Journal of Algebraic Combinatorics (2022) 56:1023–1030
https://doi.org/10.1007/s10801-022-01143-0

Betti numbers of fat forests and their Alexander dual

Ralf Fröberg[1]

Received: 10 December 2021 / Accepted: 23 April 2022 / Published online: 20 May 2022

Abstract

Let k be a field and $R = k[x_1, \dots, x_n]/I = S/I$ a graded ring. Then R has a t-linear resolution if I is generated by homogeneous elements of degree t, and all higher syzygies are linear. Thus, R has a t-linear resolution if $\mathrm{Tor}^S_{i,j}(S/I, k) = 0$ if $j \neq i+t-1$. For a graph G on $\{1, \dots, n\}$, the edge algebra is $k[x_1, \dots, x_n]/I$, where I is generated by those $x_i x_j$ for which $\{i, j\}$ is an edge in G. We want to determine the Betti numbers of edge rings with 2-linear resolution. But we want to do that by looking at the edge ring as a Stanley–Reisner ring. For a simplicial complex Δ on $[\mathbf{n}] = \{1, \dots, n\}$ and a field k, the Stanley–Reisner ring $k[\Delta]$ is $k[x_1, \dots, x_n]/I$, where I is generated by the squarefree monomials $x_{i_1} \dots x_{i_k}$ for which $\{i_1, \dots, i_k\}$ does not belong to Δ. Which Stanley–Reisner rings that are edge rings with 2-linear resolution are known. Their associated complexes has had different names in the literature. We call them fat forests here. We determine the Betti numbers of many fat forests and compare our result with what is known. We also consider Betti numbers of Alexander duals of fat forests.

Keywords Stanley–Reisner ring · Edge ring · Betti numbers · Hilbert series

1 Background

The simplicial complexes we will consider have had different names. They are called generalized forests [12], quasiforests [23], or fat forests [2]. We will call them fat forests. They are recursively defined as follows. A d-simplex F_1 of dimension ≥ 0 (i.e. with $d+1$ vertices) is a fat forest. If F_i, $i = 1, \dots, k$, are simplices and $G_{k-1} = F_1 \cup \dots \cup F_{k-1}$ is a fat forest, then $G_{k-1} \cup F_k$ is a fat forest if $H = G_{k-1} \cap F_k$ is a simplex, $\dim H \geq -1$. (If $\dim H = -1$, then G_{k-1} and F_k are disjoint.)A fat forest a called a fat tree if it is connected, so if $\dim G_{k-1} \cap F_k \geq 0$ for all k, but here it is not necessary to treat fat trees separately.

✉ Ralf Fröberg
frobergralf@gmail.com

1 Mathematics Department, Stockholm University, Stockholm, Sweden

Let G be a graph on $[\mathbf{n}]$ and let $S = k[x_1, \ldots, x_n]$, where k is a field. The edge ring of G is S/I,where I is generated by all $x_i x_j$ for which $\{i, j\}$ is an edge of G. Let G^c be the complement graph to G, i.e. the graph on $[\mathbf{n}]$ with edges $\{k, l\}$ for which $\{k, l\}$ is not an edge of G. It is shown in [12] that the edge ring of G has a 2-linear resolution, meaning $\mathrm{Tor}^S_{i,j}(S/I, k) = 0$ if $j \neq i + 1$, if and only if the complementary graph G^c (having those edges which are not edges in G) is chordal. A graph is chordal if every k-cycle, $k \geq 4$, has a chord. This theorem has been reproved in different ways in [10, 11, 13, 21].

An edge ring $k[x_1, \ldots, x_n]/I$ can also be seen as a Stanley–Reisner ring $k[\Delta]$. Then I is generated by those squarefree monomials that do not belong to a simplicial complex Δ. The 1-skeleton of a simplicial complex Δ consists of all faces of Δ that have dimension ≤ 1. Dirac has shown in [8, Theorem 1 and 2], that a graph G is chordal if and only if it is the 1-skeleton of a fat forest. There is an algebraic proof of Dirac's theorems in [15]. Dirac's theorem gives easily, see [12], that the Stanley–Reisner ring $k[\Delta]$ has a 2-linear resolution if and only if Δ is a fat forest.

Edge rings of Ferrer's graphs have 2-linear resolution. Their resolution has been determined in [6]. Other classes of monomial rings with 2-linear resolution are treated in, for example, [1, 7, 16, 17]. Finally minimal resolution of all edge rings with 2-linear resolution was determined in [5]. This gives a (rather complicated) way to determine the Betti numbers. In this note we will show that if one is only interested in the Betti numbers of edge rings with 2-linear resolution, there is in many cases a very easy way to determine them using the description of the rings as Stanley–Reisner rings. Monomial rings with 2-linear resolution are also treated in, for example, [14, 18, 22].

2 Hilbert series and Betti numbers of fat forests

If S/I has a 2-linear resolution it looks like this:

$$0 \leftarrow S/I \leftarrow S \leftarrow S[-2]^{b_1} \leftarrow S[-3]^{b_2} \leftarrow \cdots \leftarrow S[-p-1]^{b_p} \leftarrow 0$$

where $S[-i]$ means that we have shifted degrees of S i steps. Using that the alternating sum of the k-dimensions in each degree is 0, we get that the Hilbert series of $k[\Delta]$ with 2-linear resolution equals $\frac{1-\beta_{1,2}t^2+\beta_{2,3}t^3-\cdots(-1)^p\beta_{p,p+1}t^{p+1}}{(1-t)^n}$, where $\beta_{i,j}$ are the graded Betti numbers $\dim_k \mathrm{Tor}^S_{i,j}(k[\Delta], k)$, and n is the number of vertices in Δ. We are primarily interested in the Betti numbers $\beta_{i,j} = \dim_k \mathrm{Tor}^S_{i,j}(S/I, k)$ of Stanley–Reisner rings of fat trees, but the Hilbert series contains the same information as the set of Betti numbers.

Example Let Δ be the simplicial complex with facets (maximal faces) $\{1, 2\}$, $\{2, 3, 4\}$, and $\{5\}$. This is a fat forest which can be built in the following way. Start with the simplex $\{2, 3, 4\}$ which has Hilbert series $\frac{1}{(1-t)^3}$. Then adjoin $\{1, 2\}$ in $\{2\}$. We add the Hilbert series $\frac{1}{(1-t)^2}$ for the face $\{1, 2\}$ and subtract $\frac{1}{1-t}$ for the face $\{2\}$ which has been counted twice. Finally adjoin $\{5\}$ in $\emptyset$, that is, we add $\frac{1}{1-t}$ and subtract 1. Thus, the Hilbert series of Δ is $\frac{1}{(1-t)^3} + \frac{1}{(1-t)^2} - \frac{1}{1-t} + \frac{1}{1-t} - 1 = \frac{1-6t^2+9t^3-5t^4+t^5}{(1-t)^5}$, and the

Betti numbers of $k[\Delta]$ are $b_{1,2} = 6$, $b_{2,3} = 9$, $b_{3,4} = 5$, and $b_{4,5} = 1$. This ring can also be seen as the edge ring of a graph with edges {1,3}, {1,4}, {1,5}, {2,5}, {3,5}, and {4,5}.

Theorem 1 *Let $F = F_1 \cup \cdots \cup F_k$ be a fat tree with F_i a simplex of dimension d_i and $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ a simplex of dimension r_j. Then the Hilbert series of $k[F]$ is $\sum_{i=1}^k \frac{1}{(1-t)^{d_i+1}} - \sum_{i=2}^k \frac{1}{(1-t)^{r_i+1}}$. The projective dimension is $\sum_{i=1}^k d_i - \sum_{i=2}^k r_i + 1 - \min\{r_i\} - 2$. The depth of $k[F]$ is $\min\{r_i\} + 2$, and F is CM (Cohen–Macaulay) if and only if there is a d such that $d_i = d$ for all i and $r_i = d - 1$ for all i.*

Proof The definition of fat forests directly gives the Hilbert series. The number of vertices of F is $\sum_{i=1}^k (d_i + 1) - \sum_{i=2}^k (r_i + 1) = \sum_{i=1}^k d_i - \sum_{i=2}^k r_i + 1 = n$, so the degree of the numerator $p(t)$ of the Hilbert series $\frac{p(t)}{(1-t)^n}$ of $k[F]$ is $n - \min\{r_i\} - 1$ so the projective dimension is $n - \min\{r_i\} - 2$, and the depth of $k[F]$ is $\min\{r_i\} + 2$ by the Auslander–Buchsbaum formula. We have $\dim k[F] = 1 + \max\{d_i\}$, $\operatorname{depth} k[F] = \min\{r_i\} + 2$, and $d_i > r_i$ for all i. The only possibility for $\dim k[F] = \operatorname{depth} k[F]$ is that there is a d such that $d_i = r_i + 1 = d$ for all i.

Remark The characterization of CM fat trees is not new. With another (more complicated) proof it is found in [12].

In the remaining part of this section, we give some examples of results we can achieve.

Jacques has determined the Betti numbers of the complete bipartite graph $K_{m,n}$, [17]. The result is that the only nonzero Betti numbers are $\beta_{i,i+1}(K_{m,n}) = \sum_{\substack{j+l=i+1 \\ j,l\geq 1}} \binom{m}{j}\binom{n}{l}$. We give an alternative proof.

Theorem 2 *The edge ring of $K_{m,n}$ has a 2-linear resolution and $\beta_{i,i+1}(K_{m,n}) = \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}$.*

Proof The edge ring of $K_{m,n}$ is the Stanley–Reisner ring of $K_m \sqcup K_n$, the disjoint union of two complete graphs, so the resolution is 2-linear. The Hilbert series is $\frac{1}{(1-t)^m} + \frac{1}{(1-t)^n} - 1 = \frac{(1-t)^n + (1-t)^m - (1-t)^{m+n}}{(1-t)^{m+n}}$. Thus, $\beta_{i,i+1} = \binom{n+m}{i+1} - \binom{n}{i+1} - \binom{m}{i+1}$.

Corollary 1 $\sum_{\substack{j+l=i+1 \\ j,l\geq 1}} \binom{m}{j}\binom{n}{l} = \binom{n+m}{i+1} - \binom{n}{i+1} - \binom{m}{i+1}$.

Also the complete multipartite graph $K_{n_1,\ldots,n_s}$ with parts of size $n_1, \ldots, n_s$ is treated in [17]. The result there is

$$\beta_{i,i+1}(K_{n_1,\ldots,n_s}) = \sum_{l=2}^{s} (l-1) \sum_{\substack{\alpha_1+\cdots+\alpha_l=i+1 \\ \alpha_1,\ldots,\alpha_l \geq 1 \quad j_1<\cdots<j_l}} \binom{n_{j_1}}{\alpha_1} \cdots \binom{n_{j_l}}{\alpha_l}.$$

With our method we get

Theorem 3 $\beta_{i,i+1}(K_{n_1,\ldots,n_s}) = \sum_{i=1}^s \binom{N-n_i}{i+1} - \binom{N}{i+1}$, *where $N = \sum_{i=1}^s n_i$.*

Proof The edge ring of $K_{n_1,\dots,n_s}$ is the Stanley–Reisner ring of the disjoint union of $K_{n_1},\dots,K_{n_s}$ with Hilbert series $\sum_{i=1}^{s}\frac{1}{(1-t)^{n_i}}-(s-1)=\frac{\sum_{i=1}^{s}(1-t)^{N-n_i}-(s-1)(1-t)^N}{(1-t)^N}$. The result follows as before.

Corollary 2

$$\sum_{l=2}^{s}(l-1)\sum_{\substack{\alpha_1+\cdots+\alpha_l=i+1\\ j_1<\cdots<j_l \quad \alpha_1,\dots,\alpha_l\geq 1}}\binom{n_{j_1}}{\alpha_1}\cdots\binom{n_{j_l}}{\alpha_l}=\sum_{i=1}^{s}\binom{N-n_i}{i+1}-\binom{N}{i+1}.$$

In [1], squarefree lexsegments ideal with q-linear resolution are studied. They show that ideals generated by initial segment of squarefree monomials in degree q in lexicographic order have q-linear resolution. For $q=2$, this means that ideals $L(a,b)$ generated by all squarefree monomials of degree 2 that are larger than or equal to $x_a x_b$ for some (a,b) have 2-linear resolution.

Theorem 4 *The Betti numbers of* $k[x_1,\dots,x_n]/L(a,b)$, $a\leq b$, *are* $\beta_{i,i+1}=a\binom{b}{i+1}-a\binom{b-1}{i+1}-\binom{a}{i+1}$, $1\leq i\leq b-1$.

Proof $L(a,b)$ is the Stanley–Reisner ideal of a simplicial complex with maximal faces $\{1\},\{2\},\dots,\{a\},\{a+1,a+2,\dots,b\}$. Thus, the Hilbert series is $\frac{a}{(1-t)}+\frac{1}{(1-t)^{b-a}}-a=\frac{a(1-t)^{b-1}+(1-t)^a-a(1-t)^b}{(1-t)^b}$.

Corollary 3 $a\binom{b}{i+1}-a\binom{b-1}{i+1}-\binom{a}{i+1}=\sum_{k=0}^{a-1}(k+1)\binom{k}{i-1}+a\sum_{k=a}^{b-2}\binom{k}{i-1}$.

Proof In [1], it is shown that $\beta_{i,i+1}=\sum_{x_ix_j\geq x^ax^b}\binom{j-2}{i-1}$, $i\leq j$.

Also final squarefree segment define ideals with linear resolution. For $d=2$, this means that ideals $F(a,b)=(\{x_ix_j;\, x_ix_j\leq x_ax_b\})$ have 2-linear resolution.

Theorem 5 *The Hilbert series of* $k[x_1,\dots,x_n]/F(a,b)$, $a<b$, *is*

$$\frac{n-b-1}{(1-t)^{N-a}}+\frac{b-a-1}{(1-t)^{N-a+1}}-\frac{b-a-2}{(1-t)^{N-a}}-\frac{n-b+1}{(1-t)^{N-a+1}},$$

where $N=\binom{n-a-1}{2}+n-b+1$.

Proof $F(a,b)$ is the Stanley–Reisner ideal of a simplicial complex with maximal faces $\{1,2,\dots,a-1,i\}$, $b\leq i\leq n$, and $\{1,2,\dots,a,j\}$, $a+1\leq j\leq b-1$.

Now we treat some examples of Ferrer's ideals. A Ferrer's ideals can be indexed by a tableau $(\lambda_1,\lambda_2\dots,\lambda_m)$, $\lambda_1\geq\lambda_2\cdots\geq\lambda_m\geq 1$. Here

$$I_{(\lambda_1,\dots,\lambda_m)}=(x_1y_1,x_1y_2,\dots,x_1y_{\lambda_1},x_2y_1,\dots,x_2y_{\lambda_2},\dots,x_my_1,\dots,x_my_{\lambda_m}).$$

We start with an example from [6], namely $k[x_1,\dots,x_5,y_1,\dots,y_6]/I_{(6,4,4,2,1)}$. The fat tree with Stanley–Reisner ideal $I_{(6,4,4,2,1)}$ can be constructed like this. (We denote the simplex on $a_1,\dots,a_k$ by $[a_1,\dots,a_k]$.)

Start with $[y_1, \dots, y_6]$ and attach $[x_5, y_2, y_3, y_4, y_5, y_6]$ in $[y_2, y_3, y_4, y_5, y_6]$. Then attach $[x_4, x_5, y_3, y_4, y_5, y_6]$ in $[x_5, y_3, y_4, y_5, y_6]$. Then attach $[x_2, x_3, x_4, x_5, y_5, y_6]$ in $[x_4, x_5, y_5, y_6]$. Finally attach $[x_1, x_2, x_3, x_4, x_5]$ in $[x_2, x_3, x_4, x_5]$.

The Hilbert series of $k[x_1, \dots, x_5, y_1, \dots, y_6]/I_{(6,4,4,2,1)}$ is

$$\frac{1}{(1-t)^6} + \frac{1}{(1-t)^6} - \frac{1}{(1-t)^5} + \frac{1}{(1-t)^6} - \frac{1}{(1-t)^5} + \frac{1}{(1-t)^6}$$
$$-\frac{1}{(1-t)^4} + \frac{1}{(1-t)^5} - \frac{1}{(1-t)^4} = \frac{4(1-t)^5 - (1-t)^6 - 2(1-t)^7}{(1-t)^{11}}.$$

Thus,

$$\beta_i(k[x_1, \dots, x_5, y_1, \dots, y_6]/I_{(6,4,4,2,1)}) = 2\binom{7}{i+1} + \binom{6}{i+1} - 4\binom{5}{i+1}.$$

The formula in [6] gives

$$\beta_i(k[x_1, \dots, x_5, y_1, \dots, y_6]/I_{(6,4,4,2,1)}) = \binom{6}{i} + \binom{5}{i} + \binom{6}{i} + \binom{5}{i} + \binom{5}{i}$$
$$-\binom{5}{i+1} = 2\binom{6}{i} + 3\binom{5}{i} - \binom{5}{i+1}.$$

Denote a sequence $a, a, \dots, a$ of length b with a^b. A Ferrer's graph with tableau $\mu_1^{l_1}, \dots, \mu_k^{l_k}$ corresponds to a simplicial complex with maximal faces $[x_{L-l_k-\dots-l_i+1}, \dots, x_L, y_{\mu_i+1}, \dots, y_{\mu_1}]$, where $L = \sum_{i=1}^k l_i$, for $i = 0, \dots, k$. One could generalize the method above to any tableau, but the result becomes a bit complicated, so we refrain from doing so. Instead we give some concrete examples and compare the formulas for the Betti numbers.

First consider I_{m^n} which is the edge ring of the complete bipartite graph $K_{m,n}$. Here we have yet another expression from [6], $\beta_{i,i+1} = \sum_{j=n}^{n+m-1} \binom{j}{i} - \binom{m}{i+1}$. Thus, we get

Corollary 4 $\binom{n+m}{i+1} - \binom{n}{i+1} - \binom{m}{i+1} = \sum_{j=n}^{n+m-1} \binom{j}{i} - \binom{m}{i+1}$.

Now consider

$$I_{n,1^{m-1}} = (x_1y_1, x_1y_2, \dots, x_1y_n, x_2y_1, x_3y_1, \dots, x_my_1).$$

The fat tree with this Stanley–Reisner ideal has a maximal face $[y_1, \dots, y_n]$. Then another maximal face $[x_2, \dots, x_m, y_2, \dots, y_n]$ is attached in $[y_2, \dots, y_n]$. Finally the maximal face $[x_1, \dots, x_m]$ is attached in $[x_2, \dots, x_m]$.

Theorem 6 *The Hilbert series of* $k[x_1, \dots, x_m, y_1, \dots, y_n]/I_{n,1^{m-1}}$ *is*

$$\frac{1}{(1-t)^n} + \frac{1}{(1-t)^m} + \frac{1}{(1-t)^{m+n-2}} - \frac{1}{(1-t)^{n-1}} - \frac{1}{(1-t)^{m-1}},$$

so the Betti numbers are

$$\beta_i = \binom{n+1}{i+1} + \binom{m+1}{i+1} - \binom{n}{i+1} - \binom{m}{i+1} - \binom{2}{i+1}.$$

Corollary 5

$$\begin{aligned}&\binom{n+1}{i+1} + \binom{m+1}{i+1} - \binom{n}{i+1} - \binom{m}{i+1} - \binom{2}{i+1}\\ &\quad = \binom{n}{i} + \sum_{j=2}^{m}\binom{j}{i} - \binom{m+1}{i+1}.\end{aligned}$$

Proof The right-hand side is the formula for β_i from [6]. We can do the same in higher dimension. Consider the ideal

$$I(m,n,o) = (x_1y_i, x_1z_j, y_1x_k, y_1z_j, z_1x_k, z_1y_i, 1 \le k \le m, 1 \le i \le n, 1 \le j \le o).$$

In this case, the complex has four maximal faces,

$$[x_2,\dots,x_m, y_2,\dots,y_n, z_2,\dots,z_o], [x_1,\dots,x_m], [y_1,\dots,y_n], [z_1,\dots,z_o].$$

The three last are attached in $[x_2,\dots,x_m]$, $[y_2,\dots,y_n]$, $[z_2,\dots,z_o]$, respectively. Then $k[x_1,\dots,x_m, y_1,\dots,y_n, z_1\dots,z_o]/I(m,n,o)$ has a 2-linear resolution. The Hilbert series of $k[x_1,\dots,x_m, y_1,\dots,y_n, z_1\dots,z_o]/I(m,n,o)$ is

$$\begin{aligned}&\frac{1}{(1-t)^{m+n+o-3}} + \frac{1}{(1-t)^m} + \frac{1}{(1-t)^n} + \frac{1}{(1-t)^o}\\ &-\frac{1}{(1-t)^{m-1}} - \frac{1}{(1-t)^{n-1}} - \frac{1}{(1-t)^{o-1}},\end{aligned}$$

so the Betti numbers are

$$\begin{aligned}\beta_i = &\binom{m+n+1}{i+1} + \binom{m+o+1}{i+1} + \binom{n+o+1}{i+1}\\ &-\binom{m+n}{i+1} - \binom{m+o}{i+1} - \binom{n+o}{i+1} - \binom{3}{i+1}.\end{aligned}$$

Example Consider the Ferrer's ideal with tableau $n, n-1, n-2, \dots, 1$. Using the formula in [6] the Betti numbers are $n\binom{n}{i} - \binom{n}{i+1}$. The fat tree giving this ideal is constructed like this. Start with $[y_1,\dots,y_n]$ and attach $[x_n, y_2,\dots,y_n]$ in $[y_2,\dots,y_n]$. Then attach $[x_{n-1}, x_n, y_3,\dots,y_n]$ in $[x_n, y_3,\dots,y_n]$. Continue like this until: Attach $[x_2,\dots,x_n, y_n]$ in $[x_3,\dots,x_n,y_n]$, and finally attach $[x_1,\dots,x_n]$ in $[x_2,\dots,x_n]$. The Hilbert series is $\frac{n+1}{(1-t)^n} - \frac{n}{(1-t)^{n-1}}$, so the Betti numbers are $\beta_{i,i+1} = n\binom{n+1}{i+1} - (n+1)\binom{n}{i+1}$ and we get
$n\binom{n+1}{i+1} - (n+1)\binom{n}{i+1} = \binom{n}{i} + \sum_{j=2}^{m}\binom{j}{i} - \binom{n}{i+1}$.

3 Uniform fat forests

We will now concentrate on "uniform" fat forests. If $F = F_1 \cup \cdots \cup F_k$, where F_i is a d-simplex for $i = 1, \ldots, k$ and $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i$ is an r-simplex for $i = 2, \ldots, k$, we call F a (d, r)-forest.

Corollary 6 *The Hilbert series of a (d, r)-forest Δ with k facets is $\frac{k}{(1-t)^{d+1}} - \frac{k-1}{(1-t)^{r+1}}$. The depth of $k[\Delta]$ is $r + 2$, so $k[\Delta]$ is CM if and only if $r = d - 1$. We have that the Betti numbers $b_i(k[\Delta]) = b_{i,i+1}(k[\Delta]) = |k\binom{(k-1)(d-r)}{i+1} - (k-1)\binom{k(d-r)}{i+1}|$.*

For a simplicial complex Δ on $[\mathbf{n}]$, the Alexander dual $\Delta^\vee$ is defined as $\{F; F^c \notin \Delta\}$, where $F^c = [\mathbf{n}] \setminus F$. Alexander duals are well described in [3, 4, 9, 19]. We will now determine the Betti numbers of the Alexander dual to fat forests.

Theorem 7 *Let Δ be a (d, r)-forest with k facets. The nonzero Betti numbers of $k[\Delta^\vee]$ are $b_0 = b_{0,0} = 1$, $b_1 = b_{1,(k-1)(d-r)} = k$ and $b_2 = b_{2,k(d-r)} = k - 1$. All $k[\Delta^\vee]$ are CM, and $k[\Delta^\vee]$ has a linear resolution if and only if Δ is a $(d, d-1)$-forest.*

Proof We have that $\prod_{j=1}^{m} x_{i_j}$ is a minimal generator of I, where $k[\Delta^\vee] = k[x_1, \ldots, x_n]/I$, if and only if $[\mathbf{n}] \setminus \{i_1, \ldots, i_m\}$ is a facet in Δ. Thus, $\Delta^\vee$ has k minimal generators of degree $k(d - r) + r + 1 - (d + 1) = (k - 1)(d - r)$, so $b_1(k[\Delta^\vee]) = b_{1,(k-1)(d-r)}(k[\Delta^\vee]) = k$. The regularity of $k[\Delta]$ equals the projective dimension of $k[\Delta^\vee]$ and vice versa, [19, 20]. The projective dimension of $k[\Delta]$ is $k(d - r) - 1$ and $b_{k(d-r)-1,k(d-r)} = k - 1$. This is an extremal Betti number for $k[\Delta]$. (We have that $b_{k,l}(R)$ is an extremal Betti number for R if $b_{i,j}(R) = 0$ for $\{(i, j); i \geq k, j \geq l\} \setminus \{(k, l)\}$.) Then $b_{2,k(d-r)-1}(\Delta^\vee) = k - 1$ is an extremal Betti number, see [3]. Since $\sum_{i=0}^{2}(-1)^i b_i = 0$ we have $b_2 = k - 1$, so $b_{2,j} = 0$ if $j \neq k(d - r) - 1$. We have that $k[\Delta^\vee]$ has a linear resolution if and only if $k[\Delta]$ is CM, [9].

Acknowledgements No datasets were generated or analysed during the current study.

Funding Open access funding provided by Stockholm University.

References

1. Aramova, A., Herzog, J., Hibi, T.: Squarefree lexsegment ideals. Math. Z. **228**, 353–378 (1998)
2. Autry, J., Graves, P., Loucks, J., O'Neill, C., Ponomarenko, V., Yih, S.: Squarefree divisor complexes of certain numerical semigroup elements. arXiv:1804.06632v1
3. Bayer, D., Charalambous, H., Popescu, S.: Extremal Betti numbers and applications to monomial ideals. J. Algebra **221**, 497–512 (1999)

4. Bruns, W., Herzog, J.: Cohen–Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, pp. xii+403. Cambridge University Press, Cambridge (1993)
5. Chen, R.: Minimal free resolutions of linear edge ideals. J. Algebra **324**(12), 3591–3613 (2010)
6. Corso, A., Nagel, U.: Monomial and toric ideals associated to Ferrers graphs. Trans. Amer. Math. Soc. **361**(3), 1371–1395 (2009)
7. Corso, A., Nagel, U.: Specializations of Ferrers ideals. J. Algebraic Combin. **28**(3), 425–437 (2008)
8. Dirac, G.A.: On rigid circuit graphs. Abh. Math. Sem. Univ. Hamburg **38**, 71–76 (1961)
9. Eagon, J.A., Reiner, V.: Resolutions of Stanley–Reisner rings and Alexander duality. J. Pure Appl. Algebra **130**(3), 265–275 (1998)
10. Eisenbud, D., Green, M., Hulek, K., Popescu, S.: Restricting linear syzygies: algebra and geometry. Compos. Math. **141**(6), 1460–1478 (2005)
11. Engström, A., Stamps, M.T.: Betti diagrams from graphs. Algebra Number Theory **7**(7), 1725–1742 (2013)
12. Fröberg, R.: On Stanley-Reisner rings, Topics in algebra, part 2 (Warsaw, 1988). Banach Center Publ. **26**, 57–70 (1990)
13. Herzog, J., Hibi, T.: Monomial Ideals, Graduate Texts in Mathematics, 260. Springer, London (2011)
14. Herzog, J., Hibi, T., Zheng, X.: Monomial ideals whose powers have a linear resolution. Math. Scand. **95**(1), 23–32 (2004)
15. Herzog, J., Hibi, T., Zheng, X.: Dirac's theorem on chordal graphs and Alexander duality. European J. Combin. **25**(7), 949–960 (2004)
16. Horwitz, N.: Linear resolutions of quadratic monomial ideals. J. Algebra **318**(2), 981–1001 (2007)
17. Jacques, S.: Betti numbers of graph ideals. arXiv:0410107v1 (2004)
18. Nguyen, H., Vu, T.: Linearity defect of edge ideals and Fröberg's theorem. J. Algebraic Combin. **44**(1), 165–199 (2016)
19. Terai, N.: Generalization of Eagon–Reiner theorem and h-vectors of graded rings. Preprint
20. Terai, N.: Alexander duality theorem and Stanley–Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics (1998). Sūrikaisekikenkyūsho Kōkyūroku No. 1078, pp. 174–184 (1999)
21. Van Tuyl, A.: A beginner's guide to edge and cover ideals. Monomial ideals, computations and applications, pp. 63–94, Lecture Notes in Math., 2083. Springer, Heidelberg (2013)
22. Woodroofe, R.: Chordal and sequentially Cohen–Macaulay clutters. Electron. J. Combin. **18**(1), Paper 208 (2011)
23. Zheng, X.: Resolutions of facet ideals. Comm. Algebra **32**(6), 2301–2324 (2004)