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Modified Numerals and Split Disjunction: The First-Order Case

Maria Aloni[1] · **Peter van Ormondt**[1]

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Abstract
We present a number of puzzles arising for the interpretation of modified numerals. Following Büring and others we assume that the main difference between comparative and superlative modifiers is that only the latter convey disjunctive meanings. We further argue that the inference patterns triggered by disjunction and superlative modifiers are hard to capture in existing semantic and pragmatic analyses of these phenomena (neo-Gricean or grammatical alike), and we propose a novel account of these inferences in the framework of bilateral state-based modal logic defining a first order extension of Aloni (Semant Pragmat 15:5-EA, 2022. https://doi.org/10.3765/sp.15.5)'s BSML.

1 Introduction

Modified numerals are constructions where a numeral *n* combines with a modifier to form more complex expressions like *less than 18*, *at least 5*, *exactly 7* or *between 10 and 15*. There are many such constructions but in this paper we will focus on the pairs of expressions { at least / at most } *n* and { more than / fewer than } *n*. Following Hackl (2001), Nouwen (2010b) we call expressions in the former pair *superlative quantifiers* and the expressions in the latter pair *comparative quantifiers*.[1]

On a naive view one would perhaps expect the following expressions to be interchangeable.

[1] Other examples of modified numerals are differential quantifiers, disjunctive quantifiers, locative quantifiers, directional quantifiers (see Nouwen, 2010a; 2010b).

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✉ Peter van Ormondt
P.vanOrmondt@uva.nl

Maria Aloni
M.D.Aloni@uva.nl

1 Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, The Netherlands

(1) a. At least n A are $B \Longleftrightarrow$ More than $n-1$ A are B.
b. At most n A are $B \Longleftrightarrow$ Fewer than $n-1$ A are B.

At least since Geurts et al. (2007) it is clear that superlative and comparative quantifiers are in fact quite distinct. One of the observed differences is that superlative quantifiers generate ignorance effects, while comparative quantifiers do not (Nouwen, 2010b).[2]

(2) a. At least/at most n $\varphi \rightsquigarrow$ Speaker does not know how many φ
b. More/fewer than n $\varphi \not\rightsquigarrow$ Speaker does not know how many φ

As an illustration consider the contrast between the following two sentences from Blok (2019):

(3) a. ? I have at least three children.
b. I have more than two children.

Example (3-a) is odd because it gives rise to the unlikely implication that the speaker doesn't know how many children she has. Example (3-b) instead implicates no such thing.

The same point can be made with the downward entailing quantifiers. For instance in the following example from Nouwen (2010b):

(4) a. ? A hexagon has at most 10 sides.
b. A hexagon has fewer than 11 sides.

In the next section we will present three puzzles arising for the interpretation of modified numerals. Following Büring (2008), we will show that these puzzles can be solved if we assume that superlative quantifiers convey disjunctive meanings, while comparative ones do not. The crucial observation is that disjunctions and superlative modifiers generate the same inferences. Büring proposed to derive these inferences as Gricean conversational implicatures (Grice, 1975, 1989). Neo-Gricean (Sauerland, 2004) as well as grammatical accounts of implicatures (Chierchia, 2004; Fox, 2007; Chierchia et al., 2012), however, have difficulties in accounting for the whole range of inference patterns (Kennedy, 2015; Schwarz, 2013, 2016; Crnič et al., 2015). We will instead propose an account using the logic of pragmatic enrichment presented in Aloni (2022). The main difference with traditional grammatical or neo-Gricean approaches is that on Aloni's view such pragmatic inferences neither follow from some grammatical operation nor are the result of (complex) reasoning based on alternative expressions speakers might have used, but rather are a direct consequence of something else speakers do in conversation: when interpreting a sentence they create pictures of the world and by doing so they favour vivid and concrete representations systematically neglecting empty configurations. Let us refer to this as the *neglect zero* assumption. Aloni (2022) showed that free choice (Kamp, 1974; Zimmermann, 2000) and related inferences directly follow from *neglect zero*. Her analysis is implemented in a bilateral state-based modal logic (BSML) modelling assertion and rejection conditions rather

[2] Westera & Brasoveanu (2014) challenged this generalisation. On their view, both superlative and comparative modifiers can generate ignorance inferences and whether they do depends on the question under discussion (QUD). Cremers et al. (2019) however experimentally attested that precise knowledge negatively affects the acceptability of superlative modifiers more than the acceptability of comparative ones across all QUD types.

than truth. In this paper we will present a first order version of BSML, a quantified bilateral state-based modal logic (QBSML), and apply it to explain modified numerals and the puzzles their interpretation gives rise to.

2 Puzzles

2.1 Ignorance

Superlative and comparative quantifiers contrast in interesting ways with each other and with bare numerals in the inferences that they generate. Bare numerals generate exact quantity inferences, superlative modifiers ignorance inferences, comparative modifiers neither of the two[3]:

(5) The band has three players $\rightsquigarrow$ exactly three
(6) The band has at least three players $\rightsquigarrow$ ignorance
(7) The band has more than two players $\rightsquigarrow$ neither

The first challenge for semantic theory is to explain why these different effects are generated given that the determiners "n", "more than $(n-1)$" and "at least n" are assumed to have the same denotation in standard generalised quantifier theory (Barwise & Cooper, 1981):

(8) n / at least n / more than $(n-1) \mapsto \lambda P \lambda Q. \, |P \cap Q| \geq n$

Why do only bare numerals express an exact meaning? And why do superlative modifiers imply ignorance about exact quantity while comparative modifiers do not, given that both appear equally uninformative about the intended number?

A standard answer to the first question (Krifka, 1987; Schwarz, 2013; Kennedy, 2015) assumes that bare numerals express exact meanings as part of their conventional meaning. This is the so called two-sided analysis of numerals, which we will also adopt (even though nothing in what follows hinges on this choice):

(9) $n \mapsto \lambda P \lambda Q. \, |P \cap Q| = n$

A crucial observation towards an answer to the second question is that superlative numerals are not the only constructions that give rise to ignorance effects. The same is the case with plain disjunctions (Grice, 1989; Gazdar, 1976).

(10) Klaus married Paul or John. $\rightsquigarrow$ speaker doesn't know who
(11) Klaus has three or four children. $\rightsquigarrow$ speaker doesn't know how many

Also in the case of disjunction this ignorance effect is strong. In the introduction we observed the oddity of the following sentences:

[3] Cummins et al. (2012) observed that comparative quantifiers may receive enriched interpretation when they combine with "round" numerals as in (i):

(i) More than 90 people got married today. $\rightsquigarrow$ No more than 100 people got married today.

We will not consider these inferences here.

(3-a) ? I have at least three children.
(3-b) ? A hexagon has at most 10 sides.

Similarly, the following are odd because they suggest that the speaker doesn't know who they are married to or how many children they have:

(12) ? I married Paul or John.
(13) ? I have three or four children.

Based on these observations, we follow Büring (2008) and propose that the main difference between superlative and comparative modifiers is that only the former convey inherently disjunctive meanings:

(14) at least $n \mapsto \lambda P \lambda Q.|P \cap Q| > n \vee |P \cap Q| = n \qquad (\equiv \lambda P \lambda Q.|P \cap Q| \geq n)$
(15) more than $n \mapsto \lambda P \lambda Q.|P \cap Q| > n$
(16) at most $n \mapsto \lambda P \lambda Q.|P \cap Q| < n \vee |P \cap Q| = n \qquad (\equiv \lambda P \lambda Q.|P \cap Q| \leq n)$
(17) less than $n \mapsto \lambda P \lambda Q.|P \cap Q| < n$

What we need now is to derive the following ignorance inferences for superlative modifiers and disjunctions, while no such inference should be derived for the comparative modifier cases.

(18) Klaus married with Paul or John
$\rightsquigarrow$ (according to the speaker) it is possible that Klaus married with Paul and it is possible that Klaus married with John.
(19) The band has at least/at most three players
$\rightsquigarrow$ (according to the speaker) it is possible that the band has exactly three player and it is possible that the band has more/less than three players.

Büring proposed to derive these inferences as conversational implicatures. The following section presents a strong argument in favour of such a pragmatic account. As we will see in later sections, however, neo-Gricean (Sauerland, 2004) as well as grammatical accounts of implicatures (Chierchia, 2004; Fox, 2007; Chierchia et al., 2012) have difficulties in scaling up to account for the whole range of facts.

2.2 Obviation

It has been observed by Nouwen (2010b) and Blok (2019) that the ignorance readings we saw in the previous section can be obviated once superlative modifiers appear in the scope of certain operators. In particular, if they appear in the scope of universal quantifiers and modals. Consider first the case of the universal quantifier:

(20) Everyone read at least three books. $\not\rightsquigarrow$ ignorance

On its most prominent reading, (20) does not imply any ignorance. The example is felicitous in a context where the speaker has full knowledge of the situation and simply conveys that no individual read less than three books.

Consider now the following modal case, which as Büring (2008) observed is ambiguous:

(21) Paprika is required to read at least three books.

a. *Authoritative reading*:
It has to be the case that P reads three or more books.
$\not\rightsquigarrow$ ignorance

b. *Epistemic reading*:
Three or more is such that P has to read that many books.
$\rightsquigarrow$ ignorance

The most prevalent reading is one compatible with a situation where the speaker has full knowledge. Büring called it the authoritative reading. On this reading, Paprika is not allowed to read less than 3 books, but may read more. In such cases the speaker knows precisely what is and what is not allowed. No ignorance inference is generated.

On the less obvious epistemic reading, which Büring called speaker-insecurity reading, the speaker knows that there is some lower bound to how many books Paprika should read and that this lower bound is three or higher, but she does not know which of the two: it might be three or higher than three.

As an illustration of the authoritative reading, Büring proposes (22), encountered on a computer screen while creating a new account:

(22) The password must be at least five characters long.

Büring observes: "If it turns out that the password must in fact be seven (or more) characters long, we would object to [the sentence]".

As an illustration of the epistemic reading Büring proposes:

(23) To become a member of this club, you have to pay at least $200,000.

Büring observes "If it turns out that no donation of less than $250,000 gets you into the club's rank and file, what I said wasn't false, nor infelicitous. I merely claimed that the minimum contribution was 200K, or more".

Notice that again similar effects obtain with disjunction:

(24) a. Paprika read two or three books $\rightsquigarrow$ ignorance
b. $\varphi \vee \psi \rightsquigarrow \Diamond \varphi \wedge \Diamond \psi$

(25) a. Everyone read two or three books $\not\rightsquigarrow$ ignorance [obviation]
b. $\forall x(\varphi \vee \psi) \not\rightsquigarrow \forall x(\Diamond \varphi \wedge \Diamond \psi)$

(26) Paprika is required to read two or three books.

a. authoritative $\not\rightsquigarrow$ ignorance [obviation]
b. epistemic $\rightsquigarrow$ ignorance

The challenge here is to explain the emergence of the ignorance effect in the plain case and its obviation in these embedded cases. Semantic accounts of ignorance inferences (e.g., Geurts et al. 2007; Nouwen 2010b), that propose that superlative modifiers have an epistemic component as part of their lexical contribution, have problems in accounting for obviation, which instead is unproblematic for neo-Gricean pragmatic accounts (Büring, 2008; Schwarz, 2013; Kennedy, 2015). In pragmatic accounts, ignorance effects are treated as conversational implicatures, and therefore are expected to disappear in some embedded contexts. Pragmatic accounts however typically have difficulties in accounting for additional inferences obviation cases give rise to. These will be discussed in the following section.

2.3 Distribution and Free Choice

It has been observed that sentences with disjunction in the scope of a universal quantifier tend to give rise to distributive inferences that each of the disjuncts holds (Spector, 2006; Fox, 2007; Klinedinst, 2007).

(27) Every woman in my family has two or three children.
$\rightsquigarrow$ Some woman has two and some woman has three children.

To be more precise, this inference is only generated in contexts where the speaker is assumed to have full knowledge of the situation. In situations of partial information something weaker obtains:

(28) Every woman in my family has two or three children.
$\rightsquigarrow$ Some woman might have two and some woman might have three children.

Notice again the similarity with the case of superlative quantifiers:

(29) Every woman in my family has at least three children.

a. full knowledge $\rightsquigarrow$ Some woman has three and some woman has more than three children.
b. partial information $\rightsquigarrow$ Some woman might have three and some woman might have more than three children.

Disjunctions in the scope of a necessity modal give rise to similar distribution effects. We will call these $\Box$-free choice inferences:

(30) a. To pass this course you are required to give a presentation or write a short paper $\rightsquigarrow$ You are allowed to give a presentation and you are allowed to write a short paper
b. $\Box(\varphi \vee \psi) \rightsquigarrow \Diamond \varphi \wedge \Diamond \psi$ (NB: $\neq \Diamond (\varphi \wedge \psi)$)

Similar inferences are generated for authoritative readings of modal sentences with superlative quantifiers:

(31) a. To pass the course, you're required to read at least three books. (authoritative reading)
b. $\rightsquigarrow$ You are allowed to read three books and you are allowed to read more.

In neo-Gricean approaches, distribution and $\Box$-free choice inferences can be easily derived via negations of universal alternatives:

(32) $\forall x(\varphi \vee \psi) + \neg\forall x\varphi + \neg\forall x\psi \models \exists x\varphi \wedge \exists x\psi$
(33) $\Box(\varphi \vee \psi) + \neg\Box\varphi + \neg\Box\psi \models \Diamond \varphi \wedge \Diamond \psi$

But as experimentally attested by Crnič et al. (2015), Ramotowska et al. (2022), distributive inferences may obtain in the absence of plain negated universal inferences.[4] Consider the following sentence used in a situation where all brothers have been married to a woman, but one of the brothers married first a woman and then a man:

[4] Crnič et al. (2015) tested only the case of universal quantifier. Ramotowska et al. (2022) reached the same conclusion for the case of necessity modals either of the epistemic or the deontic kind.

(34) Every brother of mine has been married to a woman or a man.
$\rightsquigarrow$ some brother has been married to a woman and some brother has been married to a man (even in a situation where all brothers have been married to a woman)

(34) $\forall x(\varphi \vee \psi) \rightsquigarrow \exists x\varphi \wedge \exists x\psi$, even when $\not\rightsquigarrow \neg\forall x\varphi$

Furthermore eventually we also want to account for the classical cases of $\Diamond$-free choice inferences (Kamp, 1974; Zimmermann, 2000):

(35) a. You may have coffee or tea $\rightsquigarrow$ you may have coffee and you may have tea
b. $\Diamond(\varphi \vee \psi) \rightsquigarrow \Diamond\varphi \wedge \Diamond\psi$

But $\Diamond$-free choice inferences are not easy to derive by standard Gricean reasoning:

(36) $\Diamond(\varphi \vee \psi) + \neg\Diamond\varphi + \neg\Diamond\psi \not\models \Diamond\varphi \wedge \Diamond\psi$

Hence, neo-Gricean approaches have problems with both distribution inferences and free choice effects. More in general, note that the different behaviour of $\exists$ and $\Diamond$ in interaction with disjunction is surprising from a purely pragmatic (neo-Gricean) point of view:

(37) a. $\forall x(\varphi \vee \psi) \rightsquigarrow \exists x\varphi \wedge \exists x\psi$ [distribution]
b. $\exists x(\varphi \vee \psi) \not\rightsquigarrow \exists x\varphi \wedge \exists x\psi$

(38) a. $\Box(\varphi \vee \psi) \rightsquigarrow \Diamond\varphi \wedge \Diamond\psi$ [$\Box$-free choice]
b. $\Diamond(\varphi \vee \psi) \rightsquigarrow \Diamond\varphi \wedge \Diamond\psi$ [$\Diamond$-free choice]

For a natural language example illustrating (37-b) consider the following which triggers no distribution effects but rather an ignorance inference:

(39) a. Some woman in my family has two or three children.
$\not\rightsquigarrow$ some woman has two and some woman has three
$\rightsquigarrow$ some woman in my family might have two children and might have three children
b. $\exists x(\varphi \vee \psi) \not\rightsquigarrow \exists x\varphi \wedge \exists x\psi$
$\exists x(\varphi \vee \psi) \rightsquigarrow \exists x(\Diamond\varphi \wedge \Diamond\psi)$

Both distribution and free choice inferences can be captured by grammatical accounts of implicatures (Chierchia et al., 2012; Fox, 2007), which derive them by the application of *exh*, a grammaticalised operation of exhaustification. However, grammatical accounts do not allow us to derive ignorance effects for plain disjunction ($exh(\varphi \vee \psi) \not\models \Diamond\varphi \wedge \Diamond\psi$) unless we assume the presence of a covert epistemic modal operator ($exh(\Box(\varphi \vee \psi)) \models \Diamond\varphi \wedge \Diamond\psi$). It's plausible to assume that also in the case of plain disjunction the modal inference $\Diamond\phi$ and $\Diamond\psi$ obtains in the absence of the negated universal inference $\neg\Box\phi$ and $\neg\Box\psi$ (the results of recent experiments (Degano et al., 2023), a follow-up of Ramotowska et al. (2022), are in support of this assumption). To derive these facts, the grammatical approach has to assume not only that all assertions involve the presence of a covert $\Box$, but also that this covert operator induces $\Diamond$ alternatives, for which we lack an intuitive interpretation.

Another problem arising for grammatical approaches to implicatures is discussed in the following section.

2.4 Negation

One of the advantages of pragmatic approaches to ignorance inferences is that they predict their cancellation under negation. For example, disjunction under negation behaves classically and this is predicted on a Gricean view:

(40) a. Klaus didn't marry John or Bill $\rightsquigarrow$ Klaus did not marry either of the two
b. $\neg(\varphi \vee \psi) \rightsquigarrow \neg\varphi \wedge \neg\psi$

Notice that superlative quantifiers are in general infelicitous under negation[5]:

(41) ? Klaus doesn't have at least/at most 3 children.

An account where ignorance effects are predicted to systematically disappear under negation, can explain the infelicity of (41) in terms of blocking. Indeed since the following equivalences hold

$$\neg(x \leq y) = \neg x = y \wedge \neg x < y = x > y$$
$$\neg(x \geq y) = \neg x = y \wedge \neg x > y = x < y$$

example (42) would turn out be equivalent to the simpler (42), and hence would be blocked by it:

(42) Klaus has less/more than 3 children.

Since no simpler alternatives can be found for the case of plain disjunction under negation, (40) is still predicted to be felicitous on this account.[6]

This simple explanation in terms of blocking would be more difficult (if not impossible) to adopt for the grammatical view. Since *exh* is treated as a grammatical operator which can scopally interact with negation, there are more possible readings generated for example (41) and some of them might fail to be blocked by alternative forms involving simpler expressions.

2.5 Summary and Analysis of the Observed Phenomena

In the previous sections, we have discussed inference patterns generated by modified numerals and we have shown that the behaviour of superlative quantifiers share a strong resemblance to the behaviour of plain disjunctions. Given this striking similarity, we follow Büring (2008) and propose to analyse superlative quantifiers explicitly as disjunctions using the following notation:

(43) at least n $\varphi \mapsto \mathtt{n} \vee \mathtt{more}$

[5] Sentences like (41) can be used as reaction to previous utterances containing the same superlative quantifiers, but in those cases the blocking would not be warranted.

[6] We assume this competition mechanism to operate globally. As a consequence there might be embedded uses of *not at least* which might be felicitous if the global meaning cannot be expressed by a simpler form. It's hard to find such examples though. Consider for example the case of double negation: "It is not the case that Klaus doesn't have at least 3 children". Our logical system validates double negation elimination. Therefore it predicts these sentences to be blocked by the simpler "Klaus has at least 3 children" (without assuming local competition mechanisms).

Table 1 Comparison of results of different approaches

	Ignorance	Obviation	Distribution	FC and Negation
Semantic	Yes	No	?	No
Neo-Gricean	Yes	Yes	No	Yes
Grammatical view	?	Yes	Yes	?

(44) at most $n\ \varphi \mapsto \mathtt{n} \vee \mathtt{less}$

Sentences with superlative and comparative modifiers will be then translated as follows, with only the former conveying disjunctive meanings:

(45) a. The band has at least three players. [Superlative]
b. $\mathtt{three} \vee \mathtt{more}$

(46) a. The band has at most three players. [Superlative]
b. $\mathtt{three} \vee \mathtt{less}$

(47) a. The band has more than two players. [Comparative]
b. $\mathtt{more\text{-}than\text{-}two}$

(48) a. The band has less than two players. [Comparative]
b. $\mathtt{less\text{-}than\text{-}two}$

Examples (49)-(55) summarizes the inference patters discussed in the previous section, which constitute the desiderata of the formal system we will present in Sects. 3 and 4.

(49) a. Klaus has at least three children. [Ignorance]
b. $(\mathtt{three} \vee \mathtt{more}) \rightsquigarrow \Diamond\,\mathtt{three} \wedge \Diamond\,\mathtt{more}$

(50) a. Every woman in my family has at least three children. [Obviation]
b. $\forall x(\mathtt{three}(x) \vee \mathtt{more}(x)) \not\rightsquigarrow \forall x(\Diamond\,\mathtt{three}(x) \wedge \Diamond\,\mathtt{more}(x))$

(51) a. Every woman in my family has at least three children. [Distribution]
b. $\forall x(\mathtt{three}(x) \vee \mathtt{more}(x)) \rightsquigarrow \exists x\,\mathtt{three}(x) \wedge \exists x\,\mathtt{more}(x)$

(52) a. Every woman in my family has at least three children. [Distribution$^{\Diamond}$]
b. $\forall x(\mathtt{three}(x) \vee \mathtt{more}(x)) \rightsquigarrow \exists x \Diamond\,\mathtt{three}(x) \wedge \exists x \Diamond\,\mathtt{more}(x)$

(53) a. You are required to read at least three books. [$\Box$-free choice]
b. $\Box(\mathtt{three} \vee \mathtt{more}) \rightsquigarrow \Diamond\,\mathtt{three} \wedge \Diamond\,\mathtt{more}$

(54) a. You are allowed to read at least three books. [$\Diamond$-free choice]
b. $\Diamond(\mathtt{three} \vee \mathtt{more}) \rightsquigarrow \Diamond\,\mathtt{three} \wedge \Diamond\,\mathtt{more}$

(55) a. ?Klaus does not have at least three children. [Negation]
b. $\neg(\mathtt{three} \vee \mathtt{more}) \rightsquigarrow \neg\mathtt{three} \wedge \neg\mathtt{more}$

Table 1 summarizes the predictions of some previous approaches, where "?" means that the specific inference is not explicitly addressed by the relevant approach.

In the following sections we will develop a framework in which these inferences can be derived rigorously. As we have indicated, analysing superlative numerals in terms of disjunction is not new. Our analysis however will differ from previous approaches because we will use a different way of deriving pragmatic effects of disjunction using pragmatic enrichments as introduced in Aloni (2022). The ignorance and free choice effects in Examples (49), (53) and (54), and the negation fact in (55) will follow automatically by adopting Aloni (2022)'s mechanism of pragmatic intrusion (see Sect. 4.3). To account for the obviation fact in Example (50) and the distribution facts in Examples (51) and (52) we will raise the framework of Aloni (2022) to the first-order case.

3 A Logic Based Account

In the following sections we will present a logic-based account where ignorance and related inferences will follow as "reasonable inferences" in the sense of Stalnaker (1975).[7] We understand a reasonable inference not as a semantic relation but as a pragmatic one, which relates speech acts rather than propositions. To derive reasonable inference we employ a state-based modal logic modeling assertion and rejection conditions rather than truth.

Where classical modal logic interprets formulas with respect to a single possible world, state-based modal logic interprets formulas with respect to a state modelled as a set of possible worlds. In the propositional case, developed in Aloni (2022), this amounts to the following. Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model, where W is a non-empty set of worlds, $R \subseteq W \times W$ a two-place relation over W and V a world-dependent valuation function assigning truth values to propositional variables of the language.

Classical modal logic models *truth* in a possible world (an element of W) while state-based modal logic models *support* in an information state (a subset of W):

$$\mathcal{M}, w \models \varphi, \text{ where } w \in W \tag{Classical}$$

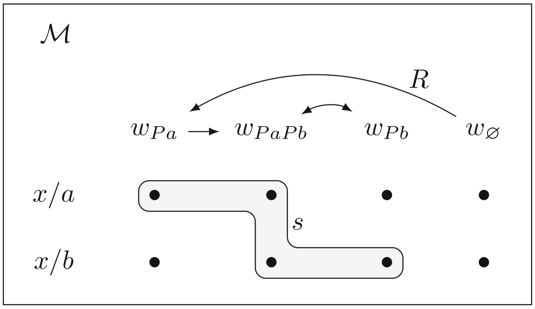
$$\mathcal{M}, s \models \varphi, \text{ where } s \subseteq W \tag{State-based}$$

Aloni (2022) employs a *bilateral* version of state-based modal logic which defines both *support* ($\models$) and *anti-support* ($=\!|$) conditions meant to capture the assertability and rejectability of a sentence in an information state,[8]

[7] To be precise, while in spirit our approach is close to Stalnaker's, empirically we make quite different predictions for example because some of the pragmatic rules Stalnaker adopts for disjunction are different from ours.

[8] Information states are then less determinate entities than possible worlds and are comparable to truthmakers (van Fraassen, 1969; Fine, 2017), possibilities (Lloyd Humberstone, 1981; Holliday, 2018) or situations (Barwise & Perry, 1983). The partial nature of an information state makes state-based systems particularly suitable for capturing phenomena at the semantics-pragmatics interface, including anaphora (Groenendijk & Stokhof, 1991; Groenendijk et al., 1996; Dekker, 2012), questions (Ciardelli & Roelofsen, 2011; Ciardelli et al., 2018), epistemic modals (Veltman, 1996).

Fig. 1 A simple model $\mathcal{M}$



We then have that

$$\begin{array}{ll} \mathcal{M}, s \models \varphi & \text{means } \varphi \text{ is assertable in } s \\ \mathcal{M}, s \mathrel{=\!|} \varphi & \text{means } \varphi \text{ is rejectable in } s \end{array}$$

In our first-order extension of state-based modal logic, the elements of a state, the possibilities or indices as we will call them, are pairs of worlds and (partial) assignments.

Before we continue let's have a look at an example to get a better understanding of our intended models.

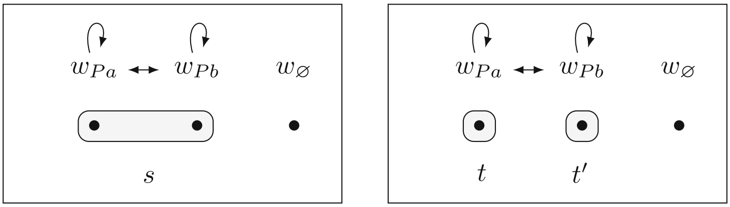
Example 3.1 (*An example of a pointed model* $\mathcal{M}, s$)
Consider the model depicted in Fig. 1. Worlds in this example (and the ones throughout this paper) will be designated by letters w, v, u, etc. together with a subscript indicating what is the case in that world. So w_{Pa} is the world w in which some object a has property P.
The arrows indicate which worlds are R-related.
In this model s consists of the following indices:

$$s = \{\langle w_{Pa}, g[x/a]\rangle, \langle w_{PaPb}, g[x/a]\rangle, \langle w_{PaPb}, g[x/b]\rangle, \langle w_{Pb}, g[x/b]\rangle\},$$

where $g[x/d]$ means the assignment function g that maps variable x onto object d in some domain D.

In state-based systems to be supported in a state you normally must be true in all possibilities. It is then easy to see for instance that $\mathcal{M}, s \models Px$ because in each $i \in s$ the value of x in i is P. It is also clear that $\mathcal{M}, s \not\models Pa$ because for instance a is not P in w_{Pb} and therefore Pa is not supported by all $i \in s$. The same is the case for Pb: $\mathcal{M}, s \not\models Pb$.

Of course, the relation R in the model does not have influence on the formulas we just evaluated and this will only become relevant when we will discuss modal formulas.



(a) $s \models [Pa \vee Pb]^+$ (b) $t, t' \models Pa \vee Pb$, $t, t' \not\models [Pa \vee Pb]^+$

Fig. 2 Ignorance inference follows from pragmatic enrichment

3.1 Split Disjunction, Non-emptiness and Pragmatic Enrichment

An important conclusion of Sect. 2 was that modified numerals, specifically superlative quantifiers, behave similarly as disjunctions do. This led to the assumption that superlative quantifiers should be analysed as disjunctions.

In previous state-based accounts of modified numerals (Coppock & Brochhagen, 2013; Blok, 2019) they were analysed in terms of *inquisitive disjunctions* (Ciardelli & Roelofsen, 2011), we will instead adopt a notion of disjunction from dependence logic and team logic, called *split (or tensor) disjunction* (Väänänen, 2007; Yang & Väänänen, 2017; Hawke & Steinert-Threlkeld, 2018; Cresswell, 2004).

We say that an information state s supports a split disjunction $\varphi \vee \psi$ iff s is the union of two substates each supporting one of the disjuncts:

$$\mathcal{M}, s \models \varphi \vee \psi \text{ iff } \exists t, t' : t \cup t' = s \text{ and } \mathcal{M}, t \models \varphi \text{ and } \mathcal{M}, t' \models \psi.$$

A result of our logic is that an empty state will support any classical formula. Note that this fact together with the notion of split disjunction above entails that whenever we have a state that supports a formula φ we can always find a substate, namely the empty state, such that the state will support the disjunction $\varphi \vee \psi$, where ψ is classical and arbitrary.

Aloni (2022) defines a *pragmatic enrichment function*, by using the non-emptiness atom NE from team logic (Yang & Väänänen, 2017), which will bar the empty substate as a possible state for evaluation. As a result, a pragmatically enriched disjunction $[\varphi \vee \psi]^+$ is supported by a state s iff there are two *non-empty* states t, t' such that $t \cup t' = s$ and $t \models \varphi$ and $t' \models \psi$.

As an example of how pragmatic enrichment will allow us to derive the ignorance effect, consider Fig. 2.

The model and state in Fig. 2a support $[Pa \vee Pb]^+$ because the state can be split into two non-empty substates, the states t and t' represented in Fig. 2b, each supporting one of the disjuncts. The modalized conjuncts $\Diamond Pa$ and $\Diamond Pb$ then follow in part by requiring the relation to be *state-based* which we will explain in Sect. 4.1.1. Notice that in each of the two singleton states in 2b classical $Pa \vee Pb$ is satisfied but its pragmatically enriched version $[Pa \vee Pb]^+$ is not.

In combination with the right notion of modality, quantification and negation, the resulting system will predict all the inferences observed in previous sections.

4 Quantified Bilateral State-based Modal Logic (QBSML)

The system we propose extends the bilateral framework of Aloni (2022) to the first-order case. Like Aloni's BSML, QBSML is a modal predicate logic with state-based semantics that defines conditions of assertion and rejection rather than conditions of truth.

We start by defining the language.

Definition 4.1 (Language) The language $\mathcal{L}$ of state-based first-order modal logic is built up from predicate constants $P^n \in \mathcal{P}^n$, with $n \in \mathbb{N}$, individual constants $c \in \mathcal{C}$ and variables $x \in \mathcal{V}$.

$$t ::= c \mid x$$
$$\varphi ::= P^n t_1 \ldots t_n \mid \mathsf{NE} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x\varphi \mid \forall x\varphi \mid \Box\varphi$$

Definition 4.2 (*Model*) A model for $\mathcal{L}$ is a quadruple $\mathcal{M} = \langle W, D, R, I \rangle$, where W is set of worlds, D is a non-empty domain, R is an accessibility relation on W, $I : W \times \mathcal{C} \cup \mathcal{P}^n \to D \cup \wp(D^n)$ is an interpretation function which assigns entities to individual constants and sets of entities to predicate letters relativized to worlds $w \in W$:

$$I(w)(\gamma) = \begin{cases} d \in D & \text{if } \gamma \in \mathcal{C}, \\ S^n \subseteq D^n & \text{if } \gamma \in \mathcal{P}^n. \end{cases}$$

Contrary to classical modal predicate logic, formulas will be interpreted with respect to states, rather than a world. A state is a set of indices. An index i is a pair $i = \langle w_i, g_i \rangle$ consisting of a world $w_i \in W$ and a partial assignment function $g_i : \mathcal{V} \to D$.[9] We may think of an information state as encoding information about the value of variables restricted to worlds. We require for all indices i, j in a state s that g_i has the same domain as g_j: $\text{dom}(g_i) = \text{dom}(g_j)$.

Given a model $\mathcal{M}$, the set of *information states* is defined as

$$\mathcal{S}_{\mathcal{M}} := \bigcup_{X \subseteq \mathcal{V}} \wp(W \times D^X).$$

This is a general definition including all subsets of $\mathcal{V}$, but usually, when modelling a dialogue this will be a union of finite sets of $\mathcal{V}$. We will use I^X to denote the set of indices $W \times D^X$ for some $X \subseteq \mathcal{V}$.

This paper models pragmatic effects that occur in discourse. We see discourse as starting out with an initial state where there is no information about the values of the variables, i.e., we have an empty assignment. See Fig. 3for an illustration of an initial information state $s = \{\langle w_{P_a P_b}, \varnothing \rangle, \langle w_{P_b}, \varnothing \rangle\}$.

When we do learn the value of a variable this information is added to the information state. In order to capture this idea we will define a number of operations on states. Let's first fix some terminology.

[9] See Heim (1982), Dekker (1993).

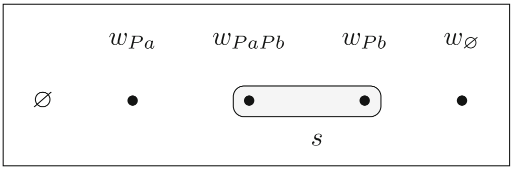


Fig. 3 Empty assignment

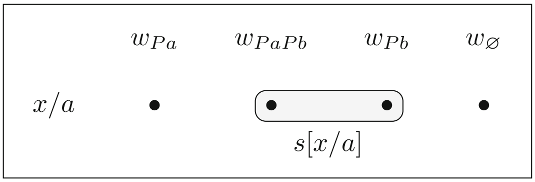


Fig. 4 Individual x-extension

Let $\mathcal{S}_{\mathcal{M}}$ be the set of information states of some model $\mathcal{M}$. We say a state s is *initial* if its domain X is empty. We say a state is of *minimal information* if it is initial and it contains all possible worlds. A state is of *maximal information* if it is a singleton set.

Next we use the following definitions for operations on states (building on Dekker 1993; Aloni 2001).

Definition 4.3

$$g[x/d] := (g \setminus \{\langle x, g(x)\rangle\}) \cup \{\langle x, d\rangle\}$$

In the course of a discourse noun phrases are associated with variables. $g[x/d]$ adds x and sets its value to d, if $x \notin \mathrm{dom}(g)$, or, otherwise, resets the value of x to d. We write $i[x/d]$ when g_i is in i and d is assigned to x.

Definition 4.4

$$i[x/d] := \langle w_i, g_i[x/d]\rangle.$$

The individual x-extension of s, $s[x/d]$, is the state resulting from s by replacing the assignment g_i in each index $i \in s$ by $g_i[x/d]$.

Definition 4.5 (*Individual x-extension of s*)

$$s[x/d] := \{i[x/d] \mid i \in s\}, \text{ for some } d \in D.$$

See Fig. 4 for an example where individual a is assigned to x extending the initial state $s = \{\langle w_{P_aP_b}, \varnothing\rangle, \langle w_{P_b}, \varnothing\rangle\}$.

The universal x-extension of s, $s[x]$, is the state which results by extending the state s with the assignment $g[x/d]$ for all $d \in D$. See Fig. 5 for an example.

Definition 4.6 (*Universal x-extension of s*)

$$s[x] := \{i[x/d] \mid i \in s \;\&\; d \in D\}.$$

A functional x-extension of s is any state t where for each index $i \in s$ there is at least one index $j \in t$ such that $j = i[x/d]$ for some $d \in D$.

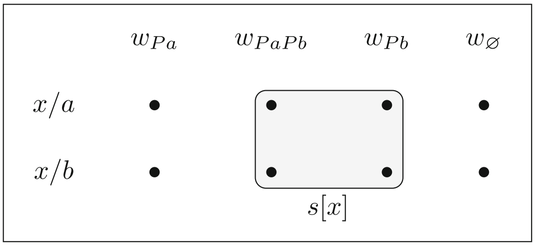


Fig. 5 Universal x-extension, where $D = \{a, b\}$

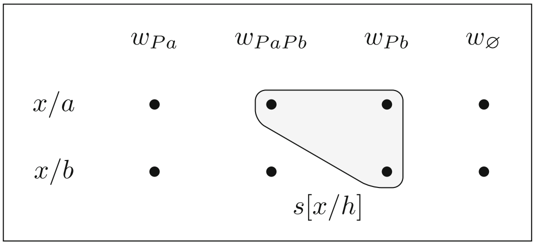


Fig. 6 A functional x-extension of s

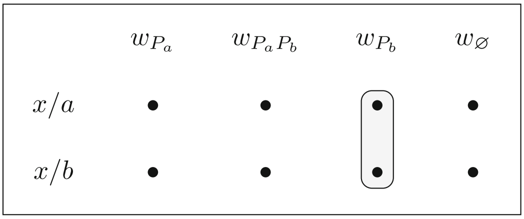


Fig. 7 Not a functional x-extension of s

Definition 4.7 (*Functional x-extension of s*)

$$s[x/h] := \{i[x/d] \mid i \in s \ \&\ d \in h(i)\}, \text{ some function } h : s \to \wp(D) \setminus \varnothing.$$

Individual extensions and universal extensions are examples of functional extensions. But also the state depicted in Fig. 6 is a functional extension of the state $s = \{\langle w_{P_a P_b}, \varnothing\rangle, \langle w_{P_b}, \varnothing\rangle\}$ with h mapping $\langle w_{P_a P_b}, \varnothing\rangle$ to $\{a\}$ and $\langle w_{P_b}, \varnothing\rangle$ to $\{a, b\}$.

However, the state represented in Fig. 7 is not a functional extension of s because it does not contain any extension of index $\langle w_{P_a P_b}, \varnothing\rangle$, such index does not "survive" in it.

We next define the interpretation of terms:

Definition 4.8 (*Terms*)

$$[\![t]\!]_{\mathcal{M},i} = \begin{cases} g_i(t) & \text{if } t \in \mathcal{V}, \\ I(w_i)(t) & \text{if } t \in \mathcal{C}. \end{cases}$$

Definition 4.9 (*Support and anti-support*) Let $\mathcal{M}$ be a model, φ a formula of our language, and s a state. We define what it means for a formula φ to be supported or anti-supported at a state s in $\mathcal{M}$ as follows.

$$
\begin{array}{lll}
\mathcal{M}, s \models P^n t_1 \ldots t_n & \text{iff} & \forall i \in s : \langle [\![t_1]\!]_{\mathcal{M},i}, \ldots, [\![t_n]\!]_{\mathcal{M},i} \rangle \in I(w_i)(P^n) \\
\mathcal{M}, s \dashv P^n t_1 \ldots t_n & \text{iff} & \forall i \in s : \langle [\![t_1]\!]_{\mathcal{M},i}, \ldots, [\![t_n]\!]_{\mathcal{M},i} \rangle \notin I(w_i)(P^n) \\
\mathcal{M}, s \models \neg\varphi & \text{iff} & \mathcal{M}, s \dashv \varphi \\
\mathcal{M}, s \dashv \neg\varphi & \text{iff} & \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \vee \psi & \text{iff} & \exists t, t' : t \cup t' = s \text{ and } \mathcal{M}, t \models \varphi \text{ and } \mathcal{M}, t' \models \psi \\
\mathcal{M}, s \dashv \varphi \vee \psi & \text{iff} & \mathcal{M}, s \dashv \varphi \text{ and } \mathcal{M}, s \dashv \psi \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \dashv \varphi \wedge \psi & \text{iff} & \exists t, t' : t \cup t' = s \text{ and } \mathcal{M}, t \dashv \varphi \text{ and } \mathcal{M}, t' \dashv \psi \\
\mathcal{M}, s \models \Box\varphi & \text{iff} & \forall i \in s : \mathcal{M}, R(w_i)[g_i] \models \varphi \\
\mathcal{M}, s \dashv \Box\varphi & \text{iff} & \forall i \in s : \exists X \subseteq R(w_i) \text{ and } X \neq \varnothing \text{ and } \mathcal{M}, X[g_i] \dashv \varphi \\
\mathcal{M}, s \models \mathrm{NE} & \text{iff} & s \neq \varnothing \\
\mathcal{M}, s \dashv \mathrm{NE} & \text{iff} & s = \varnothing \\
\mathcal{M}, s \models \forall x\varphi & \text{iff} & \mathcal{M}, s[x] \models \varphi \\
\mathcal{M}, s \dashv \forall x\varphi & \text{iff} & \mathcal{M}, s[x/h] \dashv \varphi, \text{ for some } h : s \to \wp(D) \setminus \varnothing \\
\mathcal{M}, s \models \exists x\varphi & \text{iff} & \mathcal{M}, s[x/h] \models \varphi, \text{ for some } h : s \to \wp(D) \setminus \varnothing \\
\mathcal{M}, s \dashv \exists x\varphi & \text{iff} & \mathcal{M}, s[x] \dashv \varphi
\end{array}
$$

We used the following abbreviations in the definition.

$$
\begin{aligned}
X[g_i] &= \{\langle w, g_i \rangle \mid w \in X\}, \\
R(w_i) &= \{v \in W \mid w_i R v\}.
\end{aligned}
$$

We will write $\mathcal{M}, s \not\models \varphi$ if a formula φ is not supported by s in $\mathcal{M}$.

We will use the following abbreviation: $\Diamond\varphi := \neg\Box\neg\varphi$, which gives us the following interpretation of the possibility modal.

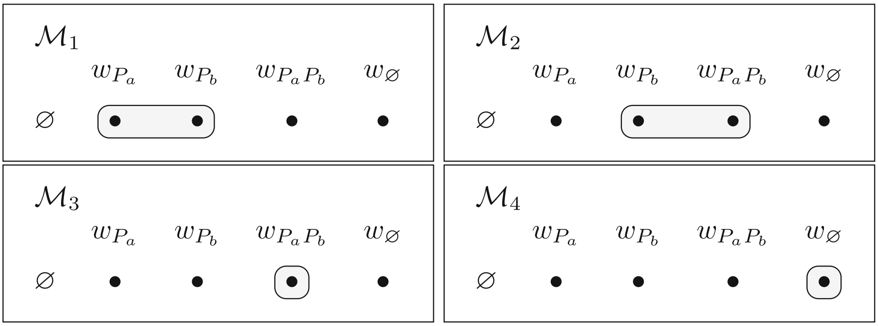
$$
\begin{array}{lll}
\mathcal{M}, s \models \Diamond\varphi & \text{iff} & \forall i \in s : \exists X \subseteq R(w_i) \text{ and } X \neq \varnothing \text{ and } \mathcal{M}, X[g_i] \models \varphi \\
\mathcal{M}, s \dashv \Diamond\varphi & \text{iff} & \forall i \in s : \mathcal{M}, R(w_i)[g_i] \dashv \varphi
\end{array}
$$

4.1 Quantifiers

The quantifiers in QBSML are defined as in team semantics[10] (Kontinen & Väänänen, 2009) and in versions of dynamic semantics. Notice that also the inquisitive existential quantifier (Ciardelli & Roelofsen, 2011; Ciardelli et al., 2018) can be defined in this framework using individual x-extensions rather than functional x-extensions. We use $\exists^1 x$ to denote this notion as in (Kontinen & Väänänen, 2009).

$$
\mathcal{M}, s \models \exists^1 x\varphi \quad \text{iff} \quad \mathcal{M}, s[x/d] \models \varphi, \text{ for some } d \in D
$$

[10] More precisely our notion of existential quantifier corresponds to the lax version discussed by Galliani (2012).



$$\mathcal{M}_1, s \models \exists x Px \qquad \mathcal{M}_2 \models \exists x Px \qquad \mathcal{M}_3, s \models \exists x Px \qquad \mathcal{M}_4, s \Dashv \exists x Px$$
$$\mathcal{M}_1, s \not\models \exists^1 x Px \qquad \mathcal{M}_2 \models \exists^1 x Px \qquad \mathcal{M}_3, s \models \exists^1 x Px \qquad \mathcal{M}_4, s \Dashv \exists^1 x Px$$
$$\mathcal{M}_1, s \Dashv \forall x Px \qquad \mathcal{M}_2 \not\models \forall x Px \qquad \mathcal{M}_3, s \models \forall x Px \qquad \mathcal{M}_4, s \Dashv \forall x Px$$

Fig. 8 Examples of quantified formulas supported and rejected

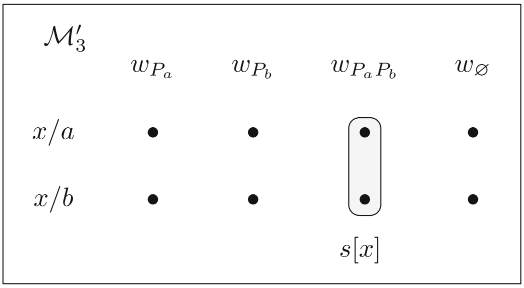


Fig. 9 Universal x-extension of $\mathcal{M}_3$

$$\mathcal{M}, s \Dashv \exists^1 x\varphi \quad \text{iff} \quad \mathcal{M}, s[x/d] \Dashv \varphi, \text{ for all } d \in D$$

Let's consider some examples. Figure 8 shows four models with states s, where $D = \{a, b\}$. The first, second and third model support the formula $\exists x Px$. Only the second and third model support the formula $\exists^1 x Px$, whereas only the third model supports $\forall x Px$.

For instance $\mathcal{M}_1, s \models \exists x Px$ because for all worlds w in s there is some $d \in D$ such that $d \in I(w)(P)$. Instead, $\mathcal{M}_1, s \not\models \exists^1 x Px$ because there is no $d \in D$ such that $d \in I(w)(P)$ in all worlds w. The reason why $\mathcal{M}_3, s \models \forall x Px$ is that we have to consider the universal x-extension of the state s in $\mathcal{M}_3$, see Fig. 9, which indeed supports Px.

4.1.1 Modal Formulas and State-based Constraints on *R*

Modals are interpreted as in BSML and inquisitive modal logic Ciardelli (2016). In these systems evaluating $\Box\varphi$ or $\Diamond\varphi$ means evaluating φ with respect to the state consisting of the set of worlds accessible from the relevant points of evaluation. Since states here are sets of world-assignment pairs we need to specify the assignment

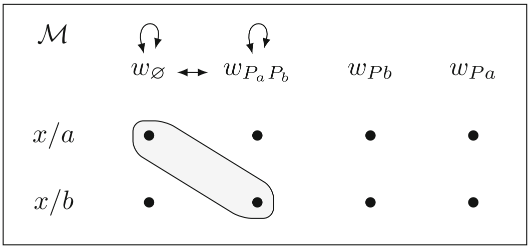


Fig. 10 An example with modal formulas

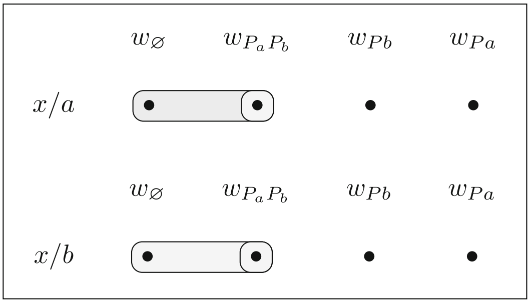


Fig. 11 Px evaluated at substates of $R(w_i)[g_i]$

parameter for example to determine how free variables would be evaluated in such a state. Therefore we interpret $\Box\,\varphi$ or $\Diamond\,\varphi$ by evaluating φ with respect to a state constructed by combining the worlds accessible from w_i with g_i, for each relevant i.

For example, $\mathcal{M}, s \models \Diamond\,\varphi$ iff for all $i \in s$ there is a non-empty subset X of $R(w_i)$ such that $X[g_i] \models \varphi$, i.e., we pair the assignment function g_i with the worlds in X.

If we look at $\mathcal{M}, s$ in Fig. 10. Then we see the following to be the case:

$$\mathcal{M}, s \not\models Px$$
$$\mathcal{M}, s \models \Diamond\, Px$$

In order to evaluate $\Diamond\, Px$ in state s, Px needs to be supported at least in a non-empty substate of the first state and in a non-empty substate of the second state depicted in Fig. 11.

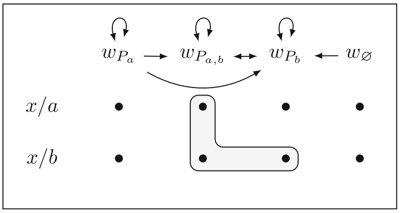
In order to capture the special characteristic of epistemic modals relevant for our ignorance inference we define properties of the accessibility relation R (as in Aloni 2022).

Definition 4.10 Let $\mathcal{M}$ be a model and s a state on $\mathcal{M}$. We define $s^{\downarrow}$ as

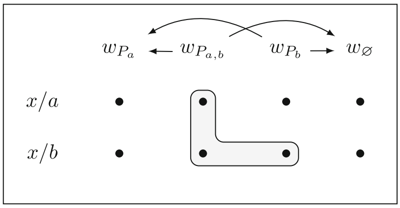
$$s^{\downarrow} := \{w \in W \mid \langle w, g\rangle \in s\}$$

and we say that R is *state-based* with respect to $\mathcal{M}, s$ iff for all $w \in s^{\downarrow}$: $R(w) = s^{\downarrow}$. We say R is *indisputable* with respect to $\mathcal{M}, s$ iff for all $w, v \in s^{\downarrow}$: $R(w) = R(v)$. We say that $\mathcal{M}, s$ is *epistemic* or *indisputable*, respectively.

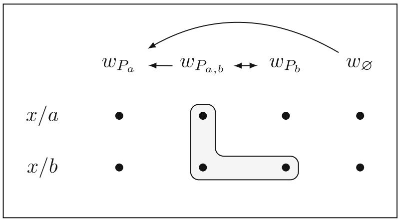
Note that if R is state-based then this implies that R is indisputable (see Fig. 12). Even though we call a model where R is state-based *epistemic*, this does not mean the modal



(a) R is state-based (and therefore indisputable)



(b) R is indisputable (but not state-based)



(c) R is neither state-based nor indisputable

Fig. 12 Models with different properties of R

should be interpreted as a knowledge operator, but rather as an epistemic possibility operator, like 'might'.

4.2 Logical Consequence and Classicality of NE-Free Fragment

The logical consequence is defined in terms of preservation of support.

Definition 4.11 (*Logical consequence*) Let $\varphi, \psi \in$ QBSML with free variables contained in the sequence $\vec{x}$. Then $\varphi \models \psi$ iff for all $\mathcal{M}, s$, such that (i) the signature of $\mathcal{M}$ contains all the non-logical symbols of φ and ψ and (ii) the domain of s contains $\vec{x}$: if $\mathcal{M}, s \models \varphi$, then $\mathcal{M}, s \models \psi$.

Definition 4.12 (*Logical consequence epistemic models*) Let $\varphi, \psi \in$ QBSML with free variables contained in the sequence $\vec{x}$. Then $\varphi \models_{\mathsf{epi}} \psi$ iff for all *epistemic* $\mathcal{M}, s$, such that (i) the signature of $\mathcal{M}$ contains all the non-logical symbols of φ and ψ and (ii) the domain of s contains $\vec{x}$: if $\mathcal{M}, s \models_{\mathsf{epi}} \varphi$, then $\mathcal{M}, s \models_{\mathsf{epi}} \psi$.

It is easy to show that the current semantics validates a number of classical laws ($\varphi \equiv \psi$ is short for $\varphi \models \psi$ and $\psi \models \varphi$):

Fact 1 (Classical QBSML validities)

$$\begin{array}{lcrl}
\varphi & \equiv & \neg\neg\varphi & \textit{(Double Negation Elimination)}\\
\neg(\varphi \vee \psi) & \equiv & \neg\varphi \wedge \neg\psi & \textit{(De Morgan Laws)}\\
\neg(\varphi \wedge \psi) & \equiv & \neg\varphi \vee \neg\psi & \\
\neg \Box \varphi & \equiv & \Diamond \neg \varphi & \textit{(Duality } \Box/\Diamond\textit{)}\\
\neg \Diamond \varphi & \equiv & \Box\neg\varphi & \\
\neg\forall x\varphi & \equiv & \exists x\neg\varphi & \textit{(Duality } \forall/\exists\textit{)}\\
\neg\exists x\varphi & \equiv & \forall x\neg\varphi &
\end{array}$$

However, not all classical validities are validated. For example, addition fails whenever NE occurs in one of the disjuncts: $\varphi \not\models \varphi \vee (\psi \wedge \mathrm{NE})$. In fact, the non-emptiness atom is the only source of non-classical behaviour.[11] As stated in Proposition 4.1, the NE-free fragment of QBSML can be reduced to classical quantified modal logic.

Proposition 4.1 *Let $\varphi(\vec{x})$ be a NE-free formula of QBSML with free variables in $\vec{x}$. And let s be a state over $\mathcal{M}$ whose domain contains $\vec{x}$. Then*

$$\mathcal{M}, s \models \varphi(\vec{x}) \textit{ iff } \mathcal{M}, w \models_g \varphi(\vec{x}), \textit{ for all } \langle w, g\rangle \in s$$

(where $\models$ on the left is the state-based support relation, and $\models$ on the right is the truth relation from classical quantified modal logic).

This result was proved for the propositional BSML by Antilla (2021, Propositions 2.2.2, 2.2.8 and 2.2.16). In order to show that it also holds for QBSML it is enough to show that, if φ is NE-free, $\forall x\varphi$, and $\exists x\varphi$ are flat, i.e., have the downward closure property, the union closure property and the empty state property. Note that given the duality of $\exists x$ and $\forall x$ all formulas of QBSML can be given in negation normal form, as was the case in BSML (Anttila 2021, Propositions 2.2.6).

Fact 2 *Let $\forall x\varphi$, $\exists x\varphi$ be formulas of QBSML such that they do not contain an occurrence of NE. Then $\forall x\varphi$ and $\exists x\varphi$ have the downward closure property, the union closure property and the empty state property.*

Proof We rely on the induction hypothesis given by Antilla (2021, Proposition 2.2.8).

$\forall x\varphi$. Downward closure property: If $\forall x\varphi$ does not contain NE we may assume by the induction hypothesis that φ is downward closed. Assume that $\mathcal{M}, s \models \forall x\varphi$ and that $t \subseteq s$. Observe that $t \subseteq s \iff t[x] \subseteq s[x]$. By $\mathcal{M}, s \models \forall x\varphi$ we have that $\mathcal{M}, s[x] \models \varphi$. By the induction hypothesis, we have that $\mathcal{M}, t[x] \models \varphi$, hence $\mathcal{M}, t \models \forall x\varphi$.

Union closure property: If $\forall x\varphi$ does not contain NE then by the induction hypothesis, φ has the union closure property. Assume that for some model $\mathcal{M}$ and some non-empty set of states S on $\mathcal{M}$ we have $\mathcal{M}, s \models \forall x\varphi$, for all $s \in S$. This means that $\mathcal{M}, s[x] \models \varphi$, for all $s \in S$. By the induction hypothesis, we have

[11] Although logical consequence in NE-free fragment is classical (e.g., it validates LEM), the logic does not satisfy bivalence since we can have states which neither support a formula neither its negation. So in a sense also the NE-free fragment displays non-classically behaviour.

that the union of the universal extensions of $s \in S$, $\bigcup_{s\in S}\{S[x]\}$, supports φ. $\bigcup_{s\in S}\{S[x]\} = \bigcup S[x]$, so $\mathcal{M}, \bigcup S[x] \models \varphi$. We conclude $\mathcal{M}, \bigcup S \models \forall x\varphi$.

Empty state property: If $\forall x\varphi$ does not contain NE, then by the induction hypothesis φ has the empty state property. Let $\mathcal{M}$ be some model $\mathcal{M}$ then by the empty state property: $\mathcal{M}, \varnothing \models \varphi$. Clearly, $\varnothing[x] = \varnothing$, hence $\mathcal{M}, \varnothing \models \forall x\varphi$.

$\exists x\varphi$. Downward closure property: If $\exists x\varphi$ does not contain NE we may assume by the induction hypothesis that φ is downward closed. Assume that $\mathcal{M}, s \models \exists x\varphi$. By $\mathcal{M}, s \models \exists x\varphi$ we have that $\mathcal{M}, s[x/h] \models \varphi$, for some $h : s \to \wp(D)\backslash\varnothing$. Observe that $t \subseteq s \Rightarrow t[x/h|_t] \subseteq s[x/h]$. By the induction hypothesis: $\mathcal{M}, t[x/h|_t] \models \varphi$. Hence $\mathcal{M}, t \models \exists x\varphi$. As t was arbitrary we conclude that $\exists x\varphi$ is downward closed.

Union closure property: If $\exists x\varphi$ does not contain NE, then by the induction hypothesis, φ has the union closure property. Assume that for some model $\mathcal{M}$ and some non-empty set of states S on $\mathcal{M}$ we have $\mathcal{M}, s \models \exists x\varphi$, for all $s \in S$. This means that for all $s \in S$: $\mathcal{M}, s[x/h_s] \models \varphi$, for some $h_s : s \to \wp(D)\backslash\varnothing$. By the induction hypothesis we have that the union of the functional extensions of $s \in S$, supports φ: $\mathcal{M}, \bigcup_{s\in S} s[x/h_s] \models \varphi$. It is easy to see that $\bigcup_{s\in S} s[x/h_s]$ is itself a x-functional extension of $\bigcup S$. We conclude $\mathcal{M}, \bigcup S \models \exists x\varphi$.

Empty state property: If $\exists x\varphi$ does not contain NE we may assume by the induction hypothesis that φ has the empty state property: $\mathcal{M}, \varnothing \models \varphi$. Clearly, $\varnothing[x/h] = \varnothing$, for any h. Hence $\mathcal{M}, \varnothing \models \exists x\varphi$. □

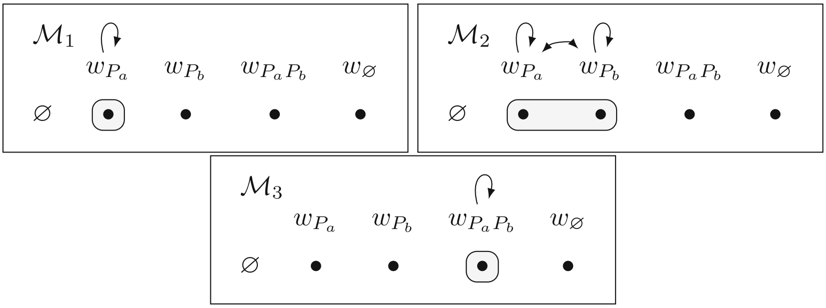
4.3 Pragmatic Intrusion

The pragmatic enrichment function is defined in terms of a systematic intrusion of NE in the recursive process of meaning composition (as in Aloni 2022).

Definition 4.13 (*Pragmatic enrichment*) A *pragmatic enrichment function* is a mapping $[.]^+$ from the NE-free fragment of $\mathcal{L}$ to $\mathcal{L}$ such that

$$\begin{aligned}
[Pt_1 \ldots t_n]^+ &= Pt_1 \ldots t_n \wedge \mathsf{NE} \\
[\neg\varphi]^+ &= \neg[\varphi]^+ \wedge \mathsf{NE} \\
[\varphi \vee \psi]^+ &= ([\varphi]^+ \vee [\psi]^+) \wedge \mathsf{NE} \\
[\varphi \wedge \psi]^+ &= ([\varphi]^+ \wedge [\psi]^+) \wedge \mathsf{NE} \\
[\Box\, \varphi]^+ &= \Box[\varphi]^+ \wedge \mathsf{NE} \\
[\Diamond\, \varphi]^+ &= \Diamond\, [\varphi]^+ \wedge \mathsf{NE} \\
[\exists x\varphi]^+ &= \exists x[\varphi]^+ \wedge \mathsf{NE} \\
[\forall x\varphi]^+ &= \forall x[\varphi]^+ \wedge \mathsf{NE}
\end{aligned}$$

Pragmatic enrichment has a crucial effect in combination with split disjunction. A state s supports a split disjunction $(\varphi \vee \psi)$ iff s can be split into two substates, each supporting one of the disjuncts. A state s supports a pragmatically enriched disjunction $[\varphi \vee \psi]^+$ iff s can be split into two *non-empty* substates, each supporting one of the disjuncts. In



$\mathcal{M}_1, s \models Pa \vee Pb$	$\mathcal{M}_1, s \not\models [Pa \vee Pb]^+$
$\mathcal{M}_2, s \models [Pa \vee Pb]^+$	$\mathcal{M}_1, s \not\dashv [Pa \vee Pb]^+$
$\mathcal{M}_3, s \models [Pa \vee Pb]^+$	$\mathcal{M}_2, s \models \Diamond Pa \wedge \Diamond Pb$

Fig. 13 Examples with pragmatically enriched formulas where models $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are state-based

Fig. 14 .

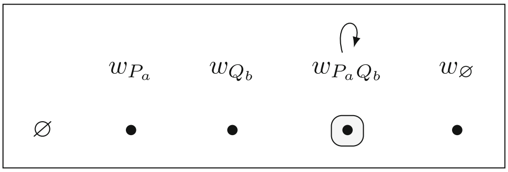
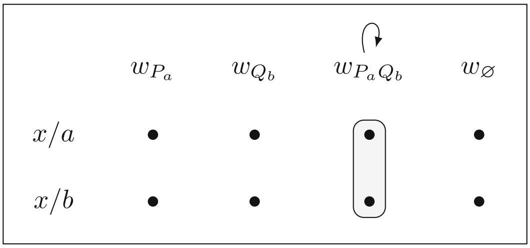


Fig. 15 .



combination with a state-based accessibility relation this will derive ignorance effects. See Fig. 13 for illustrations.

As shown in Aloni (2022), pragmatic enrichment has non-trivial effects in combination with positive disjunctions and only in these cases. This will allow us to account for ignorance, distribution and free choice inference while avoiding counterintuitive results in other configurations, in particular under negation (Figs. 14, 15).

5 Results

We are finally in a position to revisit Sect. 2.5 and show that our theory fulfills the desiderata we formulated there.

Using pragmatic enrichment we derive all our desiderata including ignorance and its obviation, distribution, free choice and classical behaviour under negation. In (56)–(58) the Diamond is interpreted epistemically (for which we assume a state-based accessibility relation as in (as in Aloni 2018, 2022).

(56) a. Klaus has at least three children. [Ignorance]
b. $[\texttt{three} \vee \texttt{more}]^+ \models \Diamond\, \texttt{three} \wedge \Diamond\, \texttt{more}$

(57) a. Every woman in my family has at least three children. [Obviation]
b. $[\forall x(\texttt{three}(x) \vee \texttt{more}(x))]^+ \not\models \forall x(\Diamond\, \texttt{three}(x) \wedge \Diamond\, \texttt{more}(x))$

(58) a. Every woman in my family has at least three children. [Distribution$^{\Diamond}$]
b. $[\forall x(\texttt{three}(x) \vee \texttt{more}(x))]^+ \models \exists x \Diamond\, \texttt{three}(x) \wedge \exists x \Diamond\, \texttt{more}(x)$

(59) a. You are required to read at least three books. [$\Box$ free choice]
b. $[\Box(\texttt{three} \vee \texttt{more})]^+ \models \Diamond\, \texttt{three} \wedge \Diamond\, \texttt{more}$

(60) a. You are allowed to read at least three books. [$\Diamond$ free choice]
b. $[\Diamond\, (\texttt{three} \vee \texttt{more})]^+ \models \Diamond\, \texttt{three} \wedge \Diamond\, \texttt{more}$

(61) a. ?Klaus does not have at least three children. [Negation]
b. $[\neg(\texttt{three} \vee \texttt{more})]^+ \models \neg\texttt{three} \wedge \neg\texttt{more}$

In the remaining of this section we give proofs of these facts.

5.1 Ignorance

The ignorance inference is derived for pragmatically enriched disjunctions, assuming a state-based accessibility relation.

Fact 3

$$[Pa \vee Pb]^+ \models_{\text{epi}} \Diamond\, Pa \wedge \Diamond\, Pb$$

Proof Assume we have a model and a state such that $\mathcal{M}, s \models [Pa \vee Pb]^+$ and assume R is state-based in $(\mathcal{M}, s)$. This means that $\mathcal{M}, s \models (Pa \wedge \text{NE}) \vee (Pb \wedge \text{NE})$. It follows that there must be non-empty t, t' such that $t \cup t' = s$ and $\mathcal{M}, t \models Pa$ and $\mathcal{M}, t' \models Pb$. Since R is state-based, it is also reflexive. By reflexivity of R we can be sure that $\mathcal{M}, t \models \Diamond\, Pa$ and $\mathcal{M}, t' \models \Diamond\, Pb$. Since $t \subseteq s$ and R is state-based, we have $\mathcal{M}, s \models \Diamond\, Pa$ and since $t' \subseteq s$ and R is state-based, we have $\mathcal{M}, s \models \Diamond\, Pb$. Hence $\mathcal{M}, s \models \Diamond\, Pa \wedge \Diamond\, Pb$. $\square$

This result can easily be generalised to arbitrary φ.

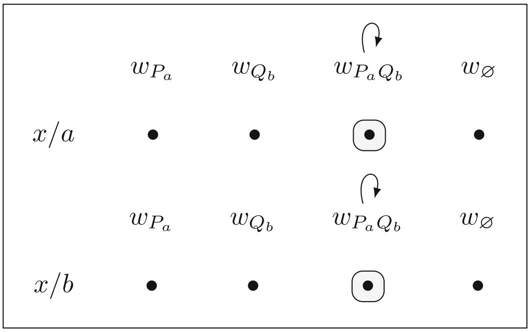
5.2 Obviation

The ignorance inference does not arise when disjunction occurs in the scope of a universal or a deontic modal operator. We show this only for the universal quantifier case. In Fact 4 we assume again that $\Diamond$ is an epistemic modal, relying on a state-based accessibility relation:

Fact 4

$$[\forall x Px \vee Qx]^+ \not\models \forall x(\Diamond\, Px \wedge \Diamond\, Qx)$$

Fig. 16 .



Proof Consider the counter-example in Fig. 14.

This state supports $[\forall x Px \vee Qx]^+$, because its universal extension supports $[Px \vee Qx]^+$. Split the state horizontally and the two non-empty substates support Px and Qx, respectively. Specifically, in the example in Fig. 15 this is true because the domain contains two objects.

But it does not support $\forall x(\Diamond Px \wedge \Diamond Qx)$, because its universal extension does not support $\Diamond Px \wedge \Diamond Qx$. The first substate in Fig. 16 below does not support $\Diamond Qx$ and the other substate does not support $\Diamond Px$. □

Notice that the ambiguity of the deontic cases between epistemic and authoritative readings is expressed as a scope ambiguity with the disjunction taking wide scope with respect to the modal operator in the epistemic case. A full approach of these cases however requires a multi-modal system where $\Box_d$ is a deontic modal and $\Diamond_e$ is an epistemic one:

(62) Paprika is required to read two or three books.

a. Authoritative: $[\Box_d Pa \vee Pb]^+ \models \Diamond_d Pa \wedge \Diamond_d Pb$
b. Epistemic: $[\Box_d Pa \vee \Box_d Pb]^+ \models \Diamond_e \Box_d Pa \wedge \Diamond_e \Box_d Pb$

5.3 Distribution

We also predict the following distribution fact. In a situation of full information $[\forall x(Px \vee Qx)]^+$ implies $\exists x Px \wedge \exists x Qx$.

Fact 5 *Assume s is a state of maximal information, i.e.,* $\text{card}(s) = 1$. *Then*

$$M, s \models [\forall x(Px \vee Qx)]^+ \Rightarrow M, s \models \exists x Px \wedge \exists x Qx$$

Proof Assume $\mathcal{M}, s \models [\forall x(Px \vee Qx)]^+$ and $\text{card}(s) = 1$. Let $s = \{i\}$. By definition, we have that $\mathcal{M}, s[x] \models (Px \wedge \text{NE}) \vee (Qx \wedge \text{NE})$. It follows that there are non-empty $t, t' \subseteq s[x]$ such that $\mathcal{M}, t \models Px$ and $\mathcal{M}, t' \models Qx$. Since $s = \{i\}$, both t and t' will contain extensions of one and the same world w_i and so we can be sure that there is some functional x-extension of s which supports Px and some which supports Qx. Hence $\mathcal{M}, s \models \exists x Px \wedge \exists x Qx$. □

Without the assumption of full information we derive something weaker.

Fact 6

$$[\forall x(Px \vee Qx)]^+ \models_{\mathsf{epi}} \exists x \Diamond Px \wedge \exists x \Diamond Qx$$

Proof Assume we have a model $\mathcal{M}$, a state s and R is state-based. Assume that $\mathcal{M}, s \models [\forall x(Px \vee Qx)]^+$. From this it follows that there are non-empty t, t' such that $t \cup t' = s[x]$ and $\mathcal{M}, t \models Px$ and $\mathcal{M}, t' \models Qx$.

1. We need to show that $\mathcal{M}, s \models \exists x \Diamond Px$. From $\mathcal{M}, t \models Px$, it follows that there is at least one index $i \in s$ and a $d \in D$ such that $i[x/d]$ supports Px. Let $X = \{i\}$ and suppose that $g_i(x) = d$. If we take the x-extension $s[x/d]$, then we can be sure that $t[x/d] \models \Diamond Px$. Since R is state-based we can access w_i from anywhere in s and it follows that for every $j \in s[x/d]$, there exists an $X \subseteq R(w_j)$ such that $X \neq \varnothing$ and $\mathcal{M}, X[g_j] \models Px$. Since individual x-extensions are particular cases of functional x-extensions, we conclude $\mathcal{M}, s \models \exists x \Diamond Px$.
2. The case for Qx is analogous.

We have shown that $\mathcal{M}, s \models \exists x \Diamond Px$ and that $\mathcal{M}, s \models \exists x \Diamond Qx$. We conclude that $\mathcal{M}, s \models \exists x \Diamond Px \wedge \exists x \Diamond Qx$. □

5.4 Free Choice

For pragmatically enriched sentences we predict $\Box$- and $\Diamond$-free choice inferences (as in Aloni 2022) but also cases of so-called universal FC, which have been attested experimentally by Chemla (2009):

(63) a. All of the boys may go to the beach or to the cinema.
$\rightsquigarrow$ All of the boys may go to the beach and all of the boys may go to the cinema.
b. $\forall x \Diamond (\varphi \vee \psi) \rightsquigarrow \forall x \Diamond \varphi \wedge \forall x \Diamond \psi$

Fact 7 ($\Box$-free choice)

$$[\Box(Pa \vee Pb)]^+ \models \Diamond Pa \wedge \Diamond Pb$$

Proof Let $\mathcal{M}$ be a model and s a state based on $\mathcal{M}$. Assume $\mathcal{M}, s \models [\Box(Pa \vee Pb)]^+$. It follows that for every $i \in s$: $\mathcal{M}, R(w_i)[g_i] \models [Pa \vee Pb]^+$. This means there are non-empty $t, t' \subseteq R(w_i)[g_i]$ and $\mathcal{M}, t \models Pa$ and $\mathcal{M}, t' \models Pb$. But this means that for every $i \in s$ there exists a non-empty $X \subseteq R(w_i)$ such that $\mathcal{M}, X[g_i] \models Pa$ and a non-empty $X' \subseteq R(w_i)$ such that $\mathcal{M}, X'[g_i] \models Pb$. We conclude that $\mathcal{M}, s \models \Diamond Pa \wedge \Diamond Pb$. □

Fact 8 ($\Diamond$-free choice)

$$[\Diamond (Pa \vee Pb)]^+ \models \Diamond Pa \wedge \Diamond Pb$$

Proof Similar to the proof of Fact 7. See also Aloni (2022, Fact 4). □

Fact 9 (Universal free choice)

$$[\forall x \Diamond (Px \vee Qx)]^+ \models \forall x \Diamond Px \wedge \forall x \Diamond Qx$$

Proof Suppose $M, s \models [\forall x \Diamond (Px \vee Qx)]^+$, which implies $M, s[x] \models \Diamond ([Px]^+ \vee [Qx]^+)$ and $s \neq \varnothing$. Let $i \in s[x]$. $M, s[x] \models \Diamond ([Px]^+ \vee [Qx]^+)$ means that there is a non-empty $X \subseteq R[w_i]$ such that $M, X[g_i] \models ([Px]^+ \vee [Qx]^+)$. Therefore there are some t_1,t_2 such that $X[g_i] = t_1 \cup t_2$ and $M, t_1 \models [Px]^+$ and $M, t_2 \models [Qx]^+$. It follows that $t_1 \neq \varnothing$ and $M, t_1 \models Px$. Since i was arbitrary we conclude $M, s[x] \models \Diamond Px$, and therefore $M, s \models \forall x \Diamond Px$. By the same reasoning we conclude $M, s \models \forall x \Diamond Qx$ and therefore $M, s \models \forall x \Diamond Px \wedge \forall x \Diamond Qx$. □

All these results can easily be generalised to arbitrary φ, ψ.

5.5 Behaviour Under Negation

We conclude the section by proving that ignorance effects disappear under negation (see Aloni, 2022, for a generalisation of this result to the case of negated modal disjunction).

Fact 10

$$[\neg(Pa \vee Pb)]^+ \models \neg Pa \wedge \neg Pb$$

Proof Assume $\mathcal{M}, s \models [\neg(Pa \vee Pb)]^+$. It follows that $s \neq \varnothing$ and $\mathcal{M}, s \dashv [Pa \vee Pb]^+$. This means that $\mathcal{M}, s \dashv Pa \wedge \mathrm{NE}$ and $\mathcal{M}, s \dashv Pb \wedge \mathrm{NE}$. Since $s = s \cup \varnothing$, and $\mathcal{M}, \varnothing \dashv \mathrm{NE}$, it follows that $\mathcal{M}, s \dashv Pa$ and $\mathcal{M}, s \dashv Pb$, which means that $\mathcal{M}, s \models \neg Pa$ and $\mathcal{M}, s \models \neg Pb$. We conclude $\mathcal{M}, s \models \neg Pa \wedge \neg Pb$. □

This means that we can account for the infelicity of (64) in terms of blocking as explained in Sect. 2.4.

(64) ? John does not have at least three children.

In other downward entailing contexts, however, superlative quantifiers can be felicitous. Consider for example the following sentence where *at least* appears in the restriction of a universal quantifier.

(65) Every candidate who has at least 2 degrees will be invited for the interview.

Our logic, extended with a proper notion of implication (cf., Flachs, 2023), can account for (65). A full investigation of the distribution of superlative quantifiers in downward entailing contexts however must be left to another occasion.

6 Conclusion

We have addressed a number of puzzles arising for the interpretation of modified numerals. Following Büring and others we have assumed that the main difference between comparative and superlative modifiers is that only the latter convey disjunctive meanings. We further argued that the inference patterns triggered by disjunction and superlative modifiers are hard to capture in existing semantic and pragmatic analyses of these phenomena (neo-Gricean or grammatical alike), and we have proposed a novel account of these inferences in the framework of bilateral state-based modal logic defining a first order extension of Aloni's BSML. In this framework, next to literal meanings (the NE-free fragment of the language, ruled by classical logic), also pragmatic factors (NE) are modelled and the additional inferences that arise from their interaction (ignorance, distribution, free choice). The intruding pragmatic factor represented by NE, connects to a tendency of language users to neglect the empty state, an abstract element comparable to the zero in mathematics. In future work we would like to seek corroboration by conducting experiments and test our predictions. This would perhaps also shed more light on the tendency to neglect the empty state and the cognitive plausibility of the framework.

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