

Invariant Measures for Dissipative Systems and Generalised Banach Limits

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Abstract Inspired by a theory due to Foias and coworkers (see, for example, Foias et al. *Navier–Stokes equations and turbulence*, Cambridge University Press, Cambridge, 2001) and recent work of Wang (Disc Cont Dyn Sys 23:521–540, 2009), we show that the generalised Banach limit can be used to construct invariant measures for continuous dynamical systems on metric spaces that have compact attracting sets, taking limits evaluated along individual trajectories. We also show that if the space is a reflexive separable Banach space, or if the dynamical system has a compact absorbing set, then rather than taking limits evaluated along individual trajectories, we can take an ensemble of initial conditions: the generalised Banach limit can be used to construct an invariant measure based on an arbitrary initial probability measure, and any invariant measure can be obtained in this way. We thus propose an alternative to the classical Krylov–Bogoliubov construction, which we show is also applicable in this situation.

Keywords Invariant measure · Generalised Banach limit · Krylov–Bogoliubov procedure · Global attractor

1 Introduction

In this article we will investigate the existence of invariant measures in deterministic dynamical systems, using both the classical Krylov–Bogoliubov procedure [12] and an approach

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based on that developed by Foias and later collaborators for the Navier–Stokes equations [7–9] which relies on the generalised Banach limit to compute time averages. (For a discussion of invariant measures for infinite-dimensional stochastic systems, see [6].)

We treat dynamical systems on a metric space X , where the trajectory through an initial condition $u_0 \in X$ is given by a continuous nonlinear semigroup $S(\cdot)$, i.e. the solution at time t that starts at u_0 is $S(t)u_0$. Our main assumption is that $S(\cdot)$ has a compact attracting set or, equivalently, has a global attractor. We define a continuous semigroup precisely, and prove the equivalence of the existence of a compact attracting set and the existence of a global attractor, in Sect. 2.

By a finite measure on X we will mean a positive measure μ on the σ -algebra of Borel subsets of X , such that $\mu(X) < \infty$. We will denote by $\text{supp } \mu$ the support of μ , i.e. the smallest closed subset C of X with measure $\mu(C) = \mu(X)$. When $\mu(X) = 1$, we will say that μ is a probability measure on X .

Given an initial probability measure μ^0 on X , its image under the map $S(t)$ is given by the measure μ_t , where for any Borel set $B \subset X$

$$\mu_t(B) = \mu^0(S(t)^{-1}B). \tag{1}$$

A measure μ is called invariant if

$$\mu(B) = \mu(S(t)^{-1}B)$$

for all $t \geq 0$ and any Borel set $B \subset X$. One can recast these definitions in a weak form, with a measure μ invariant if for every $\varphi \in C_b(X)$ (continuous bounded functions on X)

$$\int_X \varphi(v) \, d\mu(v) = \int_X \varphi(S(t)v) \, d\mu(v).$$

The proof that these two formulations of invariance are equivalent can be found in Theorem 15.1 of [1].

In order to construct invariant measures, given an initial probability measure μ^0 , it is natural to consider the time-averages

$$\bar{\mu}_t := \frac{1}{t} \int_0^t \mu_s \, ds \tag{2}$$

(μ_s is defined as in (1)). While these may not converge, any weak limit point of these measures is invariant, i.e. if

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \int_0^{t_n} \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds = \int_X \varphi(v) \, d\mu(v) \quad \text{for all } \varphi \in C_b(X)$$

then μ is an invariant probability measure. This is essentially the Krylov–Bogoliubov method, and we show in Sect. 5.1 that one can generate invariant measures in this way within our framework, i.e. that the set of weak limit points of (2) is non-empty. Moreover, if μ^0 is chosen to be an invariant measure then $\mu = \mu^0$: this procedure has the key property that it can be used to generate any possible invariant measure.

Another possibility, when the support of μ^0 is contained in the set U of quasi-regular points of the attractor (roughly points where the time average converges, see Theorem 8 for a precise definition), is to construct the invariant probability measure μ by the formula

$$\int_X \varphi(v) \, d\mu(v) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds$$

for all $\varphi \in C_b(X)$. This is another variant of the Krylov–Bogoliubov method, and we also show in Sect. 5.1 that with the above equality one can generate every invariant probability measure on X .

The idea used by Foias et al. to circumvent the fact that the time average in (2) may not converge is to construct invariant measures using generalised Banach limits (we introduce these in Sect. 3). With a view to applications to the three-dimensional Navier–Stokes equations, for which the uniqueness of solutions is currently an open problem, they show that for any trajectory $u(t)$ there exists a measure μ such that

$$\int_X \varphi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(u(t)) \, dt, \tag{3}$$

where $\text{LIM}_{t \rightarrow \infty}$ denotes any generalised limit, and that this measure is invariant. While this convergence holds for all $\varphi \in C(X)$ in the two-dimensional case, for the three-dimensional equations one has to restrict to $\varphi \in C(X_w)$, where X_w denotes X equipped with the weak topology.

Wang [20] showed that for any dissipative system (i.e. one that has a global attractor) on a reflexive separable Banach space with $S(\cdot)$ weak-to-weak continuous, one can construct invariant probability measures as the Banach limits of time averages along individual trajectories. In Sect. 4 we provide a new proof of a similar result that is valid in any metric space and requires only continuity of $S(\cdot)$.

In Sect. 5.2 we apply a method similar to the method used by Foias et al., but start with an arbitrary initial probability measure μ^0 . Under the assumption that there exists a compact attracting set for $S(\cdot)$ we prove that if X is a reflexive separable Banach space, and $S(t)$ is weak-to-weak continuous, then there exists a unique probability measure μ on X such that

$$\int_X \varphi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds \tag{4}$$

for all $\varphi \in C_b(X)$, and that this measure μ is again invariant. In particular, if μ^0 is invariant then this process recovers μ^0 . In this way we provide an alternative to the Krylov–Bogoliubov construction: we give a recipe by which one can construct invariant measures, and by which any invariant measure can be constructed. When X is a general separable and complete metric space, we also prove that if there exists a compact absorbing set for $S(\cdot)$, then (4) can be used to obtain an invariant probability measure on X : in this case we require only continuity of $S(\cdot)$ (rather than weak-to-weak continuity).

2 Omega Limit Sets and Attractors

Throughout this section we work in a general metric space X , with metric $d(\cdot, \cdot)$. We consider a dynamical system on X specified by a continuous semigroup $S : \mathbb{R}_+ \times X \rightarrow X$, satisfying

- (a) $S(0)v = v$ for all $v \in X$;
- (b) $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$ ('the semigroup property'); and
- (c) $S : (t, v) \in \mathbb{R}_+ \times X \mapsto S(t)v \in X$ is a continuous mapping.

We now recall some standard definitions (see [4, 10, 15, 17] for example).

Let $\text{dist}(C_1, C_2)$ denote the Hausdorff semi-distance between two subsets, C_1 and C_2 , of X ,

$$\text{dist}(C_1, C_2) := \sup_{x \in C_1} \inf_{y \in C_2} d(x, y).$$

A non-empty subset K of X is *invariant* if

$$S(t)K = K \quad t \geq 0;$$

is *absorbing* if for any bounded subset $B \subset X$ there exists a time $t_{K,B} \geq 0$ such that

$$S(t)B \subset K \quad t \geq t_{K,B};$$

attracts a subset B of X if

$$\lim_{t \rightarrow +\infty} \text{dist}(S(t)B, K) = 0;$$

and is *attracting* if it attracts every bounded subset of X .

For any bounded subset $B \subset X$, we define the ω -limit set of B , $\omega(B)$, as

$$\omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}. \tag{5}$$

A global attractor for $S(\cdot)$ is a compact, invariant, attracting subset of X . If such a set exists it is unique, since if \mathcal{A}_1 and \mathcal{A}_2 are both global attractors,

$$\text{dist}(\mathcal{A}_1, \mathcal{A}_2) = \lim_{t \rightarrow \infty} \text{dist}(S(t)\mathcal{A}_1, \mathcal{A}_2) = 0,$$

and similarly $\text{dist}(\mathcal{A}_2, \mathcal{A}_1) = 0$. Since \mathcal{A}_1 and \mathcal{A}_2 are compact, $\mathcal{A}_1 = \mathcal{A}_2$. This set is also characterised as the maximal compact invariant set, and the minimal closed attracting set.

We now give a theorem on the existence of the global attractor with the minimal hypothesis. The majority of this result is not new (see Theorem 5.4.1 in [11] or Theorem 2.2 in Chap. 2 of [3], for example), but we could not find the proof that the attractor is given as $\omega(K)$ in the literature.

Theorem 1 *Suppose that $S(\cdot)$ possesses a compact attracting set K . Then for any bounded set B , $\omega(B)$ is non-empty, compact, invariant, and attracts B . The set $\mathcal{A} := \omega(K)$ is the global attractor.*

Proof First we note that if there exists a compact attracting set K , then $S(\cdot)$ is ‘asymptotically compact’, i.e. any sequence $\{S(t_n)b_n\}$ with $t_n \rightarrow \infty$ and $\{b_n\}$ bounded has a convergent subsequence whose limit lies in K . To see this, for each n find a $k_n \in K$ such that

$$d(S(t_n)b_n, k_n) = \text{dist}(S(t_n)b_n, K).$$

Since K is compact, k_n has a convergent subsequence, $k_{n_j} \rightarrow k \in K$; it follows that $S(t_{n_j})b_{n_j} \rightarrow k$ too.

Since K is attracting, any neighbourhood U of K is a bounded absorbing set for $S(\cdot)$, i.e. for any bounded B , there exists a time t_B such that

$$S(t)B \subseteq U \quad \text{for all } t \geq t_B.$$

That $\omega(B)$ has the properties listed in the theorem for an asymptotically compact semigroup with a bounded absorbing set is standard, see [10] or [17].

Now set

$$\mathcal{A} = \overline{\bigcup_{B \text{ bounded}} \omega(B)}. \tag{6}$$

Clearly \mathcal{A} is the global attractor: it is compact (it is a closed subset of K), invariant, and attracts all bounded sets. It remains only to show that in fact $\mathcal{A} = \omega(K)$. It is immediate from (6) that $\mathcal{A} \supseteq \omega(K)$. On the other hand, since \mathcal{A} is the minimal closed set that attracts bounded sets, $\mathcal{A} \subseteq K$, and consequently $\mathcal{A} = \omega(\mathcal{A}) \subseteq \omega(K)$, from which the result follows. \square

It follows that the existence of the global attractor and the existence of a compact attracting set are equivalent (invariance is the extra property enjoyed by the attractor).

Under the condition of this theorem, one can also consider the ‘point attractor’, \mathcal{A}_p , which is the minimal closed set that attracts all orbits, i.e. for every $u_0 \in X$,

$$\text{dist}(S(t)u_0, \mathcal{A}_p) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

While it is natural to make dynamical assumptions in terms of the global attractor, any invariant measure is in fact supported by the point attractor [5].

There are two corollaries of the existence of the global attractor that will prove useful below. The first is a general result about continuity of functions in a neighbourhood of a compact set (cf. Exercise 10.1 in [15]).

Lemma 2 *Let (X_1, d_1) and (X_2, d_2) be metric spaces, K a compact subset of X_1 , and $f : X_1 \rightarrow X_2$ a continuous function. Then f is ‘uniformly continuous near K ’, in the sense that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$x \in K, y \in X_1, \text{ and } d_1(y, x) \leq \delta \implies d_2(f(y), f(x)) \leq \varepsilon.$$

Proof If not then there exists an $\epsilon_0 > 0$ and a sequence of pairs (y_n, x_n) with $x_n \in K$ and $d_1(y_n, x_n) \rightarrow 0$ such that $d_2(f(y_n), f(x_n)) > \epsilon_0$. Since K is compact, there is a subsequence of the x_n (which we relabel) that converges to some $x^* \in K$. It follows that $d_1(y_n, x^*) \leq d_1(y_n, x_n) + d_1(x_n, x^*) \rightarrow 0$, but that

$$d_2(f(y_n), f(x^*)) \geq d_2(f(y_n), f(x_n)) - d_2(f(x_n), f(x^*)) \geq \epsilon_0/2$$

for all n sufficiently large (using the continuity of f at x^* for the second term on the right-hand side). But this contradicts the continuity of f at x^* . \square

One of the main tools we will use in the proof of our result in Sect. 4—and the key ingredient that allows us to bypass the use of the weak topology employed in [20]—is the following simple tracking lemma proved in [13]. For the sake of completeness we reproduce the proof here.

Lemma 3 *Suppose that $S(\cdot)$ is a continuous semigroup on the metric space X that has a global attractor \mathcal{A} . Then given an initial condition $u_0 \in X$ there exists a sequence of positive times θ_n with $\theta_{n+1} > \theta_n$ and $\theta_{n+1} - \theta_n \rightarrow \infty$, a decreasing sequence $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$, and a sequence $v_n \in \mathcal{A}$ such that*

$$d(S(t)u_0, S(t - \theta_n)v_n) \leq \varepsilon_n \quad \text{for all } t \in [\theta_n, \theta_{n+1}].$$

Proof For each $n \in \mathbb{N}$, choose $\delta_n > 0$ such that $d(y, x) < \delta_n$ with $x \in \mathcal{A}$ implies that $d(S(t)y, S(t)x) < 1/n$ for all $t \in [0, n]$. Lemma 2 with $X_1 = [0, n] \times X$, $X_2 = X$, $f(t, x) = S(t)x$, and $K = [0, n] \times \mathcal{A}$, guarantees that this can be done.

Since $\text{dist}(S(t)u_0, \mathcal{A}) \rightarrow 0$ as $t \rightarrow \infty$, there exists a time $T_n \geq n$ such that $\text{dist}(S(t)u_0, \mathcal{A}) \leq \delta_n$ for all $t \geq T_n$. Set

$$\theta_0 = 0, \quad \theta_1 = T_1, \quad \text{and} \quad \varepsilon_0 = \sup_{t \in [0, T_1]} \text{dist}(S(t)u_0, \mathcal{A}),$$

and $v_0 \in \mathcal{A}$ chosen arbitrarily.

Then, for any $k \geq 2$, if $T_j \leq \theta_{k-1} < T_{j+1}$, take $\theta_k = \theta_{k-1} + j$, $\varepsilon_{k-1} = 1/j$, and v_{k-1} a point in \mathcal{A} such that $d(S(\theta_{k-1}u_0, v_{k-1}) = \text{dist}(S(\theta_{k-1})u_0, \mathcal{A})$. □

3 Generalised Banach Limits

Our approach is based on the use of the generalised Banach limit. We recall here the definition of such a limit along with some of its properties. We write \mathcal{B}_+ for the space of all bounded real-valued functions on $[0, \infty)$ with sup norm.

Definition 4 A generalised Banach limit is any linear functional, denoted $\text{LIM}_{t \rightarrow \infty}$, on \mathcal{B}_+ such that

- (i) $\text{LIM}_{t \rightarrow \infty} g(t) \geq 0$ for all $g \in \mathcal{B}_+$ with $g(s) \geq 0$ for all $s \geq 0$; and
- (ii) $\text{LIM}_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty} g(t)$ for all $g \in \mathcal{B}_+$ for which the usual limit exists.

For the existence of such generalised limits see [9], where it is also proved that

- (iii) for every $g \in \mathcal{B}_+$, $|\text{LIM}_{t \rightarrow \infty} g(t)| \leq \limsup_{t \rightarrow \infty} |g(t)|$, and
- (iv) for every $f \in L^\infty(0, \infty)$ and every $a \geq 0$

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t+a} \int_0^{t+a} f(s) \, ds = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) \, ds.$$

Note that (iv) is a particular case of the translation invariance property $\text{LIM}_{t \rightarrow \infty} g(t) = \text{LIM}_{t \rightarrow \infty} g(t+h)$ in the case that $g(t) = \frac{1}{t} \int_0^t f(s) \, ds$. We can require this more general translation invariance property of generalised limits if we wish, but it does not follow from (i) and (ii) alone (unlike (iv)).

In the proof of our results we will also require

- (v) for any $f \in L^\infty(0, \infty)$ and $\tau > 0$,

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) \, ds = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\tau + r) \, dr, \quad \text{and}$$

- (vi) for any $f, g \in L^\infty(0, \infty)$ such that $\lim_{t \rightarrow \infty} (f(t) - g(t)) = 0$,

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) \, ds = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t g(s) \, ds.$$

Property (v) follows immediately from the linearity of $\text{LIM}_{t \rightarrow \infty}$ and (then) property (iii) after noting that

$$\left| \frac{1}{t} \int_0^t f(\tau + r) \, dr - \frac{1}{t} \int_0^t f(s) \, ds \right| = \frac{1}{t} \left| \int_t^{t+\tau} f(s) \, ds - \int_0^\tau f(s) \, ds \right| \leq \frac{2\tau}{t} \|f\|_{L^\infty(0, \infty)},$$

while property (vi) is a consequence the linearity of $\text{LIM}_{t \rightarrow \infty}$ and properties (v) and (iii).

4 Invariant Measures Via Averages Over a Single Trajectory

In this section we give a new proof of a result due to Wang [20], that in any dissipative system (i.e. one that has a global attractor) on a reflexive and separable Banach space one can construct invariant measures as the (generalised) time averages along trajectories. Our proof is valid in any metric space (we avoid the use of weakly continuous functions) and only requires the semigroup to be continuous.¹

Theorem 5 *Let X be a metric space. Assume that there exists a global attractor \mathcal{A} for a continuous semigroup $S(\cdot)$ in X . Fix a generalised Banach limit $\text{LIM}_{t \rightarrow \infty}$. Then for any $u_0 \in X$ there exists a unique probability measure μ on X which is supported on \mathcal{A} and such that for all $\varphi \in C(X)$,*

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)u_0) \, ds = \int_{\mathcal{A}} \varphi(v) \, d\mu(v). \tag{7}$$

This measure satisfies

$$\int_X \varphi(v) \, d\mu(v) = \int_X \varphi(S(t)v) \, d\mu(v) \text{ for any } t > 0, \text{ for all } \varphi \in C(X) \tag{8}$$

and hence μ is an invariant measure.

Proof Notice that equation (7) has sense: due to the existence of the global attractor the function $\varphi(S(t)u_0)$ is bounded on the positive semiaxis $t \geq 0$ for every continuous function φ on X , and therefore the function

$$f_\varphi(t) = \frac{1}{t} \int_0^t \varphi(S(s)u_0) \, ds$$

(with the convention $0/0 = 0$), is also bounded on the positive semiaxis $t \geq 0$. In fact, if $\varphi(S(t)u_0)$ is not bounded on the positive semiaxis $t \geq 0$, then there exists a sequence $s_n \rightarrow \infty$ such that

$$|\varphi(S(s_n)u_0)| \geq n \text{ for each } n \geq 1. \tag{9}$$

¹ Wang’s proof in fact requires $S(\cdot)$ to be ‘weak to weak’ continuous: if $x_n \rightharpoonup x$ then $S(t)x_n \rightharpoonup S(t)x$. This is not made explicit, but is required in the calculations at the bottom of page 525, where $\psi(\cdot) = \varphi(S(t)\cdot)$ should be in $C_w(X)$.

But then, as $\text{dist}(S(t)u_0, \mathcal{A}) \rightarrow 0$ as $t \rightarrow \infty$, we can find a subsequence $\{s'_m\} \subset \{s_n\}$ and a sequence $\{v_m\} \subset \mathcal{A}$ such that $d(S(s'_m)u_0, v_m) \leq 1/m$ for each $m \geq 1$. Then, as \mathcal{A} is compact, there is a subsequence of the v_m (which we relabel) such that $v_m \rightarrow v \in \mathcal{A}$. But then also $S(s'_m)u_0 \rightarrow v$, and therefore $\varphi(S(s'_m)u_0) \rightarrow \varphi(v)$ as $m \rightarrow \infty$, which contradicts (9).

Now let $\{\theta_n\}$, $\{v_n\}$, and $\{\varepsilon_n\}$ be as in Lemma 3. Let τ_n be an increasing sequence such that $\tau_n \rightarrow \infty$, $\tau_n/\tau_{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Define a function $n(t) = \max\{n : \tau_n \leq t\}$ for $t \geq \tau_2$.

Let

$$G_\varphi(t) = \frac{1}{t} \int_0^{\tau_{n(t)-1}} \varphi(S(s)u_0) \, ds$$

for $t \geq \tau_2$, and $G_\varphi = 0$ on the interval $[0, \tau_2)$, and similarly,

$$H_\varphi(t) = \frac{1}{t} \int_{\tau_{n(t)-1}}^t \varphi(S(s)u_0) \, ds$$

Then we have $f_\varphi(t) = G_\varphi(t) + H_\varphi(t)$ for $t \geq \tau_2$, and

$$|G_\varphi(t)| \leq \frac{1}{t} \tau_{n(t)-1} M \leq \frac{\tau_{n(t)-1}}{\tau_{n(t)}} M \rightarrow 0, \quad t \rightarrow \infty,$$

where $M = \sup_{t \geq 0} |\varphi(S(t)u_0)|$.

The function $H_\varphi(t)$ is bounded, and we have $\lim_{t \rightarrow \infty} f_\varphi(t) = \lim_{t \rightarrow \infty} H_\varphi(t)$.

Let $[\tau_{n(t)-1}, t] \subset \cup_{k=k(t)}^{l(t)} [\theta_k, \theta_{k+1}]$, where $k(t) = \max\{k : \theta_k \leq \tau_{n(t)-1}\}$, $l(t) = \max\{k : \theta_k < t\}$. Then

$$H_\varphi(t) = \frac{1}{t} \left\{ \int_{\tau_{n(t)-1}}^{\theta_{k(t)+1}} + \sum_{k=k(t)+1}^{l(t)-1} \int_{\theta_k}^{\theta_{k+1}} + \int_{\theta_{l(t)}}^t \right\} \varphi(S(s)u_0) \, ds.$$

Let

$$\begin{aligned} J_\varphi(t) &= \frac{1}{t} \int_{\tau_{n(t)-1}}^{\theta_{k(t)+1}} \varphi(S(s - \theta_{k(t)})v_{k(t)}) \, ds \\ &\quad + \frac{1}{t} \sum_{k=k(t)+1}^{l(t)-1} \int \varphi(S(s - \theta_k)v_k) \, ds \\ &\quad + \frac{1}{t} \int_{l(t)}^t \varphi(S(s - \theta_{l(t)})v_{l(t)}) \, ds \end{aligned}$$

be defined for $t \geq \tau_2$, and let $J_\varphi = 0$ on interval $[0, \tau_2)$. Notice that the points v_k are in the global attractor and the times θ_k do not depend on the choice of φ . Let $|\varphi(S(s)u_0) - \varphi(S(s - \theta_k)v_k)| \leq \varepsilon(\varepsilon_k)$ for $s \in [\theta_k, \theta_{k+1}]$ and for a decreasing sequence $\varepsilon(\varepsilon_k) \rightarrow 0$.

We shall show that $\lim_{t \rightarrow \infty} H_\varphi(t) = \lim_{t \rightarrow \infty} J_\varphi(t)$. Indeed, we have

$$\begin{aligned} |H_\varphi(t) - J_\varphi(t)| &\leq \frac{1}{t} (t - \tau_{n(t)-1}) \varepsilon(\varepsilon_{k(t)}) \\ &\leq \varepsilon(\varepsilon_{k(t)}) \rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

Thus $\text{LIM}_{t \rightarrow \infty} f_\varphi(t) = \text{LIM}_{t \rightarrow \infty} H_\varphi(t) = \text{LIM}_{t \rightarrow \infty} J_\varphi(t)$.

Observe that $J_\varphi(t)$, and thus $\text{LIM}_{t \rightarrow \infty} J_\varphi(t)$, only depends on the values of φ on \mathcal{A} . Consequently, for every $\psi \in C(\mathcal{A})$ we can define without ambiguity $L(\psi) := \text{LIM}_{t \rightarrow \infty} J_{\tilde{\psi}}(t)$, where $\tilde{\psi} \in C(X)$ and is any extension of ψ (this extension exists by Tietze extension theorem). We then obtain a linear positive functional L on $C(\mathcal{A})$, with $L(1) = 1$. By the Kakutani–Riesz (or Radon–Riesz) Representation Theorem [9, 14] there exists a probability measure μ on \mathcal{A} such that

$$L(\psi) = \int_{\mathcal{A}} \psi(v) \, d\mu(v) \quad \text{for all } \psi \in C(\mathcal{A}).$$

The measure μ can be extended to a probability measure on X with support contained in \mathcal{A} , by defining $\mu(E) = \mu(E \cap \mathcal{A})$ for all Borel subsets of X , and then, for any $\varphi \in C(X)$ one has

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \int_{\mathcal{A}} \varphi|_{\mathcal{A}}(v) \, d\mu(v) \\ &= L(\varphi|_{\mathcal{A}}) \\ &= \text{LIM}_{t \rightarrow \infty} J_{\tilde{\varphi}}(t) \\ &= \text{LIM}_{t \rightarrow \infty} f_\varphi(t), \end{aligned}$$

i.e. (7) holds.

The weak invariance (8) is an easy consequence of property (v) of LIM. Indeed, let us fix $\tau > 0$ and $\varphi \in C(X)$. Then from (7), the fact that μ is supported on \mathcal{A} , and property (v) of the generalised limit, we have

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(r + \tau)v) \, dr \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(r)v) S(\tau)v \, dr, \end{aligned}$$

where we have also used the semigroup property of $S(\cdot)$. Now, if we define $\psi(v) = \varphi(S(\tau)v)$, the function ψ belongs to $C(X)$, and so

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \psi(S(r)v) \, dr \\ &= \int_X \psi(v) \, d\mu(v) \\ &= \int_X \varphi(S(\tau)v) \, d\mu(v), \end{aligned}$$

i.e. (8) holds, and this is equivalent to the fact that μ is an invariant measure (see Corollary 12 in the Appendix). □

Now, we present a different and significantly simpler proof of the same result in the case where the semigroup acts in a uniformly convex Banach space. This therefore applies, in particular, when X is a Hilbert space or $X = L^p$ for any $1 < p < \infty$.

Theorem 6 *Let $S(\cdot)$ be a continuous semigroup on a uniformly convex Banach space X that has a global attractor \mathcal{A} . Let us denote by P the ‘closest point mapping’ from X onto the closed convex hull of \mathcal{A} . Fix a generalised Banach limit $\text{LIM}_{t \rightarrow \infty}$. Then for any $u_0 \in H$ there exists a unique invariant probability Borel measure μ on X which is supported on \mathcal{A} and such that for all $\varphi \in C(X)$,*

$$\begin{aligned} \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(PS(s)u_0) \, ds &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)u_0) \, ds \\ &= \int_{\mathcal{A}} \varphi(v) \, d\mu(v). \end{aligned} \tag{10}$$

This measure satisfies (8), and is therefore invariant.

Proof Let us denote by K the closed convex hull of the global attractor. As \mathcal{A} is compact, it is known that K is also a compact subset of X (see [1], Theorem 5.35, p. 185). As K is compact and X is uniformly convex, for each $x \in X$ there exists a unique $k_x \in K$ such that $\|x - k_x\| = \inf_{k \in K} \|x - k\|$. The projection operator $P : x \in X \mapsto k_x \in K$ is continuous (see Lemma 13 in the Appendix).

Given $u_0 \in X$ consider $t \mapsto P(S(t)u_0)$, the projection onto K of the trajectory $t \mapsto S(t)u_0$. Since K is compact the function $[0, \infty) \ni t \rightarrow \varphi(P(S(t)u_0)) \in \mathbb{R}$ is continuous and bounded for $\varphi \in C(X)$.

Moreover,

$$|\varphi(S(s)u_0) - \varphi(P(S(s)u_0))| \rightarrow 0 \quad \text{as } s \rightarrow \infty. \tag{11}$$

Indeed, if not there exists $\varepsilon > 0$ and a sequence $t_n \rightarrow \infty$ such that

$$|\varphi(S(t_n)u_0) - \varphi(P(S(t_n)u_0))| \geq \varepsilon \quad \text{for all } n \geq 1. \tag{12}$$

But

$$\begin{aligned} \|S(t_n)u_0 - P(S(t_n)u_0)\| &= \text{dist}(S(t_n)u_0, K) \\ &\leq \text{dist}(S(t_n)u_0, \mathcal{A}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{13}$$

As K is compact, from the sequence $\{P(S(t_n)u_0)\}$ we can extract a convergent subsequence, and then, from (13) and the continuity of φ we obtain a contradiction with (12). Thus (11) holds.

Now, from (11) and property (vi) of generalised Banach limits we conclude that

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)u_0) \, ds = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(P(S(s)u_0)) \, ds.$$

The right-hand side defines a linear positive functional on $C(K)$. Exactly as in the proof of Theorem 5 we prove that the measure associated with this functional via the Kakutani–Riesz Representation theorem satisfies (10), which ends the proof. □

5 Construction of an Invariant Probability Measure on X from an Arbitrary Initial Probability Measure

In this section we study how to construct an invariant measure based on a given initial probability measure. The constructions we consider have the property that any invariant measure can be obtained in this way.

5.1 The Krylov–Bogoliubov Procedure

Ideally one would construct an invariant measure as the time average of a measure evolving under the flow, but such a time average need not converge. The classical Krylov–Bogoliubov approach ([12]; or see [19] for a more modern treatment) obtains invariant measures as the weak limit points of such time averages. Here we show that this construction works within our framework.

Theorem 7 *Suppose that $S(\cdot)$ is a continuous semigroup on a complete separable metric space X that has a compact set \mathcal{A}_0 (not necessarily invariant) that attracts every compact subset of X . For any given initial probability measure μ^0 on X let μ_t be the measure defined by*

$$\mu_t(B) = \mu^0(S(t)^{-1}B)$$

for any Borel subset B of X , and denote the time-averaged measure by

$$\bar{\mu}_t = \frac{1}{t} \int_0^t \mu_s \, ds.$$

Then for any sequence of $t_n \rightarrow \infty$, there exists a subsequence t_{n_j} such that $\bar{\mu}_{t_{n_j}}$ converges weakly to an invariant probability measure μ . Furthermore, any invariant measure can be arrived at in this way.

Of course, the weak convergence of $\bar{\mu}_{t_{n_j}}$ to μ means that

$$\lim_{j \rightarrow \infty} \frac{1}{t_{n_j}} \int_0^{t_{n_j}} \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds = \int_X \varphi(v) \, d\mu(v)$$

for every $\varphi \in C_b(X)$.

Proof Since X is a Polish space, for each $n \geq 1$ there exists a compact set $K_n \subset X$ such that

$$\mu^0(X \setminus K_n) \leq 1/n.$$

Since \mathcal{A}_0 is attracting, for each n there exists a $t_{\delta,n}$ such that

$$S(t)K_n \subset \mathcal{A}_\delta \quad \text{for all } t \geq t_{\delta,n},$$

where \mathcal{A}_δ is the open δ -neighbourhood of \mathcal{A}_0 . Clearly this implies that

$$\mu_t(\mathcal{A}_\delta) \geq 1 - (1/n) \quad \text{for all } t \geq t_{\delta,n}.$$

It follows that

$$\bar{\mu}_t(\mathcal{A}_\delta) = \frac{1}{t} \left[\int_0^{t_{\delta,n}} \mu_s(\mathcal{A}_\delta) \, ds + \int_{t_{\delta,n}}^t \mu_s(\mathcal{A}_\delta) \, ds \right] \geq \left[1 - \frac{t_{\delta,n}}{t} \right] \left[1 - \frac{1}{n} \right],$$

and hence $\bar{\mu}_t$ is asymptotically tight. It follows from Prohorov’s Theorem (e.g. Theorem 1.3.8 in [18]) that any sequence $t_n \rightarrow \infty$ has a subsequence t_{n_j} such that $\bar{\mu}_{t_{n_j}}$ converges weakly to a limiting probability measure μ . The proof that μ is invariant is standard, see [19], for example.

Finally, if μ^0 is invariant then $\bar{\mu}_t = \mu^0$ for all t , and one recovers μ^0 . □

Another classical possibility is to construct invariant probability measures using an initial probability measure with full measure in the set of quasi-regular points in the attractor (see [14]).

Theorem 8 *Suppose that $S(\cdot)$ is a continuous semigroup on a metric space X that has a global attractor \mathcal{A} . Let us denote by U the set of quasi-regular points of the restriction of $S(\cdot)$ to \mathcal{A} , i.e. the set of all points $p \in \mathcal{A}$ such that the limit*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds \tag{14}$$

exists for any $\varphi \in C_b(X)$.

Then, for any given initial probability measure μ^0 on X such that $\mu^0(U) = 1$ there exists a unique invariant probability measure μ on X such that for any $\varphi \in C_b(X)$

$$\int_{\mathcal{A}} \varphi(v) \, d\mu(v) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_U \varphi(S(s)p) \, d\mu^0(p) \, ds. \tag{15}$$

Furthermore, any invariant probability measure on X can be arrived at in this way.

In particular, any invariant probability measure μ on X is of the form

$$\mu(E) = \int_U \mu_p(E) \, d\mu^0(p) \tag{16}$$

for all Borel sets E in X , where μ_p is the invariant probability measure on X associated to p by Theorem 5.

Proof Observe that by Corollary 12 and Theorem 14 (in the Appendix) every probability measure ν on X , invariant with respect to $S(\cdot)$, is a probability measure on \mathcal{A} which is invariant with respect to the restriction $S(\cdot)|_{\mathcal{A}}$ of $S(\cdot)$ to \mathcal{A} . Thus, by Theorem 9.12, p. 496 in [14], the set U of quasi-regular points of $S(\cdot)|_{\mathcal{A}}$ is an invariant subset of \mathcal{A} and satisfies that $\nu(U) = 1$ for every invariant probability measure ν on X .

Let $\varphi \in C_b(X)$. By the definition of U , we know that the limit in (14) exists. On the other, by Theorem 5, we know that for each $p \in X$ there exists a unique invariant probability measure μ_p on X such that

$$\text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds = \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v).$$

Thus, from property (ii) of generalised Banach limits, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds = \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v) \quad \text{for all } p \in U. \tag{17}$$

In particular, from this equality and the continuity of $S(\cdot)$, we deduce that the mapping

$$p \in U \mapsto \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v) \in \mathbb{R}$$

is Borel-measurable.

Let μ^0 be a probability measure on X such that $\mu^0(U) = 1$. Integrating in (17) with respect to μ^0 , we obtain

$$\int_U \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds \, d\mu^0(p) = \int_U \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v) \, d\mu^0(p). \tag{18}$$

Now observe that by Fubini’s theorem and the Lebesgue dominated convergence theorem,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_U \varphi(S(s)p) \, d\mu^0(p) \, ds \\ &= \lim_{t \rightarrow \infty} \int_U \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds \, d\mu^0(p) \\ &= \int_U \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) \, ds \, d\mu^0(p), \end{aligned}$$

and therefore, by (18),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_U \varphi(S(s)p) \, d\mu^0(p) \, ds = \int_U \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v) \, d\mu^0(p). \tag{19}$$

The right hand side of (19) defines a linear positive functional L on $C(\mathcal{A})$, with $L(1) = 1$. Then, the probability measure μ on X , with support contained in \mathcal{A} , associated to L by the Kakutani–Riesz Representation theorem satisfies (15).

From (15) and the semigroup property, for any $\varphi \in C_b(X)$ and $\tau > 0$

$$\begin{aligned} \int_X \varphi(S(\tau)v) \, d\mu(v) &= \int_{\mathcal{A}} \varphi(S(\tau)v) \, d\mu(v) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_U \varphi(S(s + \tau)p) \, d\mu^0(p) \, ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\tau}^{t+\tau} \int_U \varphi(S(s)p) \, d\mu^0(p) \, ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_U \varphi(S(s)p) \, d\mu^0(p) \, ds \\ &= \int_X \varphi(v) \, d\mu(v), \end{aligned}$$

and therefore, by Corollary 12, μ is invariant.

On the other hand, if μ^0 is invariant with $\mu^0(U) = 1$, then

$$\int_U \varphi(S(s)p) \, d\mu^0(p) = \int_U \varphi(p) \, d\mu^0(p),$$

and consequently the invariant measure μ associated to μ^0 by (15) satisfies

$$\int_{\mathcal{A}} \varphi(v) \, d\mu(v) = \int_U \varphi(p) \, d\mu^0(p),$$

for any $\varphi \in C_b(X)$, and therefore, by Theorem 11 from the Appendix, $\mu = \mu^0$.

Finally, by the construction of μ ,

$$\int_{\mathcal{A}} \varphi(v) \, d\mu(v) = \int_U \int_{\mathcal{A}} \varphi(v) \, d\mu_p(v) \, d\mu^0(p)$$

for any $\varphi \in C_b(X)$. Passing to characteristic functions in this last equation, we obtain (16). □

5.2 Construction of an Invariant Probability Measure on X Via the Banach Limit

We now show that one can use the generalised limit to construct an invariant measure based on an arbitrary given initial probability measure μ^0 on X .

Note that if we only assume the existence of a compact attracting set then our proof requires the weak-to-weak continuity of $S(t)$ (as did that in [20]). While this assumption is verified, for example, for the two-dimensional Navier–Stokes equations [16], and for reaction-diffusion equations on bounded and unbounded domains [2], it would be preferable to assume only continuity of $S(t)$.

In Theorem 10 we show that if we assume the existence of a compact absorbing set (rather than only a compact attracting set) then we can prove a similar result with $S(t)$ continuous (rather than weak-to-weak continuous).

Theorem 9 *Suppose that $S(\cdot)$ is a continuous semigroup on a separable, reflexive Banach space X that has a global attractor \mathcal{A} , and that for every $t \geq 0$, $S(t)$ is continuous from X_w into X_w (i.e. $u_n \rightharpoonup u_0$ implies that $S(t)u_n \rightharpoonup S(t)u_0$). Fix a generalised Banach limit $\text{LIM}_{t \rightarrow \infty}$. Then, for any given initial probability measure μ^0 on X there exists a unique invariant probability measure μ on X such that for any $\varphi \in C_b(X)$*

$$\int_X \varphi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds. \tag{20}$$

Furthermore, any invariant measure can be arrived at in this way.

Proof. Step One. First we assume that μ^0 is a finite positive measure on X (not necessarily a probability measure) and that $\text{supp } \mu^0$ is a bounded subset of X . We adapt the argument from [9] to show the existence of an invariant measure such that (20) holds for all $\varphi \in C_b(X_w)$, i.e. all $\varphi : X \rightarrow \mathbb{R}$ bounded such that $x_n \rightharpoonup x$ implies that $\varphi(x_n) \rightarrow \varphi(x)$ (clearly this is a strict subset of $C_b(X)$).

Since $S(\cdot)$ has a global attractor \mathcal{A} , there exists a closed ball B and a time $t_B > 0$ such that

$$S(s)(\text{supp } \mu^0) \subset B \quad \text{for all } s \geq t_B. \tag{21}$$

Let us denote by B_w the set B with the topology of the Hausdorff space X_w . Since B is convex, closed, and bounded, and X is reflexive, B_w is compact. For each $\psi \in C(B_w)$, let us define

$$f_\psi(t) = \begin{cases} \frac{1}{t} \int_{t_B}^t \int_{\text{supp } \mu^0} \psi(S(s)v) \, d\mu^0(v) \, ds, & \text{if } t > t_B; \\ 0, & \text{if } 0 \leq t \leq t_B \end{cases}$$

(Theorem 17 in the Appendix guarantees that the Borel sets for the strong and the weak topologies in X are the same). Evidently the function f_ψ is well defined, and belongs to \mathcal{B}_+ . Let us define

$$L(\psi) := \text{LIM}_{t \rightarrow \infty} f_\psi(t), \quad \psi \in C(B_w). \tag{22}$$

Note that L defines a positive linear functional on $C(B_w)$.

It follows from the Kakutani–Riesz Representation Theorem that there exists a positive measure μ on B defined on some σ -algebra \mathcal{M} that contains all weak Borel subsets of B , such that $\mu(B) = \mu^0(B) = \mu^0(X)$ and

$$\int_B \psi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} f_\psi(t) \tag{23}$$

for all $\psi \in C(B_w)$.

The measure μ can be extended to a measure on X by setting $\mu(E) = \mu(E \cap B)$ for any Borel subset E of X . Clearly $\mu(X \setminus B) = 0$, and therefore the support of μ is contained in B .

Now, if $\varphi \in C_b(X_w)$, then evidently

$$\int_X \varphi(S(s)v) \, d\mu^0(v) = \int_{\text{supp } \mu^0} \varphi(S(s)v) \, d\mu^0(v),$$

for all $s \geq 0$, and therefore by the properties of $\text{LIM}_{t \rightarrow \infty}$, the fact that $\varphi|_B \in C(B_w)$ and (23), we obtain

$$\begin{aligned} \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_{t_B}^t \int_{\text{supp } \mu^0} \varphi(S(s)v) \, d\mu^0(v) \, ds \\ &= \text{LIM}_{t \rightarrow \infty} f_{\varphi|_B}(t) \\ &= \int_B \varphi(v) \, d\mu(v) \\ &= \int_X \varphi(v) \, d\mu(v). \end{aligned}$$

Thus, (20) holds for all $\varphi \in C_b(X_w)$.

We now prove that for any $\tau \geq 0$, the measure μ satisfies

$$\int_X \varphi(v) \, d\mu(v) = \int_X \varphi(S(\tau)v) \, d\mu(v) \tag{24}$$

for all $\varphi \in C_b(X_w)$. Let us fix $\tau \geq 0$ and $\varphi \in C_b(X_w)$. Then from the definition (20) and property (iv) of the generalised limit we have

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(r + \tau)v) \, d\mu^0(v) \, dr \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(\tau)S(r)v) \, d\mu^0(v) \, dr, \end{aligned}$$

where we have also used the semigroup property of $S(\cdot)$. Now, if we define $\psi(v) = \varphi(S(\tau)v)$, the function ψ belongs to $C(X_w)$ because $S(\tau) : X_w \rightarrow X_w$ is continuous, and so

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \psi(S(r)v) \, d\mu^0(v) \, dr \\ &= \int_X \psi(v) \, d\mu(v) \\ &= \int_X \varphi(S(\tau)v) \, d\mu(v), \end{aligned}$$

i.e. (24) holds for all $\varphi \in C_b(X_w)$.

Consequently, by Corollary 19 the measure μ is invariant, and in particular its support is contained in \mathcal{A} .

Step Two. We now show that if μ^0 is a finite positive measure on X with bounded support then (20) holds for all $\varphi \in C_b(X)$ (and not just in $C_b(X_w)$), following [20].

Let $\varphi \in C_b(X)$ be given. Due to the compactness of \mathcal{A} , and the fact that X is a separable and reflexive Banach space, the function $\varphi|_{\mathcal{A}} \in C(\mathcal{A}_w)$. We can assume that the closed ball B satisfying (21) satisfies $\mathcal{A} \subset B$. As B_w is Hausdorff and compact, it is a normal space,

and therefore, by Tietze extension Theorem there exists a function $\tilde{\varphi} \in C(B_w)$ such that $\tilde{\varphi}|_{\mathcal{A}} = \varphi|_{\mathcal{A}}$.

To extend (20) to $\varphi \in C_b(X)$, the key observation (cf. [20]) is that

$$\sup_{v \in \text{supp } \mu^0} |\varphi(S(t)v) - \tilde{\varphi}(S(t)v)| \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{25}$$

Indeed, if not then there exists a $\delta > 0$, $v_n \in \text{supp } \mu^0$, and $t_n \rightarrow \infty$ such that

$$|\varphi(S(t_n)v_n) - \tilde{\varphi}(S(t_n)v_n)| \geq \delta. \tag{26}$$

Since \mathcal{A} is compact and attracting, there exists a subsequence (which we relabel) such that $S(t_n)v_n \rightarrow v_\infty$, with $v_\infty \in \mathcal{A}$. Since $\varphi \in C(X)$, $\tilde{\varphi} \in C(X_w)$, and they coincide on \mathcal{A} , it follows that

$$\lim_{n \rightarrow \infty} \varphi(S(t_n)v_n) = \varphi(v_\infty) = \tilde{\varphi}(v_\infty) = \lim_{n \rightarrow \infty} \tilde{\varphi}(S(t_n)v_n).$$

contradicting (26). From (25) it follows that

$$\lim_{t \rightarrow \infty} \int_{\text{supp } \mu^0} (\varphi(S(t)v) - \tilde{\varphi}(S(t)v)) \, d\mu^0(v) = 0. \tag{27}$$

Therefore, by Step One

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \int_{\mathcal{A}} \varphi(v) \, d\mu(v) \\ &= \int_{\mathcal{A}} \tilde{\varphi}(v) \, d\mu(v) \\ &= \int_B \tilde{\varphi}(v) \, d\mu(v) \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_{tB} \int_{\text{supp } \mu^0} \tilde{\varphi}(S(s)v) \, d\mu^0(v) \, ds \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_{tB} \int_{\text{supp } \mu^0} \varphi(S(s)v) \, d\mu^0(v) \, ds \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds, \end{aligned}$$

using (27) and property (vi) of the generalised limit. In this way we obtain (20) for all $\varphi \in C_b(X)$.

Step Three. We now assume that μ^0 is a general probability measure on X .

As X is a Polish space, for each $n \geq 1$ there exists a compact set $K_n \subset X$ such that

$$\mu^0(X \setminus K_n) \leq 1/n.$$

If for each $n \geq 2$ and E a Borel set of X we define

$$\mu_n^0(E) = \mu^0(E \cap K_n),$$

we obtain a sequence $\{\mu_n^0\}_{n \geq 1}$ of positive measures on X such that $\mu_n^0(X) = \mu^0(K_n) \leq 1$, and $\text{supp } \mu_n^0 \subset K_n$.

Thus, according to Step Two, there exists a sequence $\{\mu_n\}_{n \geq 1}$ of finite invariant positive measures on X such that for any $\varphi \in C_b(X)$

$$\int_X \varphi(v) \, d\mu_n(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu_n^0(v) \, ds. \tag{28}$$

In particular,

$$\text{supp } \mu_n \subset \mathcal{A}, \tag{29}$$

for all $n \geq 2$. Observe also that

$$\mu_n(X) = \mu_n^0(X) \leq 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu_n(X) = 1.$$

From (28) and the fact that $\text{supp } \mu_n^0 \subset K_n$, and μ_n^0 and μ^0 coincide on K_n ,

$$\begin{aligned} \int_X \varphi(v) \, d\mu_n(v) &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{K_n} \varphi(S(s)v) \, d\mu^0(v) \, ds \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds \\ &\quad - \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{X \setminus K_n} \varphi(S(s)v) \, d\mu^0(v) \, ds. \end{aligned} \tag{30}$$

But observe that

$$\begin{aligned} \left| \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{X \setminus K_n} \varphi(S(s)v) \, d\mu^0(v) \, ds \right| &\leq \mu^0(X \setminus K_n) \sup_{v \in X} |\varphi(v)| \\ &\leq \frac{1}{n} \sup_{v \in X} |\varphi(v)|, \end{aligned}$$

and consequently, from (30) we deduce that

$$\lim_{n \rightarrow \infty} \int_X \varphi(v) \, d\mu_n(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds, \tag{31}$$

for any $\varphi \in C_b(X)$.

On the other hand, let us set

$$\tilde{\mu}_n(B) = \mu_n(B) / \mu_n^0(X) \quad n \geq 1,$$

for each Borel subset B of X . We obtain a sequence $\{\tilde{\mu}_n\}_{n \geq 1}$ of probability measures on X which, due to (29), is tight. Thus, by Prohorov’s theorem, there exists a subsequence $\{\tilde{\mu}_{n'}\}_{n' \geq 1} \subset \{\tilde{\mu}_n\}_{n \geq 1}$ and a probability measure μ on X such that

$$\lim_{n' \rightarrow \infty} \int_X \psi(v) \, d\tilde{\mu}_{n'}(v) = \int_X \psi(v) \, d\mu(v),$$

for any $\psi \in C_b(X)$.

Consequently, since $\mu_n^0(X) \rightarrow 1$ as $n \rightarrow \infty$, we also have

$$\lim_{n' \rightarrow \infty} \int_X \psi(v) \, d\mu_{n'}(v) = \int_X \psi(v) \, d\mu(v)$$

for any $\psi \in C_b(X)$.

From this last equality and (31) we deduce that we have a probability measure μ on X such that

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \lim_{n \rightarrow \infty} \int_X \varphi(v) \, d\mu_n(v) \\ &= \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds, \end{aligned} \tag{32}$$

for all $\varphi \in C_b(X)$.

Setting $\psi(v) = \varphi(S(\tau)v)$, from the invariance of μ_n and (32) we deduce that

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \lim_{n \rightarrow \infty} \int_X \varphi(v) \, d\mu_n(v) \\ &= \lim_{n \rightarrow \infty} \int_X \varphi(S(\tau)v) \, d\mu_n(v) \\ &= \lim_{n \rightarrow \infty} \int_X \psi(v) \, d\mu_n(v) \\ &= \int_X \psi(v) \, d\mu(v), \\ &= \int_X \varphi(S(\tau)v) \, d\mu(v) \end{aligned} \tag{33}$$

for any $\varphi \in C(X)$, $\tau \geq 0$, and therefore μ is invariant.

Finally we observe that if μ^0 is an invariant probability measure then for any $s \geq 0$

$$\int_X \varphi(S(s)v) \, d\mu^0(v) = \int_X \varphi(v) \, d\mu^0(v),$$

and so

$$\frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds = \int_X \varphi(v) \, d\mu^0(v).$$

Using property (ii) of LIM it follows that

$$\int_X \varphi(v) \, d\mu^0(v) = \int_X \varphi(v) \, d\mu(v)$$

for all $\varphi \in C_b(X)$, and therefore, by Theorem 11, $\mu^0 = \mu$. □

We have chosen to treat the problem without assuming the existence of a compact absorbing set; but the existence of such a set would allow one to relax our other hypotheses and to greatly simplify the proof. We state the result under this stronger dynamical assumption, and give a brief indication of the simplifications to the proof this produces.

Theorem 10 *Suppose that $S(\cdot)$ is a continuous semigroup on a complete separable metric space X that has a compact absorbing set K . Fix a generalised Banach limit $\text{LIM}_{t \rightarrow \infty}$. Then, for any given initial probability measure μ^0 on X there exists a unique invariant probability measure μ on X such that for any $\varphi \in C_b(X)$*

$$\int_X \varphi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds. \tag{34}$$

Furthermore, any invariant measure can be arrived at in this way.

Proof In Step One, replace B in (21) by K , and work with $\varphi \in C_b(X)$ throughout. Omit Step Two, and continue through Step Three as above. \square

6 Conclusion

We have shown how the generalised Banach limit can be used in a variety of situations to construct invariant measures for systems that possess a global attractor.

Theorem 5 (invariant measures from averages along individual trajectories) generalises a recent result of [20], being valid in any metric space and requiring only that the semigroup is continuous. The proof of Theorem 6 provides a very concise argument to obtain the same conclusion in the more restrictive case when the phase space is a uniformly convex Banach space.

We have also shown in Theorem 9 that one can in fact start with an arbitrary initial probability measure, and construct an invariant measure ‘based on’ this initial choice, again using a particular Banach limit,

$$\int_X \varphi(v) \, d\mu(v) = \text{LIM}_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_X \varphi(S(s)v) \, d\mu^0(v) \, ds,$$

assuming that $S(\cdot)$ has a global attractor and that $S(\cdot)$ is weak-to-weak continuous. Since any invariant measure can be arrived at in this way, this offers an alternative to the classical Krylov–Bogoliubov construction. Theorem 10 provides the same conclusion in a complete separable metric space, with $S(\cdot)$ only required to be continuous but making the stronger assumption that there is a compact absorbing set. It is natural to conjecture that Theorem 9 remains true under the weaker assumption that $S(\cdot)$ is only continuous.

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Appendix: Proof of Technical Results

For the sake of completeness, we provide the proof here of several technical results used in the main part of the article.

An Equivalent Formulation of Invariance

We use the following result on equality of measures in the proof of Theorem 8; the proof can be found in [1, Theorem 15.1].

Theorem 11 *Let X be a metric space, and let μ and ν be two finite measures on X such that*

$$\int_X \varphi(v) \, d\mu(v) = \int_X \varphi(v) \, d\nu(v) \quad \text{for any } \varphi \in C_b(X). \tag{35}$$

Then

$$\mu = \nu.$$

As a consequence of this theorem, we obtain the following alternative characterisation of invariance, which is used in Theorems 5 and 8.

Corollary 12 *Let X be a metric space, and $S : X \rightarrow X$ a Borel-measurable mapping. Assume that μ is a finite measure on X such that*

$$\int_X \varphi(Sv) \, d\mu(v) = \int_X \varphi(v) \, d\mu(v) \quad \text{for any } \varphi \in C_b(X). \tag{36}$$

Then,

$$\mu(E) = \mu(S^{-1}(E)) \quad \text{for any Borel subset } E \text{ of } X.$$

Proof If we define $\nu(E) := \mu(S^{-1}(E))$ for any Borel subset E of X , then ν is a finite Borel measure on X , and

$$\int_X \varphi(v) \, d\nu(v) = \int_X \varphi(Sv) \, d\mu(v).$$

Thus, by (36), the equality (35) holds. □

Continuity of the Projection Operator

The proof of Theorem 6 relies on the following simple lemma, which establishes (in particular) the continuity of the mapping from a uniformly convex Banach spaces onto the unique closest point in a closed convex set.

Lemma 13 *Let X be a metric space and $K \subset X$ a nonempty compact subset. Assume that for each $x \in X$ the point $k_x \in K$ satisfying $d(x, k_x) = \min_{k \in K} d(x, k)$ is unique. Then, the mapping $P : x \in X \mapsto k_x \in K$, is continuous.*

Proof Let $\{x_n\}_{n \geq 1} \subset X$ a sequence such that $x_n \rightarrow x$ in X when n goes to infinity. Let $\{x_{n'}\}_{n' \geq 1} \subset \{x_n\}_{n \geq 1}$ an arbitrary subsequence. Then, as $\{Px_{n'}\}_{n' \geq 1} \subset K$, we can extract a subsequence $\{x_{n''}\}_{n'' \geq 1} \subset \{x_{n'}\}_{n' \geq 1}$ such that $Px_{n''} \rightarrow \widehat{k}$ when n'' , goes to infinity for some $\widehat{k} \in K$.

Let $k \in K$ be arbitrarily chosen. From $d(x_{n''}, Px_{n''}) \leq d(x_{n''}, k)$, making $n'' \rightarrow \infty$, we get $d(x, \widehat{k}) \leq d(x, k)$. Consequently, by the uniqueness of k_x , we deduce $\widehat{k} = k_x = Px$.

This argument shows that the complete sequence Px_n converges to Px when n goes to infinity, and therefore P is a continuous operator from X onto K . □

The Support of an Invariant Measure is Contained in the Attractor

The following result is used in the proof of Theorem 8.

Theorem 14 *Let $S(\cdot)$ be a continuous semigroup on a metric space X , and let μ be a finite measure on X which is invariant for this semigroup. If there exists the global attractor \mathcal{A} for $S(\cdot)$ then the support of μ is contained in \mathcal{A} .*

The proof of this result can be found in [9] in the context of 2D Navier-Stokes equations, and is given here for the sake of completeness.

Lemma 15 *Let μ be a finite measure on X which is invariant for the semigroup $S(\cdot)$ on X , and let B_a be a bounded subset of X that is absorbing for $S(\cdot)$. Then, $\mu(X \setminus B_a) = 0$.*

Proof Let us fix $x_0 \in X$, and for each $r > 0$ let $B_r = \{x \in X : d(x, x_0) \leq r\}$. Since B_a is absorbing for $S(t)$, there exists a $t_r \geq 0$ such that $S(t)B_r \subset B_a$ for all $t \geq t_r$. Hence $B_r \subset S(t)^{-1}B_a$, and hence, as μ is invariant,

$$\mu(B_r) \leq \mu(S(t)^{-1}B_a) = \mu(B_a) \quad \text{for all } t \geq t_r.$$

Letting $r \rightarrow \infty$ from the last inequality we obtain $\mu(X) \leq \mu(B_a)$, and therefore $\mu(X) \leq \mu(B_a)$, i.e. $\mu(X \setminus B_a) = 0$. □

Lemma 16 *Assume that there exists the global attractor \mathcal{A} for the semigroup $S(\cdot)$ on X , and let B_a be a bounded subset of X that is absorbing for $S(\cdot)$. Let $t_k : k \geq 1$ be a sequence of positive numbers such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then,*

$$\mathcal{A} = \bigcap_{k \geq 1} S(t_k)B_a.$$

Proof Let us set $\mathcal{A}' = \bigcap_{k \geq 1} S(t_k)B_a$.

As \mathcal{A} is invariant and B_a is absorbing, $\mathcal{A} \subset B_a$, but then, again by invariance, $\mathcal{A} = S(t_k)\mathcal{A} \subset S(t_k)B_a$, for all $k \geq 1$, and consequently $\mathcal{A} \subset \mathcal{A}'$.

On the other hand, we know that

$$\mathcal{A} = \omega(B_a) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)B_a}.$$

But,

$$S(t_k)B_a \subset \overline{\bigcup_{s \geq t} S(s)B_a} \quad \forall 0 \leq t \leq t_k, \quad \forall k \geq 1.$$

Hence, since $t_k \rightarrow \infty$,

$$\begin{aligned} \mathcal{A}' &\subset \bigcap_{k \geq 1} \bigcap_{0 \leq t \leq t_k} \overline{\bigcup_{s \geq t} S(s)B_a} \\ &= \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)B_a} = \mathcal{A}. \end{aligned}$$

□

Proof (of Theorem 14) Let $B(0, R)$ a closed ball such that $\mathcal{A} \subset B(0, R)$. Evidently, $B_a = B(0, 2R)$ is absorbing for the semigroup $S(t)$. Let $t_* > 0$ be such that $S(t)B_a \subset B_a$ for all $t \geq t_*$. Then, the sequence of sets $\{S(kt_*)B_a\}$ is decreasing, and therefore, by Lemma 16 we have

$$\mu(\mathcal{A}) = \lim_{k \rightarrow \infty} \mu(S(kt_*)B_a). \tag{37}$$

But, since B_a is absorbing, the set $S(kt_*)B_a \subset B_a$ is also a bounded absorbing set, and therefore by Lemma 15 it follows that $\mu(S(kt_*)B_a) = \mu(X)$ for all $k \geq 1$. Thus, by (37) we deduce that $\mu(\mathcal{A}) = \mu(X)$, and therefore, as \mathcal{A} , the support of μ is contained in \mathcal{A} . □

Equivalence of Strong Borel and Weak Borel Subsets in a Separable Banach Space

In separable Banach spaces, strong and weak Borel subsets are the same. This is a key component of the proof of Theorem 9.

Theorem 17 *If X is a separable Banach space, then a set $E \subset X$ is a Borel subset of X if and only if is a Borel subset of X_w .*

Proof (a) Every open subset of X_w is an open subset of X , thus every Borel subset of X_w is a Borel subset of X .

(b) To finish, it is enough to prove that if $E \subset X$ is an open subset of X , then E is a Borel subset of X_w . This will be proved in two steps.

(b1) If $E \subset X$ is an open subset of X , then E can be written as the union of a countable collection of closed balls in X .

To see this, let $C = \{x_n\}_{n \geq 1}$ a sequence dense in X , and denote \mathcal{C} the countably family of all closed balls $B(x_n; 1/m)$ centred in x_n with radius $1/m$, with $n, m \geq 1$.

Given $x \in E$, there exist $m(x)$ such that $B(x; 1/m(x)) \subset E$, and $n(x)$ such that $x \in B(x_{n(x)}; 1/(2m(x)))$. But then, $B(x_{n(x)}; 1/(2m(x))) \subset B(x; 1/m(x)) \subset E$, and therefore

$$E = \bigcup_{x \in E} B(x_{n(x)}; 1/(2m(x))),$$

i.e. E can be written as the union of a sub-collection of balls of \mathcal{C} .

(b2) Every closed ball in X is closed in X_w , and therefore a Borel subset of X_w . This is a consequence of the well known fact that a convex subset of X is closed in X if and only if it is closed in X_w .

□

In a Reflexive and Separable Banach Space, Weakly Invariant Measures with Bounded Support are Invariant Measures

The equivalence of weakly invariant measures and invariant measures, when the measure has bounded support, is a simple corollary (Corollary 19) of the following theorem. This equivalence is used in the proof of Theorem 9.

Theorem 18 *Let X be a reflexive and separable Banach space, and B a closed ball of X . Let μ and ν be two finite measures on X such that $\text{supp } \mu \subset B$, and*

$$\int_X \varphi(v) \, d\mu(v) = \int_X \varphi(v) \, d\nu(v) \quad \text{for any } \varphi \in C_b(X_w). \tag{38}$$

Then, $\mu = \nu$.

Proof We will prove our result in two steps:

Step 1. $\text{supp } \nu \subset B$.

Both μ and ν are regular. In particular, as the set $X \setminus B$ is a Borel subset of X ,

$$\nu(X \setminus B) = \sup\{\nu(K) : K \subset X \setminus B, \quad K \text{ weakly compact in } X\}. \tag{39}$$

Let $K \subset X \setminus B$ be a weakly compact set. Then, B and K are disjoint weakly compact sets, and evidently there exists $R_K > 0$ such that $B \cup K \subset B(0, R_K)$. As $B(0, R_K)$ with the weak topology is metrisable, we deduce that there exists $\psi \in C(B(0, R_K)_w)$ such that $0 \leq \psi \leq 1$, $\psi|_K \equiv 1$ and $\psi|_B \equiv 0$. Taking into account that for any $n \geq 2$ the closed ball $B(0, nR_K)$ is also metrisable, we deduce that there exists $\varphi \in C(X_w)$ with $0 \leq \varphi \leq 1$, such that $\varphi|_{B(0, R_K)} \equiv \psi$.

Then, from (38) and the fact that $\text{supp } \mu \subset B$, we deduce that

$$\nu(K) \leq \int_X \varphi(v) \, d\nu(v) = \int_X \varphi(v) \, d\mu(v) = \int_B \varphi(v) \, d\mu(v) = 0.$$

Consequently, $\nu(K) = 0$, and from (39) we deduce that $\nu(X \setminus B) = 0$. This proves that $\text{supp } \nu \subset B$.

Step 2. $\mu(E) = \nu(E)$ for all Borel subset E of X .

In order to prove this, observe that as $\mu(E) = \mu(E \cap B)$ and $\nu(E) = \nu(E \cap B)$, it is enough to prove the equality $\mu(E) = \nu(E)$ for all Borel subset of X such that $E \subset B$.

Let $E \subset B$ a Borel subset of X , and fix $\varepsilon > 0$. As μ and ν are regular, from (A3) and (A4) on page 220 in [9], we deduce that there exist a weakly compact set K of X , and a weakly open set \mathcal{O} of X , with $K \subset E \subset \mathcal{O}$, such that

$$\mu(E \setminus K) \leq \varepsilon, \quad \nu(\mathcal{O} \setminus E) \leq \varepsilon.$$

Thus, taking $\mathcal{O} = \mathcal{O} \cap B$, we obtain a set $E \subset \mathcal{O} \subset B$ which is open in the relative weak topology of B , such that

$$\nu(\mathcal{O} \setminus E) \leq \varepsilon.$$

Now, as B is metrisable, there exists $\psi \in C(B_w)$ such that $0 \leq \psi \leq 1$, $\psi|_K \equiv 1$, and $\text{supp } \psi \subset \mathcal{O}$. Reasoning as in the proof of a), we can deduce that there exists $\varphi \in C(X_w)$,

with $0 \leq \varphi \leq 1$, and $\varphi|_B \equiv \psi$. Then, as $\text{supp } \mu \subset B$,

$$\begin{aligned} \int_X \varphi(v) \, d\mu(v) &= \int_B \psi(v) \, d\mu(v) \\ &\geq \int_K \psi(v) \, d\mu(v) \\ &= \mu(K) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

Thus, by (38) and the fact that $\text{supp } \nu \subset B$, we obtain

$$\begin{aligned} \mu(E) &\leq \varepsilon + \int_X \varphi(v) \, d\mu(v) \\ &= \varepsilon + \int_X \varphi(v) \, d\nu(v) \\ &= \varepsilon + \int_B \varphi(v) \, d\nu(v) \\ &= \varepsilon + \int_{\mathcal{O}} \varphi(v) \, d\nu(v) \\ &\leq \varepsilon + \nu(\mathcal{O}) \\ &\leq 2\varepsilon + \nu(E). \end{aligned}$$

Consequently, as $\varepsilon > 0$ is arbitrary, we obtain $\mu(E) \leq \nu(E)$. Interchanging the roles of μ and ν , we deduce that $\nu(E) \leq \mu(E)$, and therefore $\nu(E) = \mu(E)$. \square

As a consequence of the preceding theorem, we obtain:

Corollary 19 *Let X be a reflexive and separable Banach space, and $S : X \rightarrow X$ a Borel-measurable mapping. Assume that μ is a finite measure on X with bounded support, such that*

$$\int_X \varphi(Sv) \, d\mu(v) = \int_X \varphi(v) \, d\mu(v) \quad \text{for any } \varphi \in C_b(X_w). \tag{40}$$

Then,

$$\mu(E) = \mu(S^{-1}(E)) \quad \text{for any Borel subset } E \text{ of } X.$$

Proof If we define $\nu(E) := \mu(S^{-1}(E))$ for any Borel subset E of X , then ν is a Borel measure on X , and

$$\int_X \varphi(v) \, d\nu(v) = \int_X \varphi(Sv) \, d\mu(v).$$

Thus, by (38), $\mu = \nu$ and the equality (40) holds. \square

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