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Existence and Stability of a Spike in the Central Component for a Consumer Chain Model

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Abstract We study a three-component consumer chain model which is based on Schnakenberg type kinetics. In this model there is one consumer feeding on the producer and a second consumer feeding on the first consumer. This means that the first consumer (central component) plays a hybrid role: it acts both as consumer and producer. The model is an extension of the Schnakenberg model suggested in Gierer and Meinhardt (Kybernetik 12:30–39, 1972) and Schnakenberg (J Theoret Biol 81:389–400, 1979) for which there is only one producer and one consumer. It is assumed that both the producer and second consumer diffuse much faster than the central component. We construct single spike solutions on an interval for which the profile of the first consumer is that of a spike. The profiles of the producer and the second consumer only vary on a much larger spatial scale due to faster diffusion of these components. It is shown that there exist two different single spike solutions if the feed rates are small enough: a large-amplitude and a small-amplitude spike. We study the stability properties of these solutions in terms of the system parameters. We use a rigorous analysis for the linearized operator around single spike solutions based on nonlocal eigenvalue problems. The following result is established: If the time-relaxation constants for both producer and second consumer vanish, the large-amplitude spike solution is stable and the small-amplitude spike solution is unstable. We also derive results on the stability of solutions when these two time-relaxation constants are small. We show a *new effect*: if the time-relaxation constant of the second consumer is very small, the large-amplitude spike solution becomes unstable. To the best of our knowledge this phenomenon has not been observed before for the stability of spike patterns. It seems that this behavior is not possible for two-component reaction–diffusion systems but that at least three components are required. Our main motivation to

Dedicated to the memory of Professor Klaus Kirchgässner, with deep gratitude.

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study this system is mathematical since the novel interaction of a spike in the central component with two other components results in new types of conditions for the existence and stability of a spike. This model is realistic if several assumptions are made: (i) cooperation of consumers is prevalent in the system, (ii) the producer and the second consumer diffuse much faster than the first consumer, and (iii) there is practically an unlimited pool of producer. The first assumption has been proven to be correct in many types of consumer groups or populations, the second assumption occurs if the central component has a much smaller mobility than the other two, the third assumption is realistic if the consumers do not feel the impact of the limited amount of producer due to its large quantity. This chain model plays a role in population biology, where consumer and producer are often called predator and prey. This system can also be used as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir.

Keywords Pattern formation · Consumer chain model · Predator–prey model · Autocatalytic reaction · Reaction–diffusion systems · Spiky solutions · Stability

Mathematics Subject Classification Primary 35B35, 92C40 · Secondary 35B40

1 Introduction

We consider a reaction–diffusion system which serves as a cooperative consumer chain model. It takes into account the interaction of three components, one pure producer, one pure consumer and a central component which acts as both producer and consumer. These three components supply each other in a linear chain. This model is an extension of the Schnakenberg model introduced in [12,27] which possesses only one producer and one consumer. In the model under investigation we have a central component which plays a hybrid role: it consumes the pure producer and it is consumed by the second consumer. It is assumed that both the producer and second consumer diffuse much faster than the central component.

The system can be written as follows:

$$\begin{cases} \tau \dfrac{\partial S}{\partial t} = D_1 \Delta S + 1 - \dfrac{a_1}{\epsilon} S u_1^2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_1}{\partial t} = \epsilon^2 \Delta u_1 - u_1 + S u_1^2 - a_2 u_1 u_2^2, & x \in \Omega,\ t > 0, \\ \tau_1 \dfrac{\partial u_2}{\partial t} = D_2 \Delta u_2 - u_2 + \dfrac{1}{\epsilon} u_1 u_2^2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where S and u_i denote the concentrations of the producer (food source) and the two consumers, respectively. Here $0 < \epsilon^2 \ll 1$ and $0 < D_1,\ 0 < D_2$ are three positive diffusion constants. The constants $a_1,\ a_2$ (positive) for the feed rates and $\tau,\ \tau_1$ (nonnegative) for the time relaxation constants will be treated as parameters and their choices will distinguish between stability and instability of steady-state solutions.

We choose as domain the interval $\Omega = (-1, 1)$ and consider Neumann boundary conditions

$$\begin{aligned} &\frac{dS}{dx}(-1,0) = \frac{dS}{dx}(1,0) = 0, \quad \frac{du_1}{dx}(-1,0) = \frac{du_1}{dx}(1,0) = 0, \\ &\frac{du_2}{dx}(-1,0) = \frac{du_2}{dx}(1,0) = 0. \end{aligned} \tag{1.2}$$

These type of boundary conditions are also called “reflective” boundary conditions and model a system which does not have exchange to the outside world by permeation through the boundary.

Remark 1.1 Our choice of diffusion constants is essential for the type of spike solutions under consideration. We need to have a very small diffusion constant for the central component to get a spike and much larger diffusion constants for the other two components resulting in profiles on the order unity scale only.

Remark 1.2 The choice of the coefficients $-\frac{a_1}{\epsilon}$, 1, $-a_2$, $\frac{1}{\epsilon}$ of the nonlinear reaction terms in (1.1) allows us to have spiky solutions for which all three components have an amplitude of order $O(1)$ as $\epsilon \to 0$. Other choices of parameters in the model are possible, but they would result in amplitudes which are not of order $O(1)$. In that case, a rescaling of amplitudes is possible which will lead to the scaling we used in (1.1) and amplitudes of order $O(1)$. For this reason we have used the system in the form (1.1) as our starting point.

The interaction of a spike in the central component of a consumer chain model with two other components, one preceding it and the other succeeding it, results in new types of conditions for the existence and stability of a spike. This was the main motivation for us to study this problem in detail.

We first prove the existence of single spike solutions in an interval. It is shown that such a pattern exists if the feed rates a_1, a_2 are small enough. We prove that there are two such spiky solutions, one with a large-amplitude spike and the other with a small-amplitude spike.

We show that the large-amplitude solution can be stable for $\tau_1 = 0$, whereas the small-amplitude solution is always unstable. However, for $0 < \epsilon \ll \tau_1 \ll 1$ the large-amplitude solution is unstable due to an eigenvalue of order $O(\frac{1}{\tau_1})$ which has a positive real part (see Corollary 2.1).

We expect that for $0 < \tau_1 \ll \epsilon \ll 1$ the system will be stable, i.e. the instability will vanish if the time-relaxation constant τ_1 of the last component is very small compared to the square root of the diffusion constant of the spike component.

This is a *new effect* which to the best of our knowledge has not been observed before for the stability of spike patterns. It seems that this behavior is not possible for two-component reaction–diffusion systems but that at least three components are required.

We use a rigorous analysis for the linearized operator around a single spike solution based on nonlocal eigenvalue problems.

Models involving a chain of components play an important role in biology, chemistry, social sciences and many other fields. Well-known examples include consumer chains, predator-prey systems, food chains, genetic signaling pathways, autocatalytic chemical reactions and nuclear chain reactions. For food chains it is commonly assumed that there is only limited supply of resources which leads to a saturation effect and the solutions remain bounded for all times. On the other hand, for autocatalytic chemical or nuclear chain reactions the interaction of the components in the chain has a self-enforcing effect and the solutions can grow and become unbounded. In our model the cooperation of consumers is accounted for by superlinear nonlinearities. In general we do not know the exact shape of the nonlinearities, which will depend on more details of the application considered, and so for simplicity we choose quadratic nonlinearities. This choice can be motivated for chemical reaction systems by the mass balance law in the case of binary reactions. It can also be derived using mathematical principles by expanding a general nonlinearity for small amplitudes around zero and will then play a role in understanding solutions with small amplitudes.

In this respect, it is interesting to consider the work of Bettencourt and West [2] who collected extensive empirical data on typical activities in cities such as scientific publications, patents, GDP, the number of educational institutions but also on crime, traffic congestion or certain diseases indicating that they grow at a superlinear rate with population size. They established a universal growth rate which applies to most of the activities in major cities independent of geographic location, ethnicity of the population or cultural background which corresponds to a power law with power of around 1.15. Although this is less than the quadratic power law considered in this paper we expect that many of our results will not change qualitatively if we replace the quadratic law by this smaller power growth. The general explanation behind this superlinear growth in societies is that they are able to attract those people which will be most suitable to interact with the pre-existing population successfully.

In our model we further assume that the limited amount of resources is not felt which is realistic if resources are plentiful or if consumption is practised wisely to use the remaining supplies in a sustainable way.

We refer to the recent work [18] in which the stability of food chains was analyzed under the assumption that supply of resources is limited.

In biological populations consumer and producer are often called predator and prey. For more background on predator-prey models we refer to [21]. Our system can also be used as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir. Similar models have been suggested, see e.g. Chapter 8 of [29] and the references therein.

Our main results are generalizations of similar statements for the Schnakenberg model. Let us briefly recall some related results: In [15,30] the existence and stability of spiky patterns on bounded intervals is established. In [39] similar results are shown for two-dimensional domains. In [1] it is shown how the degeneracy of the Turing bifurcation [28] can be lifted using spatially varying diffusion coefficients. In [22–24] spikes are considered rigorously for the shadow system.

For a closely related system, the Gray-Scott model introduced in [13,14], some of the results are the following: In [4–7] the existence and stability of spike patterns on the real line is proved. The two-dimensional case is studied in [32,33,36,37]. In [16,17] different regimes for the Gray-Scott system are considered and the existence and stability of spike patterns in an interval is shown. In [25,26] a skeleton structure and separators for the Gray-Scott model are established.

Other “large” reaction diffusion systems (more than two components) with spiky patterns include the hypercycle of Eigen and Schuster [8–11,34,35], and Meinhardt and Gierer’s model of mutual exclusion and segmentation [19,20,40]. These results have been summarized and reviewed in [42].

The paper [41] is a companion to the current one. In that work the diffusion constants are chosen as follows: the diffusivity for the first component is much larger than for the second, and for the second it is much larger than for the third. Results on the existence and stability of a spiky cluster solution have been derived. That solution has a spike for the last component which acts on a very small scale, for the central component there are two partial spikes glued together acting on an intermediate scale, and for the first component there is a profile which changes on the order unity scale only. This spiky solution can be stable, but to achieve stability a fine balance is required between the three components.

The structure of this paper is as follows:

In Sect. 2, we state and explain the main theorems on existence and stability.

In Sects. 3 and 4, we prove the main existence result, Theorem 2.1. In Sect. 3, we compute the amplitudes of the spikes. In Sect. 4, we give a rigorous existence proof.

In Sects. 5 and 6, we prove the main stability results, Theorem 2.2 and Corollary 2.1. In Sect. 5, we derive a nonlocal eigenvalue problem (NLEP) and determine the stability of the $O(1)$ eigenvalues. In Sect. 6, we study the stability of the $o(1)$ eigenvalues.

Throughout this paper, the letter C will denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon\to 0} \frac{A}{B} = 1$ and $A = O(B)$ is defined as $|A| \le C|B|$ for some $C > 0$.

2 Main Results: Existence and Stability of a Single Spike Solution

We now state the main results of this paper on existence and stability. We first construct stationary spike solutions to (1.1), i.e. spike solutions to the system

$$\begin{cases} D_1 \Delta S + 1 - \dfrac{a_1}{\epsilon} S u_1^2 = 0, & x \in \Omega,\ t > 0, \\ \epsilon^2 \Delta u_1 - u_1 + S u_1^2 - a_2 u_1 u_2^2 = 0, & x \in \Omega,\ t > 0, \\ D_2 \Delta u_2 - u_2 + \dfrac{1}{\epsilon} u_1 u_2^2 = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{2.1}$$

with the Neumann boundary conditions given in (1.2).

We will construct solutions of (2.1) which are even:

$$\begin{aligned} S &= S(|x|) \in H_N^2(\Omega), \\ u_1 &= u_1(|y|) \in H_N^2(\Omega_\epsilon), \quad y = \frac{x}{\epsilon} \\ u_2 &= u_2(|x|) \in H_N^2(\Omega), \end{aligned}$$

where

$$\begin{aligned} H_N^2(\Omega) &= \left\{ v \in H^2(\Omega)\ :\ v'(-1) = v'(1) = 0 \right\}, \\ \Omega_\epsilon &= \left(-\frac{1}{\epsilon}, \frac{1}{\epsilon} \right), \\ H_N^2(\Omega_\epsilon) &= \left\{ v \in H^2(\Omega_\epsilon)\ :\ v'\left(-\frac{1}{\epsilon}\right) = v'\left(\frac{1}{\epsilon}\right) = 0 \right\}. \end{aligned}$$

Before stating the main results, we introduce some necessary notations and assumptions. Let w be the unique solution of the problem

$$\begin{cases} w_{yy} - w + w^2 = 0, & w > 0 \ \text{ in } \mathbb{R}, \\ w(0) = \max_{y\in\mathbb{R}} w(y), & w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{2.2}$$

The ODE problem (2.2) can be solved explicitly and w can be written as

$$w(y) = \frac{3}{2\cosh^2 \frac{y}{2}}. \tag{2.3}$$

We now state the main existence result.

Theorem 2.1 *Assume that*

$$D_1 = const., \quad \epsilon \ll 1, \quad D_2 = const. \tag{2.4}$$

Let G_{D_1} and G_{D_2} be the Green's functions defined in (7.1) and (7.18), respectively. Assume that

$$a_1^2 a_2 < \frac{|\Omega|^2}{4} G_{D_2}^2(0,0) - \delta_0. \tag{2.5}$$

(*Expressed more precisely,* (2.4) *means that* ϵ *is small enough;* (2.5) *means the following: there are positive numbers* δ_0 *and* ϵ_0 *such that* (2.5) *is valid for all* ϵ *with* $0 < \epsilon < \epsilon_0$.)

Then problem (2.1) *admits two "single-spike" solutions* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *and* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *with the following properties:*

(i) *all components are even functions.*

(ii)

$$S^{s,l}_\epsilon(x) = c^{s,l}_{1,\epsilon} G_{D_1}(x,0) + O(\epsilon), \tag{2.6}$$

$$u_{1,\epsilon}(x) = \xi^{s,l}_\epsilon w\left(\frac{|x|\sqrt{1+\alpha^{s,l}_\epsilon}}{\epsilon}\right) + O(\epsilon), \tag{2.7}$$

$$u_{2,\epsilon}(x) = c^{s,l}_{2,\epsilon} G_{D_2}(x,0) + O(\epsilon), \tag{2.8}$$

where w *is the unique solution of* (2.2),

$$\left(\xi^l_\epsilon\right)^2 = \frac{|\Omega|^2 + \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G^{-2}_{D_2}(0,0)}}{72a_1^2} + O(\epsilon), \tag{2.9}$$

$$\left(\xi^s_\epsilon\right)^2 = \frac{|\Omega|^2 - \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G^{-2}_{D_2}(0,0)}}{72a_1^2} + O(\epsilon), \tag{2.10}$$

$$c^{s,l}_{1,\epsilon} = \frac{1+\alpha^{s,l}_\epsilon}{\xi^{s,l}_\epsilon G_{D_1}(0,0)} + O(\epsilon), \quad c^{s,l}_{2,\epsilon} = \frac{\sqrt{1+\alpha^{s,l}_\epsilon}}{6\xi^{s,l}_\epsilon G^2_{D_2}(0,0)} + O(\epsilon), \tag{2.11}$$

where $\alpha^{s,l}_\epsilon$ *is given by* (3.8).

(iii) *If* ϵ *is small enough and*

$$a_1^2 a_2 > \frac{|\Omega|^2}{4} G^2_{D_2}(0,0) + \delta_0.$$

for some $\delta_0 > 0$ *independent of* ϵ (*in the same sense as in* (2.5)) *then there are no single-spike solutions which satisfy* (i) *and* (ii).

Remark 2.1 We choose to keep the factor $|\Omega|$ in the estimate (2.5) although of course in our scaling we have $|\Omega| = 2$.

Theorem 2.1 will be proved in Sects. 3 and 4.

The second goal of this paper is to study the stability properties of the single-spike solution constructed in Theorem 2.1. We now state our main results on stability.

Theorem 2.2 *Assume that* (2.4) *and* (2.5) *are satisfied. Suppose that* $\tau = \tau_1 = 0$. *Then we have the following:*

(1) (Stability) *The large-amplitude solution* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *is linearly stable. There is a small eigenvalue of exact order* $O(\epsilon^2)$ *with negative real part which is given in* (6.23).
(2) (Instability) *The small-amplitude solution* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *is linearly unstable. There is a large eigenvalue of exact order* $O(1)$ *with positive real part. There is also a small eigenvalue of exact order* $O(\epsilon^2)$ *with negative real part which is given in* (6.23).

For the case of τ and τ_1 positive and small we have the following result:

Corollary 2.1 *Assume that* (2.4) *and* (2.5) *are satisfied.*

(1) (Stability/Instability) *There exists a constant* $\tau_0 > 0$ *independent of* ϵ *such that for* $0 \le \tau \le \tau_0$ *and* $\tau_1 = 0$ *the stability properties of the large-amplitude solution* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *and the small-amplitude solution* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *are the same as in the case* $\tau = \tau_1 = 0$. *There is also small eigenvalue of exact order* $O(\epsilon^2)$ *with negative real part which is given in (6.23).*

(2) (Instability) *There exists a constant* $\tau_0 > 0$ *independent of* ϵ *such that for* $0 \le \tau \le \tau_0$ *and* $0 < \epsilon \ll \tau_1 \ll 1$ *for both the large-amplitude solution* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *and the small-amplitude solution* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *there is an eigenvalue*

$$\lambda_\epsilon = \frac{\rho_0}{\tau_1} + O(1)$$

with corresponding eigenfunction

$$\phi_\epsilon = w + O(\tau_1).$$

Thus both solutions $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *and* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *are unstable. There is also small eigenvalue of exact order* $O(\epsilon^2)$ *with negative real part which is given in* (6.23).

We would like to make a few remarks on the stability results.

Remark 2.2 This result can be interpreted as follows: to have this type of spiky solution, the feed rates a_1 and a_2, in particular their combination $a_1^2 a_2$, must be small enough. Otherwise the food source S and the hybrid u_1 will not be able to sustain u_1 and u_2, respectively. Instead, among others, one of the following three behaviors can happen:

(i) The consumer u_2 dies out resulting in the long-term limit $u_{2,\epsilon} = 0$ and a spike for the two-component Schnakenberg model remains for which only the components S_ϵ and $u_{1,\epsilon}$ are non-vanishing. We get the same solution by setting $\alpha_\epsilon = 0$ in Theorem 2.1. This solution has been analyzed in [15].

(ii) The component u_2 dies out and u_1, S will both approach positive constants. It can easily be seen that we have

$$S = \frac{a_1}{\epsilon}, \quad u_1 = \frac{\epsilon}{a_1}.$$

(iii) The components approach a positive homogeneous steady state which solves

$$S = \frac{\epsilon}{a_1 u_1^2}, \quad u_1^2 - \frac{\epsilon}{a_1} u_1 + a_2 \epsilon^2 = 0, \quad u_2 = \frac{\epsilon}{u_1}.$$

Remark 2.3 In the proof of Corollary 2.1 we expand the eigenvalue and eigenfunction further, see (5.16) and (5.17).

Remark 2.4 We do not rigorously study the dynamics of this model. Instead we analyze the stability or instability of the steady states. Then the dynamics can be understood locally near the equilibrium points by using the fact that the unstable eigenfunctions will grow in amplitude, whereas the stable eigenfunctions will decay to zero as time progresses.

Next we plot the large-amplitude and small-amplitude spike solutions.

Figure 1 shows the spatial profiles of the large-amplitude spike $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$, i.e. u_1 is large.

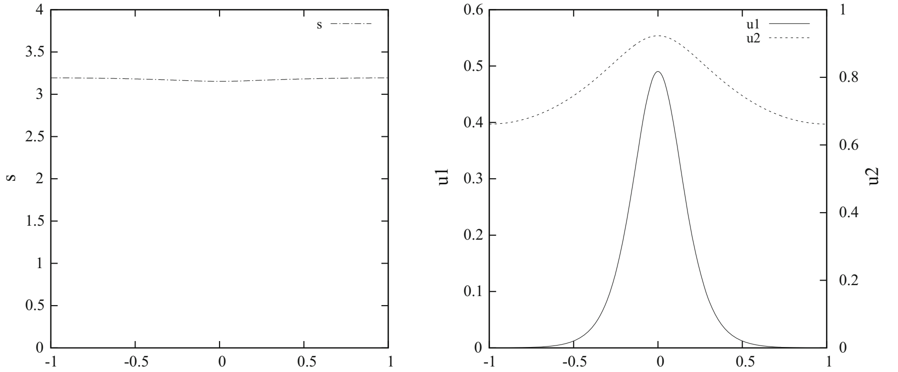

Fig. 1 The spatial profiles of the large-amplitude spike steady state $(S^l_\epsilon,\ u^l_{1,\epsilon},\ u^l_{2,\epsilon})$ for parameters $D_1 = 10,\ \epsilon^2 = 0.01,\ D_2 = 1,\ a_1 = 1,\ a_2 = 0.04$

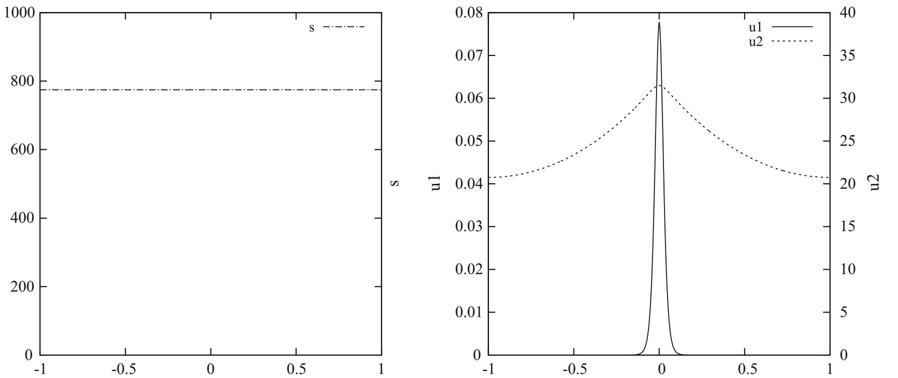

Fig. 2 The spatial profiles of the small-amplitude spike steady state $(S^s_\epsilon,\ u^s_{1,\epsilon},\ u_{2,\epsilon})$ for parameters $D_1 = 10,\ \epsilon^2 = 0.01,\ D_2 = 1,\ a_1 = 1,\ a_2 = 0.04$

Figure 2 shows the spatial profiles of the small-amplitude spike $(S^s_\epsilon,\ u^s_{1,\epsilon},\ u^s_{2,\epsilon})$, i.e. u_1 is small.

Here by the choice of parameters the amplitudes of the three components are very different.

We will rigorously derive the existence result Theorem 2.1 in Sects. 3 and 4. The stability results Theorem 2.2 and Corollary 2.1 will be proved in Sects. 5 and 6.

3 Existence I: Computation of the Amplitudes in Leading Order

In this section and the next, we will show the existence of spike solutions to (2.1) and prove Theorem 2.1. We begin by computing the amplitudes in leading order and will give a rigorous existence proof in the next section.

Proof of Theorem 2.1 We will show the existence of spike solutions to (2.1) which in leading order are given by (2.6)–(2.8). More precisely, we choose the second component of the approximate solution as follows:

$$\tilde{u}_{1,\epsilon}(x) = \xi_\epsilon \, w \left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon} \right) \chi(|x|) \tag{3.1}$$

for some positive constants ξ_ϵ and α_ϵ. Here χ is a smooth cutoff function which satisfies

$$\chi \in C_0^\infty(-1,1), \quad \chi(x) = 1 \text{ for } |x| \le \frac{5}{8}, \quad \chi(x) = 0 \text{ for } |x| \ge \frac{3}{4}. \tag{3.2}$$

The main reason for using the cut-off function (3.2) in the ansatz (3.1) is that Neumann boundary conditions are satisfied exactly. □

We set

$$y = \frac{x}{\epsilon},$$

and consider the limit

$$\epsilon \to 0.$$

We substitute (2.7) into the second equation of (2.1) and, using (2.2), we note that $w\left(y\sqrt{1+\alpha_\epsilon}\right)$ satisfies

$$w_{yy} - (1+\alpha_\epsilon)w + (1+\alpha_\epsilon)w^2 = 0. \tag{3.3}$$

Comparing coefficients between the second equation and (3.3) gives

$$\alpha_\epsilon = a_2 u_{2,\epsilon}^2(0) + O(\epsilon), \tag{3.4}$$

$$\xi_\epsilon = \frac{1+\alpha_\epsilon}{S_\epsilon(0)} + O(\epsilon). \tag{3.5}$$

We remark that in leading order $S_\epsilon u_{1,\epsilon}^2$ agrees with $S_\epsilon(0)u_{1,\epsilon}^2$ since $u_{1,\epsilon}$ decays rapidly away from 0.

Substituting (2.7) into the third equation of (2.1) and using (2.2), we get

$$u_{2,\epsilon}(x) = G_{D_2}(x,0)u_{2,\epsilon}^2(0) \, \frac{\xi_\epsilon}{\sqrt{1+\alpha_\epsilon}} \int_{\mathbb{R}} w(y)\,dy + O(\epsilon),$$

where G_{D_2} has been defined in (7.18). This implies

$$u_{2,\epsilon}(0) = \frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)\xi_\epsilon \int w(y)\,dy} + O(\epsilon), \tag{3.6}$$

$$u_{2,\epsilon}(x) = \frac{G_{D_2}(x,0)\sqrt{1+\alpha_\epsilon}}{G_{D_2}^2(0,0)\xi_\epsilon \int w(y)\,dy} + O(\epsilon). \tag{3.7}$$

In the next step, we will derive two conditions, by substituting (3.1), (3.6) with (3.4), (3.5) in (2.1). Then we will solve these two conditions to determine α_ϵ and ξ_ϵ.

Integrating the first equation in (2.1), using the Neumann boundary condition and balancing the last two terms, we get the first condition

$$|\Omega| = a_1 S_\epsilon(0) \, \frac{\xi_\epsilon^2}{\sqrt{1+\alpha_\epsilon}} \int_{\mathbb{R}} w^2(y)\,dy + O(\epsilon).$$

From (3.4), we compute

$$\alpha_\epsilon = \frac{a_2(1+\alpha_\epsilon)}{\xi_\epsilon^2 G_{D_2}^2(0,0)(\int_{\mathbb{R}} w(y)\,dy)^2} + O(\epsilon).$$

Summarizing these results, $(\alpha_\epsilon, \xi_\epsilon)$ solve the system

$$\alpha_\epsilon = \frac{a_2}{\xi_\epsilon^2 G_{D_2}^2(0,0)(\int_{\mathbb{R}} w(y)\,dy)^2 - a_2} + O(\epsilon), \tag{3.8}$$

$$|\Omega| = a_1 \xi_\epsilon \int_{\mathbb{R}} w^2(y)\,dy\,\sqrt{1+\alpha_\epsilon} + O(\epsilon). \tag{3.9}$$

Using

$$\int_{\mathbb{R}} w(y)^2\,dy = \int_{\mathbb{R}} w(y)\,dy = 6,$$

the system (3.8), (3.9) can be rewritten as a quadratic equation in ξ_ϵ^2

$$36^2 a_1^2 G_{D_2}(0,0)^2 \xi_\epsilon^4 - 36 G_{D_2}(0,0)^2 \xi_\epsilon^2 |\Omega|^2 + a_2 |\Omega|^2 = O(\epsilon)$$

which has the two solutions

$$\left(\xi_\epsilon^{s,l}\right)^2 = \frac{|\Omega|^2 \pm \sqrt{|\Omega|^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}(0,0)^{-2}}}{72 a_1^2} + O(\epsilon) \tag{3.10}$$

under the condition

$$a_1^2 a_2 < \frac{|\Omega|^2}{4} G_{D_2}(0,0)^2.$$

The last condition states that, all other constants being equal, the combination $a_1^2 a_2$ must be small enough.

This implies that under the condition

$$a_1^2\, a_2 < \frac{|\Omega|^2}{4} G_{D_2}(0,0)^2 - \delta_0 \quad \text{for some } \delta_0 > 0$$

there are two solutions for ξ_ϵ which satisfy

$$0 < \xi_\epsilon^s < \frac{|\Omega|^2}{72 a_1^2} < \xi_\epsilon^l.$$

On the other hand, if

$$a_1^2\, a_2 > \frac{|\Omega|^2}{4} G_{D_2}(0,0)^2 + \delta_0 \quad \text{for some } \delta_0 > 0,$$

then there are no such solutions.

Resulting from the two solutions ξ_ϵ^s and ξ_ϵ^l there are also two solutions for α_ϵ^s and α_ϵ^l which are computed from (3.8).

Now we show that

$$\alpha_\epsilon^l < 1 \text{ and } \alpha_\epsilon^s > 1. \tag{3.11}$$

Substituting (2.10) and (2.9) in (3.8), we get

$$\alpha_\epsilon = \frac{2a_1^2 a_2}{|\Omega|^2 G_{D_2}(0,0)^2 \pm \sqrt{|\Omega|^4 G_{D_2}(0,0)^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}(0,0)^2} - 2a_1^2 a_2}$$

Thus it remains to show that

$$|\Omega|^2 G_{D_2}(0,0)^2 - 4a_1^2 a_2 < \sqrt{|\Omega|^4 G_{D_2}(0,0)^4 - 4a_1^2 a_2 |\Omega|^2 G_{D_2}(0,0)^2}$$

which follows easily after taking squares on both sides.

Finally, this results in the two single-spike solutions $(S^s_\epsilon,\ u^s_{1,\epsilon}, u^s_{2,\epsilon})$ and $(S^l_\epsilon,\ u^l_{1,\epsilon}, u^l_{2,\epsilon})$ of (2.1). In the next section we will rigorously prove the existence of these two solutions.

4 Existence II: Rigorous Proofs

In this section we show the existence of solutions of (2.1) for which the central component has a spike. As we have shown in the previous section, there are two such solutions, $(S^s_\epsilon,\ u^s_{1,\epsilon}, u^s_{2,\epsilon})$ and $(S^l_\epsilon,\ u^l_{1,\epsilon}, u^l_{2,\epsilon})$ which differ by the size of the amplitude. The existence proof applies to both of them. Therefore we will not write the superscripts ^{s} and ^{l} in this section.

The second component of the approximate spike solution introduced in (3.1) is given by

$$\tilde{u}_{1,\epsilon}(x) = \xi_\epsilon\, w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right)\chi(|x|) + O(\epsilon),$$

where ξ_ϵ and α_ϵ have been computed to leading order in (3.8), (3.10), and χ has been introduced in (3.2).

Further, $\tilde{S}_\epsilon$ and $\tilde{u}_{2,\epsilon}$ solve a partial differential equation which depends on $\tilde{u}_{1,\epsilon}$ only. Therefore we denote $\tilde{S}_\epsilon = T_1[\tilde{u}_{1,\epsilon}]$ and $\tilde{u}_{2,\epsilon} = T_2[\tilde{u}_{1,\epsilon}]$, respectively. We insert this approximate spike solution into (2.1) and compute its error.

The LHS of the second equation in (2.1) at $(\tilde{S}_\epsilon,\ \tilde{u}_{1,\epsilon},\ \tilde{u}_{2,\epsilon}) = (T_1[\tilde{u}_{1,\epsilon}],\ \tilde{u}_{1,\epsilon},\ T_2[\tilde{u}_{1,\epsilon}])$ is calculated as follows:

$$\begin{aligned}\Delta_y\tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon\tilde{u}^2_{1,\epsilon} - a_2\tilde{u}_{1,\epsilon}\tilde{u}^2_{2,\epsilon} &= \Delta_y\tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon(0)\tilde{u}^2_{1,\epsilon} - a_2\tilde{u}_{1,\epsilon}\tilde{u}^2_{2,\epsilon}(0)\\ &\quad + \big[\tilde{S}_\epsilon - \tilde{S}_\epsilon(0)\big]\tilde{u}^2_{1,\epsilon} - a_2\tilde{u}_{1,\epsilon}2\big(\tilde{u}_{2,\epsilon}\\ &\quad - \tilde{u}_{2,\epsilon}(0)\big)\tilde{u}_{2,\epsilon}(0) + O\big(\epsilon^2\big)\\ &=: E_1 + E_2 + E_3 + O\big(\epsilon^2\big)\end{aligned}$$

in $L^2(\Omega_\epsilon)$, where $\Omega_\epsilon = \left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right)$.

We compute

$$E_1 = O(\epsilon)$$

by the definition of ξ_ϵ and α_ϵ in (3.4) and (3.5). Computing $\tilde{S}_\epsilon(x)$, using the Green's function G_{D_1} defined in (7.1), we derive the following estimate:

$$\begin{aligned}E_2 &= \big[\tilde{S}_\epsilon(\epsilon y) - \tilde{S}_\epsilon(0)\big]\tilde{u}^2_{1,\epsilon}(\epsilon y)\\ &= -\tilde{u}^2_{1,\epsilon}(\epsilon y)a_1\int_{-1/\epsilon}^{1/\epsilon}\big[G_{D_1}(\epsilon y,\epsilon z) - G_D(0,\epsilon z)\big]\,\tilde{S}_\epsilon(z)\tilde{u}^2_{1,\epsilon}(z)\,dz\,\big(1+O(\epsilon)\big)\\ &= a_1\frac{\tilde{u}^2_{1,\epsilon}(\epsilon y)}{\tilde{S}_\epsilon(0)}\epsilon\big(1+\alpha_\epsilon\big)^2\int_{\mathbb{R}}\left(\frac{1}{2D_1}|y-z| - \frac{1}{2D_1}|z|\right)w^2\big(z\sqrt{1+\alpha_\epsilon}\big)\,dz\big(1+O\left(\epsilon|y|\right)\big)\\ &\quad + a_1\big(1+\alpha_\epsilon\big)^{3/2}\frac{\tilde{u}^2_{1,\epsilon}(\epsilon y)}{\tilde{S}_\epsilon(0)}\epsilon^2y^2H_{D_1,xx}(0,0)6\big(1+O\left(\epsilon|y|\right)\big) = O\big(\epsilon|y|\big)\tilde{u}^2_{1,\epsilon}.\end{aligned}$$

Thus we have

$$E_2 = O(\epsilon)\quad \text{in } L^2(\Omega_\epsilon).$$

Here we have used that $H_{D_1,x}(0,0) = 0$ by (7.4).

Similarly, we compute

$$\begin{aligned}
E_3 &= -a_2 \tilde{u}_{1,\epsilon}(\epsilon y) 2\big(\tilde{u}_{2,\epsilon}(\epsilon y) - \tilde{u}_{2,\epsilon}(0)\big)\tilde{u}_{2,\epsilon}(0) \\
&= 2a_2 \tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}^3(0) \int_{-1/\epsilon}^{1/\epsilon} \big[G_{D_2}(\epsilon y, \epsilon z) \\
&\quad -G_D(0,\epsilon z)\big]\tilde{u}_{1,\epsilon}(\epsilon z)\, dz \,\big(1 + O(\epsilon)\big) \\
&= 2\alpha_\epsilon(1+\alpha_\epsilon)\tilde{u}_{1,\epsilon}(\epsilon y)\frac{\tilde{u}_{2,\epsilon}(0)}{\tilde{S}_\epsilon(0)} \int_{\mathbb{R}} \Big(K_{D_2}\big(\epsilon|y-z|\big) \\
&\quad -K_{D_2}\big(\epsilon|z|\big)\Big)\, w(z\sqrt{1+\alpha_\epsilon})\, dz\, \big(1 + O(\epsilon|y|)\big) \\
&\quad -2\alpha_\epsilon\sqrt{1+\alpha_\epsilon}\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}(0)\epsilon^2 y^2 H_{D_2,xx}(0,0)\left(\int_{\mathbb{R}} w\, dy\right)\big(1 + O(\epsilon|y|)\big) \\
&= -2\alpha_\epsilon(1+\alpha_\epsilon)\tilde{u}_{1,\epsilon}(\epsilon y)\frac{\tilde{u}_{2,\epsilon}(0)}{\tilde{S}_\epsilon(0)} \int_{\mathbb{R}} \Big(K_{D_2}\big(\epsilon|y-z|\big) \\
&\quad -K_{D_2}\big(\epsilon|z|\big)\Big)\, w\big(z\sqrt{1+\alpha_\epsilon}\big)\, dz\, (1 + O(\epsilon|y|)) \\
&\quad -2\alpha_\epsilon\sqrt{1+\alpha_\epsilon}\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}(0)\epsilon^2 y^2\, H_{D_2,xx}(0,0)6\,(1 + O(\epsilon|y|)) \\
&= O(\epsilon|y|)\tilde{u}_{1,\epsilon}.
\end{aligned}$$

Thus we have

$$E_3 = O(\epsilon) \quad \text{in } L^2(\Omega_\epsilon).$$

By definition, the first and third equations of (2.1) are solved exactly and so do not contribute to the error.

Writing the system (2.1) in the form $R_\epsilon(S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon}) = 0$, we have now shown the estimate

$$\Big\|R_\epsilon\big(T_1[\tilde{u}_{1,\epsilon}], \tilde{u}_{1,\epsilon}, T_2[\tilde{u}_{1,\epsilon}]\big)\Big\|_{L^2(\Omega_\epsilon)} = O\,(\epsilon)\,. \tag{4.1}$$

Next, we investigate the linearized operator $\tilde{\mathcal{L}}_\epsilon$ around the approximate solution $(\tilde{S}_\epsilon,\ \tilde{u}_{\epsilon,1},\ \tilde{u}_{\epsilon,2})$. It is defined as follows:

$$\tilde{\mathcal{L}}_\epsilon\, : H_N^2(\Omega) \times H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega) \times L^2(\Omega_\epsilon) \times L^2(\Omega),$$

$$\tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \psi_{1,\epsilon} \\ \phi_\epsilon \\ \psi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} D_1\Delta\psi_{1,\epsilon} - 2\frac{a_1}{\epsilon}\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon - \frac{a_1}{\epsilon}\psi_{1,\epsilon}\tilde{u}_{1,\epsilon}^2 \\ \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon + \psi_{1,\epsilon}\tilde{u}_{1,\epsilon}^2 - a_2\phi_\epsilon\tilde{u}_{2,\epsilon}^2 - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\psi_{2,\epsilon} \\ D_2\Delta\psi_{2,\epsilon} - \psi_{2,\epsilon} + \frac{1}{\epsilon}\phi_\epsilon\tilde{u}_{2,\epsilon}^2 + \frac{2}{\epsilon}\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\psi_{2,\epsilon} \end{pmatrix}. \tag{4.2}$$

We will show this operator will lead to a uniformly invertible one for ϵ small enough.

To study the kernel of $\tilde{\mathcal{L}}_\epsilon$, we first solve its first and third components. Therefore, we have $\psi_{1,\epsilon} = T_1'[\tilde{u}_{1,\epsilon}]\phi_\epsilon$ and $\psi_{2,\epsilon} = T_2'[\tilde{u}_{1,\epsilon}]\phi_\epsilon$, where $T_1'[\tilde{u}_{1,\epsilon}]$ and $T_2'[\tilde{u}_{1,\epsilon}]$ are linearized operators which can be expressed by the Green's functions G_{D_1} and G_{D_2} defined in (7.1) and (7.18), respectively. Substituting these expressions into $\tilde{\mathcal{L}}_\epsilon$, the first and third components vanish and it only remains to consider the second component. We obtain the following operator:

$$\begin{aligned}\bar{\mathcal{L}}_\epsilon \ &: \ H_2^N(\Omega_\epsilon) \to L_2(\Omega_\epsilon),\\ \bar{\mathcal{L}}_\epsilon(\phi_\epsilon) \ &= \ \Delta_y\phi_\epsilon - \phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_\epsilon + \Big(T_1'[\tilde{u}_{1,\epsilon}]\phi_\epsilon\Big)\tilde{u}_{1,\epsilon}^2 - a_2\phi_\epsilon\tilde{u}_{2,\epsilon}^2\\ &\quad -2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\Big(T_2'[\tilde{u}_{2,\epsilon}]\phi_\epsilon\Big). \end{aligned} \tag{4.3}$$

In order to introduce a uniformly invertible operator, we define approximate kernel and co-kernel as

$$\mathcal{K}_\epsilon = \text{span}\big\{\tilde{u}_{1,\epsilon}'\big\} \subset H_N^2(\Omega_\epsilon),$$
$$\mathcal{C}_\epsilon = \text{span}\big\{\tilde{u}_{1,\epsilon}'\big\} \subset L^2(\Omega_\epsilon).$$

Then the linear operator $\mathcal{L}_\epsilon$ is defined by

$$\begin{aligned}&\mathcal{L}_\epsilon \ : \mathcal{K}_\epsilon^\perp \to \mathcal{C}_\epsilon^\perp,\\ &\mathcal{L}_\epsilon = \pi \circ \bar{\mathcal{L}}_\epsilon\big(\phi_\epsilon\big)\end{aligned} \tag{4.4}$$

where $\mathcal{K}_\epsilon^\perp$ and $\mathcal{C}_\epsilon^\perp$ denote the orthogonal complement with the scalar product of $L^2(\Omega_\epsilon)$ to $\mathcal{K}_\epsilon$ and $\mathcal{C}_\epsilon$, respectively, and π is the L^2-projection from $L^2(\Omega_\epsilon)$ into $\mathcal{C}_\epsilon^\perp$.

We will show that this operator is uniformly invertible for ϵ small enough. In fact, we have the following result:

Proposition 4.1 *There exist positive constants $\bar{\epsilon}$, λ such that for all $\epsilon \in (0, \bar{\epsilon})$,*

$$\|\mathcal{L}_\epsilon\phi\|_{L_2(\Omega_\epsilon)} \geq \lambda\|\phi\|_{H^2(\Omega_\epsilon)} \quad \text{for all } \phi \in \mathcal{K}_\epsilon^\perp. \tag{4.5}$$

Further, the linear operator $\mathcal{L}_\epsilon$ is surjective.

Proof of Proposition 4.1: We give an indirect proof. Suppose (4.5) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\phi^k\}$ with $\epsilon_k \to 0$, $\phi^k = \phi_{\epsilon_k}$, $k = 1, 2, \ldots$ such that

$$\big\|\mathcal{L}_{\epsilon_k}\phi^k\big\|_{L^2(\Omega_\epsilon)} \to 0 \quad \text{as } k \to \infty, \tag{4.6}$$
$$\big\|\phi^k\big\|_{H^2(\Omega_\epsilon)} = 1, \qquad k = 1, 2, \ldots. \tag{4.7}$$

By using the cut-off function χ defined in (3.2), we define the following functions:

$$\begin{aligned}\phi_{1,\epsilon}(y) &= \phi_\epsilon(y)\chi\big(|x|\big), && y \in \Omega_\epsilon.\\ \phi_{2,\epsilon}(y) &= \phi_\epsilon(y)\big(1 - \chi(|x|)\big), && y \in \Omega_\epsilon.\end{aligned} \tag{4.8}$$

At first the functions $\phi_{1,\epsilon}$, $\phi_{2,\epsilon}$ are only defined in Ω_ϵ. However, by a standard extension result, $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ can be extended to $\mathbb{R}$ such that the norms of $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ in $H^2(\mathbb{R})$ are bounded by a constant independent of ϵ for all ϵ small enough. In the following we shall study this extension. For simplicity, we use the same notation for the extension. Since for $i = 1, 2$ each sequence $\{\phi_i^k\} := \{\phi_{i,\epsilon_k}\}$ $(k = 1, 2, \ldots)$ is bounded in $H_{loc}^2(\mathbb{R})$ it has a weak limit in $H_{loc}^2(\mathbb{R})$, and therefore also a strong limit in $L_{loc}^2(\mathbb{R})$ and $L_{loc}^\infty(\mathbb{R})$. We call these limits ϕ_i.

Taking the limit $\epsilon \to 0$ in (4.4), then $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ satisfies

$$\int_{\mathbb{R}} \phi_1 w_y \, dy = 0 \tag{4.9}$$

and it solves the system

$$\mathcal{L}\phi_1 = 0, \tag{4.10}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\phi_1 = \Delta_y\phi_1 - (1+\alpha)\phi_1 + 2(1+\alpha)w\phi_1 - 2(1+\alpha)\frac{\int_{\mathbb{R}} w\phi_1\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2 + 2\alpha\frac{\int_{\mathbb{R}}\phi_1\,dy}{\int_{\mathbb{R}} w\,dy}w.$$

In Lemma 5.1 below we will show that the system (4.9), (4.10) has only the solution $\phi_1 = 0$ in $\mathbb{R}$.

Further, trivially, $\phi_2 = 0$ in $\mathbb{R}$.

By standard elliptic estimates we get $\|\phi_{i,\epsilon_k}\|_{H^2(\mathbb{R})} \to 0$ for $i = 1,\ 2$ as $k \to \infty$.

This contradicts the assumption that $\|\phi^k\|_{H^2(\Omega_\epsilon)} = 1$.

To complete the proof of Proposition 4.1, we need to show that the adjoint operator of $\mathcal{L}_\epsilon$ (denoted by $\mathcal{L}_\epsilon^*$) is injective from $\mathcal{K}_\epsilon{}^\perp$ to $\mathcal{C}_\epsilon{}^\perp$. We first pass to the limit $\epsilon \to 0$ for the adjoint operator $\mathcal{L}_\epsilon^*$. This limiting process follows along the same lines as for $\mathcal{L}_\epsilon$ and is therefore omitted. Then we have to show that the limiting adjoint operator $\mathcal{L}^*$ has only the trivial kernel. This will be done in Lemma 5.2 below. □

Finally, we solve the system (2.1). It can be written as

$$R_\epsilon\big(\tilde{S}_\epsilon + \psi_1, \tilde{u}_{1,\epsilon} + \phi, \tilde{u}_{2,\epsilon} + \psi_2\big) = R_\epsilon\big(U_\epsilon + \Phi\big) = 0, \tag{4.11}$$

where $U_\epsilon = \left(\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon}\right)$, $\Phi = (\psi_1, \phi, \psi_2)$. Since $\mathcal{L}_\epsilon$ is uniformly invertible if ϵ is small enough, we can write (4.11) in function space with even Φ as

$$\Phi = -\mathcal{L}_\epsilon^{-1}R_\epsilon\big(U_\epsilon\big) - \mathcal{L}_\epsilon^{-1}N_\epsilon(\Phi) =: M_\epsilon(\Phi), \tag{4.12}$$

where $\mathcal{L}_\epsilon^{-1}$ is the inverse of $\mathcal{L}_\epsilon$ and

$$N_\epsilon(\Phi) = R_\epsilon\big(U_\epsilon + \Phi\big) - R_\epsilon\big(U_\epsilon\big) - R_\epsilon'\big(U_\epsilon\big)\Phi. \tag{4.13}$$

Note that the operator M_ϵ defined by (4.12) is a mapping from $H_N^2(\Omega) \times H_N^2(\Omega_\epsilon) \times H_N^2(\Omega)$ into itself. We are going to show that the operator M_ϵ is a contraction on

$$B_{\epsilon,\delta} \equiv \left\{\Phi \in H_N^2(\Omega) \times H_N^2(\Omega_\epsilon) \times H_N^2(\Omega)\ :\ \Phi \text{ even},\ \|\Phi\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} < \delta\right\}$$

if ϵ is small enough and δ is suitably chosen. By (4.1) and Proposition 4.1, we have

$$\begin{aligned}\|M_\epsilon(\Phi)\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} &\le \lambda^{-1}\Big(\|N_\epsilon(\Phi)\|_{L^2(\Omega)\times L^2(\Omega_\epsilon)\times L^2(\Omega)} \\ &\quad + \|R_\epsilon(U_\epsilon)\|_{L^2(\Omega)\times L^2(\Omega_\epsilon)\times L^2(\Omega)}\Big) \\ &\le \lambda^{-1}C_0\big(c(\delta)\delta + \epsilon\big),\end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$, $\epsilon > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly, we show

$$\|M_\epsilon(\Phi_1) - M_\epsilon(\Phi_2)\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} \le \lambda^{-1}C_0\big(c(\delta)\delta\big)\|\Phi_1 - \Phi_2\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)},$$

where $c(\delta) \to 0$ as $\delta \to 0$. Choosing $\delta = C_1\epsilon$ for $\lambda^{-1}C_0 < C_1$ and taking ϵ small enough, then M_ϵ maps from $B_{\epsilon,\delta}$ into $B_{\epsilon,\delta}$ and it is a contraction mapping in $B_{\epsilon,\delta}$. The existence of a fixed point $\Phi_\epsilon \in B_{\epsilon,\delta}$ now follows from the standard contraction mapping principle, and Φ_ϵ is a solution of (4.12).

We have thus proved

Lemma 4.1 *There exists $\bar{\epsilon} > 0$ such that for every ϵ with $0 < \epsilon < \bar{\epsilon}$ there is an even $\Phi_\epsilon \in H^2_N(\Omega) \times H^2(\Omega_\epsilon) \times H^2_N(\Omega)$ satisfying $R_\epsilon(U_\epsilon + \Phi_\epsilon) = 0$. Furthermore, we have the estimate*

$$\|\Phi_\epsilon\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} \le C\epsilon. \tag{4.14}$$

In this section we have constructed two exact spikes solution of the form $U_\epsilon + \Phi_\epsilon = (S_\epsilon, u_{\epsilon,1}, u_{\epsilon,2})$. We are now going to study their stability.

5 Stability I: Derivation, Rigorous Deduction and Analysis of a NLEP

We study a small perturbation of a single-spike steady state $(S_\epsilon,\ u_{\epsilon,1},\ u_{\epsilon,2})$ which could be either the small-amplitude solution $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ or the large-amplitude solution $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$.

We linearize (1.1) around the single-spike solution we derive in leading order $S_\epsilon + \psi_{1,\epsilon} e^{\lambda_\epsilon t}$, $u_{\epsilon,1} + \phi_\epsilon e^{\lambda_\epsilon t}$, $u_{\epsilon,2} + \psi_{2,\epsilon} e^{\lambda_\epsilon t}$, where the three perturbations $\psi_{1,\epsilon} \in H^2_N(\Omega)$, $\phi_\epsilon \in H^2_N(\Omega_\epsilon)$ $\psi_{2,\epsilon} \in H^2_N(\Omega)$ are small in their respective norms Then the perturbations in leading order satisfy the eigenvalue problem

$$\tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \psi_{1,\epsilon} \\ \phi_\epsilon \\ \psi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} \tau\lambda_\epsilon \psi_{1,\epsilon} \\ \lambda_\epsilon \phi_\epsilon \\ \tau_1 \lambda_\epsilon \psi_{2,\epsilon} \end{pmatrix}, \tag{5.1}$$

where $\tilde{\mathcal{L}}_\epsilon$ denotes the linearized operator around the steady state steady state $(S_\epsilon,\ u_{\epsilon,1},\ u_{\epsilon,2})$ which has been defined in (4.4) and has the domain $H^2_N(\Omega) \times H^2_N(\Omega_\epsilon) \times H^2_N(\Omega)$. Here we have $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers.

We say that a spike solution is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in a left half plane $\{\lambda \in \mathbb{C} :\ \mathrm{Re}(\lambda) \le -c_0\}$ for some $c_0 > 0$. A spike solution is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\mathrm{Re}(\lambda_\epsilon) > 0$.

We first consider the case $\tau = 0$ and $\tau_1 = 0$ and show stability. Then we study the stability for $\tau \ge 0$ small or $\tau_1 \ge 0$ small. We will show that for $\tau \ge 0$ small and $\tau_1 = 0$ we still have stability, but for $\tau \ge 0$ small and $0 < \epsilon \ll \tau_1 \ll 1$ the solution will be unstable.

Writing down $\tilde{\mathcal{L}}_\epsilon$ explicitly and expressing $\psi_{i,\epsilon} = T'_i[u_{i,\epsilon}]\phi_\epsilon$, $i = 1, 2$, using Green's functions G_{D_i} defined in (7.1) and (7.18), respectively, we can rewrite (5.1) as

$$\epsilon^2\phi_{\epsilon.xx} - \phi_\epsilon + 2S_\epsilon u_{1,\epsilon}\phi_\epsilon + \big(T'_1[u_{1,\epsilon}]\phi_\epsilon\big)u^2_{1,\epsilon} - a_2\phi_\epsilon u^2_{2,\epsilon} - 2a_2 u_{1,\epsilon}u_{2,\epsilon}\big(T'_2[u_{2,\epsilon}]\phi_\epsilon\big) = \lambda_\epsilon\phi_\epsilon. \tag{5.2}$$

Then, arguing as in the proof of Proposition 4.1, a subsequence of the sequence ϕ_ϵ converges to a limit which we denote by ϕ. Next we derive an eigenvalue problem for ϕ.

Integrating the first equation of (5.1), we get

$$\psi_{1,\epsilon}(0)\int_{-1}^{1} u^2_{1,\epsilon}\,dx = -2S_\epsilon(0)\int_{-1}^{1} u_{1,\epsilon}\phi_\epsilon\,dx + O(\epsilon)$$

which implies

$$\psi_{1,\epsilon}(0) = -\frac{2S_\epsilon(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}\big(1 + O(\epsilon)\big) \tag{5.3}$$

This gives

$$\psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = -2\frac{S_\epsilon(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}\xi_\epsilon^2 w^2\left(1+O(\epsilon)\right)$$
$$= -2(1+\alpha_\epsilon)\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2\left(1+O(\epsilon)\right) \quad \text{in } H^2\big(\Omega_\epsilon\big),$$

using (3.5). We also derive from (2.11) that

$$u_{2,\epsilon}(0) = \frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)6\xi_\epsilon} + O(\epsilon)$$

and compute

$$\psi_{2,\epsilon}(0) = G_{D_2}(0,0)\left[u_{2,\epsilon}^2(0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}}\phi\,dy\right.$$
$$\left.+2\psi_{2,\epsilon}(0)u_{2,\epsilon}(0)\frac{\xi_\epsilon}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}} w\,dy\right]\left(1+O(\epsilon)\right)$$
$$= u_{2,\epsilon}(0)G_{D_2}(0,0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\left[u_{2,\epsilon}(0)\int_{\mathbb{R}}\phi\,dy + 2\psi_{2,\epsilon}(0)\xi_\epsilon\int_{\mathbb{R}} w\,dy\right]$$

which implies

$$\psi_{2,\epsilon}(0) = -\frac{u_{2,\epsilon}(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}\left(1+O(\epsilon)\right).$$

Finally, we get

$$\psi_{2,\epsilon}(0) = -\frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)6\xi_\epsilon^2}\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}\left(1+O(\epsilon)\right). \tag{5.4}$$

Therefore, we compute

$$-a_2u_{1,\epsilon}2u_{2,\epsilon}\psi_{2,\epsilon} = -a_2u_{1,\epsilon}2u_{2,\epsilon}(0)\psi_{2,\epsilon}(0)\left(1+O(\epsilon)\right)$$
$$= -2\alpha_\epsilon w\frac{\xi_\epsilon\psi_{2,\epsilon}(0)}{u_{2,\epsilon}(0)}\left(1+O(\epsilon)\right)$$
$$= +2\alpha_\epsilon\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}w\left(1+O(\epsilon)\right) \quad \text{in } H^2(\Omega_\epsilon),$$

using (3.4).

Putting all these expressions into (5.2) and taking the limit $\epsilon \to 0$, we derive the NLEP

$$\mathcal{L}\phi = \Delta_y\phi - (1+\alpha)\phi + 2(1+\alpha)w\phi - 2(1+\alpha)\frac{\int_{\mathbb{R}} w\phi\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2 + 2\alpha\frac{\int_{\mathbb{R}}\phi\,dy}{\int_{\mathbb{R}} w\,dy}w = \lambda\phi, \tag{5.5}$$

where $\alpha = \lim_{\epsilon\to 0}\alpha_\epsilon$.

Although this derivation has been only made formally, we can rigorously prove the following separation of eigenvalues.

Theorem 5.1 *Let λ_ϵ be an eigenvalue of (5.2) for which $Re(\lambda_\epsilon) > -a_0$ for some suitable constant a_0 fixed independent of ϵ.*

(1) *Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the NLEP given in (5.5).*

(2) *Let* $\lambda_0 \neq 0$ *be an eigenvalue of the NLEP given in* (5.5). *Then for all* ϵ *sufficiently small, there is an eigenvalue* λ_ϵ *of* (5.2) *with* $\lambda_\epsilon \to \lambda_0$ *as* $\epsilon \to 0$.

Remark From Theorem 5.1 we see rigorously that the eigenvalue problem (5.2) is reduced to the study of the NLEP (5.5).

Now we prove Theorem 5.1.

Proof of Theorem 5.1: Part (1) follows by an asymptotic analysis combined with passing to the limit as $\epsilon \to 0$ which is similar to the proof of Proposition 4.1.

Part (2) follows from a compactness argument by Dancer introduced in Sect. 2 of [3]. It was applied in [38] to a related situation, therefore we omit the details.

The stability or instability of the large eigenvalues follows from the following results:

Theorem 5.2 [31] *Consider the nonlocal eigenvalue problem*

$$\phi'' - \phi + 2w\phi - \gamma \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \alpha\phi. \tag{5.6}$$

(1) *If* $\gamma < 1$, *then there is a positive eigenvalue to* (5.6).
(2) *If* $\gamma > 1$, *then for any nonzero eigenvalue* λ *of* (5.6), *we have*

$$Re(\lambda) \leq -c_0 < 0.$$

(3) *If* $\gamma \neq 1$ *and* $\lambda = 0$, *then* $\phi = c_0 w'$ *for some constant* c_0.

In our applications to the case when $\tau > 0$ or $\tau_1 > 0$, we need to handle the situation when the coefficient γ is a complex function of $\tau\lambda$. Let us suppose that

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\lambda)| \leq C \ \text{ for } \lambda_R \geq 0, \ \tau \geq 0, \tag{5.7}$$

where C is a generic constant independent of τ, λ. Then we have

Theorem 5.3 (Theorem 3.2 of [38])
Consider the nonlocal eigenvalue problem

$$\phi'' - \phi + 2w\phi - \gamma(\tau\lambda) \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \lambda\phi, \tag{5.8}$$

where $\gamma(\tau\lambda)$ *satisfies* (5.7). *Then there exists* $\tau_0 > 0$ *such that for all* $0 \leq \tau < \tau_0$,

(1) if $\gamma(0) < 1$, *then there is a positive eigenvalue to* (5.8);
(2) if $\gamma(0) > 1$, *then for any nonzero eigenvalue* λ *of* (5.8), *we have*

$$Re(\lambda) \leq -c_0 < 0.$$

Now we consider the stability of the eigenvalue problem (5.5).

Lemma 5.1 (1) *If* $\alpha < 1$, *the eigenvalue problem* (5.5) *has only stable eigenvalues, i.e. for any nonzero eigenvalue of (5.5), we have*

$$Re(\lambda) \leq -c_0 < 0.$$

If $\alpha > 1$, *the eigenvalue problem* (5.5) *has an eigenvalue with* $Re(\lambda) > 0$.
(2) *If* $\alpha \neq 1$ *and* $\lambda = 0$, *then* $\phi = c_0 w'$ *for some constant* c_0.

Proof of Lemma 5.1
Proof of (1): Integrating (5.5), we derive

$$(\lambda + 1 - \alpha) \int_{\mathbb{R}} \phi \, dy = 0.$$

Then for all the eigenvalues we have (i) $\lambda + 1 - \alpha = 0$ or the corresponding eigenfunction satisfies (ii) $\int_{\mathbb{R}} \phi \, dy = 0$.

Let us first consider case (i). If $\alpha < 1$ then (i) implies that $\lambda < 0$ and this eigenvalue λ is stable for (5.5). If $\alpha > 1$, then we construct an eigenfunction ϕ with eigenvalue $\lambda = \alpha - 1 > 0$ as follows and the eigenvalue problem (5.5) is unstable: first we set

$$\phi = (L + 1 - \alpha)^{-1} \left[c_1 w^2 + c_2 w \right], \tag{5.9}$$

where

$$\begin{aligned} L \; &: K^{\perp} \to C^{\perp}, \quad L\phi := \Delta\phi - (1+\alpha)\phi + 2(1+\alpha)w\phi, \\ K^{\perp} &= \left\{ v \in H^2(\mathbb{R}) : \int v w_y \, dy = 0 \right\}, \quad C^{\perp} = \left\{ v \in L^2(\mathbb{R}) : \int v w_y \, dy = 0 \right\}, \\ c_1 &= \frac{2(1+\alpha) \int_{\mathbb{R}} w\phi \, dy}{\int_{\mathbb{R}} w^2 \, dy}, \quad c_2 = -\frac{2\alpha \int_{\mathbb{R}} \phi \, dy}{\int_{\mathbb{R}} w \, dy}. \end{aligned}$$

Then we multiply (5.9) by w and 1, respectively, and integrating we get a linear system for the coefficients ($\int_{\mathbb{R}} w\phi \, dy$, $\int_{\mathbb{R}} \phi \, dy$) which has a unique nontrivial solution. Solving this system, using the identities

$$Lw = (1+\alpha)w^2, \quad L\left(\frac{y\sqrt{\alpha+1}}{2} w_y + w \right) = (1+\alpha)w,$$

we can eliminate ϕ in the definitions of c_1 and c_2. We finally get

$$\begin{aligned} c_1 &= \int_{\mathbb{R}} w \big(L + 1 - \alpha \big)^{-1} w \, dy, \\ c_2 &= -\int_{\mathbb{R}} w \big(L + 1 - \alpha \big)^{-1} w^2 \, dy + \frac{3}{1-\alpha}. \end{aligned}$$

Thus the eigenvalue problem is unstable for $\alpha > 1$.

Next we consider case (ii). Rescaling the spatial variable, NLEP (5.5) reduces to the familiar NLEP considered in Theorem 5.2 with $\gamma = 2$ which implies that the real parts of all eigenvalues are strictly negative and we have stability.

Proof of (2) Integrating (5.5), we derive

$$\int_{\mathbb{R}} \phi \, dy = 0.$$

Rescaling the spatial variable, NLEP (5.5) reduces to the familiar NLEP considered in Theorem 5.2 with $\gamma = 2$ and we derive $\phi = c_0 w'$ for some constant c_0. □

Proof of Theorem 2.2 By (3.11) we have $\alpha_\epsilon^l < 1$ and $\alpha_\epsilon^s > 1$. Then the theorem follows by combining the results of Theorem 5.1 and Lemma 5.1. □

We also need to consider the adjoint operator $\mathcal{L}_\epsilon^*$ to the linear operator $\mathcal{L}_\epsilon$. Expressing $\mathcal{L}_\epsilon^*$ explicitly, we can rewrite the adjoint eigenvalue problem as follows:

$$\begin{cases} D_1 \Delta \psi_{1,\epsilon} + \frac{1}{\epsilon}(\phi_\epsilon - a_1 \psi_{1,\epsilon}) u_{1,\epsilon}^2 = \tau \lambda_\epsilon \psi_{1,\epsilon}, \\ \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + 2 S_\epsilon u_{1,\epsilon} \big(\phi_\epsilon - a_1 \psi_{1,\epsilon}\big) + \big(\psi_{2,\epsilon} - a_2 \phi_\epsilon\big) u_{2,\epsilon}^2 = \lambda_\epsilon \phi_\epsilon, \\ D_2 \Delta \psi_{2,\epsilon} - \psi_{2,\epsilon} + \frac{2}{\epsilon} u_{1,\epsilon} u_{2,\epsilon} \big(\psi_{2,\epsilon} - a_2 \phi_\epsilon\big) = \tau_1 \lambda_\epsilon \psi_{2,\epsilon}. \end{cases} \tag{5.10}$$

We need to consider the kernel of this adjoint eigenvalue problem. (In the proof of Proposition 4.1 we need the result that this kernel is trivial.) Taking the limit $\epsilon \to 0$ as in the proof of Proposition 4.1, we derive the following nonlocal linear operator which is the adjoint operator of (5.5):

$$\mathcal{L}^* \phi = \Delta_y \phi - (1+\alpha)\phi + 2(1+\alpha) w \phi - 2(1+\alpha) \frac{\int_{\mathbb{R}} w^2 \phi \, dy}{\int_{\mathbb{R}} w^2 \, dy} w + 2\alpha \frac{\int_{\mathbb{R}} w \phi \, dy}{\int_{\mathbb{R}} w \, dy} = 0. \tag{5.11}$$

We are now going to show the following Lemma:

Lemma 5.2 *The kernel of the operator* (5.11) *is trivial.*

Proof of Lemma 5.2: Integrating (5.11), we derive $\int_{\mathbb{R}} w \phi \, dy = 0$ since otherwise there is an unbounded term. Further, we get the relation

$$\int_{\mathbb{R}} \phi \, dy + 2 \int_{\mathbb{R}} w^2 \phi \, dy = 0. \tag{5.12}$$

Multiplying (5.11) by w and integrating, we derive

$$\int_{\mathbb{R}} w^2 \phi \, dy = 0. \tag{5.13}$$

Then from (5.12) we get $\int_{\mathbb{R}} \phi \, dy = 0$. Finally, going back to (5.11), all the nonlocal terms vanish and by Theorem 5.2 in the special case $\gamma = 0$ we derive $\phi = c_0 w'$ for some constant c_0. Thus the kernel of $\mathcal{L}^*$ is trivial. □

Now we extend the consideration of the stability problem for the linearized operator to the conditions $\tau \geq 0$ or $\tau_1 \geq 0$ and prove Corollary 2.1.

Proof of Corollary 2.1 To emphasize the possible different behaviors if $\tau \geq 0$ or $\tau_1 \geq 0$, we consider the cases separately:

Proof of (1): $0 \leq \tau \leq \tau_0$ for some $\tau_0 > 0$ and $\tau_1 = 0$.

We first compute, using (3.5), (7.8),

$$\begin{aligned} \psi_{1,\epsilon}(0) &= -\frac{a_1}{\epsilon} \int_{-1}^{1} G_{D_1,\tau\lambda} \left[\psi_{1,\epsilon} u_{1,\epsilon}^2 + 2 S_\epsilon u_{1,\epsilon} \phi_\epsilon \right] dx \\ &= -\frac{a_1}{\epsilon} G_{D_1,\tau\lambda}(0,0) \left[\psi_{1,\epsilon}(0) \int_{-1}^{1} u_{1,\epsilon}^2 \, dx + 2 S_\epsilon(0) \int_{-1}^{1} u_{1,\epsilon} \phi_\epsilon \, dx \right] \big(1 + O(\epsilon)\big) \\ &= -a_1 G_{D_1,\tau\lambda}(0,0) \left[\psi_{1,\epsilon}(0) \frac{\xi_\epsilon^2}{\sqrt{1+\alpha_\epsilon}} \int_{\mathbb{R}} w^2 \, dy + 2\sqrt{1+\alpha_\epsilon} \int_{\mathbb{R}} w \phi_\epsilon \, dy \right] \big(1 + O(\epsilon)\big). \end{aligned}$$

This implies

$$\psi_{1,\epsilon}(0) = -\frac{2a_1 G_{D_1,\tau\lambda}(0,0)\sqrt{1+\alpha_\epsilon}\int_{\mathbb{R}} w\phi_\epsilon\,dy}{1 + a_1 G_{D_1,\tau\lambda}(0,0)\frac{\xi_\epsilon^2}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}} w^2\,dy}$$
$$= -\frac{2(1+\alpha_\epsilon)\int_{\mathbb{R}} w\phi_\epsilon\,dy}{\frac{\sqrt{1+\alpha_\epsilon}}{a_1 G_{D_1,\tau\lambda}(0,0)} + \xi_\epsilon^2\int_{\mathbb{R}} w^2\,dy}\big(1+O(\epsilon)\big). \tag{5.14}$$

Putting everything together, we compute

$$\psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = -\frac{2(1+\alpha_\epsilon)\int_{\mathbb{R}} w\phi_\epsilon\,dy}{\frac{\sqrt{1+\alpha_\epsilon}}{a_1 G_{D_1,\tau\lambda}(0,0)} + \xi_\epsilon^2\int_{\mathbb{R}} w^2\,dy}\xi_\epsilon^2 w^2\left(1+O(\epsilon)\right)$$
$$= -\frac{2(1+\alpha_\epsilon)}{1+\frac{\sqrt{1+\alpha_\epsilon}}{6a_1 G_{D_1,\tau\lambda}(0,0)\xi_\epsilon^2}}\frac{\int_{\mathbb{R}} w\phi_\epsilon\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2\left(1+O(\epsilon)\right)$$
$$= -\frac{2(1+\alpha_\epsilon)}{1+c_{3,\epsilon}\tau\lambda}\frac{\int_{\mathbb{R}} w\phi_\epsilon\,dy}{\int_{\mathbb{R}} w^2\,dy}w^2\left(1+O(\epsilon+|\tau\lambda|)\right) \quad \text{in } H^2(\Omega_\epsilon),$$

where $c_{3,\epsilon} = \frac{\sqrt{1+\alpha_\epsilon}}{3a_1\xi_\epsilon^2} > 0$, using formula (7.9). In particular, the factor

$$\frac{-2(1+\alpha_\epsilon)}{1+\frac{\sqrt{1+\alpha_\epsilon}}{6a_1 G_{D_1,\tau\lambda}(0,0)\xi_\epsilon^2}}$$

is bounded if $\mathrm{Re}(\lambda) \ge 0$. Therefore, by Theorem 5.3, both the stability and instability result extend from $\tau = 0$ to a range $0 \le \tau < \tau_0$ (for some constant $\tau_0 > 0$).

Proof of (2) *We prove this case in two stages. In the first stage we only allow* τ_1 *to be nonzero, i.e. we assume* $\tau = 0$ *and* $0 < \epsilon \ll \tau_1 \ll 1$.

Similar to the derivation of (5.14), we have

$$\psi_{2,\epsilon}(0) = G_{D_2,\tau_1\lambda}(0,0)\left[u_{2,\epsilon}^2(0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}}\phi_\epsilon\,dy\right.$$
$$\left.+2\psi_{2,\epsilon}(0)u_{2,\epsilon}(0)\frac{\xi_\epsilon}{\sqrt{1+\alpha_\epsilon}}\int_{\mathbb{R}} w\,dy\right]\left(1+O(\epsilon)\right)$$
$$= u_{2,\epsilon}(0)G_{D_2,\tau_1\lambda}(0,0)\frac{1}{\sqrt{1+\alpha_\epsilon}}\left[u_{2,\epsilon}(0)\int_{\mathbb{R}}\phi_\epsilon\,dy + 2\psi_{2,\epsilon}(0)\xi_\epsilon\int_{\mathbb{R}} w\,dy\right]$$

which implies

$$\psi_{2,\epsilon}(0)\left(\frac{2G_{D_2,\tau_1\lambda}(0,0)}{G_{D_2}(0,0)} - 1\right) = -\frac{G_{D_2,\tau_1\lambda}(0,0)}{G_{D_2}(0,0)}\,\frac{u_{2,\epsilon}(0)}{\xi_\epsilon}\,\frac{\int_{\mathbb{R}}\phi_\epsilon\,dy}{\int_{\mathbb{R}} w\,dy}\big(1+O(\epsilon)\big).$$

Thus we have

$$\psi_{2,\epsilon}(0) = -\frac{G_{D_2,\tau_1\lambda}(0,0)}{2G_{D_2,\tau_1\lambda}(0,0) - G_{D_2}(0,0)}\,\frac{\sqrt{1+\alpha_\epsilon}}{G_{D_2}(0,0)6\xi_\epsilon^2}\,\frac{\int_{\mathbb{R}}\phi_\epsilon\,dy}{\int_{\mathbb{R}} w\,dy}\big(1+O(\epsilon)\big). \tag{5.15}$$

Finally, we get

$$-a_2 u_{1,\epsilon}2u_{2,\epsilon}\psi_{2,\epsilon} = 2\alpha_\epsilon\frac{G_{D_2,\tau_1\lambda}(0,0)}{2G_{D_2,\tau_1\lambda}(0,0) - G_{D_2}(0,0)}\,\frac{\int_{\mathbb{R}}\phi_\epsilon\,dy}{\int_{\mathbb{R}} w\,dy}\,w\left(1+O(\epsilon)\right) \quad \text{in } H^2(\Omega_\epsilon).$$

It is now essential to study the asymptotic behavior of the function

$$
\begin{aligned}
f(\tau_1\lambda, D_2) &= \frac{G_{D_2,\tau_1\lambda}(0,0)}{2G_{D_2,\tau_1\lambda}(0,0) - G_{D_2}(0,0)} \\
&= \frac{1}{2 - \frac{G_{D_2}(0,0)}{G_{D_2,\tau_1\lambda}(0,0)}} \\
&= \frac{1}{2 - \frac{\sqrt{1+\tau_1\lambda}\coth\theta_2}{\coth(\theta_2\sqrt{1+\tau_1\lambda})}},
\end{aligned}
$$

using the formulas (7.19), (7.20), (7.24). Thus we have

$$
\begin{aligned}
& f(0, D_2) = 1, \qquad f(\rho, D_2) \to 0 \quad \text{as } \rho \to \infty; \\
& f(\rho, D_2) \to \pm\infty \quad \text{as } \rho \to \mp\rho_0,
\end{aligned}
$$

where ρ_0 is the unique positive solution of

$$
\sqrt{1+\rho_0}\coth\theta_2 = 2\coth(\theta_2\sqrt{1+\rho_0}).
$$

We expand the eigenvalue problem (5.1) with respect to τ_1 for $|\tau_1\lambda - \rho_0| = O(\tau_1)$, Thus we get the expansions

$$
\begin{aligned}
f(\tau_1\lambda) &= \frac{f_1}{\rho_0 - \tau_1\lambda} + f_2 + O(\tau_1), \\
\lambda &= \frac{\rho_0}{\tau_1} + \lambda_1 + \lambda_2\tau_1 + O\left(\tau_1^2\right), \\
\phi_\epsilon &= w + \phi_1\tau_1 + O\left(\tau_1^2\right) \quad \text{in } H^2(\Omega_\epsilon),
\end{aligned}
$$

which satisfy

$$
\begin{aligned}
& 2\alpha_\epsilon\left(\frac{f_1}{\rho_0 - \tau_1\lambda} + f_2\right) w + (1+\alpha_\epsilon)w^2 - \frac{2(1+\alpha_\epsilon)}{1 + c_{3,\epsilon}\tau\lambda} w^2 \\
& \quad + \Delta\phi_1 - (1+\alpha_\epsilon)\phi_1 + 2(1+\alpha_\epsilon)w\phi_1 - \frac{2(1+\alpha_\epsilon)}{1 + c_{3,\epsilon}\tau\lambda} \frac{\int_{\mathbb{R}} w\phi_1\,dy}{\int_{\mathbb{R}} w^2\,dy} w^2 \\
& \quad + 2\alpha_\epsilon\left(\frac{f_1}{\rho_0 - \tau_1\lambda} + f_2\right)\frac{\int_{\mathbb{R}} \phi_\epsilon\,dy}{\int_{\mathbb{R}} w\,dy} w - \frac{2(1+\alpha_\epsilon)}{1 + c_{3,\epsilon}\tau\lambda} \frac{\int_{\mathbb{R}} w\phi_1\,dy}{\int_{\mathbb{R}} w^2\,dy} w^2 \\
& = \left(\frac{\lambda_0}{\tau_1} + \lambda_1 + \lambda_2\tau_1 + O(\tau_1^2)\right)\left(w + \phi_1\tau_1 + O(\tau_1^2)\right) \quad \text{in } H^2(\Omega_\epsilon).
\end{aligned}
$$

Comparing powers of τ_1, we get

$$
f(\tau, \lambda) = -\frac{f_1}{\tau_1\lambda_1} + f_1\frac{\lambda_2}{\lambda_1^2} + f_2 + O(\tau_1),
$$

the eigenvalue

$$
\lambda = \frac{\rho_0}{\tau_1} + \lambda_1 + \lambda_2\tau_1 + O\left(\tau_1^2\right), \tag{5.16}
$$

where

$$
\lambda_1 = -\frac{2\alpha_\epsilon f_1}{\rho_0}, \quad \lambda_2 = \frac{\lambda_1^2}{2\alpha_\epsilon f_1} - \frac{f_2}{f_1}\lambda_1^2 + \frac{\int_R \phi_1\,dy}{\int_R w\,dy}\lambda_1,
$$

and the eigenfunction

$$\phi_\epsilon = w + \frac{1+\alpha_\epsilon}{\lambda_0}\left(1 - \frac{2}{1+c_{3,\epsilon}\tau\lambda_\epsilon}\right) w^2\tau_1 + O\left(\tau_1^2\right) \quad \text{in } H^2(\Omega_\epsilon). \tag{5.17}$$

In conclusion, the eigenvalue problem (5.1) is always unstable for $0 < \tau_1$ small enough, although it is stable for $\tau_1 = 0$.

This behavior stands in marked contrast (1), where in the regime $0 < \tau < \tau_0$ (for some $\tau_0 > 0$) the stability behavior is the same as for $\tau = 0$.

In the second stage we allow both τ and τ_1 to be nonzero. We assume $0 \le \tau < \tau_0$ for some $\tau_0 > 0$ small enough and $0 < \epsilon \ll \tau_1 \ll 1$.

Combining the formulas in the proofs of (1) and (2), it follows that now we have the same behavior as in (2) since the leading terms in (2) which are of exact order $\frac{1}{\tau_1}$ dominate those in (1) which are of exact order 1.

The analysis in the proof has been performed considering the limiting eigenvalue problem for $\epsilon = 0$ and then letting $\tau_1 \to 0$. The proof extends to the case $0 < \epsilon \ll \tau_1 \ll 1$ in (5.1) by a perturbation argument as in Theorem 5.1. □

Remark 5.1 We expect that in the regime $0 < \tau_1 \ll \epsilon \ll 1$ and $0 \le \tau < \tau_0$ for some $\tau_0 > 0$ small enough the system will be stable.

6 Stability II: Computation of the Small Eigenvalues

We now compute the small eigenvalues of the eigenvalue problem (5.1), i.e. we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We will prove that these eigenvalues satisfy $\lambda_\epsilon = O(\epsilon^2)$. We emphasize that the analysis in this section applies to both $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ and $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$. Further, it includes nonzero values for τ or τ_1 , i.e. we assume $0 \le \tau < \tau_0$, where $\tau_0 > 0$ is a constant which is small enough and may be chosen independent of ϵ, and $0 \le \tau_1 \ll 1$. Let us define

$$\tilde{u}_{1,\epsilon}(x) = \chi\left(|x|\right)u_{1,\epsilon}(x). \tag{6.1}$$

Then it follows easily that

$$u_{1,\epsilon}(x) = \tilde{u}_{1,\epsilon}(x) + \text{e.s.t.} \quad \text{in } H^2(\Omega_\epsilon). \tag{6.2}$$

Taking the derivative of the system (2.1) w.r.t. y, we compute

$$\tilde{u}'''_{1,\epsilon} - \tilde{u}'_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\tilde{u}'_{1,\epsilon} + \epsilon S'_\epsilon u_{1,\epsilon}^2 - a_2\tilde{u}'_{1,\epsilon}u_{2,\epsilon}^2 - 2\epsilon a_2 u_{1,\epsilon}u_{2,\epsilon}\tilde{u}'_{2,\epsilon} = \text{e.s.t.}. \tag{6.3}$$

Here $'$ denotes derivative w.r.t. the variable of the corresponding function, i.e. it means derivative w.r.t. x for S_ϵ and $u_{2,\epsilon}$, and w.r.t. y for $u_{1,\epsilon}$.

Let us now decompose the eigenfunction $(\psi_{1,\epsilon},\ \phi_\epsilon,\ \psi_{2,\epsilon})$ as follows:

$$\phi_\epsilon = a^\epsilon \tilde{u}'_{1,\epsilon} + \phi_\epsilon^\perp \tag{6.4}$$

where a^ϵ is a complex number to be determined and

$$\phi_\epsilon^\perp \perp \mathcal{K}_\epsilon = \text{span}\left\{\tilde{u}'_{1,\epsilon}\right\} \subset H_N^2\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right).$$

We decompose the eigenfunction $\psi_{1,\epsilon}$ as follows:

$$\psi_{1,\epsilon} = a^\epsilon \psi_{1,\epsilon}^0 + \psi_{1,\epsilon}^\perp,$$

where $\psi_{1,\epsilon}^0$ satisfies

$$\begin{cases} D_1\Delta\psi_{1,\epsilon}^0 - \frac{a_1}{\epsilon}\psi_{1,\epsilon}^0 u_{1,\epsilon}^2 - 2\frac{a_1}{\epsilon}S_\epsilon u_{1,\epsilon}\tilde{u}_{1,\epsilon}' = \tau\lambda_\epsilon\psi_{1,\epsilon}^0, \\ {\psi_{1,\epsilon}^0}'(\pm 1) = 0 \end{cases} \tag{6.5}$$

and $\psi_{1,\epsilon}^\perp$ is given by

$$\begin{cases} D_1\Delta\psi_{1,\epsilon}^\perp - \frac{a_1}{\epsilon}\psi_{1,\epsilon}^\perp u_{1,\epsilon}^2 - 2\frac{a_1}{\epsilon}S_\epsilon u_{1,\epsilon}\phi_\epsilon^\perp = \tau\lambda_\epsilon\psi_{1,\epsilon}^\perp, \\ {\psi_{1,\epsilon}^\perp}'(\pm 1) = 0. \end{cases} \tag{6.6}$$

Similarly, we decompose the eigenfunction $\psi_{2,\epsilon}$ as follows:

$$\psi_{2,\epsilon} = a^\epsilon\psi_{2,\epsilon}^0 + \psi_{2,\epsilon}^\perp,$$

where $\psi_{2,\epsilon}^0$ satisfies

$$\begin{cases} D_2\Delta\psi_{2,\epsilon}^0 - \psi_{2,\epsilon}^0 + \frac{2}{\epsilon}u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^0 + \frac{1}{\epsilon}\tilde{u}_{1,\epsilon}' u_{2,\epsilon}^2 = \tau_1\lambda_\epsilon\psi_{2,\epsilon}^0, \\ {\psi_{2,\epsilon}^0}'(\pm 1) = 0 \end{cases} \tag{6.7}$$

and $\psi_{2,\epsilon}^\perp$ is given by

$$\begin{cases} D_2\Delta\psi_{2,\epsilon}^\perp - \psi_{2,\epsilon}^\perp + \frac{2}{\epsilon}u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^\perp + \frac{1}{\epsilon}\phi_\epsilon^\perp u_{2,\epsilon}^2 = \tau_1\lambda_\epsilon\psi_{2,\epsilon}^\perp, \\ {\psi_{2,\epsilon}^\perp}'(\pm 1) = 0. \end{cases} \tag{6.8}$$

Note that $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ can be uniquely expressed in terms of ϕ_ϵ by solving the first and third equation using the Green's functions $G_{D_1,\tau\lambda_\epsilon}$ and $G_{D_2,\tau\lambda_\epsilon}$ defined in (7.8) and (7.23), respectively,

$$\psi_{1,\epsilon} = a^\epsilon\psi_{1,\epsilon}^0 + \psi_{1,\epsilon}^\perp = a^\epsilon T_{1,\tau\lambda_\epsilon}'\tilde{u}_{1,\epsilon}' + T_{1,\tau\lambda_\epsilon}'\phi_\epsilon^\perp. \tag{6.9}$$

$$\psi_{2,\epsilon} = a^\epsilon\psi_{2,\epsilon}^0 + \psi_{2,\epsilon}^\perp = a^\epsilon T_{2,\tau_1\lambda_\epsilon}'\tilde{u}_{1,\epsilon}' + T_{2,\tau_1\lambda_\epsilon}'\phi_\epsilon^\perp. \tag{6.10}$$

Using the Green's function G_{D_1} defined in (7.1) we compute S_ϵ' near zero. We get

$$\begin{aligned} \epsilon S_\epsilon'(\epsilon y) - \epsilon S_\epsilon'(0) &= a_1\epsilon\int_{-1/\epsilon}^{1/\epsilon}\Bigg[\frac{1}{2D_1}\big(\mathrm{sgn}(y-z) - \mathrm{sgn}(-z)\big) \\ &\quad + H_{D_1,x}(\epsilon y,\epsilon z) - H_{D_1,x}(0,\epsilon z)\Bigg] S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz + O\big(\epsilon^3|y|^2\big) \\ &= a_1\frac{\epsilon}{D_1}\int_0^y S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz \\ &\quad + a_1\epsilon^2 y\int_{-1/\epsilon}^{1/\epsilon} H_{D_1,xx}(0,0)S_\epsilon(\epsilon z)u_{1,\epsilon}^2(\epsilon z)\,dz + O\big(\epsilon^3|y|^2\big) \\ &= \frac{a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\frac{\epsilon}{D_1}\left[\int_0^y w^2(z)\,dz - \frac{\epsilon y}{2}\int_{\mathbb{R}} w^2(z)\,dz\right] + O\big(\epsilon^3|y|^2\big) \\ &= \frac{a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\frac{\epsilon}{D_1}\left[\int_0^y w^2(z)\,dz - 3\epsilon y\right] + O\big(\epsilon^3|y|^2\big), \end{aligned} \tag{6.11}$$

where sgn is the sign function ($\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(0) = 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$.)

Similarly, we compute using the Green's function $G_{D_1,\tau\lambda_\epsilon}$ defined in (7.8) that

$$\begin{aligned}\psi_{1,\epsilon}^0(\epsilon y)-\psi_{1,\epsilon}^0(0) &= -a_1\epsilon\int_{\Omega_\epsilon}\left[G_{D_1,\tau\lambda_\epsilon}(\epsilon y,\epsilon z)-G_{D_1,\tau\lambda_\epsilon}(0,\epsilon z)\right]2S_\epsilon u_{1,\epsilon}(z)\frac{1}{\epsilon}\tilde{u}_{1,\epsilon}'(\epsilon z)\,dz\\ &\quad+O\big(\epsilon^3|y|^2\big)\\ &=\frac{\epsilon a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}\left[\int_{-1/\epsilon}^{1/\epsilon}\frac{1}{D_1}\epsilon\big(|y-z|-|z|\big)zww'\,dz\right.\\ &\quad\left.+2\underbrace{H_{D_1,xz}(0,0)}_{=0}\epsilon y\int_{\mathbb{R}}zww'\,dz\right]\Big(1+O\big((\tau+\tau_1)|\lambda_\epsilon|\big)+O(\epsilon|y|)\Big).\end{aligned} \tag{6.12}$$

Note that from (6.5), we derive

$$\psi_{1,\epsilon}^0(0)=O\big(\epsilon+\tau|\lambda_\epsilon|\big). \tag{6.13}$$

Adding the contributions from (6.11) and (6.12), we get

$$\begin{aligned}&\frac{d}{dy}\left[S_\epsilon(\epsilon y)-S_\epsilon(0)\right]-\left[\psi_{1,\epsilon}(\epsilon y)-\psi_{1,\epsilon}(0)\right]\\ &=\epsilon^2\left(H_{D_1,xx}(0,0)+H_{D_1,xz}(0,0)\right)\frac{6a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}y\left(1+O\big(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|\big)\right)\\ &=-\frac{\epsilon^2}{D_1}\frac{3a_1(1+\alpha_\epsilon)^2}{S_\epsilon(0)}y\left(1+O\big(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|\big)\right).\end{aligned} \tag{6.14}$$

Similarly, we from (6.7) we get

$$\psi_{2,\epsilon}^0(0)=O\big(\epsilon+\tau_1|\lambda_\epsilon|\big). \tag{6.15}$$

Using G_{D_2}, we compute that

$$\begin{aligned}&\frac{d}{dy}\left[u_{2,\epsilon}(\epsilon y)-u_{2,\epsilon}(0)\right]-\left[\psi_{2,\epsilon}(\epsilon y)-\psi_{2,\epsilon}(0)\right]\\ &=\epsilon^2\left(H_{D_2,xx}(0,0)+H_{D_2,xz}(0,0)\right)\frac{6u_{2,\epsilon}^2(0)(1+\alpha_\epsilon)}{S_\epsilon(0)}y\Big(1+O\big(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|\big)\Big)\\ &=-\frac{\epsilon^2}{D_2}\frac{3(1+\alpha_\epsilon)}{S_\epsilon(0)}u_{2,\epsilon}^2(0)\,\theta_2(\coth\theta_2-\tanh\theta_2)y\Big(1+O\big(\epsilon|y|+(\tau+\tau_1)|\lambda_\epsilon|\big)\Big),\end{aligned} \tag{6.16}$$

where $\theta_i=\frac{1}{\sqrt{D_i}}$, $i=1,2$.

Suppose that ϕ_ϵ satisfies $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}=1$. Then $|a^\epsilon|\le C$.

Substituting the decompositions of $\psi_{1,\epsilon}$, ϕ_ϵ and $\psi_{2,\epsilon}$ into (5.2) and subtracting (6.3), we have

$$\begin{aligned}&a^\epsilon u_{1,\epsilon}^2\left(\psi_{1,\epsilon}-\epsilon S_\epsilon'\right)-a^\epsilon 2a_2u_{1,\epsilon}u_{2,\epsilon}\left(\psi_{2,\epsilon}-\epsilon u_{2,\epsilon}'\right)\\ &\quad+\left(\phi_\epsilon^\perp\right)''-\phi_\epsilon^\perp+2u_{1,\epsilon}S_\epsilon\phi_\epsilon^\perp+u_{1,\epsilon}^2\psi_{1,\epsilon}^\perp-2a_2u_{1,\epsilon}u_{2,\epsilon}\psi_{2,\epsilon}^\perp-2a_2\phi_\epsilon^\perp u_{2\epsilon}^2-\lambda_\epsilon\phi_\epsilon^\perp\\ &=\lambda_\epsilon a^\epsilon\tilde{u}_{1,\epsilon}'.\end{aligned} \tag{6.17}$$

Let us first compute, using (6.13) and (6.14),

$$I_1 := a^\epsilon u_{1,\epsilon}^2 \left(\psi_{1,\epsilon} - \epsilon S'_\epsilon\right) = \epsilon^2 a^\epsilon \frac{a_1(1+\alpha_\epsilon)}{D_1}(\xi_\epsilon)^3 y w^2(y) 3 \Big(1 + O\big(\epsilon|y| + (\tau+\tau_1)|\lambda_\epsilon|\big)\Big). \tag{6.18}$$

Similarly, we compute from (6.15) and (6.16),

$$I_2 := -a^\epsilon 2a_2 u_{1,\epsilon} u_{2,\epsilon} \left(\psi_{2,\epsilon} - \epsilon u'_{2,\epsilon}\right) = \epsilon^2 a^\epsilon \frac{2a_2}{D_2}(\xi_\epsilon)^2 u_{2,\epsilon}^3(0)\theta_2(\coth\theta_2 - \tanh\theta_2) y w(y) 3 \Big(1 + O\big(\epsilon|y| + (\tau+\tau_1)|\lambda_\epsilon|\big)\Big). \tag{6.19}$$

We now estimate the orthogonal part of the eigenfunction which is given by $(T'_{1,\tau\lambda_\epsilon}\phi_\epsilon^\perp, \phi_\epsilon^\perp, T'_{2,\tau_1\lambda_\epsilon}\phi_\epsilon^\perp)$. Expanding, we get

$$\bar{\mathcal{L}}_\epsilon \phi_\epsilon^\perp = g_{1,\epsilon} + g_{2,\epsilon}$$

where

$$\|g_{1,\epsilon}\|_{L^2(\Omega_\epsilon)} = O\big(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|\big).$$

and

$$g_{2,\epsilon} \perp \mathcal{C}_\epsilon^\perp.$$

By Proposition 4.1 we conclude that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = O\big(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|\big). \tag{6.20}$$

This implies that

$$\|T'_{1,\tau\lambda_\epsilon}\phi_\epsilon^\perp\|_{H^2(\Omega)} = O\big(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|\big) \tag{6.21}$$

and

$$\|T'_{2,\tau_1\lambda_\epsilon}\phi_\epsilon^\perp\|_{H^2(\Omega)} = O\big(\epsilon^3 + \epsilon(\tau+\tau_1)|\lambda_\epsilon|\big) \tag{6.22}$$

Multiplying the eigenvalue problem (5.2) by w' and integrating, we get

$$\begin{aligned}
\text{LHS} &= \int_{\mathbb{R}} (I_1 + I_2) w'\, dy \\
&= \epsilon^2 a^\epsilon \frac{a_1(1+\alpha_\epsilon)}{D_1}(\xi_\epsilon)^3 3 \Big(1 + O\big(\epsilon + (\tau+\tau_1)|\lambda_\epsilon|\big)\Big) \int_{\mathbb{R}} y w^2(y) w'(y)\, dy \\
&\quad + \epsilon^2 a^\epsilon \frac{2a_2}{D_2}(\xi_\epsilon)^2 u_{2,\epsilon}^3(0)\theta_2 \,(\coth\theta_2 - \tanh\theta_2)\, 3 \\
&\quad \big(1 + O\big(\epsilon + (\tau+\tau_1)|\lambda_\epsilon|\big)\big) \int_{\mathbb{R}} y w(y) w'(y)\, dy \\
&= -\epsilon^2 a^\epsilon (\xi_\epsilon)^2 \left[\frac{7.2 a_1(1+\alpha_\epsilon)}{D_1}\xi_\epsilon + \frac{18 a_2}{D_2} u_{2,\epsilon}^3(0)\theta_2\big(\coth\theta_2 - \tanh\theta_2\big)\right] \big(1 + O(\epsilon)\big).
\end{aligned}$$

Here we have used the elementary computations

$$\int_{\mathbb{R}} y w^2(y) w'(y)\, dy = -\int_{\mathbb{R}} \frac{w^3}{3}\, dy_2 = -\int_{\mathbb{R}} \frac{9}{8\cosh^6 \frac{y}{2}}\, dy = -2.4,$$

$$\int_{\mathbb{R}} y w(y) w'(y)\, dy = -\int_{\mathbb{R}} \frac{w^2}{2}\, dy_2 = -\int_{\mathbb{R}} \frac{9}{8\cosh^4 \frac{y}{2}}\, dy = -3.$$

Further, the contributions to LHS which coming from orthogonal part of the eigenfunction can be estimated by $O(\epsilon^3 + \epsilon(\tau + \tau_1)|\lambda_\epsilon|)$, using (6.20)–(6.22).

Further, we compute

$$\begin{aligned} \text{RHS} &= \lambda_\epsilon a^\epsilon \int_{\mathbb{R}} (w')^2 \, dy \left(1 + O(\epsilon)\right) \\ &= 1.2 a^\epsilon \lambda_\epsilon \left(1 + o(1)\right). \end{aligned}$$

Note that in the previous calculation

$$(\tau + \tau_1)|\lambda_\epsilon| = O(\epsilon^2)$$

and thus the error terms involving τ or τ_1 can be neglected. Therefore

$$\lambda_\epsilon = -\epsilon^2 \xi_\epsilon^2 \left[\frac{6a_1(1+\alpha_\epsilon)}{D_1} \xi_\epsilon + \frac{15a_2}{D_2} u_{2,\epsilon}^3(0) \theta_2 \big(\coth\theta_2 - \tanh\theta_2\big) \right] + o\big(\epsilon^2\big).$$

We summarize our result on the small eigenvalues in the following theorem.

Theorem 6.1 *The eigenvalues of* (5.1) *with* $\lambda_\epsilon \to 0$ *satisfy*

$$\lambda_\epsilon = -\epsilon^2 \xi_\epsilon^2 \left[\frac{6a_1(1+\alpha_\epsilon)}{D_1} \xi_\epsilon + \frac{15a_2}{D_2} u_{2,\epsilon}^3(0) \theta_2 \big(\coth\theta_2 - \tanh\theta_2\big) \right] + o\big(\epsilon^2\big). \tag{6.23}$$

In particular these eigenvalues are stable.

This completes the proof of Theorem 2.2. □

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Appendix: Two Green's Functions

Let $G_{D_1}(x, z)$ be the Green's function of the Laplace operator with Neumann boundary conditions:

$$\begin{cases} D_1 G_{D_1,xx}(x,z) - \frac{1}{2} + \delta_z(x) = 0 \quad \text{in } (-1,1), \\ \int_{-1}^{1} G_{D_1}(x,z)\, dx = 0, \\ G_{D_1,x}(-1,z) = G_{D_1,x}(1,z) = 0. \end{cases} \tag{7.1}$$

Here $\delta_z(x)$ denotes the Dirac delta distribution concentrated at the point z.

We can decompose $G_{D_1}(x, z)$ as follows:

$$G_{D_1}(x,z) = -\frac{1}{2D_1}|x - z| - H_{D_1}(x,z), \tag{7.2}$$

where H_{D_1} is the regular part of G_{D_1}.

Written explicitly, we have

$$G_{D_1}(x,z) = \begin{cases} -\frac{1}{D_1}\left[\frac{1}{3} - \frac{(x+1)^2}{4} - \frac{(1-z)^2}{4}\right], & -1 < x \le z < 1, \\ -\frac{1}{D_1}\left[\frac{1}{3} - \frac{(z+1)^2}{4} - \frac{(1-x)^2}{4}\right], & -1 < z \le x < 1. \end{cases} \tag{7.3}$$

By simple computations, we have

$$H_{D_1}(x,z) = -\frac{1}{2D_1}\left[\frac{1}{3} + \frac{x^2}{2} + \frac{z^2}{2}\right]. \tag{7.4}$$

For $x \neq z$, we calculate

$$\nabla_x \nabla_z G_{D_1}(x,z) = 0.$$

Further, we have

$$\nabla_x \nabla_z G_{D_1}(x,z) = 0, \quad \nabla_x G_{D_1}(x,z) = \begin{cases} \frac{x+1}{2D_1}, & -1 < x < z < 1, \\ \frac{x-1}{2D_1} & -1 < z < x < 1. \end{cases} \tag{7.5}$$

We further have

$$\left\langle \nabla_x G_{D_1}(x,z)|_{x=z} \right\rangle = -\nabla_x H_{D_1}(x,z)|_{x=z} = \frac{z}{2D_1}, \tag{7.6}$$

where $\langle \cdot \rangle$ denotes the average of the limits from both sides.

Taking another derivative, we get

$$\begin{aligned} G_{D_1,xx}(0,0) &= \frac{1}{2D_1}, \\ G_{D_1,xz}(0,0) &= 0. \end{aligned}$$

Note that in particular

$$G_{D_1,xx}(0,0) + G_{D_1,xz}(0,0) = \frac{1}{2D_1} > 0. \tag{7.7}$$

Next we define

$$\begin{cases} D_1 G_{D_1,\tau\lambda,xx}(x,z) - \tau\lambda G_{D_1,\tau\lambda}(x,z) + \delta_z(x) = 0 \ \text{ in } (-1,1), \\ G_{D_1,\tau\lambda,x}(-1,z) = G_{D_1,\tau\lambda,x}(1,z) = 0. \end{cases} \tag{7.8}$$

We calculate explicitly

$$G_{D_1,\tau\lambda}(x,z) = \begin{cases} \dfrac{\theta_1}{\sqrt{\tau\lambda}\sinh(2\theta_1\sqrt{\tau\lambda})}\cosh\left[\theta_1\sqrt{\tau\lambda}(1+x)\right]\cosh\left[\theta_1\sqrt{\tau\lambda}(1-z)\right], & -1 < x \le z < 1, \\ \dfrac{\theta_1}{\sqrt{\tau\lambda}\sinh(2\theta_1\sqrt{\tau\lambda})}\cosh\left[\theta_1\sqrt{\tau\lambda}(1-x)\right]\cosh\left[\theta_1\sqrt{\tau\lambda}(1+z)\right], & -1 < z \le x < 1, \end{cases} \tag{7.9}$$

where

$$\theta_1 = \frac{1}{\sqrt{D_1}}. \tag{7.10}$$

We can decompose $G_{D_1,\tau\lambda}(x,z)$ as follows:

$$G_{D_1,\tau\lambda}(x,z) = -\frac{1}{2D_1}|x-z| - H_{D_1,\tau\lambda}(x,z), \tag{7.11}$$

where $H_{D_1,\tau\lambda}$ is the regular part of $G_{D_1,\tau\lambda}$.

Closely related, let $\tilde{G}_{D_1,\tau\lambda}(x,z)$ be the Green's function given by

$$\begin{cases} D_1\tilde{G}_{D_1,\tau\lambda,x}(x,z) - \tau\lambda\tilde{G}_{D_1,\tau\lambda}(x,z) - \frac{1}{2} + \delta_z(x) = 0 \text{ in } (-1,1), \\ \tilde{G}_{D_1,\tau\lambda,x}(-1,z) = \tilde{G}_{D_1,\tau\lambda,x}(1,z) = 0. \end{cases} \tag{7.12}$$

We calculate explicitly

$$\tilde{G}_{D_1,\tau\lambda}(x,z) = \begin{cases} \frac{\theta_1}{\sqrt{\tau\lambda}\sinh(2\theta_1\sqrt{\tau\lambda})}\cosh\left[\theta_1\sqrt{\tau\lambda}(1+x)\right]\cosh\left[\theta_1\sqrt{\tau\lambda}(1-z)\right] - \frac{1}{2\tau\lambda}, & -1 < x \le z < 1, \\ \frac{\theta_1}{\sqrt{\tau\lambda}\sinh(2\theta_1\sqrt{\tau\lambda})}\cosh\left[\theta_1\sqrt{\tau\lambda}(1-x)\right]\cosh\left[\theta_1\sqrt{\tau\lambda}(1+z)\right] - \frac{1}{2\tau\lambda}, & -1 < z \le x < 1, \end{cases} \tag{7.13}$$

We can decompose $\tilde{G}_{D_1,\tau\lambda}(x,z)$ as follows:

$$\tilde{G}_{D_1,\tau\lambda}(x,z) = \frac{1}{2D_1}|x-z| - \frac{1}{2\tau\lambda} - H_{D_1,\tau\lambda}(x,z), \tag{7.14}$$

where $H_{D_1,\tau\lambda}$ is the regular part of $\tilde{G}_{D_1,\tau\lambda}$. Then an elementary computation shows that

$$\left| H_{D_1}(x,z) - H_{D_1,\tau\lambda}(x,z) - \frac{1}{2\tau\lambda} \right| \le C|\tau\lambda| \tag{7.15}$$

uniformly for all $(x,z) \in \Omega \times \Omega$. For the first two derivatives we have

$$\left| \nabla\left[H_{D_1}(x,z) - H_{D_1,\tau\lambda}(x,z)\right] \right| \le C|\tau\lambda| \tag{7.16}$$

uniformly for all $(x,z) \in \Omega \times \Omega$ and

$$\left| \nabla^2\left[H_{D_1}(x,z) - H_{D_1,\tau\lambda}(x,z)\right] \right| \le C|\tau\lambda| \tag{7.17}$$

uniformly for all $(x,z) \in \Omega \times \Omega$, where ∇ in (7.16) and (7.17) can mean derivative w.r.t. to x or z.

Further, let $G_{D_2}(x,z)$ be the following Green's function:

$$\begin{cases} D_2 G_{D_2,xx}(x,z) - G_{D_2}(x,z) + \delta_z(x) = 0 \text{ in } (-1,1), \\ G_{D_2,x}(-1,z) = G_{D_2,x}(1,z) = 0. \end{cases} \tag{7.18}$$

We calculate

$$G_{D_2}(x,z) = \begin{cases} \frac{\theta_2}{\sinh(2\theta_2)}\cosh\left[\theta_2(1+x)\right]\cosh\left[\theta_2(1-z)\right], & -1 < x \le z < 1, \\ \frac{\theta_2}{\sinh(2\theta_2)}\cosh\left[\theta_2(1-x)\right]\cosh\left[\theta_2(1+z)\right], & -1 < z \le x < 1, \end{cases} \tag{7.19}$$

where

$$\theta_2 = \frac{1}{\sqrt{D_2}}. \tag{7.20}$$

We set

$$K_{D_2}(|x-z|) = \frac{\theta_2}{2}e^{-\theta_2|x-z|} \tag{7.21}$$

to be the singular part of $G_{D_2}(x,z)$. Then we decompose

$$G_{D_2}(x,z) = K_{D_2}(x,z) - H_{D_2}(x,z), \quad (x,z) \in \Omega \times \Omega.$$

Note that G_{D_2} is C^∞ for $(x, z) \in \Omega \times \Omega \setminus \{x = z\}$ and H_{D_2} is C^∞ for all $(x, z) \in \Omega \times \Omega$. Explicitly, we calculate

$$H_{D_2,xx}(0, 0) = -\frac{\theta_2^3}{2} \coth \theta_2,$$
$$H_{D_2,xz}(0, 0) = \frac{\theta_2^3}{2} \tanh \theta_2.$$

Note that in particular

$$H_{D_2,xx}(0, 0) + H_{D_2,xz}(0, 0) = \frac{\theta_2^3}{2}\big(- \coth \theta_2 + \tanh \theta_2\big) < 0. \tag{7.22}$$

Closely related, let $G_{D_2,\tau_1\lambda}(x, z)$ be the Green's function defined by

$$\begin{cases} D_2 G_{D_2,\tau_1\lambda,xx}(x, z) - (1 + \tau_1\lambda) G_{D_2,\tau_1\lambda}(x, z) + \delta_z(x) = 0 \ \text{ in } (-1, 1), \\ G_{D_2,\tau_1\lambda,x}(-1, z) = G_{D_2,\tau_1\lambda,x}(1, z) = 0. \end{cases} \tag{7.23}$$

We calculate explicitly

$$G_{D_2,\tau_1\lambda}(x, z) = \begin{cases} \frac{\theta_2}{\sqrt{1+\tau_1\lambda}\sinh(2\theta_2\sqrt{1+\tau_1\lambda})} \cosh\Big[\theta_2\sqrt{1+\tau_1\lambda}(1+x)\Big] \cosh\Big[\theta_2\sqrt{1+\tau_1\lambda}(1-z)\Big], & -1 < x \le z < 1, \\ \frac{\theta_2}{\sqrt{1+\tau_1\lambda}\sinh(2\theta_2\sqrt{1+\tau_1\lambda})} \cosh\Big[\theta_2\sqrt{1+\tau_1\lambda}(1-x)\Big] \cosh\Big[\theta_2\sqrt{1+\tau_1\lambda}(1+z)\Big], & -1 < z \le x < 1. \end{cases} \tag{7.24}$$

We can decompose $G_{D_2,\tau_1\lambda}(x, z)$ as follows

$$G_{D_2,\tau_1\lambda}(x, z) = K_{D_2}\big(|x - z|\big) - H_{D_2,\tau_1\lambda}(x, z), \tag{7.25}$$

where $H_{D_2,\tau_1\lambda}$ is the regular part of $G_{D_2,\tau_1\lambda}$. Then an elementary computation shows that

$$\big|H_{D_2}(x, z) - H_{D_2,\tau_1\lambda}(x, z)\big| \le C|\tau_1\lambda| \tag{7.26}$$

uniformly for all $(x, z) \in \Omega \times \Omega$,

$$\Big|\nabla\big[H_{D_2}(x, z) - H_{D_2,\tau_1\lambda}(x, z)\big]\Big| \le C|\tau_1\lambda| \tag{7.27}$$

uniformly for all $(x, z) \in \Omega \times \Omega$, and

$$\Big|\nabla^2\big[H_{D_2}(x, z) - H_{D_2,\tau_1\lambda}(x, z)\big]\Big| \le C|\tau_1\lambda| \tag{7.28}$$

uniformly for all $(x, z) \in \Omega \times \Omega$, where ∇ in (7.27) and (7.28) can mean derivative w.r.t. to x or z.

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