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2-Rainbow domination number of Cartesian products: $C_n \square C_3$ and $C_n \square C_5$

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Abstract A function $f : V(G) \rightarrow \mathcal{P}(\{1, \ldots, k\})$ is called a k-rainbow dominating function of G (for short $kRDF$ of G) if $\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\}$, for each vertex $v \in V(G)$ with $f(v) = \varnothing$. By $w(f)$ we mean $\sum_{v \in V(G)} |f(v)|$ and we call it the weight of f in G. The minimum weight of a $kRDF$ of G is called the k-rainbow domination number of G and it is denoted by $\gamma_{rk}(G)$. We investigate the 2-rainbow domination number of Cartesian products of cycles. We give the exact value of the 2-rainbow domination number of $C_n \square C_3$ and we give the estimation of this number with respect to $C_n \square C_5$, $(n \geq 3)$. Additionally, for $n = 3, 4, 5, 6$, we show that $\gamma_{r2}(C_n \square C_5) = 2n$.

Keywords Domination · Rainbow domination · Cartesian product of graphs

1 Introduction

For notation and graph theory terminology not given here, we follow Diestel (1997) and also Haynes et al. (1998). Let $G = (V(G), E(G))$ be a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex v is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For two subsets A, B of $V(G)$, $E(A, B) = \{ab \in E(G) : a \in A, b \in B\}$.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We restrict our attention to the Cartesian product of C_n and C_m, $n, m \geq 3$. Let $V(C_n) = \{1, 2, \ldots, n\}$, $E(C_n) = \{i(i+1), 1n :$

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$i = 1, 2, \dots, n-1\}$ and $V(C_m) = \{1, 2, \dots, m\}$, $E(C_m) = \{j(j+1), 1m : j = 1, 2, \dots, m-1\}$. Let (i, j) be a vertex of $C_n \Box C_m$—instead of (i, j) we write v_{ij}. By $\mathcal{C}^i = \{v_{i1}, v_{i2}, \dots, v_{im}\}$ we mean the ith column of $C_n \Box C_m$.

A function $f : V(G) \to \mathcal{P}(\{1, \dots, k\})$ is called a *k-rainbow dominating function of G* (for short $kRDF$ of G), if $\bigcup_{u \in N(v)} f(u) = \{1, \dots, k\}$ for each vertex $v \in V(G)$ with $f(v) = \emptyset$. By $w(f)$ we mean $\sum_{v \in V(G)} |f(v)|$ and we call it the *weight of a function f* in G. The minimum weight of a $kRDF$ of G is called the *k-rainbow domination number of G* and it is denoted by $\gamma_{rk}(G)$. If f is a $2RDF$ function of G and $w(f) = \gamma_{r2}(G)$, then f is called a *γ_{r2}-function*. Let $X \subset V(G)$. By $w(f(X))$ we mean $\sum_{v \in X} |f(v)|$. Thus $w(f) = w(f(V(G)))$. For more information about rainbow domination we refer the reader to Brešar and Šumenjak (2007), Chunling et al. (2009), Wu and Rad (2010), Xu (2009).

The concept of rainbow domination seems to be of independent interest and it attracted several authors. In particular, Hartnell and Rall (2004) obtained a couple of observations about rainbow domination, for instance, $\gamma_{rk}(G) \leq k\gamma(G)$, where $\gamma(G)$ is the domination number of G. Moreover, the concept of 2-rainbow domination of a graph G coincides with the ordinary domination of the prism $G \Box K_2$ (Brešar et al. 2008). Since for any graph H, $\gamma(H) \geq |V(H)| / (\Delta(H) + 1)$ we have $\gamma_{r2}(G) = \gamma(G \Box K_2) \geq 2|V(G)| / (\Delta(G) + 2)$. As a consequence, we have

$$\frac{nm}{3} \leq \gamma_{r2}(C_n \Box C_m) \leq 2\gamma(C_n \Box C_m). \tag{1}$$

In this paper, we show that these bounds are attained for some classes of cycles.

For a $2RDF$ of $C_n \Box C_m$, instead of $f(v_{ij}) = \emptyset$, $f(v_{ij}) = \{1\}$, $f(v_{ij}) = \{2\}$, we simply write $f(v_{ij}) = 0$, $f(v_{ij}) = 1$ or $f(v_{ij}) = 2$, respectively.

Further, instead of, for example,

$$f\begin{pmatrix} v_{13} & v_{23} & \dots & v_{n3} \\ v_{12} & v_{22} & \dots & v_{n2} \\ v_{11} & v_{21} & \dots & v_{n1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 2 \\ 0 & 2 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

we simply write

$$f(V(C_n \Box C_3)) = \begin{pmatrix} 0 & 0 & \dots & 2 \\ 0 & 2 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

It is clear from the context that (in our example) $f(v_{11}) = 1$, $f(v_{12}) = 0$, $f(v_{13}) = 0$ and so on.

2 2-Rainbow domination number of $C_n \Box C_3$

Lemma 1 *For $n \geq 3$,*

$$n \leq \gamma_{r2}(C_n \Box C_3) \leq \begin{cases} n, & if\, n \equiv 0 \mod 6, \\ n+1, & if\, n \equiv 1, 2, 3, 5 \mod 6, \\ n+2, & if\, n \equiv 4 \mod 6. \end{cases}$$

Proof The lower bound follows from (1) for $m = 3$. To show the upper bound we define the functions f as follows:

For $l \geq 1$,

$$f(V(C_{6l}\Box C_3)) = \begin{pmatrix} 0&0&1&0&0&2|&\\ 0&2&0&0&1&0|&-\\ 1&0&0&2&0&0|& \end{pmatrix};$$

$$f(V(C_{6l+1}\Box C_3)) = \begin{pmatrix} 0&0&1&1&0&0&2|0&0&1&0&0&2|&\\ 0&2&2&0&0&1&0|0&2&0&0&1&0|&-\\ 1&0&0&0&2&0&0|1&0&0&2&0&0|& \end{pmatrix};$$

$$f(V\ (C_{6l+2}\Box C_3)) = \begin{pmatrix} 0&2|0&0&1&0&0&2|&\\ 1&0|0&2&0&0&1&0|&-\\ 1&0|1&0&0&2&0&0|& \end{pmatrix}.$$

For $l \geq 0$,

$$f(V\ (C_{6l+3}\Box C_3)) = \begin{pmatrix} 0&0&2|0&0&1&0&0&2|&\\ 0&\{1,2\}&0|0&2&0&0&1&0|&-\\ 1&0&0|1&0&0&2&0&0|& \end{pmatrix};$$

$$f(V(C_{6l+4}\Box C_3)) = \begin{pmatrix} 0&2&0&2|0&0&1&0&0&2|&\\ 0&2&1&0|0&2&0&0&1&0|&-\\ 1&0&0&2|1&0&0&2&0&0|& \end{pmatrix};$$

$$f(V\ (C_{6l+5}\Box C_3)) = \begin{pmatrix} 0&0&1&0&2|0&0&1&0&0&2|&\\ 0&2&0&0&1|0&2&0&0&1&0|&-\\ 1&0&0&2&0|1&0&0&2&0&0|& \end{pmatrix};$$

where "–" means that we repeat the block

$$\begin{vmatrix} 0&0&1&0&0&2\\ 0&2&0&0&1&0\\ 1&0&0&2&0&0 \end{vmatrix}$$

at most $l - 1$ times. It is not difficult to observe that f is a 2RDF of $C_n\Box C_3$ and

$$w(f) = \begin{cases} n, & if n \equiv 0 \mod 6,\\ n+1, & if n \equiv 1,2,3,5 \mod 6,\\ n+2, & if n \equiv 4 \mod 6, \end{cases}$$

proving the result. □

To show that $w(f)$ in Lemma 1 equals the 2-rainbow domination number of $C_n\Box C_3$ we use the concept introduced in Chunling et al. (2009).

Let f be any $2RDF$ of $C_n\Box C_m$ and let

$$V_0 = \{v \in C_n\Box C_m : f(v) = \varnothing\},$$

$$\begin{aligned} V_1 &= \{v \in C_n \Box C_m : f(v) = \{1\} \text{ or } f(v) = \{2\}\}, \\ V_2 &= \{v \in C_n \Box C_m : f(v) = \{1, 2\}\}, \\ V_{i_1 i_2} &= \{v \in V_0 : |N(v) \cap V_t| = i_t, t = 1, 2\}, \\ E_1 &= \{uv \in E(C_n \Box C_m) : u, v \in V_1\}, \\ E_2 &= \{uv \in E(C_n \Box C_m) : u, v \in V_2\}, \\ E_{12} &= \{uv \in E(C_n \Box C_m) : u \in V_1, v \in V_2\}. \end{aligned}$$

Obviously $V = V(C_n \Box C_m) = V_0 \cup V_1 \cup V_2$, $V_i \cap V_j = \varnothing, i, j = 0, 1, 2, i \neq j$.
Let

$$\mathcal{W} = \{V_{01}, V_{02}, V_{03}, V_{04}, V_{11}, V_{12}, V_{13}, V_{20}, V_{21}, V_{22}, V_{30}, V_{31}, V_{40}\}.$$

Observe that the collection $\mathcal{W}$ is pairwise disjoint and $V_0 = \bigcup_{S \in \mathcal{W}} S$.
Now, we establish $|E(V_0, V_1)|$. On the one hand,

$$|E(V_0, V_1)| = 4|V_1| - |E_{12}| - 2|E_1|$$

and on the other hand,
$|E(V_0, V_1)| = |V_{11}| + |V_{12}| + |V_{13}| + 2|V_{20}| + 2|V_{21}| + 2|V_{22}| + 3|V_{30}| + 3|V_{31}| + 4|V_{40}|$.
Similarly, $|E(V_0, V_2)| = 4|V_2| - |E_{12}| - 2|E_2|$ and $|E(V_0, V_2)| = |V_{01}| + |V_{11}| + |V_{21}| + |V_{31}| + 2|V_{02}| + 2|V_{12}| + 2|V_{22}| + 3|V_{03}| + 3|V_{13}| + 4|V_{04}|$.
Therefore,
$|E(V_0, V_1)| + 2|E(V_0, V_2)| = 4|V_1| + 8|V_2| - 2|E_1| - 3|E_{12}| - 4|E_2|$
and
$|E(V_0, V_1)| + 2|E(V_0, V_2)| = 2(|V| - |V_1| - |V_2|) + |V_{11}| + 3|V_{12}| + 5|V_{13}| + 2|V_{21}| + 4|V_{22}| + |V_{30}| + 3|V_{31}| + 2|V_{40}| + 2|V_{02}| + 4|V_{03}| + 6|V_{04}|$.
Hence (because of $|V| = mn$)
$6|V_1| + 12|V_2| - 3|E_{12}| - 2|E_1| - 4|E_2| = 2mn + 2|V_2| + |V_{11}| + 3|V_{12}| + 5|V_{13}| + 2|V_{21}| + 4|V_{22}| + |V_{30}| + 3|V_{31}| + 2|V_{40}| + 2|V_{02}| + 4|V_{03}| + 6|V_{04}|$.
Since $w(f) = |V_1| + 2|V_2|$, thus
$6w(f) = 2mn + 2|V_2| + |V_{11}| + 3|V_{12}| + 5|V_{13}| + 2|V_{21}| + 4|V_{22}| + |V_{30}| + 3|V_{31}| + 2|V_{40}| + 2|V_{02}| + 4|V_{03}| + 6|V_{04}| + 3|E_{12}| + 2|E_1| + 4|E_2|$.
Let

$$\begin{aligned} \beta = 2|V_2| + |V_{11}| + 3|V_{12}| + 5|V_{13}| + 2|V_{21}| + 4|V_{22}| + |V_{30}| \\ +3|V_{31}| + 2|V_{40}| + 2|V_{02}| + 4|V_{03}| + 6|V_{04}| \\ +3|E_{12}| + 2|E_1| + 4|E_2|, \end{aligned} \tag{2}$$

so

$$6w(f) = 2nm + \beta. \tag{3}$$

Lemma 2 *Let f be a $2RDF$ of $C_n \Box C_3$. If $w(f(\mathcal{C}^i)) = 0$ for some $i \in \{1, 2, \dots, n\}$, then $w(f) \geq n + 2$.*

Proof We may assume without loss of generality that $w(f(\mathcal{C}^2)) = 0$. Consider the following cases:

(1) If $\left|\left(\mathcal{C}^1 \cup \mathcal{C}^3\right) \cap V_2\right| \geq 3$, then $|V_2| \geq 3$, $|E_2| \geq 1$ and by (2), $\beta \geq 2|V_2| + 4|E_2| \geq 10$.
(2) If $\left|\left(\mathcal{C}^1 \cup \mathcal{C}^3\right) \cap V_2\right| = 2$, then $|E_{12}| \geq 2$ and $\beta \geq 2|V_2| + 3|E_{12}| \geq 10$.
(3) If $\left|\left(\mathcal{C}^1 \cup \mathcal{C}^3\right) \cap V_2\right| = 1$, then $|E_{12}| \geq 2$, $|E_1| \geq 2$ and $\beta \geq 2|V_2| + 2|E_1| + 3|E_{12}| \geq 12$.
(4) If $\left|\left(\mathcal{C}^1 \cup \mathcal{C}^3\right) \cap V_1\right| = 6$, then $|E_1| \geq 6$ and $\beta \geq 2|E_1| \geq 12$.

Further from (3), $w(f) = \left\lceil n + \frac{\beta}{6} \right\rceil \geq \left\lceil n + \frac{10}{6} \right\rceil = n + 2$, as desired. □

Theorem 3

$$\gamma_{r2}(C_n \Box C_3) = \begin{cases} n, & if n \equiv 0 \mod 6, \\ n+1, & if n \equiv 1, 2, 3, 5 \mod 6, \\ n+2, & if n \equiv 4 \mod 6. \end{cases}$$

Proof From Lemma 1 we have that $\gamma_{r2}(C_n \Box C_3) = n$ for $n \equiv 0 \bmod 6$.

For the proof it suffices to show that one cannot construct a $2RDF$ f of $C_n \Box C_3$ with $w(f) = n$ for $n \equiv 1, 2, 3, 5 \bmod 6$ and with $w(f) \leq n + 1$ for $n \equiv 4 \bmod 6$.

Suppose that $n \equiv 1, 2, 3, 5 \bmod 6$, and suppose f is $2RDF$ of $C_n \Box C_3$ such that $w(f) = n$. Lemma 2 implies that $w(f(\mathcal{C}^i)) = 1$ for $i = 1, 2, \dots, n$. Without loss of generality assume that $f(v_{11}) = 1$.

First note that $f(v_{21}) = 0$. Indeed, if $f(v_{21}) \in V_1$, then $w(f(\mathcal{C}^3)) \geq 2$ (otherwise v_{22} and v_{23} would not be dominated).

So either $v_{22} \in V_1$ or $v_{23} \in V_1$. Suppose that $v_{22} \in V_1$, then $f(v_{22}) = 2$ (otherwise $w(f(C^3)) \geq 2$). Observe that it must be that $f(v_{33}) = 1$, since v_{23} must be dominated. Continuing in this way, we obtain that

$$\left(\begin{array}{c||cccccccccccc} 3 & 0 & 0 & 1 & 0 & 0 & 2 & \dots & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots & 0 & 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & \dots & 1 & 0 & 0 & 2 & 0 \\ \hline\hline & 1 & 2 & 3 & 4 & 5 & 6 & \dots & 6l+1 & 6l+2 & 6l+3 & 6l+4 & 6l+5 \end{array}\right).$$

This shows that f is not a $2RDF$ of $C_n \Box C_3$ for $n \equiv 1, 2, 3, 4, 5 \bmod 6$, because of $v_{13} \in V_0$ but $2 \notin \bigcup_{x \in N(v_{13})} f(x)$.

The similar fact holds for $f(v_{23}) = 2$. Namely, we have the following situation:

$$\left(\begin{array}{c||cccccccccccc} 3 & 0 & 2 & 0 & 0 & 1 & 0 & \dots & 0 & 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 & 2 & \dots & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & \dots & 1 & 0 & 0 & 2 & 0 \\ \hline\hline & 1 & 2 & 3 & 4 & 5 & 6 & \dots & 6l+1 & 6l+2 & 6l+3 & 6l+4 & 6l+5 \end{array}\right).$$

Therefore, $\gamma_{r2}(C_n \Box C_3) > n$ for $n \equiv 1, 2, 3, 4, 5 \bmod 6$. By Lemma 1 we have that $\gamma_{r2}(C_n \Box C_3) = n + 1$ for $n \equiv 1, 2, 3, 5 \bmod 6$.

Finally, suppose that $\gamma_{r2}(C_n \Box C_3) = n+1$ for $n \equiv 4 \bmod 6$. By Lemma 2, we may assume that $w(f(\mathcal{C}^i)) = 1$ for $i = 1, 2, \ldots, n-1$. Suppose that $f(v_{11}) = 1$. By above we have

$$\left(\begin{array}{c||ccccccccccc} 3 & 0 & 0 & 1 & 0 & 0 & 2 & \ldots & 0 & 0 & 1 & . \\ 2 & 0 & 2 & 0 & 0 & 1 & 0 & \ldots & 0 & 2 & 0 & . \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & \ldots & 1 & 0 & 0 & . \\ \hline\hline & 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 6l+1 & 6l+2 & 6l+3 & 6l+4 \end{array}\right)$$

or

$$\left(\begin{array}{c||ccccccccccc} 3 & 0 & 2 & 0 & 0 & 1 & 0 & \ldots & 0 & 2 & 0 & . \\ 2 & 0 & 0 & 1 & 0 & 0 & 2 & \ldots & 0 & 0 & 1 & . \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & \ldots & 1 & 0 & 0 & . \\ \hline\hline & 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 6l+1 & 6l+2 & 6l+3 & 6l+4 \end{array}\right)$$

and it is easy to check that one cannot dominate all vertices of $\mathcal{C}^1 \cup \mathcal{C}^{6l+3} \cup \mathcal{C}^{6l+4}$ to obtain a $2RDF$ with $w(f) = n+1$. So, by Lemma 1 we have that $\gamma_{r2}(C_n \Box C_3) = n+2$ for $n \equiv 4 \bmod 6$. □

Roughly speaking, $\gamma_{r2}(C_n \Box C_3)$ is very close to the general lower bound in (1). Note that for $C_{6l} \Box C_3$ the lower bound in (1) is attained.

3 2-Rainbow domination number of $C_n \Box C_5$

Now, we give an upper bound of the 2-rainbow domination number of $C_n \Box C_5$. Moreover, we show that this bound is attained for small n. We do believe that it is also attained for any $n \geq 3$. First we give the useful result.

Lemma 4 *For any $n \geq 8$ there exist nonnegative integers a, b such that $n = 5a + 3b$.*

In general, the above fact is known as Frobenius problem.

Theorem 5 *For $n \geq 3$, $\gamma_{r2}(C_n \Box C_5) \leq 2n$.*

Proof For $n \leq 7$, let us define the functions f as follows

$$f_3 = f(V(C_3 \Box C_5)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \quad f(V(C_4 \Box C_5)) = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$f_5 = f(V(C_5 \Box C_5)) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix}.$$

Observe that

$$f_3|f_3 \stackrel{df}{=} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

is a 2RDF of $C_6 \Box C_5$. Further,

$$f(V(C_7 \Box C_5)) = \begin{pmatrix} 2 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

By above, it is an easy exercise to check that $w(f) = 2n$, for $n \le 7$.

Additionally, note that the functions $f_5|f_5$, $f_5|f_3$ and $f_3|f_5$ are also 2RDFs of $C_{10}\Box C_5$, $C_8 \Box C_5$, $C_8 \Box C_5$, respectively.

Let $n \ge 8$, then by Lemma 4 we have $n = 5a + 3b$, $a, b \ge 0$. Let $f : V(C_n) \to \mathcal{P}(\{1, 2\})$ be defined as follows:

$$f(V(C_n \Box C_5)) = \overbrace{f_5 \,|\cdots|\, f_5}^{a \text{ times}}|\overbrace{f_3 \,|\cdots|\, f_3}^{b \text{ times}}.$$

it is easy to verify that f is a $2RDF$ of $C_n \Box C_5 (n \ge 8)$ and $w(f) = 2n$. Therefore, $\gamma_{r2}(C_n \Box C_5) \le 2n (n \ge 3)$, as desired. □

Our aim is to show that for $n = 3, 4, 5, 6$, $\gamma_{r2}(C_n \Box C_5) = 2n$. For this purpose, we prove the following Lemma.

Lemma 6 *Let f be a $2RDF$ of $C_n \Box C_5$ and $\mathcal{C}^r, \mathcal{C}^s, \mathcal{C}^t$ be three consecutive columns of $C_n \Box C_5$.*

(i) *If $w(f(\mathcal{C}^s)) = 0$, then $w(f\left(\mathcal{C}^r \cup \mathcal{C}^t\right)) \ge 10$.*
(ii) *If $w(f(\mathcal{C}^s)) = 1$, then $w(f\left(\mathcal{C}^r \cup \mathcal{C}^t\right)) \ge 6$.*

Proof The proof follows immediately from the definition of a $2RDF$. □

Theorem 7 *For $n = 3, 4, 5, 6$, $\gamma_{r2}(C_n \Box C_5) = 2n$.*

Proof By Theorem 3 $\gamma_{r2}(C_5 \Box C_3) = \gamma_{r2}(C_3 \Box C_5) = 6$, as required. Further, by Theorem 5, it suffices to show that $\gamma_{r2}(C_n \Box C_5) \ge 2n$ for $n = 4, 5, 6$. Let f be a γ_{r2}-function of $C_n \Box C_5$ for $n = 4, 5, 6$. If $w(f(\mathcal{C}^i)) \ge 2$ for any $i \in \{1, 2, \ldots, n\}$, then $\gamma_{r2}(C_n \Box C_5) \ge 2n$. Otherwise, there exists $k \in \{1, 2, \ldots, n\}$ such that $w(f(\mathcal{C}^k)) \le 1$. Without loss of generality suppose that $k = 2$.

First assume that $w(f(\mathcal{C}^2)) = 0$. By Lemma 6 (i) for $s = 2$, $w\left(f\left(\mathcal{C}^1 \cup \mathcal{C}^3\right)\right) \ge 10$. Thus for $n = 4, 5$ we have $\gamma_{r2}(C_n \Box C_5) \ge 2n$. Further, suppose that

$\gamma_{r2}(C_5 \Box C_6) = w\left(f\left(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3\right)\right) + w\left(f\left(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6\right)\right) \leq 11$. This implies that $w\left(f\left(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6\right)\right) = 1$ but then f is not a $2RDF$.

It remains to consider the case where $w(f(\mathcal{C}^2)) = 1$ and $w(f(\mathcal{C}^i)) \neq 0$ for other i. From Lemma 6 (ii) for $s = 2$, we have $w\left(f\left(\mathcal{C}^1 \cup \mathcal{C}^3\right)\right) \geq 6$. Further $w\left(f\left(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3\right)\right) \geq 7$. Therefore, $\gamma_{r2}(C_5 \Box C_4) \geq 2n = 8$.

Let $n = 5$. Suppose that $w(f) = w(f\left(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3\right)) + w(f\left(\mathcal{C}^4 \cup \mathcal{C}^5\right)) \leq 9$, so $w(f(\mathcal{C}^4)) = 1 = w(f(\mathcal{C}^5))$. Applying Lemma 6 (ii) to $\mathcal{C}^4$ and $\mathcal{C}^5$ we get $w(f(\mathcal{C}^3)) \geq 5$ and $w(f(\mathcal{C}^1)) \geq 5$. Thus we obtain $\gamma_{r2}(C_5 \Box C_5) \geq 5 + 1 + 5 + 1 + 1 = 13$. However, this contradicts our assumption.

Let $n = 6$. Suppose that $\gamma_{r2}(C_5 \Box C_6) = w(f(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3 \cup \mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6)) = w(f(\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^3)) + w(f(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6)) \leq 11$. Thus we have $w(f(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6)) = 3$ or 4. The condition $w(f(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6)) = 3$ implies that $w(f(\mathcal{C}^4)) = w(f(\mathcal{C}^5)) = w(f(\mathcal{C}^6)) = 1$ but it is impossible by Lemma 6 (ii). For $w(f(\mathcal{C}^4 \cup \mathcal{C}^5 \cup \mathcal{C}^6)) = 4$ the possibilities for $w(f(\mathcal{C}^4))$, $w(f(\mathcal{C}^5))$, $w(f(\mathcal{C}^6))$, are (i) $1, 1, 2$, (ii) $2, 1, 1$, (iii) $1, 2, 1$. Cases (i) and (ii) can be eliminated because of Lemma 6 (ii). In case (iii), applying Lemma 6 (ii) to $\mathcal{C}^4$ and $\mathcal{C}^5$ we get $w(f(\mathcal{C}^3)) \geq 4$ and $w(f(\mathcal{C}^1)) \geq 4$. Thus we obtain $\gamma_{r2}(C_5 \Box C_6) \geq 4 + 1 + 4 + 1 + 2 + 1 = 13$. However, this contradicts our assumption. □

Since $\gamma(C_5 \Box C_5) = 5$, we have that $\gamma_{r2}(C_5 \Box C_5) = 10 = 2\gamma(C_5 \Box C_5)$, see Klavžar and Seifter (1995). Thus for this graph the upper bound in (1) is attained.

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