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A tight analysis of Brown-Baker-Katseff sequences for online strip packing

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Abstract We study certain adversary sequences for online strip packing which were first designed and investigated by Brown, Baker and Katseff (Acta Inform. 18:207–225) and determine the optimal competitive ratio for packing such Brown-Baker-Katseff sequences online. As a byproduct of our result, we get a new lower bound of $\rho \geq 3/2 + \sqrt{33}/6 \approx 2.457$ for the competitive ratio of online strip packing.

Keywords Online algorithm · Strip packing · Competitive ratio

1 Introduction

In the two-dimensional strip packing problem a number of rectangles have to be packed without rotation or overlap into a strip such that the height of the strip used is minimum. The width of the rectangles is bounded by 1 and the strip has width 1 and infinite height. Baker et al. (1980) show that this problem is NP-hard.

We study the online version of this packing problem. In the online version the rectangles are given to the online algorithm one by one from a list, and the next rectangle is given as soon as the current rectangle is irrevocably placed into the strip. To evaluate the performance of an online algorithm we employ competitive analysis. For a list of rectangles L, the height of a strip used by online algorithm A and by the optimal solution is denoted by $A(L)$ and $OPT(L)$, respectively. The optimal solution is not restricted in any way by the ordering of the rectangles in the list. Competitive analysis measures the absolute worst-case performance of online algorithm A by its competitive ratio $\sup_L\{A(L)/OPT(L)\}$ (cf. Pruhs et al. 2004).

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Regarding the upper bound on the competitive ratio for online strip packing, recent advances have been made by Ye et al. (2009) and Hurink and Paulus (2008). Independently they show that a modification of the well-known shelf algorithm yields an online algorithm with competitive ratio $7/2 + \sqrt{10} \approx 6.6623$. We refer to these two papers for a more extensive overview of the literature.

In the early 80's, Brown et al. (1982) derived a lower bound $\rho \geq 2$ on the competitive ratio of any online algorithm by constructing certain (adversary) sequences in a fairly straightforward way (cf. Sect. 2). These sequences were further studied by Johannes (2006) and Hurink and Paulus (2008), who derived improved lower bounds of 2.25 and 2.43, resp. (Both results are computer aided and presented in terms of online parallel machine scheduling, a closely related problem.) The paper of Hurink and Paulus (2008) also presents an upper bound of $\rho \leq 2.5$ for packing such "Brown-Baker-Katseff sequences". The purpose of our present paper is to close the gap between 2.43 and 2.5 by presenting a tight analysis, showing that Brown-Baker-Katseff sequences can be packed online with competitive ratio $\rho = 3/2 + \sqrt{33}/6$ and that this is best possible. As a byproduct, we obtain a new lower bound $\rho \approx 2.457$ for online strip packing.

The result of this paper has been presented at the CTW 2010 (cf. Kern 2010). Meanwhile, in a joint work with R. Harren, we tried to analyze a modified version of Brown-Baker-Katseff sequences, yielding a slightly better lower bound (but no exact analysis as here), cf. Harren and Kern (2011).

2 The instance construction

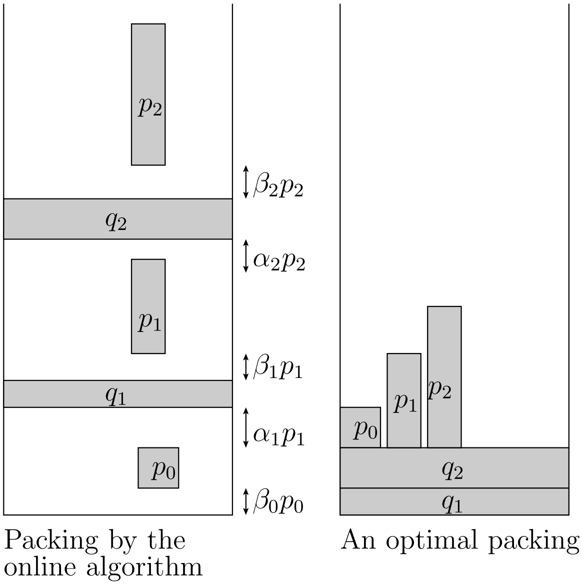
In this section we describe the construction of Brown-Baker-Katseff sequences L_n according to Brown et al. (1982). In addition, we present an online algorithm for packing the sequences L_n online with ratio $\hat{\rho} = 3/2 + \sqrt{33}/6$. For convenience, let throughout this note $\hat{\rho} = 3/2 + \sqrt{33}/6$.

We define L_n as the list of rectangles $(p_0, q_1, p_1, q_2, p_2, \ldots, q_n, p_n)$, where p_i denotes a rectangle of height p_i and negligible width (no more than $1/(n+1)$), and q_i denotes a rectangle of height q_i and width 1. The rectangle heights are defined such that, when the items are packed online, each item must be packed on top of the preceding ones. More precisely, we let

$$\begin{aligned}
p_0 &= 1, \\
p_i &= \beta_{i-1} p_{i-1} + p_{i-1} + \alpha_i p_i + \epsilon \quad \forall i \geq 1, \\
q_1 &= \beta_0 p_0 + \epsilon, \\
q_i &= \max\{\alpha_{i-1} p_{i-1}, q_{i-1}, \beta_{i-1} p_{i-1}\} + \epsilon \quad \forall i \geq 2,
\end{aligned}$$

where $\alpha_i p_i$ and $\beta_i p_i$ are distances the online algorithm has placed between earlier rectangles, and ϵ is a small positive value. The value $\alpha_i p_i$ denotes the vertical distance between rectangles p_{i-1} and q_i, and the value $\beta_i p_i$ denotes the vertical distance between q_i and p_i. This is illustrated in Fig. 1. The values α_i and β_i completely characterize the behavior of the online algorithm when processing L_n. For consistency we define in addition $\alpha_0 = 0$.

Fig. 1 Online and optimal packing of L_2



By definition of the rectangles' heights and widths, an online algorithm can only pack the rectangles one above the other in the same order as the rectangles appear in the list L_n. An optimal offline packing is obtained by first packing the rectangles q_i on top of each other and then pack all p_i next to each other on top of the q-rectangles. The sole function of the positive term ϵ is to ensure this structure on any online packing. From now on we assume that ϵ is small enough to be omitted from the analysis.

We start with the (simpler) upper bound:

Theorem 1 *Each list L_n can be packed online with competitive ratio $\hat{\rho} = \frac{3}{2} + \frac{\sqrt{33}}{6}$.*

Proof Consider the online algorithm A that chooses $\beta_0 = \hat{\rho} - 1$, $\alpha_2 = 1/(\hat{\rho} - 1)$, and all other gaps equal to 0. So $p_0 = 1, q_1 = q_2 = \hat{\rho} - 1, p_1 = \hat{\rho}$, and p_2 can be computed from $p_2 = p_1 + \alpha_2 p_2$: We get $p_2 = \hat{\rho}/(1 - \frac{1}{\hat{\rho}-1}) = \frac{(\hat{\rho}-1)\hat{\rho}}{\hat{\rho}-2}$, or $p_2 = (\hat{\rho} - 1)(3\hat{\rho} - 2) = 4\hat{\rho} - 2$ by our choice of $\hat{\rho}$.

We claim that the resulting algorithm is $\hat{\rho}$-competitive when presented with L_n.

- After packing $p_0 = 1$ we have $A(L_0) = \hat{\rho}$ and $OPT(L_0) = 1$. Thus, the competitive ratio is exactly $\hat{\rho}$ at this point.
- After packing rectangle q_1 the online and optimal packing increase by the same amount. Thus the competitive ratio decreases.
- After $p_1 = \hat{\rho}$ we have $A(L_1) = \beta_0 p_0 + p_0 + q_1 + p_1 = \hat{\rho} - 1 + 1 + \hat{\rho} - 1 + \hat{\rho}$, while $OPT(L_1) = q_1 + p_1 = 2\rho - 1$. Hence $A(L_1)/OPT(L_1) = (3\hat{\rho} - 1)/(2\hat{\rho} - 1) < \hat{\rho}$.
- After q_2 we have $A(L_1 q_2) = A(L_1) + q_2 + \alpha_2 p_2 = 3\hat{\rho} - 1 + \hat{\rho} - 1 + 3\hat{\rho} - 2 = 7\hat{\rho} - 4$, while $OPT(L_1 q_2) = p_1 + q_1 + q_2 = 3\rho - 2$. So $A(L_1 q_2)/OPT(L_1 q_2) = (7\hat{\rho} - 4)/(3\hat{\rho} - 2) = \hat{\rho}$. Again the competitive ratio is exactly $\hat{\rho}$ at this point (by definition of $\hat{\rho}$).

- After p_2 we have $A(L_2) = A(L_1 q_2) + p_2 = 7\hat{\rho} - 4 + 4\hat{\rho} - 2 = 11\hat{\rho} - 6$, while $OPT(L_2) = q_1 + q_2 + p_2 = 2\hat{\rho} - 2 + 4\hat{\rho} - 2 = 6\hat{\rho} - 4$. Thus $A(L_2)/OPT(L_2) = (11\hat{\rho} - 6)/(6\hat{\rho} - 4) < \hat{\rho}$.
- For $i \geq 3$ there are no more gaps introduced by online algorithm A. In particular, $p_2 = p_3 = p_4 = \cdots = (\hat{\rho} - 1)(3\hat{\rho} - 2)$ and $q_i = \alpha_2 p_2 = 3\hat{\rho} - 2$ for all $i \geq 3$. When packing q_i, the online and optimal packing increase by the same amount and, thus, ρ-competitiveness is not violated. After packing p_{i+1}, however, we have $OPT(L_{i+1}) = OPT(L_i) + q_{i+1}$ and $A(L_{i+1}) = A(L_i) + q_{i+1} + p_{i+1} = A(L_i) + \hat{\rho} q_{i+1}$. The height of the online packing grows exactly $\hat{\rho}$ times as fast as the optimal packing.

So, online algorithm A is $\hat{\rho}$-competitive for the list of rectangles L_n. □

3 Lower bound on the competitive ratio

In this section we prove a lower bound of $\hat{\rho} = 3/2 + \sqrt{33}/6$ on the competitive ratio for online packing Brown-Baker-Katseff sequences—and hence for online strip packing in general. The outline of the proof is as follows.

Assume that there exists a ρ-competitive online algorithm A with $\rho < \hat{\rho}$. We present this algorithm with the list L_n, with n arbitrarily large. To obtain a contradiction we define a potential function Φ_i on the state of the online packing after packing rectangle p_i. We argue that this potential function is both bounded from below and that it decreases to $-\infty$, giving us the required contradiction.

After packing the rectangle p_i, we measure with γ_i how much online algorithm A improves upon the ρ-competitiveness bound: We define γ_i through

$$A(L_i) + \gamma_i p_i = \rho OPT(L_i).$$

The potential function Φ_i is defined (after packing rectangle p_i) by

$$\Phi_i := \frac{\gamma_i + \beta_i - (\rho - 2)\alpha_i}{1 - \alpha_i}.$$

We admit that the potential function looks rather involved. Its exact form is simply motivated by the technical analysis that follows. In particular, the potential function is designed such that a certain "shift invariance" (cf. Lemma 2 below) holds, which helps a lot in simplifying the analysis. (Yet, in Harren and Kern 2011 we analyze "extended" Brown-Baker-Katseff sequences with a simpler potential $\Phi_i = \gamma_i + \beta_i$ and considerable technical effort.)

The values of α_i and β_i are nonnegative by definition and γ_i is nonnegative by the ρ-competitiveness of online algorithm A. Observe that shifting p_i, say, upward, increases β_i and decreases γ_i by the same amount, so that shifting p_i has actually no effect on Φ_i. In Lemma 2 we will see that the same holds w.r.t. shifting q_i. Thus the decisions of the online algorithm in "phase i" have no effect on Φ_i, but rather on subsequent potential values. (This phenomenon is observed also elsewhere, cf., e.g. Fuchs et al. 2005 or Harren and Kern 2011.) The crucial result is Lemma 6, stating that $\rho < \hat{\rho}$ forces a significant decrease of the potential in each step. Thus

$\Phi_{i+1} \to -\infty$ for $i \to \infty$. On the other hand, in Lemma 5 we show among other things that $\alpha_i < 1/(\hat{\rho} - 1)$. As a consequence, $\Phi_i > -1$ for all i. This contradiction finally implies $\rho \geq \hat{\rho}$, the main result of this note:

Theorem 2 *The best possible ratio for packing Brown-Baker-Katseff sequences is* $\hat{\rho} = 3/2 + \sqrt{33}/6$. *This value also provides a lower bound for online strip packing.*

The remainder of this note is concerned with the proofs of the afore mentioned lemmata. We start by providing some basic properties (Lemmas 1 and 2 below) of the potential function Φ_i. In the following, ρ is assumed to be suitably close to $\hat{\rho}$ whenever this is necessary.

First observe that the following equation defines p_{i+1} according to the construction rule for the list L_n. We will use it at several places during our analysis:

$$(\beta_i + 1)p_i = (1 - \alpha_{i+1})p_{i+1}. \tag{1}$$

Lemma 1 *If* $\Phi_i \leq \rho - 1$ *then* $\gamma_i + \beta_i + \alpha_i \leq \rho - 1$.

Proof

$$\begin{aligned} \Phi_i \leq \rho - 1 \quad &\Rightarrow \quad \gamma_i + \beta_i - (\rho - 2)\alpha_i \leq (\rho - 1)(1 - \alpha_i) \\ &\Rightarrow \quad \gamma_i + \beta_i + \alpha_i \leq \rho - 1. \end{aligned}$$ □

Lemma 2 *The potential* Φ_i *is invariant under shifting* p_i *and/or* q_i.

Proof Shifting rectangle p_i, say, upward by one unit does not affect $OPT(L_i)$ nor α_i. Furthermore, $A(L_i)$ increases by one unit and, hence, $\gamma_i p_i$ decreases by one unit. At the same time $\beta_i p_i$ increases by one unit, so that $\gamma_i + \beta_i$ remains constant and Φ_i is invariant under shifting p_i.

To show that Φ_i is invariant under shifting q_i, assume w.l.o.g. that $\beta_i = 0$, i.e., that p_i has been shifted down to q_i, and that we shift the concatenated rectangles $q_i p_i$ simultaneously, say, upward. This causes an increase of p_i by one unit, so $A(L_i)$ increases in total by 2 units. On the other hand, $OPT(L_i)$ increases by just one unit, so $\gamma_i p_i$ will increase by $\rho - 2$ units. Clearly, $\alpha_i p_i$ increases by one unit and $(1 - \alpha_i)p_i = (1 + \beta_{i-1})p_{i-1}$ remains constant. Hence

$$\Phi_i = \frac{\gamma_i p_i - (\rho - 2)\alpha_i p_i}{(1 - \alpha_i)p_i}$$

will indeed remain constant. □

Clearly, shifting p_i and/or q_i *does* have an effect on subsequent values like p_{i+1} and Φ_{i+1}. Furthermore, even when we are only interested in packing L_i, shifting up p_i or shifting down the concatenated $q_i p_i$ may result in an infeasible packing. (Indeed, as we shift $q_i p_i$ down, p_i decreases and after a while the packing may cease to be ρ-competitive!) As long as we are only interested in the *value* of Φ_i, however, we may well shift p_i and q_i as we like, disregarding the competitiveness constraints. We will make extensive use of this observation further on.

Lemma 3

$$\Phi_{i+1} = \frac{\gamma_i + (\rho - 1)\beta_i - 1 + (\rho - 1)q_{i+1}/p_i}{1 + \beta_i}.$$

Proof By Lemma 2 we can shift rectangle p_{i+1} down, i.e. $\beta_{i+1} = 0$. Then

$$\begin{aligned}
(1 - \alpha_{i+1})p_{i+1}\Phi_{i+1} &= \big(\gamma_{i+1} + \beta_{i+1} - (\rho - 2)\alpha_{i+1}\big)p_{i+1} \\
&=_{(\text{Lem. } 2)} \big(\gamma_{i+1} - (\rho - 2)\alpha_{i+1}\big)p_{i+1} \\
&= \rho OPT(L_{i+1}) - A(L_{i+1}) - (\rho - 2)\alpha_{i+1}p_{i+1} \\
&= \rho\big(OPT(L_i) + q_{i+1} + \beta_i p_i + \alpha_{i+1}p_{i+1}\big) \\
&\quad - \big(A(L_i) + \alpha_{i+1}p_{i+1} + q_{i+1} + p_{i+1}\big) \\
&\quad - (\rho - 2)\alpha_{i+1}p_{i+1} \\
&= \gamma_i p_i + \rho\beta_i p_i - (1 - \alpha_{i+1})p_{i+1} + (\rho - 1)q_{i+1} \\
&=_{(1)} \big(\gamma_i + (\rho - 1)\beta_i - 1\big)p_i + (\rho - 1)q_{i+1}.
\end{aligned}$$

By (1) we can divide the left hand side by $(1 - \alpha_{i+1})p_{i+1}$ and the right hand side by $(1 + \beta_i)p_i$ to obtain the result. □

Lemma 4 *If* $q_{i+1} = \max\{\alpha_i p_i, q_i\}$, *then we may assume w.l.o.g. that* $\beta_i = 0$.

Proof Shifting rectangle p_i down decreases the distance $\beta_i p_i$ and increases $\alpha_{i+1}p_{i+1}$. However, when we keep all other distances equal it does not affect p_j with $j > i$. Due to the increase in $\alpha_{i+1}p_{i+1}$ some q_j with $j > i$ may increase, but this is only in favor of the online algorithm since the optimal value increases by exactly the same amount. So the alternative online algorithm that schedules p_i earlier and leaves all other distances unchanged is also feasible. □

Lemma 5 *For* $i \geq 0$, $\Phi_i \leq \rho - 1$, *and for* $i \geq 1$, $\alpha_i \leq 1/(\hat{\rho} - 1)$ *and* $q_i/p_i \leq 1(\hat{\rho} - 1)$. *In case* $q_i = \alpha_{i-1}p_{i-1}$ *or* $q_i = q_{i-1}$, *we even have* $q_i/p_i \leq (1 - \alpha_i)/(\hat{\rho} - 1)$.

Proof By induction: The claim holds for $i = 0$ since $\alpha_0 = 0$ by definition, $\gamma_0 + \beta_0 = \rho - 1$ and thus $\Phi_0 = \rho - 1$. We assume the lemma holds up to i, and prove it for $i + 1$ by case distinction on the way the height of rectangle q_{i+1} is determined. (For $i = 1$, case 2 applies.)

Case 1: $q_{i+1} = \alpha_i p_i$.

By Lemma 4 we may assume $\beta_i = 0$. By (1), this further implies $p_i = (1 - \alpha_{i+1})p_{i+1}$. Hence

$$\frac{q_{i+1}}{p_{i+1}} = \frac{\alpha_i p_i}{p_i + \alpha_{i+1}p_{i+1}} = \frac{\alpha_i(1 - \alpha_{i+1})p_{i+1}}{(1 - \alpha_{i+1})p_{i+1} + \alpha_{i+1}p_{i+1}} = (1 - \alpha_{i+1})\alpha_i \leq \frac{1 - \alpha_{i+1}}{\hat{\rho} - 1}$$

by induction.

The online algorithm A is by assumption ρ-competitive after packing rectangle q_{i+1}, which means that the distance between rectangles q_{i+1} and p_i is not too large, i.e., $\alpha_{i+1} p_{i+1} \le \gamma_i p_i + (\rho-1) q_{i+1} = \gamma_i p_i + (\rho-1)\alpha_i p_i$. Together with Lemma 1 this gives

$$\alpha_{i+1} = \frac{\alpha_{i+1} p_{i+1}}{p_{i+1}} = \frac{\alpha_{i+1} p_{i+1}}{p_i + \alpha_{i+1} p_{i+1}} \le \frac{\gamma_i + (\rho-1)\alpha_i}{1 + \gamma_i + (\rho-1)\alpha_i}$$
$$\le \frac{(\rho-1)^2}{1+(\rho-1)^2} < \frac{1}{\hat{\rho}-1}$$

for $\rho \le \hat{\rho}$. (This upper bound for α_{i+1}, obtained by setting $\gamma_i = 0$ and $\alpha_i = \rho - 1$ (cf. Lemma 1) is rather weak, but sufficient for our purposes.)
Finally, by Lemma 3, the induction assumption and Lemma 1 we get

$$\Phi_{i+1} = \gamma_i - 1 + (\rho-1) q_{i+1}/p_i = \gamma_i - 1 + (\rho-1)\alpha_i < \gamma_i \le \rho - 1.$$

Case 2: $q_{i+1} = \beta_i p_i$.
By Lemma 1 we have $\beta_i \le \rho - 1$ and thus,

$$\frac{q_{i+1}}{p_{i+1}} = \frac{\beta_i p_i}{(1+\beta_i) p_i + \alpha_{i+1} p_{i+1}} \le \frac{\beta_i}{1+\beta_i} \le \frac{\rho-1}{\rho} \tag{2}$$

which is less than $1/(\hat{\rho}-1)$ for $\rho \le \hat{\rho}$.
The online algorithm A is by assumption ρ-competitive after packing rectangle q_{i+1}, which means that the distance between rectangles q_{i+1} and p_i is not too large, i.e. $\alpha_{i+1} p_{i+1} \le \gamma_i p_i + (\rho-1) q_{i+1} = \gamma_i p_i + (\rho-1)\beta_i p_i$. This, together with $\beta_i + \gamma_i \le \rho - 1$ (by Lemma 1) gives

$$\alpha_{i+1} = \frac{\alpha_{i+1} p_{i+1}}{p_{i+1}} = \frac{\alpha_{i+1} p_{i+1}}{(1+\beta_i) p_i + \alpha_{i+1} p_{i+1}} \le \frac{\gamma_i + (\rho-1)\beta_i}{(1+\beta_i) + \gamma_i + (\rho-1)\beta_i}$$
$$= \frac{\gamma_i + (\rho-1)\beta_i}{1 + \gamma_i + \rho\beta_i} \le \frac{(\rho-1)}{1+(\rho-1)} < \frac{1}{\hat{\rho}-1}$$

for $\rho \le \hat{\rho}$. (In the upper bound computation for α_i above we assumed that γ_i is as large as possible and β_i is as small as possible. This is justified by the fact that $\frac{\rho-1}{\rho} < \frac{1}{\hat{\rho}-1}$.)
For the potential, Lemma 3 gives

$$\Phi_{i+1} = \frac{\gamma_i + 2(\rho-1)\beta_i - 1}{1+\beta_i} \le \frac{2(\rho-1)^2 - 1}{\rho}, \tag{3}$$

which is strictly less than $\rho - 1$ for ρ sufficiently close to $\hat{\rho}$. (Again, for the last inequality in (3), note that increasing β_i as much as possible instead of γ_i is justified: If we let $f(\beta,\gamma) = \frac{\gamma + 2(\rho-1)\beta - 1}{1+\beta}$, then

$$\frac{\partial f}{\partial \gamma} = \frac{1}{1+\beta} < \frac{\partial f}{\partial \beta} = \frac{2(\rho-1) - \gamma + 1}{(1+\beta)^2} \quad \Leftrightarrow \quad \beta + \gamma < 2\rho - 1.$$

The latter, however, is true as we assume $\beta + \gamma \le \rho - 1$.)
Case 3: $q_{i+1} = q_i$.
Induction yields

$$\frac{q_{i+1}}{p_{i+1}} = \frac{q_i}{p_{i+1}} \le (1 - \alpha_{i+1}) \frac{q_i}{p_i} \le \frac{1 - \alpha_{i+1}}{\hat{\rho} - 1}.$$

By Lemma 4 we can assume $\beta_i = 0$, and thus

$$\Phi_{i+1} = \gamma_i - 1 + (\rho - 1)\frac{q_{i+1}}{p_i} = \gamma_i - 1 + (\rho - 1)\frac{q_i}{p_i}$$
$$\leq \gamma_i - 1 + (\rho - 1)\frac{1}{\hat{\rho} - 1} \leq \gamma_i \leq \rho - 1.$$

To argue that $\alpha_{i+1} \leq 1/(\hat{\rho} - 1)$ we shift the concatenated rectangles q_i, p_i down as far as possible, i.e., until either $\gamma_i = 0$ or $\alpha_i = 0$ (thereby increasing α_{i+1}). By shifting q_i, p_i down, the length of p_i decreases, therefore γ_i can become 0. At the same time p_{i+1} increases, causing the optimal and online solution to increase by the same amount. So the online algorithm is still ρ-competitive after this shift.
If $\gamma_i = 0$, then $\alpha_{i+1} p_{i+1} \leq (\rho - 1) q_{i+1} = (\rho - 1) q_i \leq p_i$. Thus $\alpha_{i+1} \leq 1/2 \leq 1/(\hat{\rho} - 1)$.
If $\alpha_i = 0$, the rectangles p_{i-1}, q_i, p_i are concatenated. To show $\alpha_{i+1} \leq 1/(\hat{\rho} - 1)$ for this case, we distinguish three subcases.

Case 3a: $q_{i+1} = q_i = \alpha_{i-1} p_{i-1}$.
By Lemma 4 we can assume $\beta_{i-1} = 0$ (and hence $p_i = p_{i-1}$). Thus we conclude that $\gamma_i p_i = \gamma_{i-1} p_{i-1} + (\rho - 1) q_i - p_i \leq \gamma_{i-1} p_{i-1}$. Thus (using Lemma 1 again),

$$\begin{aligned}
\alpha_{i+1} p_{i+1} &\leq (\rho - 1) q_{i+1} + \gamma_i p_i \\
&\leq (\rho - 1)\alpha_{i-1} p_{i-1} + \gamma_{i-1} p_{i-1} \\
&\leq (\rho - 1)\alpha_{i-1} p_{i-1} + (\rho - 1 - \alpha_{i-1}) p_{i-1} \\
&\leq (\rho - 1) p_{i-1} + (\rho - 2)\alpha_{i-1} p_{i-1},
\end{aligned}$$

and therefore

$$\alpha_{i+1} = \frac{\alpha_{i+1} p_{i+1}}{p_i + \alpha_{i+1} p_{i+1}} \leq \frac{\rho - 1 + (\rho - 2)\alpha_{i-1}}{\rho + (\rho - 2)\alpha_{i-1}}$$
$$\leq \frac{\rho - 1 + (\rho - 2)/(\hat{\rho} - 1)}{\rho + (\rho - 2)/(\hat{\rho} - 1)} < \frac{1}{\hat{\rho} - 1}$$

for $\rho \leq \hat{\rho}$.
Case 3b: $q_{i+1} = q_i = \beta_{i-1} p_{i-1}$.

$$\alpha_{i+1} p_{i+1} \leq (\rho - 1) q_{i+1} + \gamma_i p_i = (\rho - 1) q_i + \gamma_i p_i$$
$$\leq_{\text{by (2)}} \left(\frac{(\rho - 1)^2}{\rho} + \gamma_i\right) p_i.$$

Since $\alpha_i = 0$ and $\beta_i = 0$ we have $\Phi_i = \gamma_i \leq (2(\rho - 1)^2 - 1)/\rho$ (cf. (3)). Thus,

$$\alpha_{i+1} = \frac{\alpha_{i+1} p_{i+1}}{p_i + \alpha_{i+1} p_{i+1}} \leq \frac{\frac{(\rho-1)^2}{\rho} + \gamma_i}{1 + \frac{(\rho-1)^2}{\rho} + \gamma_i}$$
$$\leq \frac{(\rho - 1)^2/\rho + 2((\rho - 1)^2 - 1)/\rho}{1 + (\rho - 1)^2/\rho + 2((\rho - 1)^2 - 1)/\rho}.$$

The latter is increasing in ρ, so we conclude (after multiplying the enumerator and denominator with ρ) that

$$\alpha_{i+1} \leq \frac{(\hat{\rho}-1)^2 + 2((\hat{\rho}-1)^2 - 1)}{\hat{\rho} + (\hat{\rho}-1)^2 + 2((\hat{\rho}-1)^2 - 1)} = \frac{1}{\hat{\rho}-1}$$

by definition of $\hat{\rho}$. (Recall that $3\hat{\rho}^2 - 9\hat{\rho} + 4 = 0$.)

Case 3c: $q_{i+1} = q_i = q_{i-1}$. By Lemma 4 we may assume $\beta_{i-1} = 0$, so that actually $q_{i-1}, p_{i-1}, q_i, p_i$ are concatenated and $p_i = p_{i-1}$. Since $\gamma_i p_i = \gamma_{i-1} p_{i-1} + (\rho-1) q_i - p_i < \gamma_{i-1} p_{i-1}$, i.e., the improvement of A upon ρ-competitiveness decreases, the value of α_{i+1} is smaller than α_i could at most be, thus, in particular, less than $1/(\hat{\rho}-1)$. □

Lemma 6 $\Phi_{i+1} \leq \Phi_i - \frac{\hat{\rho}-\rho}{\hat{\rho}-1}$.

Proof By case distinctions:

Case 1: $q_{i+1} = \alpha_i p_i$.

By Lemma 4 we can assume $\beta_i = 0$. Furthermore, Lemma 2 allows us to shift p_{i+1} resp. the concatenated $q_{i+1} p_{i+1}$ down until $\alpha_{i+1} = \beta_{i+1} = 0$ without affecting Φ_{i+1} (although this might result in a negative value of γ_{i+1} in case Φ_{i+1} is negative). Summarizing, let us assume that $q_i, p_i, q_{i+1}, p_{i+1}$ are concatenated.

We seek to analyze how Φ_i and Φ_{i+1} vary as the concatenated $q_i, p_i, q_{i+1}, p_{i+1}$ are shifted upward. By Lemma 2, Φ_i remains unchanged. As to Φ_{i+1}, observe that shifting $q_i, p_i, q_{i+1}, p_{i+1}$ upward by one unit will increase $\alpha_i p_i$ and hence q_{i+1} as well as p_i and p_{i+1} by one unit each. Hence $A(L_{i+1})$ increases by 4. On the other hand, $OPT(L_{i+1})$ increases by 2, so that $\gamma_{i+1} p_{i+1}$ increases by $(2\rho - 4)$. Hence shifting $q_i, p_i, q_{i+1}, p_{i+1}$ upward will increase Φ_{i+1} as long as $\Phi_{i+1} < 2\rho - 4$. So we are led to distinguish the following two cases:

$\Phi_{i+1} > 2\rho - 4$: In this case, Φ_{i+1} will increase as we shift $q_i, p_i, q_{i+1}, p_{i+1}$ downward. Doing so, we decrease q_{i+1}. So we eventually end up in a situation where either $q_{i+1} = q_i$ (which will be treated in Case 3 below) or γ_i becomes zero (revealing that $\Phi_i = -(\rho-2)\alpha/(1-\alpha)$ must be negative). But $\gamma_i = 0$ implies

$$\begin{aligned}\gamma_{i+1} p_{i+1} &= \gamma_i p_i + (\rho-1) q_{i+1} - p_{i+1} \\ &= (\rho-1)\alpha_i p_i - p_{i+1} \leq (\rho-1)\alpha_i p_{i+1} - p_{i+1}.\end{aligned}$$

So $\Phi_{i+1} = \gamma_{i+1} = (\rho-1)\alpha_i - 1 < 0$, contradicting our assumption that $\Phi_{i+1} > 2\rho - 4$.

$\Phi_{i+1} \leq 2\rho - 4$: In this case Φ_{i+1} increases as we shift $q_i, p_i, q_{i+1}, p_{i+1}$ upward until either q_i gets tight in the sense that $A(L_i - p_i) = \rho\, OPT(L_i - p_i)$ or $\Phi_{i+1} = 2\rho - 4$ is reached. The latter is impossible: Since $\gamma_{i+1} p_{i+1} = \gamma_i p_i + (\rho-1) q_{i+1} - p_{i+1}$, we conclude (using Lemma 1) that

$$\begin{aligned}\Phi_{i+1} = \gamma_{i+1} &\leq \gamma_i + (\rho-1)\alpha_i - 1 \leq \rho - 1 - \alpha_i + (\rho-1)\alpha_i - 1 \\ &\leq (\rho-2) + \frac{\rho-2}{\hat{\rho}-1} < 2\rho - 4,\end{aligned}$$

contradicting our assumption that $\Phi_{i+1} = 2\rho - 4$.

Finally, assume that q_i gets tight. In this case, $\gamma_i p_i = \rho(p_i - p_{i-1}) - p_i \geq \rho\alpha_i p_i - p_i$. So $\gamma_i \geq \rho\alpha_i - 1$. Similarly, $\gamma_{i+1}p_{i+1} = \gamma_i p_i + (\rho - 1)q_{i+1} - p_{i+1}$. Dividing by $p_{i+1} = p_i$, we obtain $\gamma_{i+1} = \gamma_i + (\rho - 1)\alpha_i - 1$. Hence

$$\Delta\Phi = \Phi_{i+1} - \Phi_i = \gamma_{i+1} - \Phi_i = \gamma_i + (\rho - 1)\alpha_i - 1 - \frac{\gamma_i - (\rho - 2)\alpha_i}{1 - \alpha_i}$$

is maximized when γ_i is as small as possible, i.e. $\gamma_i = \rho\alpha_i - 1$. Thus

$$\Delta\Phi \leq 2\rho\alpha_i - \alpha_i - 2 - \frac{2\alpha_i - 1}{1 - \alpha_i} = (2\rho - 1)\alpha_i - \frac{1}{1 - \alpha_i}.$$

This shows that $\Delta\Phi$ is indeed strictly negative since

$$\alpha_i(1 - \alpha_i) \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} < \frac{1}{2\rho - 1}$$

for $\rho \leq \hat{\rho}$.

Case 2: $q_{i+1} = \beta_i p_i$.

By Lemma 3 we have

$$\Delta\Phi = \Phi_{i+1} - \Phi_i = \frac{\gamma_i + 2(\rho - 1)\beta_i - 1}{1 + \beta_i} - \frac{\gamma_i + \beta_i - (\rho - 2)\alpha_i}{1 - \alpha_i}.$$

The derivative with respect to γ_i of the above is $1/(1 + \beta_i) - 1/(1 - \alpha_i) \leq 0$, so $\Delta\Phi$ is decreasing in γ_i. Thus we may choose $\gamma_i = 0$. Additionally, we have $\alpha_i \leq \beta_i$, otherwise we are not in this case. With $\gamma_i = 0$ and under the constraints $\alpha_i \leq \beta_i \leq \rho - 1, \alpha_i + \beta_i \leq \rho - 1$ we have

$$\Phi_{i+1} - \Phi_i = \frac{2(\rho - 1)\beta_i - 1}{1 + \beta_i} - \frac{\beta_i - (\rho - 2)\alpha_i}{1 - \alpha_i} < -0.04.$$

Indeed, the function

$$\Delta = \Delta(\alpha, \beta) = \frac{2(\rho - 1)\beta - 1}{1 + \beta} - \frac{\beta - (\rho - 2)\alpha}{1 - \alpha}$$

has partial derivative $\Delta_\alpha = 0$ for $\beta = \rho - 2$. Plugging this value into Δ, we find that the resulting Δ is independent of α and equals $\rho - 2 - \frac{1}{\rho - 1} < -0.1$. On the boundary $\alpha = 0$ the function $\Delta = \Delta(\beta)$ is less than -0.04 for $\beta \in [0, \rho - 1]$ and on the boundary $\alpha = \beta$ the function $\Delta = \Delta(\beta)$ is bounded from above by -0.2 for $\beta \in [0, \frac{\rho-1}{2}]$. (Note that $\alpha = \beta$ implies $\beta \leq \frac{\rho-1}{2}$.) We omit the details.

Case 3: $q_{i+1} = q_i$.

By definition of q_{i+1}, we have $\alpha_i p_i \leq q_{i+1} = q_i$. Thus, Lemma 5 implies $\alpha_i \leq q_i/p_i \leq \frac{1-\alpha_i}{\hat{\rho}-1}$ or, equivalently, $\alpha_i \leq 1/\hat{\rho}$. Thus we conclude that

$$\Phi_i = \frac{\gamma_i - (\rho - 2)\alpha_i}{1 - \alpha_i} \geq \frac{-(\rho - 2)\alpha_i}{1 - \alpha_i} \geq \frac{-(\rho - 2)/\hat{\rho}}{1 - 1/\hat{\rho}} > \hat{\rho} - 3 \tag{4}$$

for $\rho \leq \hat{\rho}$.

Now, as in Case 1, let us assume that $\beta_i = \beta_{i+1} = 0$ and shift the concatenated $q_{i+1} p_{i+1}$ down until α_{i+1} becomes zero as well. This leaves Φ_{i+1} (and, of course, also Φ_i) invariant, but may result in a negative value of γ_{i+1} in case Φ_{i+1} ($= \gamma_{i+1}$ after the shift) is negative. (As we are only interested in $\Delta\Phi$, we do not care about negative values of γ_{i+1} here.) Thus assume that $q_i, p_i, q_{i+1}, p_{i+1}$ are concatenated. We claim that shifting the concatenated $q_i, p_i, q_{i+1}, p_{i+1}$ down increases Φ_{i+1} (while leaving Φ_i unchanged). Indeed moving down decreases both p_i and p_{i+1} by one unit, so that in total $A(L_{i+1})$ decreases by 3 units, while $OPT(L_{i+1})$ decreases by only one unit. Thus $\gamma_{i+1} p_{i+1}$ increases with $3 - \rho$ units while p_{i+1} decreases with one unit. This shows that moving $q_i, p_i, q_{i+1}, p_{i+1}$ down increases $\Phi_{i+1} = \gamma_{i+1}$ whenever $\Phi_{i+1} > \rho - 3$.

To show that Φ decreases, we may assume that $\Phi_{i+1} > \rho - 3$—else a significant decrease of at least $\hat{\rho} - \rho$ follows already immediately from (4). But then, as we have seen above, we may also assume w.l.o.g. that $q_i, p_i, q_{i+1}, p_{i+1}$ is shifted down as far as possible, i.e., until $\alpha_i = 0$. (Note that this may even result in a negative value of γ_i, but we do not care, as we are only interested in Φ-values.) When $\alpha_i = 0$, however, then

$$\Delta\Phi = \Phi_{i+1} - \Phi_i = \gamma_{i+1} - \gamma_i$$

is significantly negative since $\gamma_{i+1} p_{i+1} = \gamma_i p_i + (\rho - 1) q_{i+1} - p_{i+1}$, so that $\gamma_{i+1} \leq \gamma_i + \frac{\rho-1}{\hat{\rho}-1} - 1$.

Summarizing, in each case there is a significant decrease in the potential function, provided that $\rho < \hat{\rho}$. □

4 Conclusions

We have solved the (30 year old) problem of analysing Brown-Baker-Katseff sequences and provided an optimal online algorithm for these sequences. Up to now all previous lower bounds for online strip packing were based on Brown-Baker-Katseff sequences. A natural question to ask is whether (or to what extent) these sequences are worst case sequences for online strip packing. In Harren and Kern (2011) we have tried to analyze more generally sequences of “thin” items p_i and “blocking” items q_j, which are not necessarily alternating. We suceeded in proving lower and upper bounds close to 2.6 for such sequences. Determining the exact value appears to be involved, but in any case, our results show that to close the gap between the currently best lower bound of 2.6 and the currently best upper bound of 6.6, completely new ideas are needed.

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