambda_0 \times \Lambda_1$ with $\Lambda_i \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $i = 0, 1$, such that $G(\eta) = 0$ whenever $\eta_i \cap \Lambda_i^c \neq \emptyset$ for either of $i = 0, 1$; (c) there exists $N \in \mathbb{N}_0$ such that $G(\eta) = 0$ whenever $\max_{i=0,1} |\eta_i| > N$. Here $\Lambda_i^c := \mathbb{R}^d \setminus \Lambda_i$. By $\Lambda(G)$ and $N(G)$ we denote the smallest Λ and N with the properties just described.

By standard arguments $B_{\mathrm{bs}}(\Gamma_0^2)$ is a measure-defining class for measures on $(\Gamma_0^2, \mathcal{B}(\Gamma_0^2))$. The Lebesgue-Poisson measure λ on $(\Gamma_0^2, \mathcal{B}(\Gamma_0^2))$ is then defined by the following formula, see [5] and [9, page 130],

$$\int_{\Gamma_0^2} G(\eta)\lambda(d\eta) = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{1}{n_0! n_1!} \int_{(\mathbb{R}^d)^{n_0}} \int_{(\mathbb{R}^d)^{n_1}} G^{(n)}(x_1, \ldots, x_{n_0}; y_1, \ldots, y_{n_1}) \times dx_1 \cdots dx_{n_0} dy_1 \cdots dy_{n_1}, \tag{2.2}$$

which has to hold for all $G \in B_{\mathrm{bs}}(\Gamma_0^2)$ with the usual convention regarding the cases $n_i = 0$. The same can also be written as

$$\int_{\Gamma_0^2} G(\eta)\lambda(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\eta_0, \eta_1)(\lambda_0 \otimes \lambda_1)(d\eta_0, d\eta_1), \tag{2.3}$$

where both λ_i are the copies of the standard Lebesgue-Poisson measure on the single-component set Γ_0, see, e.g., [11]. In the sequel, both Lebesgue-Poisson measures on Γ_0^2 and on Γ_0 will be denoted by λ if no ambiguity may arise.

For $\gamma \in \Gamma^2$, by writing $\eta \Subset \gamma$ we mean that $\eta_i \Subset \gamma_i$, $i = 0, 1$, i.e., both η_i are finite subsets of the corresponding γ_i. For $G \in B_{\mathrm{bs}}(\Gamma_0^2)$, we set

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) = \sum_{\eta_0 \Subset \gamma_0} \sum_{\eta_1 \Subset \gamma_1} G(\eta_0, \eta_1), \tag{2.4}$$

see [5,9]. Note that the sums in (2.4) are finite and KG is a cylinder function on Γ^2. The latter means that it is $\mathcal{B}(\Gamma^2_{\Lambda(G)})$-measurable. Moreover, cf. [9, Eqs. (2.3) and (2.4), page 129],

$$|(KG)(\gamma)| \leq C_G \left(1 + |\gamma_0 \cap \Lambda_0(G)|\right)^{N_0(G)} \left(1 + |\gamma_1 \cap \Lambda_1(G)|\right)^{N_1(G)}. \tag{2.5}$$

2.2 Correlation Functions

In the approach we follow, see [3,6,10], the evolution of states is constructed in the next way. Let Θ denote the set of all compactly supported continuous maps $\theta = (\theta_0, \theta_1) : \mathbb{R}^d \times \mathbb{R}^d \to (-1, 0]^2$. For each $\theta \in \Theta$, the map

$$\Gamma^2 \ni \gamma \mapsto \prod_{x \in \gamma_0} (1 + \theta_0(x)) \prod_{y \in \gamma_1} (1 + \theta_1(y))$$

is measurable and bounded. Hence, for a state μ, one may define

$$B_\mu(\theta) = \int_{\Gamma^2} \prod_{x \in \gamma_0} (1 + \theta_0(x)) \prod_{y \in \gamma_1} (1 + \theta_1(y)) \mu(d\gamma), \tag{2.6}$$

– the so called Bogoliubov functional corresponding to μ, considered as a map $\Theta \to \mathbb{R}$.

Definition 2.1 By $\mathcal{P}_{\exp}(\Gamma^2)$ we denote the set of sub-Poissonian states consisting of all those $\mu \in \mathcal{P}_\pi(\Gamma^2)$ for which B_μ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^2)$.

It can be shown that a given $\mu \in \mathcal{P}_\pi(\Gamma^2)$ is sub-Poissonian if and only if B_μ can be written in the form, cf. (2.3),

$$B_\mu(\theta) = \int_{\Gamma_0^2} k_\mu(\eta) E(\theta;\eta)\lambda(d\eta), \tag{2.7}$$

cf. (2.2), with $k_\mu : \Gamma_0^2 \to [0, +\infty)$ such that $k_\mu^{(n)} \in L^\infty((\mathbb{R}^d)^{n_0} \times (\mathbb{R}^d)^{n_1} \to \mathbb{R})$ and

$$E(\theta;\eta) = e(\theta_0;\eta_0)e(\theta_1;\eta_1) := \prod_{x\in\eta_0} \theta_0(x) \prod_{y\in\eta_1} \theta_1(y). \tag{2.8}$$

This, in particular, means that k_μ is essentially bounded with respect to the Lebesgue-Poisson measure λ defined in (2.2). For the (heterogeneous) Poisson measure π_ϱ, the Bogoliubov functional is

$$B_{\pi_\varrho}(\theta) = \exp\left(\sum_{i=0,1} \int_{\mathbb{R}^d} \theta_i(x)\varrho_i(x)dx\right), \tag{2.9}$$

where $\varrho = (\varrho_0, \varrho_1)$ is the (two-component) density function. Then by (2.2) and (2.7) we have

$$k_{\pi_\varrho}(\eta) = E(\varrho;\eta) = e(\varrho_0;\eta_0)e(\varrho_1;\eta_1). \tag{2.10}$$

If one rewrites (2.6) in the form

$$B_\mu(\theta) = \int_{\Gamma^2} F_\theta(\gamma)\mu(d\gamma),$$

then the action of L on F as in (1.2) can be transformed to the action of L^Δ on k_μ according to the following rule

$$\int_{\Gamma^2} (LF_\theta)(\gamma)\mu(d\gamma) = \int_{\Gamma_0^2} (L^\Delta k_\mu)(\eta)E(\theta;\eta)\lambda(d\eta) \tag{2.11}$$

The main advantage of this is that k_μ is a function of *finite* configurations.

For $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$ and $\Lambda = (\Lambda_0, \Lambda_1)$, $\Lambda_i \in \mathcal{B}_b(\mathbb{R}^d)$, let μ^Λ be as in (2.1). Then μ^Λ is absolutely continuous with respect to the corresponding restriction λ^Λ of the measure defined in (2.2), and hence we may write

$$\mu^\Lambda(d\eta) = R_\mu^\Lambda(\eta)\lambda^\Lambda(d\eta), \qquad \eta \in \Gamma_\Lambda^2. \tag{2.12}$$

Then the correlation function k_μ and the Radon-Nikodym derivative R_μ^Λ are related to each other by, cf. (2.3),

$$\begin{aligned} k_\mu(\eta) &= \int_{\Gamma_\Lambda^2} R_\mu^\Lambda(\eta \cup \xi)\lambda^\Lambda(d\xi) \\ &= \int_{\Gamma_{\Lambda_0}} \int_{\Gamma_{\Lambda_1}} R_\mu^\Lambda(\eta_0 \cup \xi_0, \eta_1 \cup \xi_1)(\lambda_0^{\Lambda_0} \otimes \lambda_1^{\Lambda_1})(d\xi_0, d\xi_1), \qquad \eta \in \Gamma_\Lambda^2. \end{aligned} \tag{2.13}$$

Note that (2.13) relates R_μ^Λ with the restriction of k_μ to Γ_Λ^2. The fact that these are the restrictions of one and the same function $k_\mu : \Gamma_0^2 \to \mathbb{R}$ corresponds to the Kolmogorov consistency of the family $\{\mu^\Lambda\}_\Lambda$.

By (2.4), (2.1), and (2.12) we get

$$\int_{\Gamma^2} (KG)(\gamma)\mu(d\gamma) = \langle\!\langle G, k_\mu \rangle\!\rangle,$$

holding for each $G \in B_{\rm bs}(\Gamma^2_0)$ and $\mu \in \mathcal{P}_{\rm exp}(\Gamma^2)$. Here

$$\langle\!\langle G, k \rangle\!\rangle := \int_{\Gamma^2_0} G(\eta)k(\eta)\lambda(d\eta), \tag{2.14}$$

for suitable G and k. Define

$$B^\star_{\rm bs}(\Gamma^2_0) = \{G \in B_{\rm bs}(\Gamma^2_0) : (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma^2\}. \tag{2.15}$$

By [11, Theorems 6.1 and 6.2 and Remark 6.3] one can prove that the following holds.

Proposition 2.2 *Let a measurable function $k : \Gamma^2_0 \to \mathbb{R}$ have the following properties:*

$$\begin{aligned} &\text{(a)}\ \ \langle\!\langle G, k \rangle\!\rangle \geq 0, \qquad \text{for all } G \in B^\star_{\rm bs}(\Gamma^2_0);\\ &\text{(b)}\ \ k(\varnothing, \varnothing) = 1; \qquad \text{(c)}\ \ k(\eta) \leq C^{|\eta_0|+|\eta_1|}, \end{aligned} \tag{2.16}$$

with (c) holding for some $C > 0$ and λ-almost all $\eta \in \Gamma^2_0$. Then there exists a unique $\mu \in \mathcal{P}_{\rm exp}(\Gamma^2)$ for which k is the correlation function.

2.3 The Model

The model we consider is specified by the operator L given in (1.2) where the coefficients are supposed to be of the following form

$$\begin{aligned} c_0(x, y, \gamma_1) &= a_0(x - y) \exp\left(-\sum_{z\in\gamma_1} \phi_0(y - z)\right),\\ c_1(x, y, \gamma_0) &= a_1(x - y) \exp\left(-\sum_{z\in\gamma_0} \phi_1(y - z)\right), \end{aligned} \tag{2.17}$$

with jump kernels $a_i : \mathbb{R}^d \to [0, +\infty)$ such that $a_i(x) = a_i(-x)$ and

$$\int_{\mathbb{R}^d} a_i(x)dx =: \alpha_i < \infty, \qquad i = 0, 1. \tag{2.18}$$

The repulsion potentials in (2.17) $\phi_i : \mathbb{R}^d \to [0, +\infty)$ are supposed to be symmetric, $\phi_i(x) = \phi_i(-x)$, and such that

$$\int_{\mathbb{R}^d} \phi_i(x)dx =: \langle\phi_i\rangle < \infty, \qquad \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \phi_i(x) =: \bar{\phi}_i < \infty. \tag{2.19}$$

Then

$$\int_{\mathbb{R}^d} \Big(1 - \exp(-\phi_i(x))\Big)dx \leq \langle\phi_i\rangle, \qquad i = 0, 1. \tag{2.20}$$

By (1.2) and (2.11) one obtains the action of L^Δ in the following form. For $x \in \mathbb{R}^d$, we set

$$\tau^i_x(y) = \exp(-\phi_i(x - y)), \quad t^i_x(y) = \tau^i_x(y) - 1, \quad y \in \mathbb{R}^d, \ \ i = 0, 1. \tag{2.21}$$

Next, for a function $k(\eta) = k(\eta_0, \eta_1)$, cf. (2.3), we introduce the maps

$$
\begin{aligned}
(Q^0_y k)(\eta_0, \eta_1) &= \int_{\Gamma_0} k(\eta_0, \eta_1 \cup \xi) e(t^0_y; \xi) \lambda(d\xi), \\
(Q^1_y k)(\eta_0, \eta_1) &= \int_{\Gamma_0} k(\eta_0 \cup \xi, \eta_1) e(t^1_y; \xi) \lambda(d\xi),
\end{aligned} \tag{2.22}
$$

where e is as in (2.8). Then

$$
\begin{aligned}
(L^{\Delta} k)(\eta_0, \eta_1) = & \sum_{y \in \eta_0} \int_{\mathbb{R}^d} a_0(x-y) e(\tau^0_y; \eta_1) (Q^0_y k)(\eta_0 \setminus y \cup x, \eta_1) dx \\
& - \sum_{x \in \eta_0} \int_{\mathbb{R}^d} a_0(x-y) e(\tau^0_y; \eta_1) (Q^0_y k)(\eta_0, \eta_1) dy \\
& + \sum_{y \in \eta_1} \int_{\mathbb{R}^d} a_1(x-y) e(\tau^1_y; \eta_0) (Q^1_y k)(\eta_0, \eta_1 \setminus y \cup x) dx \\
& - \sum_{x \in \eta_1} \int_{\mathbb{R}^d} a_1(x-y) e(\tau^1_y; \eta_0) (Q^1_y k)(\eta_0, \eta_1) dy.
\end{aligned} \tag{2.23}
$$

This expression can be derived from the general form obtained in [9, Eqs. (4.4) and (4.5), page 142] by using the concrete form of the kernels given in (2.17). It can also be obtained directly from (1.2) and (2.11). Note that in (2.23) we use the convention $\sum_{x \in \varnothing} = 0$.

3 The Results

3.1 The Microscopic Level

As mentioned above, instead of directly studying the evolution of states by solving the problem in (1.3), we pass from μ_0 to the corresponding correlation function k_{μ_0} and then consider the problem

$$
\frac{d}{dt} k_t = L^{\Delta} k_t, \qquad k_t|_{t=0} = k_{\mu_0}, \tag{3.1}
$$

where L^{Δ} is given in (2.23). For this problem, we prove the existence of a unique global solution k_t which is the correlation function of a unique state $\mu_t \in \mathcal{P}_{\exp}(\Gamma^2)$.

We begin by defining the problem (3.1) in the corresponding spaces of functions $k : \Gamma^2_0 \to \mathbb{R}$. From the very representation (2.7), see also (2.2), it follows that $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$ implies

$$
|k_\mu(\eta)| \le C \exp\big(\vartheta\,(|\eta_0| + |\eta_1|)\big),
$$

holding for λ-almost all $\eta \in \Gamma^2_0$, some $C > 0$, and $\vartheta \in \mathbb{R}$. Keeping this in mind we set

$$
\|k\|_\vartheta = \operatorname*{ess\,sup}_{\eta \in \Gamma^2_0} \left\{ |k_\mu(\eta)| \exp\big(-\vartheta\,(|\eta_0| + |\eta_1|)\big)\right\}. \tag{3.2}
$$

Then

$$
\mathcal{K}_\vartheta := \{k : \Gamma^2_0 \to \mathbb{R} : \|k\|_\vartheta < \infty\}
$$

is a Banach space with norm (3.2) and the usual linear operations. In fact, we are going to use the ascending scale of such spaces $\mathcal{K}_\vartheta$, $\vartheta \in \mathbb{R}$, with the property

$$
\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}, \qquad \vartheta < \vartheta', \tag{3.3}
$$

where $\hookrightarrow$ denotes continuous embedding. Set, cf. (2.14) and (2.15),

$$\mathcal{K}^{\star}_{\vartheta} = \{k \in \mathcal{K}_{\vartheta} : \langle\!\langle G, k \rangle\!\rangle \geq 0 \text{ for all } G \in B^{\star}_{\mathrm{bs}}(\Gamma^2_0)\}. \tag{3.4}$$

It is a subset of the cone

$$\mathcal{K}^{+}_{\vartheta} = \{k \in \mathcal{K}_{\vartheta} : k(\eta) \geq 0 \;\text{ for } \lambda - \text{almost all } \eta \in \Gamma^2_0\}. \tag{3.5}$$

By Proposition 2.2 it follows that each $k \in \mathcal{K}^{\star}_{\vartheta}$ such that $k(\varnothing, \varnothing) = 1$ is the correlation function of a unique $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$. Then we define

$$\mathcal{K} = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_{\vartheta}, \qquad \mathcal{K}^{\star} = \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}^{\star}_{\vartheta}. \tag{3.6}$$

As a sum of Banach spaces, the linear space $\mathcal{K}$ is equipped with the corresponding inductive topology which turns it into a locally convex space.

For a given $\vartheta \in \mathbb{R}$, by (2.21)–(2.23) we define L^{Δ}_{ϑ} as a linear operator in $\mathcal{K}_{\vartheta}$ with domain

$$\mathcal{D}(L^{\Delta}_{\vartheta}) = \{k \in \mathcal{K}_{\vartheta} : L^{\Delta} k \in \mathcal{K}_{\vartheta}\}. \tag{3.7}$$

Lemma 3.1 *For each $\vartheta'' < \vartheta$, cf. (3.3), it follows that $\mathcal{K}_{\vartheta''} \subset \mathcal{D}(L^{\Delta}_{\vartheta})$.*

Proof For $\vartheta'' < \vartheta$, by (2.20), (2.21), (2.22), and (3.2) we have

$$\begin{aligned}
\left|(Q^0_y k)(\eta_0, \eta_1)\right| &\leq \|k\|_{\vartheta''} \exp\left(\vartheta''|\eta_0| + \vartheta''|\eta_1|\right) \\
&\quad \times \int_{\Gamma_0} \exp\left(\vartheta''|\xi|\right) \prod_{z \in \xi} \left(1 - \exp\left(-\phi_0(z - y)\right)\right) \lambda(d\xi) \\
&\leq \|k\|_{\vartheta''} \exp\left(\vartheta''|\eta_0| + \vartheta''|\eta_1|\right) \exp\left(\langle \phi_0 \rangle e^{\vartheta''}\right).
\end{aligned} \tag{3.8}$$

Likewise

$$\left|(Q^1_y k)(\eta_0, \eta_1)\right| \leq \|k\|_{\vartheta''} \exp\left(\vartheta''|\eta_0| + \vartheta''|\eta_1|\right) \exp\left(\langle \phi_1 \rangle e^{\vartheta''}\right). \tag{3.9}$$

Now we apply the latter two estimates together with (2.18) in (2.23) and obtain

$$\begin{aligned}
\left|(L^{\Delta} k)(\eta_0, \eta_1)\right| &\leq 2\|k\|_{\vartheta''} \exp\left(\vartheta''|\eta_0| + \vartheta''|\eta_1|\right) \\
&\quad \times \left(\alpha_0 |\eta_0| \exp\left(\langle \phi_0 \rangle e^{\vartheta''}\right) + \alpha_1 |\eta_1| \exp\left(\langle \phi_1 \rangle e^{\vartheta''}\right)\right).
\end{aligned} \tag{3.10}$$

By means of the inequality $x \exp(-\sigma x) \leq 1/e\sigma$, $x, \sigma > 0$, we get from (3.2) and (3.10) the following estimate

$$\|L^{\Delta} k\|_{\vartheta} \leq \frac{4\|k\|_{\vartheta''}}{e(\vartheta - \vartheta'')} \max_{i=0,1} \alpha_i \exp\left(\langle \phi_i \rangle e^{\vartheta''}\right), \tag{3.11}$$

which yields the proof. □

Corollary 3.2 *For each $\vartheta, \vartheta'' \in \mathbb{R}$ such that $\vartheta'' < \vartheta$, the expression in (2.23) defines a bounded linear operator $L^{\Delta}_{\vartheta\vartheta''} : \mathcal{K}_{\vartheta''} \to \mathcal{K}_{\vartheta}$ the norm of which can be estimated by means of (3.11).*

In what follows, we consider two types of operators defined by the expression in (2.23): (a) unbounded operators $(L^{\Delta}_{\vartheta}, \mathcal{D}(L^{\Delta}_{\vartheta}))$, $\vartheta \in \mathbb{R}$, with domains as in (3.7) and Lemma 3.1; (b) bounded operators $L^{\Delta}_{\vartheta\vartheta''}$ described in Corollary 3.2. These operators are related to each other in the following way:

$$\forall \vartheta'' < \vartheta \;\; \forall k \in \mathcal{K}_{\vartheta''} \qquad L^{\Delta}_{\vartheta\vartheta''}k = L^{\Delta}_{\vartheta}k. \tag{3.12}$$

By means of the bounded operators $L^{\Delta}_{\vartheta\vartheta''} : \mathcal{K}_{\vartheta''} \to \mathcal{K}_{\vartheta}$ we define also a continuous linear operator $L^{\Delta} : \mathcal{K} \to \mathcal{K}$, see (3.6). In view of this, we consider the following two equations. The first one is

$$\frac{d}{dt}k_t = L^{\Delta}_{\vartheta}k_t, \qquad k_t|_{t=0} = k_{\mu_0}, \tag{3.13}$$

considered as an equation in a given Banach space $\mathcal{K}_{\vartheta}$. The second equation is (3.1) with L^{Δ} given in (2.23) considered in the locally convex space $\mathcal{K}$.

Definition 3.3 By a solution of (3.13) on a time interval, $[0, T)$, $T \leq +\infty$, we mean a continuous map $[0, T) \ni t \mapsto k_t \in \mathcal{D}(L^{\Delta}_{\vartheta})$ such that the map $[0, T) \ni t \mapsto dk_t/dt \in \mathcal{K}_{\vartheta}$ is also continuous and both equalities in (3.13) are satisfied. Likewise, a continuously differentiable map $[0, T) \ni t \mapsto k_t \in \mathcal{K}$ is said to be a solution of (3.1) in $\mathcal{K}$ if both equalities therein are satisfied for all t. Such a solution is called global if $T = +\infty$.

Remark 3.4 The map $[0, T) \ni t \mapsto k_t \in \mathcal{K}$ is a solution of (3.1) if and only if, for each $t \in [0, T)$, there exists $\vartheta'' \in \mathbb{R}$ such that $k_t \in \mathcal{K}_{\vartheta''}$ and, for each $\vartheta > \vartheta''$, the map $t \mapsto k_t$ is continuously differentiable at t in $\mathcal{K}_{\vartheta}$ and $dk_t/dt = L^{\Delta}_{\vartheta}k_t = L^{\Delta}_{\vartheta\vartheta''}k_t$.

The main result of this subsection is contained in the following statement.

Theorem 3.5 *Assume that* (2.18) *and* (2.19) *hold. Then for each* $\mu_0 \in \mathcal{P}_{\exp}(\Gamma^2)$*, the problem* (3.1) *with* L^{Δ} *given in* (2.23) *and* $k_0 = k_{\mu_0}$ *has a unique global solution* $k_t \in \mathcal{K}^{\star} \subset \mathcal{K}$ *which has the property* $k_t(\varnothing, \varnothing) = 1$*. Therefore, for each* $t \geq 0$ *there exists a unique state* $\mu_t \in \mathcal{P}_{\exp}(\Gamma^2)$ *such that* $k_t = k_{\mu_t}$*. Moreover, let* k_0 *and* $C > 0$ *be such that* $k_0(\eta) \leq C^{|\eta_0|+|\eta_1|}$ *for* λ*-almost all* $\eta \in \Gamma^2_0$*, see* (2.16)*. Then the mentioned solution satisfies*

$$\forall t \geq 0 \qquad 0 \leq k_t(\eta) \leq C^{|\eta_0|+|\eta_1|} \exp\left\{t\left(\alpha_0|\eta_0| + \alpha_1|\eta_1|\right)\right\}. \tag{3.14}$$

3.2 The Mesoscopic Level

As is commonly recognized, see [2, Chapter 8] and [13], the comprehensive understanding of the behavior of an infinite interacting particle system requires its multi-scale analysis. In the approach which we follow, see [3] (jump dynamics) and [7] (two-component system), passing from the micro- to the mesoscopic levels amounts to considering the system at different spatial scales parameterized by $\varepsilon \in (0, 1]$ in such a way that $\varepsilon = 1$ corresponds to the micro-level, whereas the limit $\varepsilon \to 0$ yields the meso-level description in which the corpuscular structure disappears and the system turns into a (two-component) medium characterized by a density function $\varrho = (\varrho_0, \varrho_1)$, $\varrho_i : \mathbb{R}^d \to [0, +\infty)$, $i = 0, 1$. Then the evolution $\varrho_0 \mapsto \varrho_t$, obtained from a *kinetic* equation, approximates (in the mean-field sense) the evolution of the system's states as it may be seen from the mesoscopic level.

3.2.1 The Kinetic Equation

Keeping in mind that the Poissonian state π_{ϱ} is completely characterized by the density ϱ, see (2.9) and (2.10), we introduce the following notion, cf. [3, p. 1046] and [7, p. 70].

Definition 3.6 A state $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$ is said to be Poisson-approximable if: (i) there exist $\vartheta \in \mathbb{R}$ and $\varrho = (\varrho_0, \varrho_1)$, $\varrho_i \in L^\infty(\mathbb{R}^d \to \mathbb{R})$, $\varrho_i \geq 0$, $i = 0, 1$, such that both k_μ and k_{π_ϱ} lie in $\mathcal{K}_\vartheta$; (ii) for each $\varepsilon \in (0, 1]$, there exists $q_\varepsilon \in \mathcal{K}_\vartheta$ such that $q_1 = k_\mu$ and $\|q_\varepsilon - k_{\pi_\varrho}\|_\vartheta \to 0$ as $\varepsilon \to 0$.

Our aim is to show that the evolution $\mu_0 \mapsto \mu_t$ obtained in Theorem 3.5 preserves the property just defined relative to the time dependent density $\varrho_t = (\varrho_{0,t}, \varrho_{1,t})$, obtained from the following system of kinetic equations

$$\begin{cases} \frac{d}{dt}\varrho_{0,t} = (a_0 * \varrho_{0,t}) \exp\left(-(\phi_0 * \varrho_{1,t})\right) - \varrho_{0,t}\left(a_0 * \exp\left(-(\phi_0 * \varrho_{1,t})\right)\right), \\ \frac{d}{dt}\varrho_{1,t} = (a_1 * \varrho_{1,t}) \exp\left(-(\phi_1 * \varrho_{0,t})\right) - \varrho_{1,t}\left(a_1 * \exp\left(-(\phi_1 * \varrho_{0,t})\right)\right), \end{cases} \tag{3.15}$$

where $*$ denotes convolution; e.g.,

$$(a_i * \varrho_{i,t})(x) = \int_{\mathbb{R}^d} a_i(x-y)\varrho_{i,t}(y)dy, \qquad i = 0, 1.$$

Definition 3.7 By the global solution of the system of kinetic equations (3.15), subject to an initial condition, we understand a continuously differentiable map

$$[0, +\infty) \ni t \mapsto (\varrho_{0,t}, \varrho_{1,t}) \in L^\infty(\mathbb{R}^d \to \mathbb{R}^2) \tag{3.16}$$

such that both equalities in (3.15) hold. This solution is called positive if $\varrho_{i,t}(x) \geq 0$, $i = 0, 1$, for all $t \geq 0$ and Lebesgue-almost all $x \in \mathbb{R}^d$. By the positive solution of (3.15) on the time interval $[0, T]$, $0 < T < \infty$, we mean the corresponding restriction of this map.

Let $\|\cdot\|_{L^\infty}$ stand for the norm in $L^\infty(\mathbb{R}^d \to \mathbb{R})$. In Theorem 3.8, the space $L^\infty(\mathbb{R}^d \to \mathbb{R}^2)$ is equipped with the norm

$$\|\varrho\|_\infty = \max_{i=0,1} \|\varrho_i\|_{L^\infty}. \tag{3.17}$$

Theorem 3.8 *For each positive* $\varrho_0 = (\varrho_{0,0}, \varrho_{1,0}) \in L^\infty(\mathbb{R}^d \to \mathbb{R}^2)$*, the system of kinetic equations* (3.15) *with the initial condition* $(\varrho_{0,t}, \varrho_{1,t})|_{t=0} = (\varrho_{0,0}, \varrho_{1,0})$ *has a unique positive global solution such that*

$$\forall t \geq 0 \qquad \varrho_{i,t}(x) \leq \|\varrho_{i,0}\|_{L^\infty} \exp(\alpha_i t), \quad i = 0, 1, \tag{3.18}$$

where α_i *are defined in* (2.18).

The relationship between the micro- and mesoscopic descriptions is established by the following statement.

Theorem 3.9 *Let* (2.19) *hold and* k_t *and* ϱ_t *be the solutions described in Theorems 3.5 and 3.8, respectively. Assume also that the initial state* μ_0 *is Poisson-approximable by* π_{ϱ_0}*, see Definition 3.6. That is, there exist* $\vartheta_* \in \mathbb{R}$ *and* $q_{0,\varepsilon}$*,* $\varepsilon \in (0, 1]$*, such that* $k_{\mu_0} = q_{0,1}$ *and* $\|q_{0,\varepsilon} - k_{\pi_{\varrho_0}}\|_{\vartheta_*} \to 0$ *as* $\varepsilon \to 0$*. Then there exist* $\vartheta > \vartheta_*$ *and* $T > 0$ *such that*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|q_{t,\varepsilon} - k_{\pi_{\varrho_t}}\|_\vartheta = 0. \tag{3.19}$$

Theorems 3.8 and 3.9 are proved in Sect. 6 below.

3.2.2 The Stationary Solutions

Stationary solutions $\varrho_{i,t} = \varrho_i$, $t \geq 0$, of the system in (3.15) are supposed to solve the following system of equations

$$\begin{cases} (a_0 * \varrho_0) \exp\left(-(\phi_0 * \varrho_1)\right) = \varrho_0 \left(a_0 * \exp\left(-(\phi_0 * \varrho_1)\right)\right), \\ (a_1 * \varrho_1) \exp\left(-(\phi_1 * \varrho_0)\right) = \varrho_1 \left(a_1 * \exp\left(-(\phi_1 * \varrho_0)\right)\right). \end{cases} \tag{3.20}$$

It might be instructive to rewrite it in the form

$$\begin{cases} \psi_0(x) = \int_{\mathbb{R}^d} \tilde{a}_0(x,y)\psi_0(y)dy, \\ \psi_1(x) = \int_{\mathbb{R}^d} \tilde{a}_1(x,y)\psi_1(y)dy, \end{cases} \tag{3.21}$$

where

$$\tilde{a}_0(x,y) := \frac{a_0(x-y)\exp\left(-(\phi_0 * \varrho_1)(y)\right)}{\int_{\mathbb{R}^d} a_0(x-y)\exp\left(-(\phi_0 * \varrho_1)(y)\right)dy},$$
$$\tilde{a}_1(x,y) := \frac{a_1(x-y)\exp\left(-(\phi_1 * \varrho_0)(y)\right)}{\int_{\mathbb{R}^d} a_1(x-y)\exp\left(-(\phi_1 * \varrho_0)(y)\right)dy},$$

and

$$\psi_0 := \varrho_0 \exp\left(\phi_0 * \varrho_1\right), \qquad \psi_1 := \varrho_1 \exp\left(\phi_1 * \varrho_0\right). \tag{3.22}$$

For each $\widetilde{C}_i > 0$, $i = 0, 1$, the system in (3.21) has constant solutions $\psi_i \equiv \widetilde{C}_i$. Then the corresponding ϱ_i are to be found from

$$\begin{cases} \varrho_0 = \widetilde{C}_0 \exp\left(-(\phi_0 * \varrho_1)\right), \\ \varrho_1 = \widetilde{C}_1 \exp\left(-(\phi_1 * \varrho_0)\right). \end{cases} \tag{3.23}$$

Those in (3.23) may be called *birth-and-death* solutions since they solve the corresponding equation for the birth-and-death version of the Widom–Rowlinson dynamics with specific values of $\widetilde{C}_i$, expressed in terms of the model parameters, see [7, eq. (4.13)]. The translation invariant (i.e., constant) solution of (3.23) is $\varrho_i \equiv C_i$, $i = 0, 1$, with C_i satisfying, cf. (3.22),

$$\widetilde{C}_0 = C_0 \exp\left(\langle\phi_0\rangle C_1\right), \qquad \widetilde{C}_1 = C_1 \exp\left(\langle\phi_1\rangle C_0\right). \tag{3.24}$$

For given $\widetilde{C}_0, \widetilde{C}_1 > 0$, let $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$ be the set of all positive $(\varrho_0, \varrho_1) \in L^\infty(\mathbb{R}^d \to \mathbb{R}^2)$ that satisfy (3.23). Let also $\mathcal{S}_c(\widetilde{C}_0, \widetilde{C}_1)$ be the subset of $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$ consisting of constant solutions $\varrho_i \equiv C_i$, $i = 0, 1$, with C_i satisfying (3.24). The symmetric case of (3.24) with specific values of $\widetilde{C}_i$ (as mentioned above) was studied in [7, Sect. 5]. Namely, for $\langle\phi_1\rangle\widetilde{C}_0 = \langle\phi_0\rangle\widetilde{C}_1 =: a$, the set $\mathcal{S}_c(\widetilde{C}_0, \widetilde{C}_1)$ is a singleton $\{C_0, C_1\}$ whenever $a \leq e$. Here

$$C_0 = x_0/\langle\phi_1\rangle, \qquad C_1 = x_0/\langle\phi_0\rangle, \tag{3.25}$$

with some $x_0 \in (0, 1)$. This solution is a stable node for $a < e$. For $a > e$, there exist three solutions: (a) $C_0 = x_1/\langle\phi_1\rangle$, $C_1 = x_3/\langle\phi_0\rangle$; (b) $C_0 = x_3/\langle\phi_1\rangle$, $C_1 = x_1/\langle\phi_0\rangle$; (c) $C_0 = x_2/\langle\phi_1\rangle$, $C_1 = x_2/\langle\phi_0\rangle$. The first two solutions are stable nodes and $x_3 > 1$. The stability means the existence of a small neighborhood in $\mathcal{S}_c(\widetilde{C}_0, \widetilde{C}_1)$ of the mentioned solution, which does not contain any other solution.

Let us now turn to the study of the stability of the constant solutions of (3.23) with respect to perturbations $\varrho_i = C_i + \epsilon_i$, $i = 0, 1$. By (3.23) and (3.24) we conclude that the perturbations ought to satisfy

$$\begin{cases} \epsilon_0 = C_0 \left[\exp\left\{-\left(\phi_0 * \epsilon_1\right)\right\} - 1\right], \\ \epsilon_1 = C_1 \left[\exp\left\{-\left(\phi_1 * \epsilon_0\right)\right\} - 1\right]. \end{cases} \tag{3.26}$$

Theorem 3.10 *The solution $\varrho_i \equiv C_i$, $i = 0, 1$, of the system of equations in* (3.20) *is locally stable in $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$, with $\widetilde{C}_i$ and C_i satisfying* (3.24)*, whenever the following holds, cf.* (3.25)*,*

$$C_0 C_1 \langle \phi_0 \rangle \langle \phi_1 \rangle < 1. \tag{3.27}$$

This means that there exists $\delta > 0$ such that $\varrho_i \equiv C_i$, $i = 0, 1$, is the only solution in the set $K_\delta := \mathcal{S}(\widetilde{C}_0, \widetilde{C}_1) \cap \{\varrho : \|\varrho - C\|_\infty < \delta\}$, cf. (3.17)*.*

Proof Assume that $\|\epsilon_0\|_{L^\infty} > 0$. By means of the inequality $|e^{-\alpha} - 1| \leq |\alpha| e^{|\alpha|}$ we get from (3.26)

$$\|\epsilon_0\|_{L^\infty} \leq C_0 C_1 \langle \phi_0 \rangle \langle \phi_1 \rangle \exp\left[\delta\left(\langle \phi_0 \rangle + \langle \phi_1 \rangle\right)\right] \cdot \|\epsilon_0\|_{L^\infty} < \|\epsilon_0\|_{L^\infty},$$

holding for small enough δ in view of (3.27). This contradicts the assumption, and hence yields $\epsilon_0 = 0$. The corresponding estimate for $\|\epsilon_1\|_{L^\infty}$ is obtained analogously. $\square$

Assume now that both ϵ_i satisfy $\epsilon_i \in L^\infty(\mathbb{R}^d \to \mathbb{R}) \cap L^1(\mathbb{R}^d \to \mathbb{R})$. Then each solution of (3.26) is a fixed point of the nonlinear map $\Phi : L^\infty(\mathbb{R}^d \to \mathbb{R}^2) \cap L^1(\mathbb{R}^d \to \mathbb{R}^2) \to L^\infty(\mathbb{R}^d \to \mathbb{R}^2) \cap L^1(\mathbb{R}^d \to \mathbb{R}^2)$ defined by the right-hand of (3.26). Note that this Φ takes values in $L^\infty(\mathbb{R}^d \to \mathbb{R}^2) \cap L^1(\mathbb{R}^d \to \mathbb{R}^2)$ in view of (2.19). The zero solution of (3.26) gets unstable whenever there exist nonzero $\epsilon = (\epsilon_0, \epsilon_1)$ in the kernel of $I - \Phi'$, where Φ' is the Fréchet derivative of Φ at $\epsilon = (0, 0)$. By (3.26) we have

$$\Phi' \epsilon := \Phi' \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix} = \begin{pmatrix} -C_0(\phi_0 * \epsilon_1) \\ -C_1(\phi_1 * \epsilon_0) \end{pmatrix}. \tag{3.28}$$

Since Φ' contains convolutions, it can be partially diagonalized by means of the Fourier transform

$$\hat{\phi}_i(p) = \int_{\mathbb{R}^d} \phi_i(x) \exp\left(i(p, x)\right) dx, \qquad p \in \mathbb{R}^d, \ \ i = 0, 1.$$

Note that both $\hat{\phi}_i$ are uniformly continuous on $\mathbb{R}^d$ and satisfy $|\hat{\phi}_i(p)| \leq \hat{\phi}_i(0) = \langle \phi_i \rangle$, that follows from their positivity. Moreover, $|\hat{\phi}_i(p)| \to 0$ as $|p| \to +\infty$ (by the Riemann-Lebesgue lemma). Note also that $\hat{\epsilon}_i$, $i = 0, 1$, exist since ϵ_i are supposed to be integrable.

Theorem 3.11 *Assume that the following holds, cf.* (3.27)*,*

$$C_0 C_1 \langle \phi_0 \rangle \langle \phi_1 \rangle > 1. \tag{3.29}$$

Then the constant solution $\varrho_i \equiv C_i$ of (3.23)*, and hence of* (3.20)*, is unstable with respect to the perturbation $\varrho_i = C_i + \epsilon_i$, $i = 0, 1$, with $\epsilon_i \in L^\infty(\mathbb{R}^d \to \mathbb{R}) \cap L^1(\mathbb{R}^d \to \mathbb{R})$.*

Proof In view of the mentioned continuity of $\hat{\phi}_i$ and the Riemann-Lebesgue lemma, the condition in (3.29) implies the existence of $p \in \mathbb{R}^d \setminus \{0\}$ such that

$$C_0 C_1 \hat{\phi}_0(p) \hat{\phi}_1(p) = 1. \tag{3.30}$$

The instability in question takes place whenever the equation $\Phi' \epsilon = \epsilon$, cf. (3.28), has nonzero solutions in the considered space. By means of the Fourier transform it can be turned into

$$\hat{\epsilon}_i(p) = C_0 C_1 \hat{\phi}_0(p) \hat{\phi}_1(p) \hat{\epsilon}_i(p), \qquad i = 0, 1, \tag{3.31}$$

that has to hold for some $p \in \mathbb{R} \setminus \{0\}$, which is certainly the case in view of (3.30). $\square$

Given C_i, $i = 0, 1$, let $\epsilon = (\epsilon_0, \epsilon_1)$ solve (3.26). Then $\varrho = (C_0 + \epsilon_0, C_1 + \epsilon_1)$ solves (3.23) with $\widetilde{C}_i$ as in (3.24) and hence lies in $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$. Then Theorem 3.11 describes the instability of the solution $\varrho \equiv (C_0, C_1)$ in the latter set. For this reason, it is independent of the jump kernels a_i. In order to study the corresponding instability in the set of all solutions of (3.20), one has to rewrite (3.20) in the form $\Psi(\varrho) = 0$ and then to show that the Fréchet derivative Ψ' of Ψ at $\varrho \equiv (C_0, C_1)$, defined as a bounded linear self-map of $L^\infty(\mathbb{R}^d \to \mathbb{R}) \cap L^1(\mathbb{R}^d \to \mathbb{R})$, has nonzero ϵ in its kernel. By means of the arguments used in the proof of Theorem 3.11 one readily obtains that this is equivalent to, cf. (3.31),

$$\hat{\epsilon}_i(p)\left[1 - C_0C_1\hat{\phi}_0(p)\hat{\phi}_1(p)\right]\cdot\left[\alpha_i - \hat{a}_i(p)\right] = 0, \qquad i = 0, 1,$$

that has to hold for some nonzero $p \in \mathbb{R}^d$. Here $\hat{a}_i(p)$, $i = 0, 1$, are the Fourier transforms of the jump kernels, see (2.18). Thus, if both these kernels are such that $\hat{a}_i(p) < \hat{a}_i(0) = \alpha_i$ for all nonzero p, then the latter condition turns into that in (3.31).

3.3 Comments

3.3.1 The Microscopic Description

The existence of the global in time evolution stated in Theorem 3.5 is proved in the subsequent sections without any restrictions on the model parameters α_i and $\langle\phi_i\rangle$, $i = 0, 1$, see (2.18) and (2.19), respectively. That is, the global evolution exists, however, its ergodicity can hardly be expected. The analysis of the kinetic equation made in Theorem 3.11 points to the possibility of having a phase transition in the model, i.e., to the possibility of having multiple stationary states $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$.

The only work on the Widom–Rowlinson dynamics of an infinite particle system is that in [7] where a birth-and-death (rather immigration-emigration) version was studied. In that version, the particles of two types appear and disappear at random; the appearance is subject to the repulsion from the particles of the other type. The system's evolution was described by means of the corresponding initial value problem for the Bogoliubov functional. Namely, for $t < T$, where $T < \infty$ is expressed via the model parameters, in [7, Theorem 1] there was constructed the evolution $B_{\mu_0} \mapsto B_t$, where $B_t : L^1(\mathbb{R}^d \to \mathbb{R}^2) \to \mathbb{R}$ is an exponential type entire function and hence can be written down as, cf. (2.7),

$$B_t(\theta) = \int_{\Gamma_0^2} k_t(\eta) E(\theta; \eta)\lambda(d\eta).$$

However, it was not shown that B_t is the Bogoliubov functional, i.e., that k_t above is the correlation function, of some state $\mu \in \mathcal{P}_{\exp}(\Gamma^2)$. In the present work, for the jump version of the Widom–Rowlinson model we show (Theorem 3.5) that: (a) the evolution $k_{\mu_0} \mapsto k_t$, and hence also $B_{\mu_0} \mapsto B_t$, can be continued to all $t > 0$; (b) for each $t > 0$, B_t is the Bogoliubov functional of a unique sub-Poissonian state μ_t.

3.3.2 The Mesoscopic Description

In passing to the mesoscopic level of description, we use a scaling procedure described in Sect. 4 below. It is equivalent to the Lebowitz-Penrose scaling used in [7], and also to the Vlasov scaling used in [3,6]. Our Theorem 3.9 is analogous to [7, Theorem 2] proved for the birth-and-death version. Note that the convergence in (3.19) is uniform in t, whereas in the mentioned statement of [7] the convergence is point-wise.

Now we turn to the stationary solutions of (3.15) which one obtains from the system in (3.20), or, equivalently, in (3.21). The latter may have nonconstant solutions ψ_i, which then can be used to find the corresponding ϱ_i from (3.22). These solutions may depend on the jump kernels a_i. The set of all solutions of (3.20) contains the sets $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$ for each pair $\widetilde{C}_0, \widetilde{C}_1 > 0$. The corresponding solutions ϱ_i are independent of the jump kernels. Moreover, $\mathcal{S}(\widetilde{C}_0, \widetilde{C}_1)$ is exactly the set of solutions of the birth-and-death kinetic equation [7, Eq. (5.1)] corresponding to the specific values of $\widetilde{C}_i$. Thus, our Theorems 3.10 and 3.11 describe also the birth-and-death kinetic equation, which is an extension of the study in [7, Sect. 5].

4 The Rescaled Evolution

In this section, we construct the evolution $q_{0,\varepsilon} \mapsto q_{t,\varepsilon}$, $\varepsilon \in (0, 1]$, which then will be used for: (a) obtaining the evolution stated in Theorem 3.5 in the form $k_t = q_{t,1}$; (b) proving Theorem 3.9. To this end along with L^Δ defined in (2.23) we will use

$$L^{\varepsilon,\Delta} = R_\varepsilon^{-1} L_\varepsilon^\Delta R_\varepsilon, \qquad \varepsilon \in (0, 1], \tag{4.1}$$

where L_ε^Δ is obtained from L^Δ by multiplying both ϕ_i by ε, and

$$(R_\varepsilon q)(\eta_0, \eta_1) = \varepsilon^{-|\eta_0|-|\eta_1|} q(\eta_0, \eta_1).$$

We refer the reader to [3,7] for more information on deriving operators as in (4.1). Denote, cf. (2.21),

$$\tau_{x,\varepsilon}^i(y) = \exp\left(-\varepsilon\phi_i(x-y)\right), \quad t_{x,\varepsilon}^i(y) = \varepsilon^{-1}\left[\tau_{x,\varepsilon}^i(y) - 1\right], \quad i = 0, 1. \tag{4.2}$$

Observe that

$$\tau_{x,\varepsilon}^i(y) \to 1, \qquad t_{x,\varepsilon}^i(y) \to -\phi_i(x-y), \quad \text{as} \quad \varepsilon \to 0. \tag{4.3}$$

For $\varepsilon \in (0, 1]$, let $Q_{y,\varepsilon}^i$ be as in (2.22) with t_x^i replaced by $t_{x,\varepsilon}^i$ given in (4.2). Then the action of $L^{\varepsilon,\Delta}$ is given by the right-hand side of (2.23) with both Q_y^i replaced by the corresponding $Q_{y,\varepsilon}^i$ and τ_x^i replaced by $\tau_{x,\varepsilon}^i$. Note that, cf. (2.19),

$$\varepsilon^{-1}\int_{\mathbb{R}^d}\left(1 - e^{-\varepsilon\phi_i(x)}\right)dx \le \langle\phi_i\rangle, \quad i = 0, 1. \tag{4.4}$$

For each $\vartheta'' \in \mathbb{R}$, $k \in \mathcal{K}_{\vartheta''}$, and $\varepsilon \in (0, 1]$, by (4.4) both $Q_{y,\varepsilon}^i k$ satisfy the estimates as in (3.8) and (3.9). Therefore, $L^{\varepsilon,\Delta}k$ satisfies (3.10), which allows one to introduce the corresponding linear operators $L_\vartheta^{\varepsilon,\Delta} : \mathcal{D}(L_\vartheta^\Delta) \to \mathcal{K}_\vartheta$ and $L_{\vartheta'\vartheta}^{\varepsilon,\Delta} : \mathcal{K}_\vartheta \to \mathcal{K}_{\vartheta'}$, where $\mathcal{D}(L_\vartheta^\Delta)$ is defined in (3.7), see also Corollary 3.2 and (3.12). Thus, along with (3.13) we will consider the problem

$$\frac{d}{dt}q_{t,\varepsilon} = L_\vartheta^{\varepsilon,\Delta} q_{t,\varepsilon}, \qquad q_{t,\varepsilon}|_{t=0} = q_{0,\varepsilon} \in \mathcal{K}_{\vartheta_0}, \quad \vartheta_0 < \vartheta. \tag{4.5}$$

Its solutions $q_{t,\varepsilon} \in \mathcal{D}(L_\vartheta^\Delta) \subset \mathcal{K}_\vartheta$ are defined analogously as in Definition 3.3.

For $\vartheta, \vartheta' \in \mathbb{R}$ such that $\vartheta < \vartheta'$, we set, cf. (3.11),

$$T(\vartheta', \vartheta) = \frac{\vartheta' - \vartheta}{4\alpha}\exp\left(-ce^{\vartheta'}\right), \qquad \alpha = \max_{i=0,1}\alpha_i, \qquad c = \max_{i=0,1}\langle\phi_i\rangle. \tag{4.6}$$

For a fixed $\vartheta' \in \mathbb{R}$, $T(\vartheta', \vartheta))$ can be made as big as one wants by taking small enough ϑ. However, if ϑ is fixed, then

$$\sup_{\vartheta' > \vartheta} T(\vartheta', \vartheta) = \frac{\delta(\vartheta)}{4\alpha} \exp\left(-\frac{1}{\delta(\vartheta)}\right) =: \tau(\vartheta) < \infty, \tag{4.7}$$

where $\delta(\vartheta)$ is the unique positive solution of the equation

$$\delta e^{\delta} = \exp\left(-\vartheta - \log c\right). \tag{4.8}$$

Remark 4.1 The supremum in (4.7) is attained at

$$\vartheta' = \vartheta + \delta(\vartheta).$$

Note also that $\delta(\vartheta) \to 0$, and hence $\tau(\vartheta) \to 0$, as $\vartheta \to +\infty$.

Proposition 4.2 *For arbitrary $\vartheta_0 \in \mathbb{R}$ and $\varepsilon \in (0, 1]$, the problem in (4.5) with $q_{0,\varepsilon} \in \mathcal{K}_{\vartheta_0}$ and $\vartheta = \vartheta_0 + \delta(\vartheta_0)$ has a unique solution $q_{t,\varepsilon} \in \mathcal{K}_{\vartheta}$ on the time interval $[0, \tau(\vartheta_0))$.*

Proof Take $T < \tau(\vartheta_0)$ and then pick $\vartheta' \in (\vartheta_0, \vartheta_0 + \delta(\vartheta_0))$ such that $T < T(\vartheta', \vartheta_0)$. Our aim is to construct the family

$$\{S^{\varepsilon}_{\vartheta'\vartheta_0}(t) \in \mathcal{L}(\mathcal{K}_{\vartheta_0}, \mathcal{K}_{\vartheta'}) : t \in [0, T(\vartheta', \vartheta_0))\}, \tag{4.9}$$

defined by the series

$$S^{\varepsilon}_{\vartheta'\vartheta_0}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(L^{\varepsilon,\Delta}\right)^n_{\vartheta'\vartheta_0}. \tag{4.10}$$

In (4.9), $\mathcal{L}(\mathcal{K}_{\vartheta_0}, \mathcal{K}_{\vartheta'})$ stands for the Banach space of bounded linear operators acting from $\mathcal{K}_{\vartheta_0}$ to $\mathcal{K}_{\vartheta'}$ equipped with the corresponding operator norm. In (4.10), $\left(L^{\varepsilon,\Delta}\right)^0_{\vartheta'\vartheta_0}$ is the embedding operator and

$$\left(L^{\varepsilon,\Delta}\right)^n_{\vartheta'\vartheta_0} := \prod_{l=1}^{n} L^{\varepsilon,\Delta}_{\vartheta_l\vartheta_{l-1}}, \quad \vartheta_l = \vartheta_0 + l(\vartheta' - \vartheta_0)/n, \tag{4.11}$$

for $n \in \mathbb{N}$. Now we take into account that $\vartheta_l - \vartheta_{l-1} = (\vartheta' - \vartheta_0)/n$ and that $L^{\varepsilon,\Delta}$ satisfies (3.11) for all $\varepsilon \in (0, 1]$. This yields the following estimate

$$\begin{aligned} \|L^{\varepsilon,\Delta}_{\vartheta_l\vartheta_{l-1}}\| &\le \left(\frac{n}{e}\right)(\vartheta' - \vartheta_0)\left\{2\alpha_0 \exp\left(\langle\phi_0\rangle e^{\vartheta'}\right) + 2\alpha_1 \exp\left(\langle\phi_1\rangle e^{\vartheta'}\right)\right\}^{-1} \\ &\le n\big/eT(\vartheta', \vartheta_0), \end{aligned} \tag{4.12}$$

see (3.11) and (4.6). Then we apply (4.12) in (4.11) and conclude that the series in (4.10) converges in the operator norm, uniformly on $[0, T]$, to the operator-valued function $[0, T] \ni t \mapsto S^{\varepsilon}_{\vartheta'\vartheta_0}(t) \in \mathcal{L}(\mathcal{K}_{\vartheta_0}, \mathcal{K}_{\vartheta'})$ such that

$$\forall t \in [0, T] \qquad \|S^{\varepsilon}_{\vartheta'\vartheta_0}(t)\| \le \frac{T(\vartheta', \vartheta_0)}{T(\vartheta', \vartheta_0) - t}. \tag{4.13}$$

In a similar way, we get

$$\begin{aligned} \frac{d}{dt} S^{\varepsilon}_{\vartheta\vartheta_0}(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(L^{\varepsilon,\Delta}\right)^{n+1}_{\vartheta\vartheta_0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} L^{\varepsilon,\Delta}_{\vartheta\vartheta'} \left(L^{\varepsilon,\Delta}\right)^n_{\vartheta'\vartheta_0} = L^{\varepsilon,\Delta}_{\vartheta\vartheta'} S^{\varepsilon}_{\vartheta'\vartheta_0}(t), \quad t \in [0, T] \end{aligned} \tag{4.14}$$

Then

$$q_{t,\varepsilon} = S^{\varepsilon}_{\vartheta'\vartheta_0}(t)q_{0,\varepsilon} \in \mathcal{K}_{\vartheta'} \subset \mathcal{D}(L^{\varepsilon,\Delta}_{\vartheta}), \tag{4.15}$$

see Lemma 3.1, is a solution of (4.5) on the time interval $[0, \tau(\vartheta_0))$ since $T < \tau(\vartheta_0)$ has been taken in an arbitrary way and $L^{\varepsilon,\Delta}_{\vartheta\vartheta'}q_t = L^{\varepsilon,\Delta}_{\vartheta}q_t$ whenever $q_t \in \mathcal{K}_{\vartheta'}$, see (3.12).

Let us prove that the solution given in (4.15) is unique. In view of the linearity, to this end it is enough to show that the problem in (4.5) with the zero initial condition has a unique solution. Assume that $v_t \in \mathcal{D}(L^{\varepsilon,\Delta}_{\vartheta})$ is one of such solutions. Then v_t lies in $\mathcal{K}_{\vartheta''}$ for each $\vartheta'' > \vartheta$, see (3.3). Fix any such ϑ'' and then take $t < \tau(\vartheta_0)$ such that $t < T(\vartheta'', \vartheta)$. Then, cf. (3.12),

$$\begin{aligned} v_t &= \int_0^t L^{\varepsilon,\Delta}_{\vartheta''\vartheta} v_s ds \\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \left(L^{\varepsilon,\Delta}\right)^n_{\vartheta''\vartheta} v_{t_n} dt_n \cdots dt_1, \end{aligned}$$

where $n \in \mathbb{N}$ is an arbitrary number. Similarly as above we get from the latter

$$\|v_t\|_{\vartheta''} \le \frac{t^n}{n!}\left(\frac{n}{eT(\vartheta'', \vartheta)}\right)^n \sup_{s\in[0,t]} \|v_s\|_{\vartheta}.$$

Since n is an arbitrary number, this yields $v_s = 0$ for all $s \in [0, t]$. The extension of this result to all $t < \tau(\vartheta_0)$ can be done by repeating this procedure due times. □

Remark 4.3 Similarly as in obtaining (4.14) we have that, for each $\varepsilon \in (0, 1]$ and all $\vartheta_0, \vartheta_1, \vartheta_2 \in \mathbb{R}$ such that $\vartheta_0 < \vartheta_1 < \vartheta_2$, the following holds

$$S^{\varepsilon}_{\vartheta_2\vartheta_0}(t+s) = S^{\varepsilon}_{\vartheta_2\vartheta_1}(t)S^{\varepsilon}_{\vartheta_1\vartheta_0}(s), \quad t \in [0, T(\vartheta_2, \vartheta_1)), \quad s \in [0, T(\vartheta_1, \vartheta_0)). \tag{4.16}$$

5 The Proof of Theorem 3.5

With the help of Proposition 4.2 we have already obtained the unique solution of (3.13) in the form

$$k_t = S^1_{\vartheta\vartheta_0}(t)k_{\mu_0}, \qquad t < \tau(\vartheta_0), \tag{5.1}$$

where $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$ and $\vartheta \in (\vartheta_0, \vartheta_0 + \delta(\vartheta_0))$ is taken such that $t < T(\vartheta_0 + \delta(\vartheta_0), \vartheta)$. To prove Theorem 3.5 we first show (Lemma 5.1) that k_t lies in the cone (3.4) and hence is a correlation function of a unique state μ_t. Then, in Lemma 5.2, we construct an auxiliary evolution $u_0 \mapsto u_t$, with which we compare the evolution $k_{\mu_0} \mapsto k_t$ defined in (5.1). Thereby, we construct the extension of k_{μ_t} to all $t > 0$ as stated in the theorem.

5.1 The Identification Lemma

Our aim now is to show that the solution of (3.13) given in (5.1) has the property $k_t \in \mathcal{K}^{\star}_{\vartheta}$, see (3.4). By this one can identify k_t as k_{μ_t} for a unique state μ_t. Recall that bounded operators $L^{\Delta}_{\vartheta\vartheta''}$, $\vartheta'' < \vartheta$, were introduced in Corollary 3.2.

Lemma 5.1 *For arbitrary $\vartheta \in \mathbb{R}$ and $\vartheta_0 < \vartheta$, and for each $t \in [0, T(\vartheta, \vartheta_0))$, the operator defined in (4.10) has the property*

$$S^1_{\vartheta\vartheta_0}(t) : \mathcal{K}^{\star}_{\vartheta_0} \to \mathcal{K}^{\star}_{\vartheta}. \tag{5.2}$$

Proof We follow the line of arguments used in the proof of Theorem 3.8 of [3], see also [10, Lemma 4.8]. Let $\mu_0 \in \mathcal{P}_{\exp}(\Gamma^2)$ be such that $k_{\mu_0} \in \mathcal{K}^{\star}_{\vartheta_0}$, see Proposition 2.2. For $\Lambda = (\Lambda_0, \Lambda_1)$, $\Lambda_i \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $i = 0, 1$, let μ_0^Λ and $R^\Lambda_{\mu_0}$ be as in (2.12). For $N \in \mathbb{N}$, we then set

$$R_0^{\Lambda,N}(\eta) = R^\Lambda_{\mu_0}(\eta) I_N(\eta), \qquad \eta \in \Gamma_0^2, \tag{5.3}$$

where $I_N(\eta) = 1$ whenever $\max_{i=0,1} |\eta_i| \le N$ and $I_N(\eta) = 0$ otherwise. Set

$$\begin{aligned} \mathcal{R} &= L^1(\Gamma_0^2, d\lambda), \quad \mathcal{R}_\beta = L^1(\Gamma_0^2, b_\beta d\lambda), \\ b_\beta(\eta) &:= \exp\big(\beta\,(|\eta_0| + |\eta_1|)\big), \qquad \beta > 0. \end{aligned} \tag{5.4}$$

Let $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{\mathcal{R}_\beta}$ be the norms of the spaces introduced in (5.4) and $\mathcal{R}^+$ and $\mathcal{R}^+_\beta$ be the corresponding cones of positive elements (in the usual L^1-sense). For each $\beta > 0$, $R_0^{\Lambda,N}$ defined in (5.3) lies in $\mathcal{R}^+_\beta \subset \mathcal{R}^+$ and is such that $\|R_0^{\Lambda,N}\|_{\mathcal{R}} \le 1$. Then one can define a (non-normalized) measure

$$\mu_0^{\Lambda,N}(\eta) = R_0^{\Lambda,N}(\eta)\lambda(d\eta), \qquad \eta \in \Gamma_0^2.$$

Similarly as for the Kawasaki model, see [3, Sect. 3.2], it is possible to show that L^*, related by (1.4) to L given in (1.2), generates an evolution $\mu_0 \mapsto \mu_t$, $t \ge 0$, for which $0 \le \mu_t(\Gamma_0^2) \le 1$ whenever μ_0 has such a property, that is the case for $\mu_0^{\Lambda,N}$. Moreover, for each $t \ge 0$, the mentioned μ_t is absolutely continuous with respect to λ, and the equation for $R_t = d\mu_t/d\lambda$ corresponding to (1.3) can be written in the form

$$\frac{d}{dt} R_t = L^\dagger R_t, \qquad R_t|_{t=0} = R_{\mu_0}, \tag{5.5}$$

where, cf. (2.23), $L^\dagger$ is defined by the relation $L^\dagger R = d(L^*\mu)/d\lambda$, and hence acts according to the following formula

$$\begin{aligned} (L^\dagger R)(\eta_0, \eta_1) \;=\; & \sum_{y\in\eta_0} \int_{\mathbb{R}^d} a_0(x-y) e(\tau_y^0; \eta_1) R(\eta_0 \setminus y \cup x, \eta_1) dx \\ & + \sum_{y\in\eta_1} \int_{\mathbb{R}^d} a_1(x-y) e(\tau_y^1; \eta_0) R(\eta_0, \eta_1 \setminus y \cup x) dx \\ & - \Psi(\eta_0, \eta_1) R(\eta_0, \eta_1), \\ \Psi(\eta_0, \eta_1) := & \sum_{x\in\eta_0} \int_{\mathbb{R}^d} a_0(x-y) e(\tau_y^0; \eta_1) dy \\ & + \sum_{x\in\eta_1} \int_{\mathbb{R}^d} a_1(x-y) e(\tau_y^1; \eta_0) dy. \end{aligned} \tag{5.6}$$

Like in [3, Theorem 3.7], one shows that $L^\dagger$ generates a stochastic C_0-semigroup, $S_R := \{S_R(t)\}_{t\ge 0}$, on $\mathcal{R}$, which leaves invariant each $\mathcal{R}_\beta$, $\beta > 0$. Then the solution of (5.5) is $R_t = S_R(t) R_0$. For $R_0^{\Lambda,N}$ as in (5.3), we thus set

$$R_t^{\Lambda,N}(t) = S_R(t) R_0^{\Lambda,N}, \qquad t > 0. \tag{5.7}$$

Then $R_t^{\Lambda,N} \in \mathcal{R}^+_\beta \subset \mathcal{R}^+$ and $\|R_t^{\Lambda,N}\|_{\mathcal{R}} \le 1$. This yields that, for each $G \in B^\star_{\mathrm{bs}}(\Gamma_0^2)$, see (2.14) and (2.15), the following holds

$$\left\langle\!\!\left\langle KG, R_t^{\Lambda,N} \right\rangle\!\!\right\rangle \ge 0, \qquad t \ge 0. \tag{5.8}$$

The integral in (5.8) exists as $R_t^{\Lambda,N} \in \mathcal{R}_\beta$ and KG satisfies (2.5). Moreover, like in (3.11) and (5.21), for each β' such that $0 < \beta' < \beta$, we derive from (5.6) the following estimate

$$\|L^\dagger R\|_{\mathcal{R}_{\beta'}} \leq \frac{4\alpha \|R\|_{\mathcal{R}_\beta}}{e(\beta - \beta')}.$$

This allows us to define the corresponding bounded operators $(L^\dagger)^n_{\beta'\beta} : \mathcal{R}_\beta \to \mathcal{R}_{\beta'}, n \in \mathbb{N}$, cf. (4.11) and (5.23), the norms of which satisfy

$$\|(L^\dagger)^n_{\beta'\beta}\| \leq n^n \left(e\bar{T}(\beta, \beta')\right)^{-n}. \tag{5.9}$$

On the other hand, we have that

$$\begin{aligned} k_0^{\Lambda,N}(\eta) &:= \int_{\Gamma_0^2} R_0^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi) \\ &= \int_{\Gamma_0^2} R_0^{\Lambda,N}(\eta_0 \cup \xi_0, \eta_1 \cup \xi_1)(\lambda_0 \otimes \lambda_1)(d\xi_0, d\xi_1), \end{aligned} \tag{5.10}$$

cf. (2.13) and (5.3), lies in $\mathcal{K}^\star_{\vartheta_0} \subset \mathcal{K}_{\vartheta_0}$, and hence we may get

$$k_t^{\Lambda,N} = S^1_{\vartheta\vartheta_0}(t)k_0^{\Lambda,N}, \qquad t \in [0, T(\vartheta, \vartheta_0)), \tag{5.11}$$

where $S^1_{\vartheta\vartheta_0}(t) = S^\varepsilon_{\vartheta\vartheta_0}(t)|_{\varepsilon=1}$ is given in (4.10). Then the proof of (5.2) consists in showing:

$$\begin{aligned} &\text{(i)} \quad \forall G \in B^\star_{\text{bs}}(\Gamma_0^2) \qquad \left\langle\!\!\left\langle G, k_t^{\Lambda,N} \right\rangle\!\!\right\rangle \geq 0; \\ &\text{(ii)} \quad \left\langle\!\!\left\langle G, S^1_{\vartheta\vartheta_0}(t)k_0 \right\rangle\!\!\right\rangle = \lim_{\Lambda\to\mathbb{R}^d\times\mathbb{R}^d} \lim_{N\to+\infty} \left\langle\!\!\left\langle G, k_t^{\Lambda,N} \right\rangle\!\!\right\rangle. \end{aligned} \tag{5.12}$$

To prove claim (i) of (5.12), for a given $G \in B^\star_{\text{bs}}(\Gamma_0^2)$, one sets

$$\varphi_G(t) = \left\langle\!\!\left\langle KG, R_t^{\Lambda,N} \right\rangle\!\!\right\rangle, \qquad \psi_G(t) = \left\langle\!\!\left\langle G, k_t^{\Lambda,N} \right\rangle\!\!\right\rangle, \tag{5.13}$$

where ψ_G is defined for t as in (5.11). For a given $t \in (0, T(\vartheta, \vartheta_0))$, we pick $\vartheta' < \vartheta$ such that $t < T(\vartheta', \vartheta_0)$, and hence $k_s^{\Lambda,N} \in \mathcal{K}_{\vartheta'}$ for $s \in [0, t]$. Then the direct calculation based on (4.14) yields for the n-th derivative

$$\psi_G^{(n)}(t) = \left\langle\!\!\left\langle G, (L^\Delta)^n_{\vartheta\vartheta'} k_t^{\Lambda,N} \right\rangle\!\!\right\rangle, \qquad n \in \mathbb{N}.$$

As in obtaining (4.13) we then get from the latter

$$\left|\psi_G^{(n)}(t)\right| \leq A^n n^n C_{\vartheta'}(G) \sup_{s\in[0,t]} \left\|k_s^{\Lambda,N}\right\|_{\vartheta'}. \tag{5.14}$$

Here $A = 1/eT(\vartheta, \vartheta')$ and

$$C_{\vartheta'}(G) = \int_{\Gamma_0^2} |G(\eta)| \exp\left(\vartheta'|\eta_0| + \vartheta'|\eta_1|\right)\lambda(d\eta) < \infty,$$

as $G \in B_{\text{bs}}(\Gamma_0^2)$. Likewise, from (5.7) we have

$$\varphi_G^{(n)}(t) = \left\langle\!\!\left\langle KG, (L^\dagger)^n_{\beta'\beta} R_t^{\Lambda,N} \right\rangle\!\!\right\rangle$$

For the same t as in (5.14), by (5.9) we have from the latter

$$\left|\varphi_G^{(n)}(t)\right| \leq \bar{A}^n n^n \bar{C}_{\beta'}(G) \sup_{s\in[0,t]} \left\|R_s^{\Lambda,N}\right\|_{\beta'}. \tag{5.15}$$

Here $\bar{A} = 1/e\bar{T}(\beta', \beta)$ and

$$\bar{C}_{\beta'}(G) = \operatorname*{ess\,sup}_{\eta\in\Gamma_0^2} |KG(\eta)| \exp\left(-\beta'|\eta_0| - \beta'|\eta_1|\right) < \infty$$

which holds in view of (2.5). By (2.23), (5.6), and (5.10) it follows that

$$(L^\Delta k_0^{\Lambda,N})(\eta) = \int_{\Gamma_0^2} (L^\dagger R_0^{\Lambda,N})(\eta \cup \xi)\lambda(d\xi),$$

which then yields

$$\forall n \in \mathbb{N}_0 \qquad \varphi_G^{(n)}(0) = \psi_G^{(n)}(0). \tag{5.16}$$

By the Denjoy–Carleman theorem [4], (5.15) and (5.14) imply that both functions defined in (5.13) are quasi-analytic on $[0, t]$. Then (5.16) implies

$$\forall t \in [0, T(\vartheta, \vartheta_0)) \qquad \varphi_G(t) = \psi_G(t), \tag{5.17}$$

which by (5.8) yields the first line in (5.12). The convergence claimed in *(ii)* of (5.12) is proved in a standard way, see Appendix in [3]. □

Note that (5.17) yields also that

$$\forall t \in [0, T(\vartheta, \vartheta_0)) \qquad \left\langle\!\left\langle G, q_t^{\Lambda,N} \right\rangle\!\right\rangle = \left\langle\!\left\langle G, k_t^{\Lambda,N} \right\rangle\!\right\rangle, \tag{5.18}$$

where G and $k_t^{\Lambda,N}$ are as in (5.13) and

$$q_t^{\Lambda,N}(\eta) := \int_{\Gamma_0^2} R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi), \tag{5.19}$$

cf. (5.10).

5.2 An Auxiliary Evolution

The evolution which we construct now will be used to continue the solution k_t given in (5.1) to all $t > 0$ as stated in Theorem 3.5. The construction employs the operator

$$\begin{aligned}(\bar{L}k)(\eta_0, \eta_1) = &\sum_{y\in\eta_0} \int_{\mathbb{R}^d} a_0(x - y)k(\eta_0 \setminus y \cup x, \eta_1)dx \\ &+ \sum_{y\in\eta_1} \int_{\mathbb{R}^d} a_1(x - y)k(\eta_0, \eta_1 \setminus y \cup x)dx \end{aligned} \tag{5.20}$$

obtained from L^Δ given in (2.23) by putting $\phi_i = 0$, $i = 0, 1$, and then dropping the second and fourth terms. Note that $\bar{L}$ does not correspond to any Markov evolution as it describes (free) “half-jumps”. Similarly as in (3.11), we get

$$\|\bar{L}k\|_\vartheta \leq \frac{4\alpha\|k\|_{\vartheta''}}{e(\vartheta - \vartheta'')}, \tag{5.21}$$

which allows us to introduce the operators $(\bar{L}_\vartheta, \mathcal{D}(\bar{L}_\vartheta))$ and $\bar{L}_{\vartheta\vartheta''} \in \mathcal{L}(\mathcal{K}_{\vartheta''}, \mathcal{K}_\vartheta)$ such that, cf. (3.12),

$$\forall k \in \vartheta'' \qquad \bar{L}_{\vartheta\vartheta''}k = \bar{L}_\vartheta k, \qquad \vartheta'' < \vartheta.$$

Like above, we have that

$$\mathcal{K}_{\vartheta''} \subset \mathcal{D}(\bar{L}_\vartheta) := \{k \in \mathcal{K}_\vartheta : \bar{L}k \in \mathcal{K}_\vartheta\}, \qquad \vartheta'' < \vartheta.$$

Note that

$$\bar{L}_{\vartheta\vartheta''} : \mathcal{K}^+_{\vartheta''} \to \mathcal{K}^+_{\vartheta}, \qquad \vartheta'' < \vartheta, \tag{5.22}$$

see (3.5). For $n \in \mathbb{N}$, we define $(\bar{L})^n_{\vartheta'\vartheta}$ similarly as in (4.11) and denote, cf. (4.6),

$$\bar{T}(\vartheta', \vartheta) = (\vartheta' - \vartheta)/4\alpha, \qquad \vartheta < \vartheta'. \tag{5.23}$$

Our aim is to study the operator valued function defined by the series

$$\bar{S}_{\vartheta'\vartheta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\bar{L}\right)^n_{\vartheta'\vartheta}. \tag{5.24}$$

Lemma 5.2 *For each $\vartheta_0, \vartheta \in \mathbb{R}$ such that $\vartheta_0 < \vartheta$, the series in (5.24) defines a continuous function*

$$[0, \bar{T}(\vartheta, \vartheta_0)) \ni t \mapsto \bar{S}_{\vartheta\vartheta_0}(t) \in \mathcal{L}(\mathcal{K}_{\vartheta_0}, \mathcal{K}_{\vartheta}), \tag{5.25}$$

which has the following properties:

(a) *For t as in (5.25), let $\vartheta'' \in (\vartheta_0, \vartheta)$ be such that $t < \bar{T}(\vartheta'', \vartheta_0)$. Then, cf. (4.14),*

$$\frac{d}{dt}\bar{S}_{\vartheta\vartheta_0}(t) = \bar{L}_{\vartheta\vartheta''}\bar{S}_{\vartheta''\vartheta_0}(t). \tag{5.26}$$

(b) *The problem*

$$\frac{d}{dt}u_t = \bar{L}_{\vartheta}u_t, \qquad u_t|_{t=0} = u_0 \in \mathcal{K}^+_{\vartheta_0}, \tag{5.27}$$

has a unique solution $u_t \in \mathcal{K}^+_{\vartheta}$ on the time interval $[0, \bar{T}(\vartheta, \vartheta_0))$ given by

$$u_t = \bar{S}_{\vartheta''\vartheta_0}(t)u_0, \tag{5.28}$$

where, for a fixed $t \in [0, \bar{T}(\vartheta, \vartheta_0))$, ϑ'' is chosen to satisfy $t < \bar{T}(\vartheta'', \vartheta_0)$.

Proof Proceeding as in the proof of Proposition 4.2, by means of the estimate in (5.21) we prove the convergence of the series in (5.24). This allows also for proving (5.26), which yields the existence of the solution of (5.27) in the form given in (5.28). The uniqueness is proved analogously as in the case of Proposition 4.2. The stated positivity of u_t follows from (5.24) and (5.22). □

Corollary 5.3 *For a given $C > 0$, we let in (5.27) and (5.28) $\vartheta_0 = \log C$ and $u_0(\eta) = C^{|\eta_0|+|\eta_1|}$. Then the unique solution of (5.27) is*

$$u_t(\eta) = C^{|\eta_0|+|\eta_1|} \exp\left\{t(\alpha_0|\eta_0| + \alpha_1|\eta_1|)\right\}. \tag{5.29}$$

This solution can naturally be continued to all $t > 0$ for which it lies in $\mathcal{K}_{\vartheta(t)}$ with

$$\vartheta(t) = \log C + t \max_{i=0,1} \alpha_i. \tag{5.30}$$

Proof In view of the lack of interaction in (5.20), the equations for particular $u_t^{(n)}$ take the following (decoupled) form

$$\begin{aligned}
&\frac{d}{dt}u_t^{(n)}(x_1, \ldots, x_{n_0}; y_1, \ldots, y_{n_1}) \\
&\quad = \sum_{i=1}^{n_0} \int_{\mathbb{R}^d} a_0(x - x_i)u_t^{(n)}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{n_0}; y_1, \ldots, y_{n_1})dx \\
&\qquad + \sum_{i=1}^{n_1} \int_{\mathbb{R}^d} a_1(y - y_i)u_t^{(n)}(x_1, \ldots x_{n_0}; y_1, \ldots y_{i-1}, y, y_{i+1}, \ldots y_{n_1})dy, \ n \in \mathbb{N}^2,
\end{aligned}$$

which for the initial translation invariant u_0 yields (5.29). □

5.3 The Global Solution

As follows from Proposition 4.2 and Lemma 5.1, the unique solution of the problem (3.13) with $k_0 \in \mathcal{K}^{\star}_{\vartheta_0}$ lies in $\mathcal{K}^{\star}_{\vartheta}$ for $t \in (0, T(\vartheta, \vartheta_0))$. At the same time, for fixed ϑ_0, $T(\vartheta, \vartheta_0)$ is bounded, see (4.7). This means that the mentioned solution cannot be directly continued to all $t > 0$. In this subsection, by a comparison method we prove that, for $t \in (0, T(\vartheta, \vartheta_0))$, k_t satisfies (3.14) which is then used to get the continuation in question, cf. Corollary 5.3. Recall that the operators Q^i_y, $i = 0, 1$, were introduced in (2.22) and the cone $\mathcal{K}^+_{\vartheta}$ was defined in (3.5).

Lemma 5.4 *For each $k_0 \in \mathcal{K}^{\star}_{\vartheta_0}$ and $t \in (0, T(\vartheta, \vartheta_0))$, $k_t := S^1_{\vartheta\vartheta_0} k_0$ has the property*

$$k_t - e(\tau^i_y; \cdot) Q^i_y k_t \in \mathcal{K}^+_{\vartheta}, \qquad i = 0, 1, \tag{5.31}$$

holding for Lebesgue-almost all $y \in \mathbb{R}^d$.

Proof Clearly, it is enough to show that (5.31) holds for $i = 0$. For a fixed y, we denote

$$v_{t,1} = k_t - Q^0_y k_t, \quad v_{t,2} = [1 - e(\tau^0_y; \cdot)] Q^0_y k_t.$$

The proof will be done if we show that, for all $G \in B_{\mathrm{bs}}(\Gamma^2_0)$ such that $G(\eta) \geq 0$ for λ-almost all $\eta \in \Gamma^2_0$, the following holds

$$\langle\!\langle G, v_{t,j} \rangle\!\rangle \geq 0, \qquad j = 1, 2. \tag{5.32}$$

Let Λ, N, and $k^{\Lambda,N}_0$ be as in (5.10), and then $k^{\Lambda,N}_t$ be as in (5.11). Next, let $v^{\Lambda,N}_{t,j}$, $j = 1, 2$, be defined as above with k_t replaced by $k^{\Lambda,N}_t$. By (5.18) and (5.19) we then get

$$\begin{aligned} \langle\!\langle G, Q^0_y k^{\Lambda,N}_t \rangle\!\rangle &= \int_{\Gamma^2_0} \widetilde{G}(\eta) k^{\Lambda,N}_t(\eta) \lambda(d\eta) \\ &= \int_{\Gamma^2_0} \int_{\Gamma^2_0} \widetilde{G}(\eta) R^{\Lambda,N}_t(\eta \cup \xi) \lambda(d\eta) \lambda(d\xi), \end{aligned} \tag{5.33}$$

where

$$\widetilde{G}(\eta_0, \eta_1) := \sum_{\xi \subset \eta_1} e(t^0_y; \xi) G(\eta_0, \eta_1 \setminus \xi).$$

Furthermore, by (5.33) we get

$$\begin{aligned} \left\langle\!\!\left\langle G, Q^0_y k^{\Lambda,N}_t \right\rangle\!\!\right\rangle &= \int_{\Gamma^2_0} G(\eta_0, \eta_1) \\ &\quad \times \int_{\Gamma^2_0} \left(\int_{\Gamma_0} e(t^0_y; \zeta) R^{\Lambda,N}_t(\eta_0 \cup \xi_0, \eta_1 \cup \xi_1 \cup \zeta) \lambda_1(d\zeta) \right) \lambda(d\eta) \lambda(d\xi) \\ &= \int_{\Gamma^2_0} G(\eta_0, \eta_1) \int_{\Gamma^2_0} \left(\sum_{\zeta \subset \xi_1} e(t^0_y; \zeta) \right) R^{\Lambda,N}_t(\eta_0 \cup \xi_0, \eta_1 \cup \xi_1) \lambda(d\eta) \lambda(d\xi). \end{aligned} \tag{5.34}$$

By (2.21) it follows that

$$\sum_{\zeta \subset \xi_1} e(t^0_y; \zeta) = e(\tau^0_y; \xi_1).$$

We apply this in the last line of (5.34) and obtain

$$\begin{aligned}\left\langle\!\!\left\langle G, Q_y^0 k_t^{\Lambda,N}\right\rangle\!\!\right\rangle &= \int_{\Gamma_0^2} G(\eta_0,\eta_1)\int_{\Gamma_0^2} e(\tau_y^0;\xi_1) R_t^{\Lambda,N}(\eta_0\cup\xi_0,\eta_1\cup\xi_1)\lambda(d\eta)\lambda(d\xi)\\ &\le \int_{\Gamma_0^2} G(\eta_0,\eta_1)\int_{\Gamma_0^2} R_t^{\Lambda,N}(\eta_0\cup\xi_0,\eta_1\cup\xi_1)\lambda(d\eta)\lambda(d\xi)\\ &= \left\langle\!\!\left\langle G, k_t^{\Lambda,N}\right\rangle\!\!\right\rangle, \end{aligned} \tag{5.35}$$

which after the limiting transition as in (5.12) yields (5.32) for $j = 1$. For the same G, we set $\bar{G} = e(\tau_y^0;\cdot)G$. Then by (2.21) and the second line in (5.35) we get

$$\left\langle\!\!\left\langle \bar{G}, Q_y^0 k_t^{\Lambda,N}\right\rangle\!\!\right\rangle \le \left\langle\!\!\left\langle G, Q_y^0 k_t^{\Lambda,N}\right\rangle\!\!\right\rangle,$$

which after the limiting transition as in (5.12) yields (5.32) for $j = 2$. □

Lemma 5.5 *Let $C > 0$ be such that the initial condition in* (3.13) *satisfies $k_{\mu_0}(\eta) = k_0(\eta) \le C^{|\eta_0|+|\eta_1|}$. Then for all $t < T(\vartheta,\vartheta_0)$ with $\vartheta_0 = \log C$ and any $\vartheta > \vartheta_0$, the unique solution of* (3.13) *given by the formula*

$$k_t = S^1_{\vartheta\vartheta_0}(t)k_0 \tag{5.36}$$

satisfies (3.14) *for λ-almost all $\eta \in \Gamma_0^2$.*

Proof Take any $\vartheta > \vartheta_0$ and fix $t < T(\vartheta,\vartheta_0)$; then pick $\vartheta^1 \in (\vartheta_0,\vartheta)$ such that $t < T(\vartheta^1,\vartheta_0)$. Next take $\vartheta^2,\vartheta^3 \in \mathbb{R}$ such that $\vartheta^1 < \vartheta^2 < \vartheta^3$ and $t < \bar{T}(\vartheta^3,\vartheta^2)$. The latter is possible since $\bar{T}$ depends only on the difference $\vartheta_3 - \vartheta_2$, see (5.23). For the fixed t, $k_t \in \mathcal{K}^\star_{\vartheta^1} \hookrightarrow \mathcal{K}^\star_{\vartheta^3}$, and hence one can write

$$\begin{aligned} u_t &= \bar{S}_{\vartheta^3\vartheta^*}(t)u_0\\ &= (u_0 - k_0) + k_t + \int_0^t \bar{S}_{\vartheta^3\vartheta^2}(t-s)D_{\vartheta^2\vartheta^1}k_s ds, \end{aligned} \tag{5.37}$$

where

$$D_{\vartheta\vartheta''} = \bar{L}_{\vartheta\vartheta''} - L^\Delta_{\vartheta\vartheta''}, \qquad D_\vartheta = \bar{L}_\vartheta - L^\Delta_\vartheta,$$

and the latter two operators are as in (5.27) and (3.13) respectively. By Lemma 5.1, for $s \le t$, $k_s \in \mathcal{K}^\star_{\vartheta^1}$. By (2.23), (5.20), and Lemma 5.4 we have that $D_{\vartheta^2\vartheta^1} : \mathcal{K}^\star_{\vartheta^1} \to \mathcal{K}^+_{\vartheta^2}$. Then by Lemma 5.2 the third summand in the second line in (5.37) is in $\mathcal{K}^+_{\vartheta^3}$ which completes the proof since $u_0 - k_0$ is also positive. □

Proof of Theorem 3.5. According to Definition 3.3 and Remark 3.4 the map $[0,+\infty) \ni t \mapsto k_t \in \mathcal{K}^\star$ is the solution in question if: (a) $k_t(\emptyset,\emptyset) = 1$; (b) for each $t > 0$, there exists $\vartheta'' \in \mathbb{R}$ such that $k_t \in \mathcal{K}_{\vartheta''}$ and $\frac{d}{dt}k_t = L^\Delta_\vartheta k_t$ for each $\vartheta > \vartheta''$.

Let k_0 and $C > 0$ be as in the statement of Theorem 3.5. Set $\vartheta^* = \log C$. Then, for $\vartheta = \vartheta^* + \delta(\vartheta^*)$, see (4.7) and (4.8), k_t as given in (5.36) is a unique solution of (3.13) in $\mathcal{K}_\vartheta$ on the time interval $[0, T(\vartheta,\vartheta^*))$. By (2.23) we have

$$\left(\frac{d}{dt}k_t\right)(\emptyset,\emptyset) = (L^\Delta k_t)(\emptyset,\emptyset) = 0,$$

which yields that $k_t(\emptyset,\emptyset) = k_0(\emptyset,\emptyset) = 1$. By Lemma 5.1 $k_t \in \mathcal{K}^\star_\vartheta$, and hence k_t is the solution in question for $t < \tau(\vartheta^*)$. According to Lemma 5.5 k_t lies in $\mathcal{K}_{\vartheta(t)}$ with $\vartheta(t)$ given in (5.30). Fix any $\epsilon \in (0, 1)$ and then set $s_0 = 0$, $s_1 = (1-\epsilon)\tau(\vartheta^*)$, and $\vartheta_1^* = \vartheta(s_1)$. Thereafter, set $\vartheta^1 = \vartheta_1^* + \delta(\vartheta_1^*)$ and

$$k_{t+s_1} = S^1_{\vartheta^1 \vartheta_1^*}(t) k_{s_1}, \qquad t \in [0, \tau(\vartheta_1^*)).$$

Note that for t such that $t + s_1 < \tau(\vartheta^*)$,

$$k_{t+s_1} = S^1_{\vartheta^1 \vartheta^*}(t + s_1) k_0,$$

see (4.16). Thus, by Lemmas 5.1 and 5.5 the map $[0, s_1 + \tau(\vartheta_1^*)) \ni t \mapsto k_t \in \mathcal{K}_{\vartheta(t)}$ with

$$k_t = \begin{cases} S^1_{\vartheta_1^* \vartheta^*}(t) k_0 & t \le s_1; \\ S^1_{\vartheta^1 \vartheta_1^*}(t - s_1) k_{s_1} & t \in [s_1, s_1 + \tau(\vartheta_1^*)) \end{cases}$$

is the solution in question on the indicated time interval. We continue this procedure by setting $s_n = (1-\epsilon)\tau(\vartheta^*_{n-1})$, $n \ge 2$, and then

$$\vartheta_n^* = \vartheta(s_1 + \cdots + s_n), \qquad \vartheta^n = \vartheta_n^* + \delta(\vartheta_n^*). \tag{5.38}$$

This yields the solution in question on the time interval $[0, s_1 + \cdots + s_{n+1}]$ which for $t \in [s_1 + \cdots + s_l, s_1 + \cdots + s_{l+1}]$, $l = 0, \ldots, n$, is given by

$$k_t = S^1_{\vartheta^l \vartheta_l^*}(t - (s_1 + \cdots + s_l)) k_{s_l}.$$

Then the global solution in question exists whenever the series

$$\sum_{n \ge 1} s_n = (1-\epsilon) \sum_{n \ge 1} \tau(\vartheta_n^*)$$

diverges. Assume that this is not the case. Then by (5.30) and (5.38) we get that both (a) and (b) ought to be true, where (a) $\sup_{n \ge 1} \vartheta_n^* =: \bar{\vartheta} < +\infty$ and (b) $\tau(\vartheta_n^*) \to 0$ as $n \to +\infty$. However, by (4.7) and (4.8) it follows that (a) implies $\tau(\vartheta_n^*) \ge \tau(\bar{\vartheta}) > 0$, which contradicts (b). □

6 The Proof of Theorems 3.8 and 3.9

6.1 The Kinetic Equations

Here we prove Theorem 3.8. For a continuous function

$$[0, +\infty) \ni t \mapsto \varrho_t = (\varrho_{0,t}, \varrho_{1,t}) \in L^\infty(\mathbb{R}^d \to \mathbb{R}^2),$$

cf. (3.16), let us consider

$$\begin{aligned} F_{0,t}(\varrho)(x) &= \varrho_{0,0}(x) e^{-\alpha_0 t} + \int_0^t e^{-\alpha_0(t-s)} (a_0 * \varrho_{0,s})(x) \exp\left[-(\phi_0 * \varrho_{1,s})(x)\right] ds \\ &\quad + \int_0^t e^{-\alpha_0(t-s)} \varrho_{0,s}(x) \left(a_0 * \left[1 - \exp\left[-(\phi_0 * \varrho_{1,s})\right] \right] \right)(x) ds, \\ F_{1,t}(\varrho)(x) &= \varrho_{1,0}(x) e^{-\alpha_1 t} + \int_0^t e^{-\alpha_1(t-s)} (a_1 * \varrho_{1,s})(x) \exp\left[-(\phi_1 * \varrho_{0,s})(x)\right] ds \\ &\quad + \int_0^t e^{-\alpha_1(t-s)} \varrho_{1,s}(x) \left(a_1 * \left[1 - \exp\left[-(\phi_1 * \varrho_{0,s})\right] \right] \right)(x) ds. \end{aligned} \tag{6.1}$$

For a given $T > 0$, let $\mathcal{C}_T$ stand for the Banach space of continuous functions

$$[0, T] \ni t \mapsto (\varrho_{0,t}, \varrho_{1,t}) \in L^{\infty}(\mathbb{R}^d \to \mathbb{R}^2), \tag{6.2}$$

with norm

$$\|\varrho\|_T = \max_{i=0,1} \sup_{t\in[0,T]} \left\{ \|\varrho_{i,t}\|_{L^\infty} e^{-\alpha_i t} \right\}. \tag{6.3}$$

Let also $\mathcal{C}_T^+$ denote the set of all positive $\varrho \in \mathcal{C}_T$, i.e., such that $\varrho_{i,t}(x) \geq 0$ for all $i = 0, 1$, $t \in [0, T]$, and Lebesgue-almost all x. By means of $F_{i,t}$ introduced in (6.1) we then define the map

$$\mathcal{C}_T \ni \varrho \mapsto F(\varrho) = (F_0(\varrho), F_1(\varrho)) \in \mathcal{C}_T$$

such that the values of $F_i(\varrho)$ are given in the right-hand sides of (6.1). By direct inspection one concludes that both $F_{i,t}(\varrho)$, $i = 0, 1$, are continuously differentiable in t, and the function as in (6.2) is a positive solution of (3.15) on $[0, T]$ if and only if it solves in $\mathcal{C}_T^+$ the following fixed-point equation

$$\varrho = F(\varrho). \tag{6.4}$$

Let $C > 0$ be an arbitrary number and $\varrho_{i,0}$, $i = 0, 1$, be as in (3.18) and (6.1). Set

$$\Delta_C = \{\varrho \in \mathcal{C}_T^+ : (\varrho_{0,t}, \varrho_{1,t})|_{t=0} = (\varrho_{0,0}, \varrho_{1,0}), \ \text{ and } \ \|\varrho\|_T \leq C\}. \tag{6.5}$$

By (6.1) one readily gets that $F : \mathcal{C}_T^+ \to \mathcal{C}_T^+$. Let us show that

$$\forall C > 0 \qquad F : \Delta_C \to \Delta_C. \tag{6.6}$$

For $\varrho \in \Delta_C$, from the first equation in (6.1) one gets

$$\begin{aligned} \|F_{0,t}(\varrho)\|_{L^\infty} &\leq Ce^{-\alpha_0 t} + 2\alpha_0 e^{-\alpha_0 t} \int_0^t e^{\alpha_0 s} \|\varrho_{0,s}\|_{L^\infty} ds \\ &\leq Ce^{\alpha_0 t}, \qquad\qquad t \in [0, T]. \end{aligned} \tag{6.7}$$

Similarly, $\|F_{1,t}(\varrho)\|_{L^\infty} \leq Ce^{\alpha_1 t}$, which proves (6.6). To solve (6.4) we apply the Banach contraction principle. To this end we pick $T > 0$ such that F is a contraction on (6.5). We do this as follows. For $\varrho, \bar{\varrho} \in \Delta_C$, like in (6.7) we obtain

$$\begin{aligned} \|F_{0,t}(\varrho) - F_{0,t}(\bar{\varrho})\|_{L^\infty} &\leq 2\alpha_0 e^{-\alpha_0 t} \int_0^t e^{\alpha_0 s} \|\varrho_{0,s} - \bar{\varrho}_{0,s}\|_{L^\infty} ds \\ &\quad + 2\alpha_0 e^{-\alpha_0 t} \int_0^t e^{\alpha_0 s} \|\bar{\varrho}_{0,s}\|_{L^\infty} \|\varrho_{1,s} - \bar{\varrho}_{1,s}\|_{L^\infty} ds. \\ &\leq e^{\alpha_0 t} \|\varrho - \bar{\varrho}\|_T \left(1 - e^{-2\alpha_0 t} \left[1 - \frac{2}{3} C \left(e^{3\alpha_0 t} - 1 \right) \right] \right). \end{aligned}$$

The corresponding estimate for $\|F_{1,t}(\varrho) - F_{1,t}(\bar{\varrho})\|_{L^\infty}$ (with $e^{\alpha_1 t}$) can be obtained in the same way. Then according to (6.3) F is a contraction on Δ_C whenever $C > 0$ and T satisfy

$$e^{3\alpha T} < 1 + \frac{3}{2C}, \qquad \alpha := \max_{i=0,1} \alpha_i. \tag{6.8}$$

This yields the existence of the unique positive solution of (3.15) on the time interval $[0, T]$, where T is defined in (6.8) by the initial condition $(\varrho_{0,0}, \varrho_{1,0})$. This solution lies in Δ_C and hence

$$\|\varrho_{i,T}\|_{L^\infty} \leq e^{\alpha T} C, \qquad i = 0, 1. \tag{6.9}$$

Now we consider the problem (3.15) for $\varrho^{(1)}_{i,t} = \varrho_{i,T+t}$, $i = 0, 1$, where ϱ is the solution just constructed. For this new problem, by (6.9) we have

$$\|\varrho^{(1)}_{i,0}\|_{L^\infty} \leq C_1 := e^{\alpha T} C, \qquad i = 0, 1.$$

Then we repeat the above construction and obtain the solution $\varrho^{(1)}$ on the time interval $[0, T_1]$ with $T_1 > 0$ satisfying, cf. (6.8),

$$e^{3\alpha T_1} = 1 + \frac{1}{C} e^{-\alpha T} < 1 + \frac{3}{2C} e^{-\alpha T} = 1 + \frac{3}{2C_1}.$$

By further repeating this construction we obtain $\varrho^{(n)}_{i,t} = \varrho_{i,T+T_1+\cdots+T_{n-1}+t}$, $i = 0, 1$, $t \in [0, T_n]$, where the sequence $\{T_n\}_{n\in\mathbb{N}}$ is defined recursively by the condition

$$e^{3\alpha T_n} = 1 + \frac{1}{C} \exp\left[-\alpha\left(T + T_1 + \cdots + T_{n-1}\right)\right], \quad n \in \mathbb{N}. \tag{6.10}$$

Thus, the global solution in question exists if the series $\sum_n T_n$ is divergent. Assume that this is not the case. Then the right-hand side of (6.10) is bounded from below by some $b > 1$, uniformly in n. This yields that $T_n \geq \log b/3\alpha > 0$, holding for all $n \in \mathbb{N}$, which contradicts the summability of $\{T_n\}_{n\in\mathbb{N}}$ and thus completes the proof of Theorem 3.8.

6.2 The Scaling Limit

For each k and λ-almost all $\eta \in \Gamma_0^2$, we have that the following holds, cf. (2.22) and (4.3),

$$\begin{aligned}
(Q^0_{y,\varepsilon}k)(\eta_0, \eta_1) &\to (Q^0_{y,0}k)(\eta_0, \eta_1) \\
&:= \int_{\Gamma_0} k(\eta_0, \eta_1 \cup \xi) e(-\phi_0(y - \cdot); \xi)\lambda(d\xi), \quad \varepsilon \to 0, \\
(Q^1_{y,\varepsilon}k)(\eta_0, \eta_1) &\to (Q^1_{y,0}k)(\eta_0, \eta_1) \\
&:= \int_{\Gamma_0} k(\eta_0 \cup \xi, \eta_1) e(-\phi_1(y - \cdot); \xi)\lambda(d\xi), \quad \varepsilon \to 0.
\end{aligned}$$

Thus, for each k and λ-almost all $\eta \in \Gamma_0^2$,

$$(L^{\varepsilon,\Delta}k)(\eta) \to (Vk)(\eta), \qquad \text{as } \varepsilon \to 0,$$

where, cf. (2.23), (4.3)

$$\begin{aligned}
(Vk)(\eta_0, \eta_1) = &\sum_{y\in\eta_0} \int_{\mathbb{R}^d} a_0(x - y)(Q^0_{y,0}k)(\eta_0 \setminus y \cup x, \eta_1)dx \\
&- \sum_{x\in\eta_0} \int_{\mathbb{R}^d} a_0(x - y)(Q^0_{y,0}k)(\eta_0, \eta_1)dy \\
&+ \sum_{y\in\eta_1} \int_{\mathbb{R}^d} a_1(x - y)(Q^1_{y,0}k)(\eta_0, \eta_1 \setminus y \cup x)dx \\
&- \sum_{x\in\eta_1} \int_{\mathbb{R}^d} a_1(x - y)(Q^1_{y,0}k)(\eta_0, \eta_1)dy.
\end{aligned} \tag{6.11}$$

Like above, for each $\vartheta'' \in \mathbb{R}$ and $k \in \mathcal{K}_{\vartheta''}$, both $Q^i_{y,0}k$ satisfy the estimates as in (3.8) and (3.9). Then for $\vartheta, \vartheta'' \in \mathbb{R}$ such that $\vartheta'' < \vartheta$, $\|Vk\|_\vartheta$ is bounded by the right-hand side of

(3.11). This allows one to define the operators V_ϑ and $V_{\vartheta\vartheta''}$ analogous to L^Δ_ϑ and $L^\Delta_{\vartheta\vartheta''}$, respectively. For ϱ_t being the solution as in Theorem 3.8, $k_{\pi_{\varrho_t}}$ satisfies

$$\frac{d}{dt} k_{\pi_{\varrho_t}} = V_{\vartheta\vartheta''} k_{\pi_{\varrho_t}}, \qquad t > 0, \tag{6.12}$$

where $\vartheta'' \in \mathbb{R}$ is such that $k_{\pi_{\varrho_t}} \in \mathcal{K}_{\vartheta''}$, see (3.18), and $\vartheta > \vartheta''$ is arbitrary. This can be checked by direct calculations based on (6.11) and (3.15). Moreover, if we set $C = \|\varrho_0\|_\infty$, see (3.17), then $k_{\pi_{\varrho_t}}$ satisfies (3.14) with this C, which follows from (3.18). Thus, by Corollary 5.3 we conclude that $k_{\pi_{\varrho_t}} \in \mathcal{K}_{\vartheta(t)}$ for all $t > 0$.

The proof of Theorem 3.9 Let ϑ_* be as assumed. As mentioned above, we then have that $k_{\pi_{\varrho_t}} \in \mathcal{K}_{\vartheta_T}$ for all $t \in [0, T]$ with $\vartheta_T := \vartheta_* + \alpha T$ and T such that

$$T < \tau(\vartheta_* + \alpha T). \tag{6.13}$$

The latter is possible since the function $\vartheta \mapsto \tau(\vartheta)$ is continuous and $\tau(\vartheta_*) > 0$, see (4.7). Since the inequality in (6.13) is strict, we can also pick $\vartheta_1 > \vartheta_T$ such that $T < \tau(\vartheta_1)$. Thereafter, we set $\vartheta = \vartheta_1 + \delta(\vartheta_1)$, cf. Remark 4.1. For $q_{0,\varepsilon}$ with the assumed property, let $q_{t,\varepsilon}$ be the solution of (4.5) in $\mathcal{K}_\vartheta$. In view of (6.12), we then have

$$\begin{aligned} q_{t,\varepsilon} - k_{\pi_{\varrho_t}} = & \int_0^t S^\varepsilon_{\vartheta\vartheta_1}(t-s)\left(L^{\varepsilon,\Delta}_{\vartheta_1\vartheta_T} - V_{\vartheta_1\vartheta_T}\right) k_{\pi_{\varrho_s}} ds \\ & + S^\varepsilon_{\vartheta\vartheta_*}(t)\left[q_{0,\varepsilon} - k_{\pi_{\varrho_0}}\right], \qquad t \in [0, T]. \end{aligned} \tag{6.14}$$

Since $\vartheta \mapsto \tau(\vartheta)$ is decreasing, by (6.13) we have that $T < \tau(\vartheta_*)$. By (4.13) we then get

$$\forall t \in [0, T] \qquad \|S^\varepsilon_{\vartheta\vartheta_*}(t)\| \le \frac{T(\vartheta, \vartheta_*)}{T(\vartheta, \vartheta_*) - T},$$

which yields that the second term in (6.14) tends to zero uniformly on $[0, T]$. Also by (4.13) we have

$$\begin{aligned} & \left\| \int_0^t S^\varepsilon_{\vartheta\vartheta_1}(t-s)\left(L^{\varepsilon,\Delta}_{\vartheta_1\vartheta_T} - V_{\vartheta_1\vartheta_T}\right) k_{\pi_{\varrho_s}} ds \right\|_\vartheta \\ & \quad \le \|k_{\pi_{\varrho_T}}\|_{\vartheta_T} \tau(\vartheta_1) \log \frac{T(\vartheta, \vartheta_1)}{T(\vartheta, \vartheta_1) - T} \|L^{\varepsilon,\Delta}_{\vartheta_1\vartheta_T} - V_{\vartheta_1\vartheta_T}\|. \end{aligned} \tag{6.15}$$

To estimate the latter term we set

$$W^i_{y,\varepsilon} k = Q^i_{y,0} k - Q^i_{y,\varepsilon} k, \qquad i = 0, 1, \quad y \in \mathbb{R}^d. \tag{6.16}$$

By means of the inequality, cf. the proof of Theorem 4.6 in [3],

$$|b_1 \cdots b_n - a_1 \cdots a_n| \le \sum_{i=1}^n |b_i - a_i| \prod_{j \ne i} \max\{|a_j|; |b_j|\},$$

and

$$0 \le \psi(t) := (t - 1 + e^{-t})/t^2 \le 1/2, \qquad t \ge 0,$$

we obtain, cf. (3.8),

$$\begin{aligned}\left|W^0_{y,\varepsilon}k(\eta_0,\eta_1)\right| &\le \varepsilon\|k\|_{\vartheta''}\exp\left(\vartheta''|\eta_0|+\vartheta''|\eta_1|\right)\\ &\quad\times\int_{\Gamma_0}\left(e^{\vartheta''|\xi|}\sum_{x\in\xi}[\phi_0(y-x)]^2\,\psi\left(\varepsilon\phi_0(y-x)\right)\prod_{z\in\xi\setminus x}\phi_0(y-z)\right)\lambda(d\xi)\\ &\le(\varepsilon/2)\bar{\phi_0}\|k\|_{\vartheta''}\exp\left(\vartheta''|\eta_0|+\vartheta''|\eta_1|\right)\\ &\quad\times\int_{\Gamma_0}\left(|\xi|e^{\vartheta''|\xi|}\prod_{z\in\xi}\phi_0(y-z)\right)\lambda(d\xi)\\ &=(\varepsilon/2)\bar{\phi_0}\langle\phi_0\rangle\exp\left(\langle\phi_0\rangle e^{\vartheta''}\right)\|k\|_{\vartheta''}\exp\left(\vartheta''|\eta_0|+\vartheta''|\eta_1|+\vartheta''\right).\end{aligned}\tag{6.17}$$

Likewise,

$$\left|W^1_{y,\varepsilon}k(\eta_0,\eta_1)\right|\le(\varepsilon/2)\bar{\phi_1}\langle\phi_1\rangle\exp\left(\langle\phi_1\rangle e^{\vartheta''}\right)\|k\|_{\vartheta''}\exp\left(\vartheta''|\eta_0|+\vartheta''|\eta_1|+\vartheta''\right).\tag{6.18}$$

Next, by (2.23), (6.11), and (6.16) we have

$$\begin{aligned}(L^{\varepsilon,\Delta}-V)k(\eta_0,\eta_1)&=\sum_{y\in\eta_0}\int_{\mathbb{R}^d}a_0(x-y)(U^0_{y,\varepsilon}k)(\eta_0\setminus y\cup x,\eta_1)dx\\ &\quad-\sum_{x\in\eta_0}\int_{\mathbb{R}^d}a_0(x-y)(U^0_{y,\varepsilon}k)(\eta_0,\eta_1)dy\\ &\quad+\sum_{y\in\eta_1}\int_{\mathbb{R}^d}a_1(x-y)(U^1_{y,\varepsilon}k)(\eta_0,\eta_1\setminus y\cup x)dx\\ &\quad-\sum_{x\in\eta_1}\int_{\mathbb{R}^d}a_1(x-y)(U^1_{y,\varepsilon}k)(\eta_0,\eta_1)dy.\end{aligned}\tag{6.19}$$

Here we use the following notations

$$\begin{aligned}(U^0_{y,\varepsilon}k)(\eta_0,\eta_1)&=e(\tau^0_{y,\varepsilon};\eta_1)(Q^0_{y,\varepsilon}k)(\eta_0,\eta_1)-(Q^0_{y,0}k)(\eta_0,\eta_1),\\ (U^1_{y,\varepsilon}k)(\eta_0,\eta_1)&=e(\tau^1_{y,\varepsilon};\eta_0)(Q^1_{y,\varepsilon}k)(\eta_0,\eta_1)-(Q^1_{y,0}k)(\eta_0,\eta_1).\end{aligned}$$

Then, cf. (6.16),

$$\begin{aligned}\left|(U^0_{y,\varepsilon}k)(\eta_0,\eta_1)\right|&\le\left|(W^0_{y,\varepsilon}k)(\eta_0,\eta_1)\right|+\varepsilon\bar{\phi_0}|\eta_1|\left|(Q^0_{y,0}k)(\eta_0,\eta_1)\right|,\\ \left|(U^1_{y,\varepsilon}k)(\eta_0,\eta_1)\right|&\le\left|(W^1_{y,\varepsilon}k)(\eta_0,\eta_1)\right|+\varepsilon\bar{\phi_1}|\eta_0|\left|(Q^1_{y,0}k)(\eta_0,\eta_1)\right|.\end{aligned}$$

Now by (3.8), (3.9), (6.17), (6.18), and (6.19) we get

$$\begin{aligned}\left|(L^{\varepsilon,\Delta}-V)k(\eta_0,\eta_1)\right|&\le\varepsilon\alpha\|k\|_{\vartheta''}\exp\left[\vartheta''(|\eta_0|+|\eta_1|)\right]\exp\left(ce^{\vartheta''}\right)\\ &\quad\times\left(2|\eta_0||\eta_1|\left(\bar{\phi_0}+\bar{\phi_1}\right)+e^{\vartheta''}\left(\bar{\phi_0}\langle\phi_0\rangle|\eta_0|+\bar{\phi_1}\langle\phi_1\rangle|\eta_1|\right)\right)\end{aligned}$$

Like in obtaining (3.11) we then get from the latter

$$\|L^{\varepsilon,\Delta}_{\vartheta_1\vartheta_T}-V_{\vartheta_1\vartheta_T}\|\le\varepsilon\Phi(\vartheta_1,\vartheta_T),$$

where $\Phi(\vartheta_1, \vartheta_T) > 0$ depends on the choice of ϑ_1, ϑ_T and on the model parameters only, and may be calculated explicitly. Then the use of the latter estimate in (6.15) completes the proof. □

Acknowledgements The present research was supported by the European Commission under the Project STREVCOMS PIRSES-2013-612669, which is cordially acknowledged by the authors. The authors are also grateful to the referee for the comments and criticism that helped to improve the presentation of the paper.

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