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The number of cubic surfaces with 27 lines over a finite field

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Abstract
We determine the number of cubic surfaces with 27 lines over a finite field $\mathbb{F}_q$. This is based on exploiting the relationship between non-conical six-arcs in a projective plane embedded in projective three-space and cubic surfaces with 27 lines. We revisit this classical relationship, which goes back to work of Clebsch in the nineteenth century. Our result can be used as an enumerative check for a computer classification of cubic surfaces with 27 lines over finite fields.

Keywords Geometry · Cubic surface · Finite field · Counting

Mathematics Subject Classification 05B25 · 05E18 · 14E05 · 14J26 · 51E25

1 Introduction

The study of the geometry of smooth cubic surfaces, the 27 lines lying on them, their symmetries and numerous other features has a long history dating back to the nineteenth century and is still of remarkable interest, especially in some important areas in algebraic geometry (Weyl groups of type E6) and in the field of combinatorics (the Schäfli graphs). If the field of coordinates is not algebraically closed, the number of lines contained in a smooth cubic surface may be less than 27. For instance, the number of (real) lines contained in a smooth real cubic surface is one of 27, 15, 7, 3 or none at all; see Segre [22]. This gives a strong motivation for the study of cubic surfaces over a (not algebraically closed) field F which contain exactly 27 lines. The list of 27 open problems by Ranestad and Sturmfels [19] shows that there is still a lot

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of interest in finding out more about cubic surfaces. The present paper is concerned with cubic surfaces over finite fields.

One of the main topics in the study of cubic surfaces is the problem of classification up to projective equivalence. Several tools have been developed in the nineteenth century, and these tools work remarkably well in the case of finite fields also. There are the ideas of the Schläfli double-six [21], the Steiner trihedral pairs [23] and the associated arcs due to Clebsch [5]. When considering cubic surfaces over finite fields, the problem becomes finite and hence can be attacked using combinatorial methods, as well as using methods from computational group theory. Dickson [7] considered the very first case of the field of two elements. Hirschfeld and his students contributed to the problem of classification of cubic surfaces with 27 lines over finite fields [11,15,20].

The relation between non-conical six-arcs and cubic surfaces with 27 lines plays a central role in this, as is explained in Sect. 3. One of the central elements in a classification approach over finite fields is the use of group invariant relations which reduce the classification to smaller objects. In [3], the relation between Schläfli double-sixes and cubic surfaces is used. In [2], six-arcs and Steiner trihedral pairs are used. This relation was well known to nineteenth-century mathematicians. However, the action of the projective group of three-space on this relation is not obvious. We describe an invariant relation in Sect. 3.

Clebsch [5] seems to be the first to point out the relationship between cubic surfaces and six-arcs in the plane. Baker [1] mentions it only briefly. By the late nineteenth century, Felix Klein in 1873 [16] knew that the Clebsch surface [6] is related to the configuration of six lines through the origin in Euclidean space, obtained by connecting the opposite vertices in the Icosahedron. Under this relationship, the six lines are then identified with a non-conical six-arc in the real projective plane.

The present paper will exploit the above relation and develop an enumerative result for the number of cubic surfaces with 27 lines over a finite field of order q. This result can be used to confirm the correctness of computer classifications of cubic surfaces with 27 lines over finite fields. Our main result is the following:

Theorem 1 *The number of cubic surfaces with 27 lines in* $\mathrm{PG}(3, q)$ *is*

$$c_q = \frac{q^6(q^2-1)(q^3-1)(q^4-1)(q-2)(q-3)(q-5)^2}{51840}.$$

2 Background

Let us discuss some of the background material on cubic surfaces. For a deeper treatment, we refer to Hirschfeld [13] or Segre [22].

Let $\mathrm{PG}(n, \mathbb{F})$ be the n-dimensional projective space over the field $\mathbb{F}$. The elements of $\mathrm{PG}(n, \mathbb{F})$ are the subspaces of $V(n+1, \mathbb{F}) = \mathbb{F}^{n+1}$, the $n+1$-dimensional vector space over $\mathbb{F}$. We use homogeneous coordinates $X_0, \ldots, X_n$. The notation $\mathbf{v}(F)$ is used to denote the variety associated with the equation F, that is, the set of points whose coordinates evaluate to zero when substituted into F. The projective dimension of an object is one less than the vector space dimension. Thus, subspaces of ranks one are said to be projective points, subspaces of ranks two are called lines, and so on. Two lines in $\mathrm{PG}(3, \mathbb{F})$ are called skew if they do not meet.

Let $\mathbb{F}_q$ be the finite field with q elements. We write $\mathrm{PG}(n, q)$ for $\mathrm{PG}(n, \mathbb{F}_q)$. The number of k-dimensional subspaces of the vector space $\mathbb{F}_q^n$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i},$$

the q-binomial coefficient. From this, it follows that in $\mathrm{PG}(3, q)$, there are $q^2 + q + 1$ lines through a point and $q + 1$ points on each line.

A cubic surface in $\mathrm{PG}(3, \mathbb{F})$ is the zero set of a homogeneous polynomial of degree three in four variables. A line of $\mathrm{PG}(3, \mathbb{F})$ lies on the cubic surface if it is completely contained in it. This means that the equation of the surface vanishes when restricted to the coordinates of the line. Over the complex numbers, a smooth cubic surface has exactly 27 lines [4]. Over a finite field, it may have fewer lines. As there are 20 cubic monomials in four variables, 19 linearly independent conditions are required to define a cubic surface. This is because the homogeneous equation of a surface is unique up to nonzero scalars.

A plane intersects a cubic surface in a cubic. It may happen that this cubic degenerates into three lines. In this case, the plane is said to be a tritangent plane. A cubic surface with 27 lines has exactly 45 tritangent planes. Any line of the surface lies on exactly five tritangent planes. For this reason, any line of the surface is incident with exactly 10 other lines of the surface, two on each tritangent plane through it. Given two skew lines ℓ_1 and ℓ_2 of $\mathrm{PG}(3, q)$ and a point P not on either ℓ_1 or ℓ_2, there exists a unique line t through P which intersects both ℓ_1 and ℓ_2. Such a line is called a transversal. Two skew lines on the surface have exactly five common transversals on the surface.

A set of six lines on a surface, pairwise disjoint, is called a **single-six.** A set of 12 lines, partitioned into two sets of six, each forming a single six is called a **double-six** (or Schäfli double-six, in recognition of [21]) if the following condition holds: There is a bijection between the two sets such that two lines from different sets intersect if and only if they do not correspond. Double-sixes are important because they determine a cubic surface uniquely. On the other hand, one cubic surface has exactly 36 double-sixes associated with it. The Schläfli labeling a_i, b_j, c_{ij} of lines can be used to express all double-sixes in terms of one particular one. It is important to be able to build double-sixes from smaller configurations of lines. The following result of Schläfli enables this.

Theorem 2 (Schläfli [21]) *Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j ($j = 1, \ldots, 5$) has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 have a transversal a_6 also.*

Two surfaces are projectively equivalent (or isomorphic) if there is a projectivity that takes one to the other. The classification problem for cubic surfaces is the problem of determining a complete set of the pairwise inequivalent surfaces over a given field.

Several infinite families of surfaces are known, among them the examples of Fermat (see [13]), Hilbert, Cohn-Vossen (see [10]), Clebsch (see [6]) and Hirschfeld (see [14]). Families can be recognized and distinguished by a geometric invariant called the number of Eckardt points. An Eckardt point is a point on the surface where three of the lines contained in the surface meet (see [8,13]). The Hirschfeld family is recognized by the fact that its members have 45 Eckardt points (the largest it can possibly be). The Fermat, Clebsch and Hilbert, Cohn-Vossen surfaces are examples of a larger class of surfaces described by Goursat [9]. The equation of the Hilbert, Cohn-Vossen surface was described in [3], and a double-six was listed. We will use this description in some of the examples later on.

In a projective plane $\mathrm{PG}(2, \mathbb{F})$, a set of points S is called an **arc** if no line ℓ intersects S in more than two points. That is,

$$|S \cap \ell| \leq 2$$

for all lines ℓ. If we want to emphasize the size of the arc, we say that S is an n-arc where $n = |S|$. A conic is the set of points in $\mathrm{PG}(2, \mathbb{F})$ described by a non-degenerate homogeneous quadratic equation in three variables. A 5-arc determines a unique conic. A six-arc $S = \{P_1, \ldots, P_6\}$ is called **non-conical** if not all six points lie on a conic. The number of six-arcs not on a conic in $\mathrm{PG}(2, q)$ is known:

Lemma 3 (Hirschfeld [11]) *The number of six-arcs not on a conic in* $\mathrm{PG}(2, q)$ *is*

$$a_q = \frac{q^3(q+1)(q-1)^2(q-2)(q-3)(q-5)^2(q^2+q+1)}{6!}.$$

Non-conical arcs in planes have been investigated by Hirschfeld. Up to projective equivalence, the points of such an arc can be assumed in the form

$$\begin{aligned} P_1 &= P(1,0,0), \;\; P_2 = P(0,1,0), \;\; P_3 = P(0,0,1), \\ P_4 &= P(1,1,1), \;\; P_5 = P(a,b,1), \;\; P_6 = P(c,d,1), \end{aligned}$$

for some field elements a, b, c, d, subject to certain algebraic restrictions (cf. Hirschfeld [12]). We can use these arcs as a starting point for a classification of cubic surfaces with 27 lines over $\mathbb{F}_q$. This is the approach taken by Sadeh [20] and Karaoglu [15] as well as in [2].

3 The birational structure

A rational map is a function whose coordinate functions are rational. Birational maps are rational maps that are invertible as rational map almost everywhere. The birational parametrization of a circle as

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}$$

has been known since antiquity. The stereographic map can often be used to create birational maps. For cubic surfaces, the Clebsch map [5] does the trick. This has important implications for the problem of classification of cubic surfaces. There is a relation between cubic surfaces with 27 lines and non-conical six-arcs. In the algebraic geometry literature, this construction is known as the blow-up of the six points in general position in the plane. The inverse map is often called the blow-down of the surface to the plane.

The following theorem is fundamental for this work. The result seems to be part of the folklore of nineteenth-century mathematics. Theorems 20.1.1 and 20.1.5 in Hirschfeld [13] together prove the first half of the theorem. Baker [1, p. 192] discusses this relation briefly. A recent reference is the last section of [18]. Since our main result relies very much on the second half of the theorem, we include a full proof of it.

Theorem 4 *Let $\mathbb{F}$ be a field. There is a relation between non-conical six-arcs in planes embedded in* $\mathrm{PG}(3,\mathbb{F})$ *and cubic surfaces with 27 lines in* $\mathrm{PG}(3,\mathbb{F})$. *The relation is invariant under the action of the collineation group of projective space. A surface $\mathcal{F}$ and an arc S correspond whenever there is a birational map between $\mathcal{F}$ and S. A non-conical six-arc S determines a cubic surfaces with 27 lines up to isomorphism. Conversely, a cubic surface with 27 lines gives rise to a non-conical six-arc once two skew lines on the surface and a tritangent plane through one of the transversal lines of the chosen pair of lines have been chosen.*

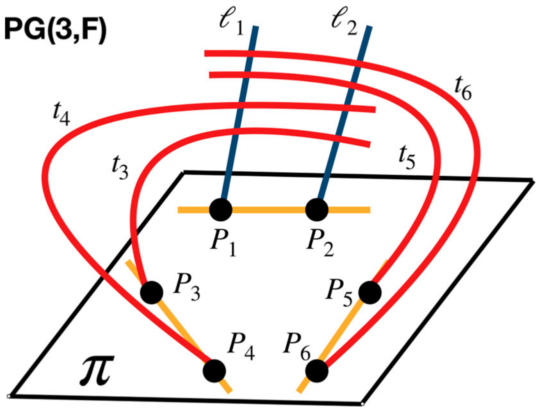
Proof Consider the following sets:

(a) Let $\mathcal{A}$ be the set of all pairs $\big(S,\pi\big)$ where π is a plane in $\mathrm{PG}(3,q)$ and S is a six-arc not on a conic in π.
(b) Let $\mathcal{B}$ be the set of cubic surfaces with 27 lines in $\mathrm{PG}(3,q)$.
(c) Let $\mathcal{R}$ be the set of all $\big(S,\pi,\mathcal{F}\big)$ where $\mathcal{F}$ is a cubic surface in $\mathrm{PG}(3,q)$ with 27 lines, π is a tritangent plane of $\mathcal{F}$ and S is a six-arc not on a conic in $\pi\cap\mathcal{F}$. We require that there exist two skew lines ℓ_1 and ℓ_2 of $\mathcal{F}$ not contained in π, such that $P_1:=\ell_1\cap\pi$ and $P_2:=\ell_2\cap\pi$ are two points of S and $t=P_1P_2$ is a line of $\mathcal{F}$. We also require that the four remaining points $P_3,\ldots,P_6$ of S are of the form $\ell\cap\pi$ where ℓ runs through the four transversals of ℓ_1 and ℓ_2 on $\mathcal{F}$ different from t.

The set $\mathcal{R}$ can be considered as a relation between $\mathcal{A}$ and $\mathcal{B}$. Simply identify $\big(S,\pi,\mathcal{F}\big)$ with $\big((S,\pi),\mathcal{F}\big)$. In the following proof, we start from a non-conical six-arc S in a plane π. After embedding this plane in $\mathrm{PG}(3,q)$, we construct a cubic surface $\mathcal{F}$ with 27 lines such that $\big(S,\pi,\mathcal{F}\big)$ is in relation $\mathcal{R}$. Once this is done, we will show how a non-conical six-arc S in a plane π can be recovered from a surface $\mathcal{F}$ assuming that $\mathcal{F}$ has 27 lines. This can be done in such a way that the pair $\big((S,\pi),\mathcal{F}\big)$ lies in the relation $\mathcal{R}$. We will then define a birational map Ψ between the plane π containing the non-conical six-arc S and the surface $\mathcal{F}$. Finally, we will see that the relation $\mathcal{R}$ is invariant under the collineation group.

Let $P_1,\ldots,P_6$ be a non-conical six-arc in $\pi=\mathrm{PG}(2,\mathbb{F})$, where π is embedded in $\mathrm{PG}(3,\mathbb{F})$. Since the group of projective space is transitive on hyperplanes, there is no loss in assuming that $\pi=\mathbf{v}(X_3)$, for instance. We also pick two skew lines ℓ_1 and ℓ_2 of $\mathrm{PG}(3,\mathbb{F})$, not contained in π, passing through P_1 and P_2, respectively. For

Fig. 1 Configuration of lines which determines a unique cubic surface



$i = 3, \ldots, 6$, let t_i be the unique transversal line of ℓ_1 and ℓ_2 through P_i. Finally, we partition the points P_3, P_4, P_5, P_6 into two pairs of two. Without loss of generality, we pair P_3 with P_4 and P_5 with P_6. Because no three points of the arc are collinear, the lines $P_1P_2,\ P_3P_4$ and P_5P_6 form a degenerate cubic curve. In total, the lines

$$\ell_1,\ \ell_2,\ t_3,\ t_4,\ t_5,\ t_6,\ P_1P_2,\ P_3P_4,\ P_5P_6$$

impose 19 linearly independent conditions on the space of cubic monomials in 4 variables. Namely, the degenerate cubic accounts for 9 independent conditions. The lines ℓ_1 and ℓ_2 add three independent conditions each. The transversals $t_3, \ldots, t_6$ contribute one further condition each, for a total of 19 linearly independent conditions. This shows that there is a unique cubic surface passing through the set of nine lines (Fig. 1). It remains to see that the surface has 27 lines. To this end, we will use Theorem 2. Recall that $P_1, \ldots, P_6$ is a non-conical six-arc. Observe that $t_3, \ldots, t_6$ are pairwise skew, for otherwise there would be a point which had two transversals to ℓ_1, ℓ_2, which is impossible. Note that the lines P_1P_2 and ℓ_1 determine a plane which intersects the cubic surface in a degenerate cubic curve. Hence, there must be a third line of the surface contained in that plane. Let this line be called t_2. By the same token, the plane through P_1P_2 and ℓ_2 contains a line of the surface which we call t_1. We claim that the lines $t_1, \ldots, t_6$ are pairwise skew. To this end, we appeal to Lemma 20.2.1 (iv) in [13]. The lines t_1 and t_2 meet π in distinct points of P_1P_2, for otherwise the point of intersection would determine a tritangent plane containing the lines P_1P_2, t_1, t_2, which is impossible. Regarding t_i and t_j with $i \leq 2$ and $j \geq 3$, we observe that line t_j intersects the tritangent plane P_1P_2, ℓ_{3-i}, t_i in ℓ_{3-i}. This point of intersections does not lie on t_i because otherwise t_i and ℓ_{3-i} would lie on the two different tritangent planes t_i, ℓ_{3-i}, t_j and t_i, ℓ_{3-i}, P_1P_2, which is impossible.

It remains to show that the configuration of lines $\ell_1, \ell_2, t_1, \ldots, t_6$ completes to a double-six, which implies that the surface has 27 lines. We verify the Schläfli property for the lines $\ell_1, t_2, \ldots, t_6$. This means that any four of $\{t_2, \ldots, t_6\}$ have a unique, distinct second transversal (different from ℓ_1). For simplicity, we look at one case. Suppose we want to show that t_2, t_3, t_4, t_5 have a unique transversal besides ℓ_1. Because

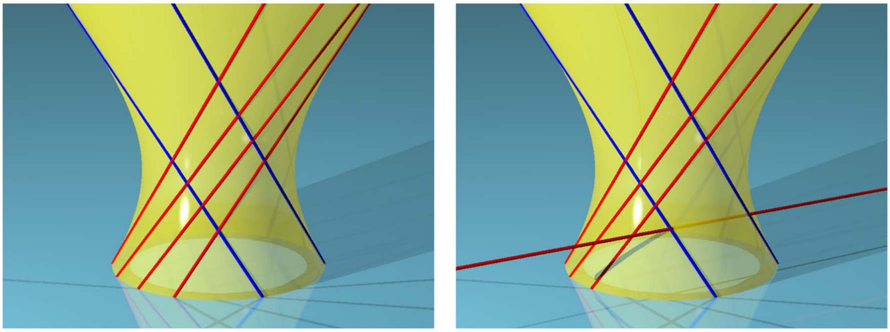


Fig. 2 The fourth line does (left) or does not (right) belong to the regulus determined by the other three

these lines are pairwise skew, any three of them determine a regulus lying on a hyperboloid $\mathscr{H}_3$. Because of the property that the points $P_1, \ldots, P_6$ are non-conical, the fourth line t_5 does not belong to the regulus generated by the lines t_2, t_3, t_4 (see Fig. 2). This means that t_5 is secant to the hyperboloid $\mathscr{H}_3$. Since it intersects the opposite regulus in ℓ_1 and since it is different from the line P_1P_2 which also has this property, it intersects $\mathscr{H}_3$ in another point. Let ℓ_6 be the line in the opposite regulus passing through that second point of intersection. This process shows that the partial double-six (lines are disjoint if and only if they are in the same row or column)

$$\begin{array}{cccccc} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ \ell_1 & \ell_2 & & & & \end{array}$$

can be completed to a double-six

$$\begin{array}{cccccc} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 \end{array}$$

and hence the cubic surface constructed has 27 lines.

Now, let $\mathcal{F}$ be a cubic surfaces with 27 lines. Let ℓ_1 and ℓ_2 be skew lines of $\mathcal{F}$ with common transversal t contained in a tritangent plane π, where t is another line of $\mathcal{F}$. The following procedure recovers a non-conical six-arc S in π associated with $\mathcal{F}$ by means of the relation $\mathcal{R}$. Let $P_i = \ell_i \cap \pi$ for $i = 1, 2$. Let t_2 be the unique third line in the tritangent plane spanned by ℓ_1 and P_1P_2. Likewise, let t_1 be the unique third line in the tritangent plane spanned by ℓ_2 and P_1P_2. Let $t_3, \ldots, t_6$ be the four transversals of ℓ_1 and ℓ_2 different from P_1P_2. Let $P_i = t_i \cap \pi$ for $i = 3, \ldots, 6$. These are four distinct points, because $t_3, \ldots, t_6$ are pairwise skew, for otherwise a point of intersection would have two distinct transversals to ℓ_1 and ℓ_2. Together, $P_1, \ldots, P_6$ are six distinct points of π. It remains to show that these six points form a non-conical six-arc.

Assume that the points $P_1, \ldots, P_6$ are on a conic. Let $\mathscr{H}_3$ be the hyperboloid generated by t_3, t_4, t_5. Then ℓ_1 and ℓ_2 belong to the opposite regulus of $\mathscr{H}_3$. The plane π intersects $\mathscr{H}_3$ in a conic which contains the five points $P_1, \ldots, P_5$. By assumption, P_6 belongs to that conic. But t_6 is a transversal to ℓ_1 and ℓ_2, to t_6 belongs to the regulus containing $t_3, \ldots, t_5$. But we have seen above that this is impossible, because this means that t_6 is not sufficiently general for ℓ_1 and $t_2, \ldots, t_6$ to generate a cubic surface with 27 lines (Fig. 2), which is a contradiction.

Regarding the arc property of $\{P_1, \ldots, P_6\}$, we see that $\ell_1, \ell_2, t_3, t_4, t_5$ lie on a $\mathscr{H}_3$, and hence, P_1, P_2, P_3, P_4, P_5 lie on a conic, so no three of these points are collinear. Considering the (different) $\mathscr{H}_3$ through $\ell_1, \ell_2, t_4, t_5, t_6$, we find that P_1, P_2, P_4, P_5, P_6 lie on a different conic, and hence, no three of those points are collinear either. Regarding the line P_3P_6, we consider two further $\mathscr{H}_3$, one through $\ell_1, \ell_2, t_3, t_5, t_6$, and the other through $\ell_1, \ell_2, t_3, t_4, t_6$. This proves the arc property of $P_1, \ldots, P_6$.

Counting intersections, a line not contained in $\mathcal{F}$ intersects $\mathcal{F}$ in three points. This allows us to define a rational map Ψ from the plane π to the surface $\mathcal{F}$. For a point $P \in \pi$, let t be the unique transversal t to ℓ_1 and ℓ_2 through P. The map sends P to the third intersection of t with the surface if t intersects $\mathcal{F}$ in three distinct points. The map is birational, i.e., the coordinate functions of the map are polynomials in terms of the coordinates of P and the coordinates of the surface equation and a basis for the two lines ℓ_1 and ℓ_2. The exceptional locus where the map is undefined is the line P_1P_2, the points of the six-arc, and the two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ where $\mathcal{C}_i = \{P_1, \ldots, P_6\} \setminus \{P_i\}$. Outside this locus, the map is one to one to the points of the surface except for the points of six lines forming a single-six, and the three lines ℓ_1, ℓ_2 and P_1P_2.

The inverse map, Ψ^{-1}, which takes the surface $\mathcal{F}$ to the plane π, can be defined similarly: For a point $Q \in \mathcal{F}$, Ψ^{-1} takes Q to $\pi \cap t$, where t is the unique transversal to ℓ_1 and ℓ_2 through Q (if it exists). Ψ^{-1} is rational, too, with exceptional locus at ℓ_1, ℓ_2 and t.

The above description shows that the points of the arc are constructed by intersecting 6 lines of the surface with the plane. Because of this, the relation is invariant under the action of the collineation group of projective space which preserves incidence relations. □

In the following, we will refer to the birational map arising in the proof of Theorem 4 as **Clebsch map**, in reference to [5].

Let us make an explicit example. We consider the Hilbert, Cohn-Vossen surface, which has the equation

$$\frac{5}{2}X_0X_1X_2 - \left(X_0^2 + X_1^2 + X_2^2\right)X_3 + X_3^3 = 0.$$

We follow the Schläfli labeling of lines given in Table 5 in [3], with parameters $a = 2$ and $b = 1$. For convenience, we reproduce this table here as Table 1. Consider the Clebsch map Ψ associated with the lines $\ell_1 = b_3$, $\ell_2 = b_4$ and the plane

$$\pi = \pi_{12,34,56} = \mathbf{v}(X_3).$$

Consider a point $P \in \pi$ with coordinates $(y_0, y_1, y_2, 0)$. The transversal line t to ℓ_1 and ℓ_2 is given in parametric form as

$$t = t(\alpha) : \left[\alpha \cdot \frac{y_0}{y_2} - \frac{4y_0^2 - y_2^2}{2y_1y_2}, \alpha \cdot \frac{y_1}{y_2} - 2 \cdot \frac{y_0}{y_2}, \alpha, 1\right].$$

Table 1 A double-six of the Hilbert, Cohn-Vossen surface

$a_1 = \mathbf{L}\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}$	$b_1 = \mathbf{L}\begin{bmatrix} 1 & -\frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & -b \end{bmatrix}$
$a_2 = \mathbf{L}\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 0 & 1 & -b \end{bmatrix}$	$b_2 = \mathbf{L}\begin{bmatrix} 1 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}$
$a_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & -\frac{1}{a} & 0 \\ 0 & 1 & 0 & -b \end{bmatrix}$	$b_3 = \mathbf{L}\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{bmatrix}$
$a_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & \frac{1}{a} & 0 \\ 0 & 1 & 0 & b \end{bmatrix}$	$b_4 = \mathbf{L}\begin{bmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & -b \end{bmatrix}$
$a_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & -a & 0 \end{bmatrix}$	$b_5 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & \frac{1}{a} & 0 \end{bmatrix}$
$a_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & a & 0 \end{bmatrix}$	$b_6 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & -\frac{1}{a} & 0 \end{bmatrix}$

The line t intersects ℓ_1 at $t(\alpha_1)$ with

$$\alpha_1 = \frac{-y_2 + 2y_0}{y_1}.$$

Likewise, the line t intersects ℓ_2 at $t(\alpha_2)$ with

$$\alpha_2 = \frac{y_2 + 2y_0}{y_1}.$$

Substituting the equation of t in the equation of the surface gives a cubic polynomial

$$a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

in α, where

$$\begin{aligned}
a &= \frac{2y_0y_1}{2y_2^2}, \\
b &= -11\frac{y_0^2}{y_2^2} + \frac{1}{4} - \frac{y_1^2}{y_2^2}, \\
c &= 14\frac{y_0^3}{y_2^2y_1} - \frac{7y_0}{2y_1} + \frac{4y_0y_1}{y_2^2}, \\
d &= -4\frac{y_0^4}{y_2^2y_1^2} + 2\frac{y_0^2}{y_1^2} - \frac{y_2^2}{4y_1^2} - 4\frac{y_0^2}{y_2^2} + 1.
\end{aligned}$$

Comparing coefficients of like terms in the expansion of

$$(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3) = \alpha^3 + \frac{b}{a}\alpha^2 + \frac{c}{a}\alpha + \frac{d}{a}$$

yields

$$-\frac{b}{a} = \alpha_1 + \alpha_2 + \alpha_3.$$

Therefore,

$$\alpha_3 = -\frac{b}{a} - \alpha_1 - \alpha_2 = \frac{4y_0^2 + 4y_1^2 - y_2^2}{10y_0y_1}$$

and the third point of intersection of t with $\mathcal{F}$ is

$$t(\alpha_3) = \Big(\frac{-8y_0^2 + 2y_1^2 + 2y_2^2}{5y_1y_2}, \frac{-16y_0^2 + 4y_1^2 - y_2^2}{10y_0y_2}, \frac{4y_0^2 + 4y_1^2 - y_2^2}{10y_0y_1}, 1\Big).$$

The Clebsch map $\Psi : P \mapsto t(\alpha_3)$ associated with b_3, b_4 and $\pi_{12,34,56}$ is given as

$$\Psi : \begin{cases} \mathrm{PG}(2, \mathbb{R}) \to \mathrm{PG}(3, \mathbb{R}), \\ [Y_0, Y_1, Y_2] \mapsto [X_0, X_1, X_2, X_3] \\ X_0 = -4(4Y_0^2 - Y_1^2 - Y_2^2)Y_0, \\ X_1 = -(16Y_0^2 - 4Y_1^2 + Y_2^2)Y_1, \\ X_2 = (4Y_0^2 + 4Y_1^2 - Y_2^2)Y_2, \\ X_3 = 10Y_0Y_1Y_2. \end{cases}$$

This is the Clebsch map from the plane to the surface. This example has been computed with help of the computer algebra system Maple [17].

The Clebsch map Ψ^{-1} from the surface down to the plane is given by

$$\Psi^{-1} : \begin{cases} \mathcal{F} \subseteq \mathrm{PG}(3, \mathbb{R}) \to \mathrm{PG}(2, \mathbb{R}), \\ [X_0, X_1, X_2, X_3] \mapsto [Y_0, Y_1, Y_2] \\ Y_0 = -2X_0X_1 + X_2X_3 \\ Y_1 = -2X_1^2 + 2X_3^2 \\ Y_2 = 4X_0X_3 - 2X_1X_2 \end{cases}$$

Let us now talk about enumerative aspects of this relation. In the finite case, when $\mathbb{F} = \mathbb{F}_q$, there are

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_q$$

hyperplanes π inside $\mathrm{PG}(3, q)$.

Lemma 5 *Let π be a plane in $\mathrm{PG}(3, q)$. Let P_1 and P_2 be two points in π. Two skew lines ℓ_1 and ℓ_2 outside π and intersecting π in P_1 and P_2, respectively, can be chosen in $q^3(q-1)$ many ways.*

Proof There are q^2 lines ℓ_1 in $\mathrm{PG}(3, q)$ passing through P_1 but not contained in π. There are $q^2 - q$ lines ℓ_2 in $\mathrm{PG}(3, q)$ passing through P_2 but not contained in π and skew to ℓ_1. □

The pointwise stabilizer of the hyperplane π is a group of transvections (see [24]). It acts sharply transitively on these configurations.

4 Proof of the main result

The proof or Theorem 1 is based on a double count of the incident pairs of the invariant relation $\mathcal{R}$ from the proof of Theorem 4. For a given element $\left(S, \pi\right)$ in $\mathcal{A}$, let $u_{S,\pi}$ be the number of surfaces $\mathcal{F}$ in $\mathcal{B}$ such that $(S, \pi, \mathcal{F})$ is in $\mathcal{R}$. For a given surface $\mathcal{F}$ in $\mathcal{B}$, let $v_{\mathcal{F}}$ be the number of (S, π) in $\mathcal{A}$ such that $\left(S, \pi, \mathcal{F}\right)$ is in $\mathcal{R}$.

Lemma 6 *The relation $\mathcal{R}$ from the proof of Theorem 4 is regular in both ways:*

1.
$$u_{S,\pi} = 45q^3(q-1)$$

2.
$$v_{\mathcal{F}} = 3240.$$

Proof For $\left(S, \pi\right)$ in $\mathcal{A}$, there are $15 = \binom{6}{2}$ ways to pick two points in S, say P_1 and P_2. The remaining four points $P_3, \ldots, P_6$ can be partitioned into two disjoint sets of size two in three ways. By Lemma 5, there are $q^3(q-1)$ ways to choose lines ℓ_1 and ℓ_2 intersecting π in points P_1 and P_2, respectively. Following Theorem 4, for each choice of ℓ_1, ℓ_2 subject to these conditions, a cubic surface $\mathcal{F}$ with 27 lines is defined with $(S, \pi, \mathcal{F})$ in $\mathcal{R}$. This shows that each pair (S, π) in $\mathcal{A}$ gives rise to

$$u_{S,\pi} = \binom{6}{2} \cdot 3 \cdot q^2(q^2 - q) = 45q^3(q-1)$$

distinct surfaces $\mathcal{F}$ in relation $\mathcal{R}$ with (S, π).

On the other hand, any surface $\mathcal{F} \in \mathcal{B}$ has 45 tritangent planes, and in each tritangent plane π, there are three ways to pick a line of the surface. For each such line t, there are $8 \cdot 6/2$ ways to pick a pair (ℓ_1, ℓ_2) of skew lines of $\mathcal{F}$ intersecting it and not contained in the given tritangent plane π. For each choice of tritangent plane π of $\mathcal{F}$, and each pair (ℓ_1, ℓ_2) of skew lines intersecting π in points P_1 and P_2 with t the unique line of $\mathcal{F}$ through P_1 and P_2 in π, a set $S = \{P_1, \ldots, P_6\}$ is defined: Let $P_3, \ldots, P_6$ be the points of the form $\ell \cap \pi$ where ℓ runs through the four transversals of ℓ_1 and ℓ_2 in $\mathcal{F}$ distinct from t. It follows from Theorem 4 that S is a six-arc not on a conic. For different choices of π, ℓ_1, ℓ_2, the arcs S which arise are all distinct, and the pair (S, π) is in relation $\mathcal{R}$ with $\mathcal{F}$. This shows that

$$v_{\mathcal{F}} = 45 \cdot 3 \cdot \frac{8 \cdot 6}{2} = 45 \cdot 72 = 3240.$$

□

Table 2 Cubic surfaces with 27 lines for $q = 13, 17, 19$

q	$\mathcal{F}_q^i$	s	e_3	$\lvert G(\mathcal{F}_q^i)\rvert$	q	$\mathcal{F}_q^i$	s	e_3	$\lvert G(\mathcal{F}_q^i)\rvert$
13	$\mathcal{F}_{13}^0$	10	4	12	19	$\mathcal{F}_{19}^0$	21	2	4
13	$\mathcal{F}_{13}^1$	7	6	24	19	$\mathcal{F}_{19}^1$	21	2	4
13	$\mathcal{F}_{13}^2$	2	9	108	19	$\mathcal{F}_{19}^2$	14	3	6
13	$\mathcal{F}_{13}^3$	2	18	648	19	$\mathcal{F}_{19}^3$	14	3	6
17	$\mathcal{F}_{17}^0$	9	1	8	19	$\mathcal{F}_{19}^4$	10	4	12
17	$\mathcal{F}_{17}^1$	14	3	6	19	$\mathcal{F}_{19}^5$	7	6	24
17	$\mathcal{F}_{17}^2$	10	4	12	19	$\mathcal{F}_{19}^6$	7	6	24
17	$\mathcal{F}_{17}^3$	10	4	12	19	$\mathcal{F}_{19}^7$	4	9	54
17	$\mathcal{F}_{17}^4$	7	6	24	19	$\mathcal{F}_{19}^8$	4	10	120
17	$\mathcal{F}_{17}^5$	7	6	24	19	$\mathcal{F}_{19}^9$	2	18	648
17	$\mathcal{F}_{17}^6$	7	6	24					

The proof of the main result is a simple double count of the incident pairs of the relation $\mathcal{R}$. The number of elements in $\mathcal{A}$ is $a_q \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q$. The number of elements in $\mathcal{B}$ is c_q. Therefore,

$$a_q \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \cdot q^3(q-1) \cdot 45 = a_q u_{S,\pi} = c_q v_{\mathcal{F}} = c_q \cdot 3240,$$

from which the result follows.

5 Numerical examples

We remark that the result of this paper was first conjectured in the Ph.D. thesis of the second author [15]. The result can been used as a mass formula to verify the classification of cubic surfaces with 27 lines over small finite fields. We present here the relevant data for the fields $\mathbb{F}_q$ where $q \in \{13, 17, 19\}$. The classification of cubic surfaces with 27 lines over the field $\mathbb{F}_{13}$ can be found in [2]. The classification of cubic surfaces with 27 lines over the fields $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$ can be found in [3]. In Table 2, these results are summarized, together with information about the order of the projectivity stabilizer $G(\mathcal{F}_q^i)$, where $\mathcal{F}_q^i$ is the representative of the ith isomorphism class. Additional information is e_3, the number of Eckardt points, and s, the number of isomorphism types of non-conical six-arcs associated with it. Some numerical data associated with the classification are shown in Table 3.

Table 3 Counting properties for $q = 13, 17, 19$

	$q = 13$	$q = 17$	$q = 19$
p_2	183	307	381
l_2	183	307	381
m_2	14	18	20
n_2	14	18	20
a_q	7, 925, 229, 312	291, 908, 606, 976	1, 253, 872, 170, 432
g_3	810, 534, 816	6, 950, 204, 928	16, 934, 047, 920
p_3	2380	5220	7240
P_3	2380	5220	7240
l_3	31, 110	89, 030	137, 922
m_3	14	18	20
n_3	183	307	381
g_4	50, 858, 076, 935, 877, 120	2, 851, 903, 720, 876, 769, 280	15, 136, 750, 711, 925, 049, 600
c_q	6, 906, 652, 423, 390, 720	1, 663, 610, 503, 844, 782, 080	15, 566, 559, 682, 757, 489, 280

p_2 := the number of points in $\mathrm{PG}(2, q)$,
l_2 := the number of lines in $\mathrm{PG}(2, q)$,
m_2 := the number of points on a line in $\mathrm{PG}(2, q)$,
n_2 := the number of lines passing through a point of $\mathrm{PG}(2, q)$,
a_q := the number of 6-arcs not on a conic in $\mathrm{PG}(2, q)$,
$g_3 := |\mathrm{PGL}(3, q)|$,
p_3 := the number of points in $\mathrm{PG}(3, q)$,
P_3 := the number of planes in $\mathrm{PG}(3, q)$,
l_3 := the number of lines in $\mathrm{PG}(3, q)$,
m_3 := the number of planes through a line in $\mathrm{PG}(3, q)$,
n_3 := the number of lines passing through a point of $\mathrm{PG}(3, q)$,
$g_4 := |\mathrm{PGL}(4, q)|$,
c_q := the number of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$

Let g_4 be the order of $\mathrm{PGL}(4, q)$. Let $G(\mathcal{F})$ be the group of projectivities of the cubic surface $\mathcal{F}$. It follows from the orbit-stabilizer theorem that

$$c_q := \sum_{\text{iso type } \mathcal{F}} \frac{g_4}{|G(\mathcal{F})|}.$$

For $q = 13$, the distribution of the automorphism group orders of the projectively distinct cubic surface with 27 lines is 12, 24, 108, 648 as shown in Table 2. Therefore, the number of cubic surfaces with twenty-seven lines in PG(3, 13) is

$$\begin{aligned} c_{13} &= |\mathrm{PGL}(4, 13)| \cdot \left(\frac{1}{12} + \frac{1}{24} + \frac{1}{108} + \frac{1}{648}\right) \\ &= 50858076935877120 \cdot \left(\frac{1}{12} + \frac{1}{24} + \frac{1}{108} + \frac{1}{648}\right) \\ &= 6906652423390720 \\ &= \frac{13^6(13^2-1)(13^3-1)(13^4-1)(13-2)(13-3)(13-5)^2}{51840}. \end{aligned}$$

For $q = 17$, the distribution of the automorphism group orders of the projectively distinct cubic surfaces is 8, 6, 12^2, 24^3 as shown in Table 2. Therefore, the number of cubic surfaces with twenty-seven lines in PG(3, 17) is

$$\begin{aligned} c_{17} &= |\mathrm{PGL}(4, 17)| \cdot \left(\frac{1}{8} + \frac{1}{6} + \frac{2}{12} + \frac{3}{24}\right) \\ &= 2851903720876769280 \cdot \left(\frac{1}{8} + \frac{1}{6} + \frac{2}{12} + \frac{3}{24}\right) \\ &= 1663610503844782080 \\ &= \frac{17^6(17^2-1)(17^3-1)(17^4-1)(17-2)(17-3)(17-5)^2}{51840}. \end{aligned}$$

For $q = 19$, the distribution of the automorphism group orders of the projectively distinct cubic surface with 27 lines is 4^2, 6^2, 12, 24^2, 54, 120, 648 as shown in Table 2. Therefore, the number of cubic surfaces with twenty-seven lines in PG(3, 19) is

$$\begin{aligned} c_{19} &= |\mathrm{PGL}(4, 19)| \cdot \left(\frac{2}{4} + \frac{2}{6} + \frac{1}{12} + \frac{2}{24} + \frac{1}{54} + \frac{1}{120} + \frac{1}{648}\right) \\ &= 15136750711925049600 \cdot \left(\frac{2}{4} + \frac{2}{6} + \frac{1}{12} + \frac{2}{24} + \frac{1}{54} + \frac{1}{120} + \frac{1}{648}\right) \\ &= 15566559682757489280 \\ &= \frac{19^6(19^2-1)(19^3-1)(19^4-1)(19-2)(19-3)(19-5)^2}{51840}. \end{aligned}$$

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