



On A_1^2 restrictions of Weyl arrangements

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Abstract

Let \mathcal{A} be a Weyl arrangement in an ℓ -dimensional Euclidean space. The freeness of restrictions of \mathcal{A} was first settled by a case-by-case method by Orlik and Terao (Tôhoku Math J 52: 369–383, 1993), and later by a uniform argument by Douglass (Represent Theory 3: 444–456, 1999). Prior to this, Orlik and Solomon (Proc Symp Pure Math Amer Math Soc 40(2): 269–292, 1983) had completely determined the exponents of these arrangements by exhaustion. A classical result due to Orlik et al. (Adv Stud Pure Math 8: 461–77, 1986) asserts that the exponents of any A_1 restriction, i.e., the restriction of \mathcal{A} to a hyperplane, are given by $\{m_1, \dots, m_{\ell-1}\}$, where $\text{exp}(\mathcal{A}) = \{m_1, \dots, m_\ell\}$ with $m_1 \leq \dots \leq m_\ell$. As a next step towards conceptual understanding of the restriction exponents, we will investigate the A_1^2 restrictions, i.e., the restrictions of \mathcal{A} to the subspaces of type A_1^2 . In this paper, we give a combinatorial description of the exponents and describe bases for the modules of derivations of the A_1^2 restrictions in terms of the classical notion of related roots by Kostant (Proc Nat Acad Sci USA 41:967–970, 1955).

Keywords Root system · Weyl arrangement · Restriction · Freeness · Exponent · Basis

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1 Introduction

Assume that $V = \mathbb{R}^\ell$ with the standard inner product (\cdot, \cdot) . Denote by Φ an irreducible (crystallographic) root system in V and by Φ^+ a positive system of Φ . Let \mathcal{A} be the Weyl arrangement of Φ^+ . Denote by $L(\mathcal{A})$ the intersection poset of \mathcal{A} . For each $X \in L(\mathcal{A})$, we write \mathcal{A}^X for the restriction of \mathcal{A} to X . Set $L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \text{codim}(X) = p\}$ for $0 \leq p \leq \ell$. Let W be the Weyl group of Φ and let m_1, \dots, m_ℓ with $m_1 \leq \dots \leq m_\ell$ be the exponents of W .

Notation 1.1 If $X \in L_p(\mathcal{A})$, then $\Phi_X := \Phi \cap X^\perp$ is a root system of rank p . A positive system of Φ_X is taken to be $\Phi_X^+ := \Phi^+ \cap \Phi_X$. Let Δ_X be the base of Φ_X associated with Φ_X^+ .

Definition 1.2 A subspace $X \in L(\mathcal{A})$ is said to be of type T (or T for short) if Φ_X is a root system of type T . In this case, the restriction \mathcal{A}^X is said to be of type T (or T).

Weyl arrangements are important examples of *free* arrangements. In other words, the module $D(\mathcal{A})$ of \mathcal{A} -derivations is a free module. Furthermore, the exponents of \mathcal{A} are the same as the exponents of W , i.e., $\text{exp}(\mathcal{A}) = \{m_1, \dots, m_\ell\}$ (e.g., [23]). It is shown by Orlik and Solomon [19], using the classification of finite reflection groups, that the characteristic polynomial of the restriction \mathcal{A}^X ($X \in L(\mathcal{A})$) of an arbitrary Weyl arrangement \mathcal{A} is fully factored. Orlik and the second author [22] proved a stronger statement that $D(\mathcal{A}^X)$ is free by a case-by-case study. Soon afterwards, Douglass [8] gave a uniform proof for the freeness using the representation theory of Lie groups.

We are interested in studying the exponents of \mathcal{A}^X and bases for $D(\mathcal{A}^X)$. For a general X , it is not an easy task. The only known general result due to Orlik, Solomon and the second author [20], asserts that for each $X \in \mathcal{A}$, i.e., X is of type A_1 , $\text{exp}(\mathcal{A}^X) = \{m_1, \dots, m_{\ell-1}\}$ and a basis for $D(\mathcal{A}^X)$ consists of the restrictions to X of the corresponding basic derivations. In this paper, we consider a “next” general class of restrictions, that is when X is of type A_1^2 . We prove that $\text{exp}(\mathcal{A}^X)$ is obtained from $\text{exp}(\mathcal{A})$ by removing either the two largest exponents, or the largest and the middle exponents, depending upon a combinatorial condition on X . Furthermore, similar to the result of [20], our method produces an explicit basis for $D(\mathcal{A}^X)$ in each case. The main combinatorial ingredient in our description is the following concept defined by Kostant:

Definition 1.3 [12] Two non-proportional roots β_1, β_2 are said to be *related* if

- (a) $(\beta_1, \beta_2) = 0$,
- (b) for any $\gamma \in \Phi \setminus \{\pm\beta_1, \pm\beta_2\}$, $(\gamma, \beta_i) = 0$ implies $(\gamma, \beta_{3-i}) = 0$ for all $i \in \{1, 2\}$.

In this case, we call the set $\{\beta_1, \beta_2\}$ *relatedly orthogonal* (RO), and the subspace $X = H_{\beta_1} \cap H_{\beta_2}$ is said to be RO.

Remark 1.4 The relatedly orthogonal sets presumably first appeared in [12], wherein Kostant required β_1, β_2 to have the same length and allowed a root is related to itself and its negative. Green called the relatedly orthogonal sets *strongly orthogonal* and defined the strong orthogonality in a more general setting [9, Definition 4.4.1]. It

should be noted that the notion of strongly orthogonal sets is probably more well-known with the definition that neither sum nor difference of the two roots is a root. For every $\beta \in \Phi$, set $\beta^\perp := \{\alpha \in \Phi \mid (\alpha, \beta) = 0\}$. Condition (b) in Definition 1.3 can be written symbolically as (b') $\beta_1^\perp \setminus \{\pm\beta_2\} = \beta_2^\perp \setminus \{\pm\beta_1\}$.

Let h be the Coxeter number of W . For $\phi \in D(\mathcal{A})$, let ϕ^X be the restriction of ϕ to X . We now formulate our main results.

Theorem 1.5 *Assume that $\ell \geq 3$. If $X \in L(\mathcal{A})$ is of type A_1^2 , then \mathcal{A}^X is free with*

$$\exp(\mathcal{A}^X) = \begin{cases} \exp(\mathcal{A}) \setminus \{h/2, m_\ell\} & \text{if } X \text{ is RO,} \\ \exp(\mathcal{A}) \setminus \{m_{\ell-1}, m_\ell\} & \text{otherwise.} \end{cases}$$

Theorem 1.6 *Assume that $\ell \geq 3$.*

- (i) *Suppose that X is A_1^2 and not RO. Let $\{\varphi_1, \dots, \varphi_\ell\}$ be a basis for $D(\mathcal{A})$ with $\deg \varphi_j = m_j$ ($1 \leq j \leq \ell$). Then, $\{\varphi_1^X, \dots, \varphi_{\ell-2}^X\}$ is a basis for $D(\mathcal{A}^X)$.*
- (ii) *Suppose that X is both A_1^2 and RO. Then, Φ must be of type D_ℓ with $\ell \geq 3$. Furthermore, a basis for $D(\mathcal{A}^X)$ is given by $\{\tau_1^X, \dots, \tau_{\ell-2}^X\}$, where*

$$\tau_i := \sum_{k=1}^{\ell} x_k^{2i-1} \partial / \partial x_k \quad (1 \leq i \leq \ell - 2).$$

Theorem 1.5 gave a little extra information: when X is RO, half of the Coxeter number is an exponent of W (hence, it lies in the “middle” of the exponent sequence). We emphasize that given [22], Theorem 1.5 can be verified by looking at the numerical results in [19] (the case of type D is rather non-trivial). It is interesting to search for a proof, free of case-by-case considerations. In this paper, we provide a conceptual proof with a minimal use of classifications of root systems: Theorem 1.5 holds true trivially for $\ell = 3$, and we use up to the classification of rank-4 root systems (for the proofs of some supporting arguments). As far as we are aware, a formula of the exponents of a restricted Weyl arrangement given under the RO condition is new. Our proof shows how the RO concept arrives at the exponent description, hoping that it will reveal a new direction for future research of the exponents through the combinatorial properties of the root system.

Theorem 1.6 seems to be, however, less straightforward even if one relies on the classification. In comparison with the classical result of [20], Theorem 1.6(i) gives a bit more flexible construction of basis for $D(\mathcal{A}^X)$. Namely, a wanted basis is obtained by taking the restriction to X of any basis for $D(\mathcal{A})$, without the need of basic derivations. Nevertheless, the remaining part of the basis construction [Theorem 1.6(ii)] cannot avoid the classification. It may happen that there are more than one derivations in a basis for $D(\mathcal{A})$ having the same degree. Hence, some additional computation is required to examine which derivation vanishes after taking the restriction to X (see Remark 2.10 and Example 5.2).

Beyond the A_1^2 restrictions, driving conceptual understanding on the restrictions in higher codimensions, or of irreducible types is much harder (see Remarks 3.15 and

4.5). The numerical results [19] say that we have a similar formula for the exponents of any A_1^k restriction ($k \geq 2$) in terms of the strongly orthogonal sets of [9]. The details are left for future research.

The remainder of this paper is organized as follows. In Sect. 2, we first review preliminary results on free arrangements and their exponents, and a recent result (Combinatorial Deletion) relating the freeness to combinatorics of arrangements (Theorem 2.8). We also prove an important result in the paper, a construction of a basis for $D(\mathcal{A}^H)$ ($H \in \mathcal{A}$) from a basis for $D(\mathcal{A})$ when \mathcal{A} and $\mathcal{A} \setminus \{H\}$ are both free (Theorem 2.9). We then review the background information on root systems, Weyl groups and Weyl arrangements. Based on the Combinatorial Deletion Theorem, we provide a slightly different proof for the result of [20] (Remark 2.15). In Sect. 3, we give an evaluation for the cardinality of every A_1^2 restriction (Proposition 3.13). This evaluation can be expressed in terms of the local and global second smallest exponents of the Weyl groups (Remark 3.14). In Sect. 4, we first provide a proof for the freeness part of Theorem 1.5, i.e., the freeness of A_1^2 restrictions (Theorem 4.4). The proof is different from (and more direct than) the proofs presented in [8,22]. We then complete the proof of Theorem 1.5 by providing a proof for the exponent part (Theorem 4.6). The proof contains two halves which we present the proof for each half in Theorems 4.8 and 4.28. We close the section by giving two results about local–global inequalities on the second smallest exponents and the largest coefficients of the highest roots (Corollary 4.34), and the Coxeter number of any irreducible component of the subsystem orthogonal to the highest root in the simply laced cases (Corollary 4.35). In Sect. 5, we present the proof of Theorem 1.6 (Theorem 5.1 and Example 5.2). In Sect. 6, we give an alternative and bijective proof of Theorem 4.30, one of the key ingredients in the proof of the exponent part of Theorem 1.5.

2 Preliminaries

2.1 Free arrangements and their exponents

For basic concepts and results of free arrangements, we refer the reader to [21].

Let \mathbb{K} be a field and let $V := \mathbb{K}^\ell$. A *hyperplane* in V is a subspace of codimension 1 of V . An *arrangement* is a finite set of hyperplanes in V . We choose a basis $\{x_1, \dots, x_\ell\}$ for V^* and let $S := \mathbb{K}[x_1, \dots, x_\ell]$. Fix an arrangement \mathcal{A} in V . The *defining polynomial* $Q(\mathcal{A})$ of \mathcal{A} is defined by

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S,$$

where $\alpha_H = a_1x_1 + \dots + a_\ell x_\ell \in V^* \setminus \{0\}$, $a_i \in \mathbb{K}$ and $H = \ker \alpha_H$. The *number* of hyperplanes in \mathcal{A} is denoted by $|\mathcal{A}|$. It is easy to see that $Q(\mathcal{A})$ is a homogeneous polynomial in S and $\deg Q(\mathcal{A}) = |\mathcal{A}|$.

The *intersection poset* of \mathcal{A} , denoted by $L(\mathcal{A})$, is defined to be

$$L(\mathcal{A}) := \{\cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\},$$

where the partial order is given by reverse inclusion. We agree that $V \in L(\mathcal{A})$ is the unique minimal element. For each $X \in L(\mathcal{A})$, we define the *localization* of \mathcal{A} on X by

$$\mathcal{A}_X := \{K \in \mathcal{A} \mid X \subseteq K\},$$

and define the *restriction* \mathcal{A}^X of \mathcal{A} to X by

$$\mathcal{A}^X := \{K \cap X \mid K \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

The *Möbius function* $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$ is formulated by

$$\mu(V) := 1, \quad \mu(X) := - \sum_{X \subsetneq Y \subseteq V} \mu(Y).$$

The *characteristic polynomial* $\chi(\mathcal{A}, t)$ of \mathcal{A} is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X)t^{\dim X}.$$

A *derivation* of S over \mathbb{K} is a linear map $\phi : S \rightarrow S$ such that for all $f, g \in S$, $\phi(fg) = f\phi(g) + g\phi(f)$. Let $\text{Der}(S)$ denote the *set* of derivations of S over \mathbb{K} . Then, $\text{Der}(S)$ is a free S -module with a basis $\{\partial_1, \dots, \partial_\ell\}$, where $\partial_i := \partial/\partial x_i$ for $1 \leq i \leq \ell$. Define an S -submodule of $\text{Der}(S)$, called the *module of \mathcal{A} -derivations*, by

$$D(\mathcal{A}) := \{\phi \in \text{Der}(S) \mid \phi(Q) \in QS\}.$$

A nonzero element $\phi = f_1\partial_1 + \dots + f_\ell\partial_\ell \in \text{Der}(S)$ is *homogeneous of degree b* if each nonzero polynomial $f_i \in S$ for $1 \leq i \leq \ell$ is homogeneous of degree b . We then write $\deg \phi = b$. The arrangement \mathcal{A} is called *free* if $D(\mathcal{A})$ is a free S -module. If \mathcal{A} is free, then $D(\mathcal{A})$ admits a basis $\{\phi_1, \dots, \phi_\ell\}$ consisting of homogeneous derivations [21, Proposition 4.18]. Such a basis is called a *homogeneous basis*. Although homogeneous basis needs not be unique, the degrees of elements of a basis are unique (with multiplicity but neglecting the order) depending only on \mathcal{A} [21, Proposition A.24]. In this case, we call $\deg \phi_1, \dots, \deg \phi_\ell$ the *exponents* of \mathcal{A} , store them in a *multiset* denoted by $\text{exp}(\mathcal{A})$ and write

$$\text{exp}(\mathcal{A}) = \{\deg \phi_1, \dots, \deg \phi_\ell\}.$$

Interestingly, when an arrangement is free, the exponents turn out to be the roots of the characteristic polynomial due to the second author.

Theorem 2.1 (Factorization) *If \mathcal{A} is free with $\text{exp}(\mathcal{A}) = \{d_1, \dots, d_\ell\}$, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

Proof See [27] or [21, Theorem 4.137]. □

For $\phi_1, \dots, \phi_\ell \in D(\mathcal{A})$, we define the $(\ell \times \ell)$ -matrix $M(\phi_1, \dots, \phi_\ell)$ as the matrix with (i, j) th entry $\phi_j(x_i)$. In general, it is difficult to determine whether a given arrangement is free or not. However, using the following criterion, we can verify that a candidate for a basis is actually a basis.

Theorem 2.2 (Saito’s criterion) *Let $\phi_1, \dots, \phi_\ell \in D(\mathcal{A})$. Then, $\{\phi_1, \dots, \phi_\ell\}$ forms a basis for $D(\mathcal{A})$ if and only if*

$$\det M(\phi_1, \dots, \phi_\ell) = cQ(\mathcal{A}) \quad (c \neq 0).$$

In particular, if ϕ_1, \dots, ϕ_ℓ are all homogeneous, then $\{\phi_1, \dots, \phi_\ell\}$ forms a basis for $D(\mathcal{A})$ if and only if the following two conditions are satisfied:

- (i) ϕ_1, \dots, ϕ_ℓ are independent over S ,
- (ii) $\sum_{i=1}^{\ell} \deg \phi_i = |\mathcal{A}|$.

Proof See [21, Theorems 4.19 and 4.23]. □

In addition to the Saito’s criterion, we have a way to check if a set of derivations is part of a homogeneous basis, and sometimes a $\{d_1, \dots, d_\ell\}_\leq$ indicates $d_1 \leq \dots \leq d_\ell$.

Theorem 2.3 *Let \mathcal{A} be a free arrangement with $\exp(\mathcal{A}) = \{d_1, \dots, d_\ell\}_\leq$. If $\phi_1, \dots, \phi_k \in D(\mathcal{A})$ satisfy for $1 \leq i \leq k$,*

- (i) $\deg \phi_i = d_i$,
- (ii) $\phi_i \notin S\phi_1 + \dots + S\phi_{i-1}$,

then ϕ_1, \dots, ϕ_k may be extended to a basis for $D(\mathcal{A})$.

Proof See [21, Theorem 4.42]. □

Definition 2.4 For $X \in L(\mathcal{A})$, let $I = I(X) := \sum_{H \in \mathcal{A}_X} \alpha_H S$ and $\bar{S} := S/I$. For $\phi \in D(\mathcal{A})$, define $\phi^X \in \text{Der}(\bar{S})$ by $\phi^X(f + I) = \phi(f) + I$. We call ϕ^X the restriction of ϕ to X .

Proposition 2.5 *If $\phi \in D(\mathcal{A})$, then $\phi^X \in D(\mathcal{A}^X)$. If $\phi^X \neq 0$, then $\deg \phi^X = \deg \phi$.*

Proof See [20, Lemma 2.12]. □

Fix $H \in \mathcal{A}$, denote $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' := \mathcal{A}^H$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ the triple with respect to the hyperplane $H \in \mathcal{A}$.

Proposition 2.6 *Define $h : D(\mathcal{A}') \rightarrow D(\mathcal{A})$ by $h(\phi) = \alpha_H \phi$ and $q : D(\mathcal{A}) \rightarrow D(\mathcal{A}'')$ by $q(\phi) = \phi^H$. The sequence*

$$0 \rightarrow D(\mathcal{A}') \xrightarrow{h} D(\mathcal{A}) \xrightarrow{q} D(\mathcal{A}'')$$

is exact.

Proof See [21, Proposition 4.45]. □

Then, the freeness of any two of the triple, under a certain condition on their exponents, implies the freeness of the third.

Theorem 2.7 (Addition–Deletion) *Let \mathcal{A} be a non-empty arrangement and let $H \in \mathcal{A}$. Then, two of the following imply the third:*

- (1) \mathcal{A} is free with $\text{exp}(\mathcal{A}) = \{d_1, \dots, d_{\ell-1}, d_\ell\}$.
- (2) \mathcal{A}' is free with $\text{exp}(\mathcal{A}') = \{d_1, \dots, d_{\ell-1}, d_\ell - 1\}$.
- (3) \mathcal{A}'' is free with $\text{exp}(\mathcal{A}'') = \{d_1, \dots, d_{\ell-1}\}$.

Moreover, all the three hold true if \mathcal{A} and \mathcal{A}' are both free.

Proof See [26] or [21, Theorems 4.46 and 4.51]. □

The Addition Theorem and Deletion Theorem above are rather “algebraic” as they rely on the conditions concerning the exponents. Recently, the first author has found “combinatorial” versions for these theorems [2–4]. In this paper, we focus on the Deletion theorem.

Theorem 2.8 (Combinatorial deletion) *Let \mathcal{A} be a free arrangement and $H \in \mathcal{A}$. Then, \mathcal{A}' is free if and only if $|\mathcal{A}_X| - |\mathcal{A}_X^H|$ is a root of $\chi(\mathcal{A}_X, t)$ for all $X \in L(\mathcal{A}^H)$.*

Proof See [2, Theorem 8.2]. □

When \mathcal{A} and \mathcal{A}' are both free, one may construct a basis for $D(\mathcal{A}'')$ from a basis for $D(\mathcal{A})$. Although the following theorem is probably well-known among experts, we give a detailed proof for the sake of completeness.

Theorem 2.9 *Let \mathcal{A} be a non-empty free arrangement and $\text{exp}(\mathcal{A}) = \{d_1, \dots, d_\ell\} \leq$. Assume further that \mathcal{A}' is also free. Let $\{\varphi_1, \dots, \varphi_\ell\}$ be a basis for $D(\mathcal{A})$ with $\text{deg } \varphi_j = d_j$ for $1 \leq j \leq \ell$. Then, there exists some p with $1 \leq p \leq \ell$ such that $\{\varphi_1^H, \dots, \varphi_\ell^H\} \setminus \{\varphi_p^H\}$ forms a basis for $D(\mathcal{A}'')$.*

Proof By Theorem 2.7, \mathcal{A}'' is also free and we may write $\text{exp}(\mathcal{A}'') = \text{exp}(\mathcal{A}) \setminus \{d_k\}$ for some k with $1 \leq k \leq \ell$. Denote $\tau_i := \varphi_i$ ($1 \leq i \leq k - 1$), and $\tau_j := \varphi_{j+1}$ ($k \leq j \leq \ell - 1$).

If $\tau_i^H \notin \overline{S}\tau_1^H + \dots + \overline{S}\tau_{i-1}^H$ for all $1 \leq i \leq \ell - 1$, then by Theorem 2.3, $\{\tau_1^H, \dots, \tau_{\ell-1}^H\} = \{\varphi_1^H, \dots, \varphi_\ell^H\} \setminus \{\varphi_k^H\}$ forms a basis for $D(\mathcal{A}'')$.

If not, there exists some p , $1 \leq p \leq \ell - 1$, such that $\tau_p^H \in \overline{S}\tau_1^H + \dots + \overline{S}\tau_{p-1}^H$. By Proposition 2.6,

$$\tau_p = f_1\tau_1 + \dots + f_{p-1}\tau_{p-1} + \alpha_H\tau,$$

where $f_i \in S$ ($1 \leq i \leq p - 1$), $\tau \in D(\mathcal{A}')$, $\text{deg } \tau = d_p - 1$. By Theorem 2.2 (Saito’s criterion), $\{\tau_1, \dots, \tau_{p-1}, \tau, \tau_{p+1}, \dots, \tau_{\ell-1}, \varphi_k\}$ is a basis for $D(\mathcal{A}')$. By Theorem 2.7, $\text{exp}(\mathcal{A}'') = \text{exp}(\mathcal{A}) \setminus \{d_p\}$. It means that $d_k = d_p$. Note that if d_k appears only once in $\text{exp}(\mathcal{A})$, then we obtain a contradiction here and the

proof is completed. Now, suppose d_k appears at least twice. Again by Theorem 2.2, $\{\tau_1, \dots, \tau_{p-1}, \alpha_H \tau, \tau_{p+1}, \dots, \tau_{\ell-1}, \varphi_k\}$ is a basis for $D(\mathcal{A})$. Since the map $q : D(\mathcal{A}) \rightarrow D(\mathcal{A}'')$ is surjective [21, Proposition 4.57], $D(\mathcal{A}'')$ is generated by $\{\tau_1^H, \dots, \tau_{p-1}^H, \tau_{p+1}^H, \dots, \tau_{\ell-1}^H, \varphi_k^H\}$. This set is the same as $\{\varphi_1^H, \dots, \varphi_{\ell}^H\} \setminus \{\varphi_p^H\}$ which indeed forms a basis for $D(\mathcal{A}'')$ by [21, Proposition A.3]. It completes the proof. \square

Remark 2.10 If d_k appears only once in $\text{exp}(\mathcal{A})$, then Theorem 2.9 gives an explicit basis for $D(\mathcal{A}'')$. However, if d_k appears at least twice, Theorem 2.9 may not be sufficient to derive an explicit basis for $D(\mathcal{A}'')$. It requires some additional computation to examine which derivation vanishes after taking the restriction to H . This observation will be useful to construct an explicit basis for $D(\mathcal{A}^X)$ when X is A_1^2 and RO (Example 5.2).

2.2 Root systems, Weyl groups and Weyl arrangements

Our standard reference for root systems and their Weyl groups is [6].

Let $V := \mathbb{R}^\ell$ with the standard inner product (\cdot, \cdot) . Let Φ be an irreducible (crystallographic) root system spanning V . The *rank* of Φ , denoted by $\text{rank}(\Phi)$, is defined to be $\dim(V)$. We fix a positive system Φ^+ of Φ . We write $\Delta := \{\alpha_1, \dots, \alpha_\ell\}$ for the simple system (base) of Φ associated with Φ^+ . For $\alpha = \sum_{i=1}^\ell d_i \alpha_i \in \Phi^+$, the *height* of α is defined by $\text{ht}(\alpha) := \sum_{i=1}^\ell d_i$. There are four classical types: A_ℓ ($\ell \geq 1$), B_ℓ ($\ell \geq 2$), C_ℓ ($\ell \geq 3$), D_ℓ ($\ell \geq 4$) and five exceptional types: E_6, E_7, E_8, F_4, G_2 . We write $\Phi = T$ if the root system Φ is of type T , otherwise, we write $\Phi \neq T$.

A *reflection* in V with respect to a vector $\alpha \in V \setminus \{0\}$ is a mapping $s_\alpha : V \rightarrow V$ defined by $s_\alpha(x) := x - 2 \frac{(x, \alpha)}{(\alpha, \alpha)} \alpha$. The *Weyl group* $W := W(\Phi)$ of Φ is a group generated by the set $\{s_\alpha \mid \alpha \in \Phi\}$. An element of the form $c = s_{\alpha_1} \dots s_{\alpha_\ell} \in W$ is called a *Coxeter element*. Since all Coxeter elements are conjugate (e.g., [6, Chapter V, §6.1, Proposition 1]), they have the same order, characteristic polynomial and eigenvalues. The order $h := h(W)$ of Coxeter elements is called the *Coxeter number* of W . For a fixed Coxeter element $c \in W$, if its eigenvalues are of the form $\exp(2\pi\sqrt{-1}m_1/h), \dots, \exp(2\pi\sqrt{-1}m_\ell/h)$ with $0 < m_1 \leq \dots \leq m_\ell < h$, then the integers m_1, \dots, m_ℓ are called the *exponents* of W (or of Φ).

Theorem 2.11 For any irreducible root system Φ of rank ℓ ,

- (i) $m_j + m_{\ell+1-j} = h$ for $1 \leq j \leq \ell$,
- (ii) $m_1 + m_2 + \dots + m_\ell = \ell h/2$,
- (iii) $1 = m_1 < m_2 \leq \dots \leq m_{\ell-1} < m_\ell = h - 1$,
- (iv) $h = 2 \mid \Phi^+ \mid / \ell$,
- (v) $h = \text{ht}(\theta) + 1$, where θ is the highest root of Φ .
- (vi) $h = 2 \sum_{\mu \in \Phi^+} (\widehat{\nu}, \widehat{\mu})^2$, for arbitrary $\gamma \in \Phi^+$. Here $\widehat{x} := x/(x, x)$.

Proof See, e.g., [6, Chapter V, §6.2 and Chapter VI, §1.11]. \square

Let $\Theta^{(r)} \subseteq \Phi^+$ be the set consisting of positive roots of height r , i.e., $\Theta^{(r)} = \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = r\}$. The *height distribution* of Φ^+ is defined as a multiset of positive

integers:

$$\{t_1, \dots, t_r, \dots, t_{h-1}\},$$

where $t_r := |\Theta^{(r)}|$. The dual partition $\mathcal{DP}(\Phi^+)$ of the height distribution of Φ^+ is given by a multiset of non-negative integers:

$$\mathcal{DP}(\Phi^+) := \{(0)^{\ell-t_1}, (1)^{t_1-t_2}, \dots, (h-2)^{t_{h-2}-t_{h-3}}, (h-1)^{t_{h-1}}\},$$

where notation $(a)^b$ means the integer a appears exactly b times.

Theorem 2.12 *The exponents of W are given by $\mathcal{DP}(\Phi^+)$.*

Proof See, e.g., [1,13,16,24]. □

Let S^W denote the ring of W -invariant polynomials. Let $\mathcal{F} = \{f_1, \dots, f_\ell\}$ be a set of basic invariants with $\deg f_1 \leq \dots \leq \deg f_\ell$. Then, $S^W = \mathbb{R}[f_1, \dots, f_\ell]$ and $m_i = \deg f_i - 1$ ($1 \leq i \leq \ell$). Let $\mathcal{D} = \{\theta_{f_1}, \dots, \theta_{f_\ell}\}$ be the set of basic derivations associated with \mathcal{F} (see, e.g., [20, Definition 2.4] and [21, Definition 6.50]). The Weyl arrangement of Φ^+ is defined by

$$\mathcal{A} = \mathcal{A}(\Phi^+) := \{(\mathbb{R}\alpha)^\perp \mid \alpha \in \Phi^+\}.$$

Theorem 2.13 *\mathcal{A} is free with $\exp(\mathcal{A}) = \{m_1, \dots, m_\ell\}$ and $\{\theta_{f_1}, \dots, \theta_{f_\ell}\}$ is a basis for $D(\mathcal{A})$.*

Proof See, e.g., [21, Theorem 6.60] and [23]. □

Recall from Proposition 2.6 the map $q : D(\mathcal{A}) \rightarrow D(\mathcal{A}'')$ defined by $q(\phi) = \phi^H$.

Theorem 2.14 *If $H \in \mathcal{A}$, then \mathcal{A}^H is free with $\exp(\mathcal{A}^H) = \{m_1, \dots, m_{\ell-1}\}$. Furthermore, $\{\theta_{f_1}^H, \dots, \theta_{f_{\ell-1}}^H\}$ is a basis for $D(\mathcal{A}^H)$.*

Proof See [20, Theorem 1.12]. □

Remark 2.15 Theorem 2.14 can be proved in a slightly different way. By [20, Theorem 3.7], $|\mathcal{A}| - |\mathcal{A}^H| = m_\ell$. Thus, $|\mathcal{A}| - |\mathcal{A}^H|$ is a root of $\chi(\mathcal{A}, t)$ by Theorems 2.13 and 2.1. By Theorem 2.8, \mathcal{A}' is free (see also Theorem 4.4 for a similar and more detailed explanation). Thus, \mathcal{A}^H is free with $\exp(\mathcal{A}^H) = \{m_1, \dots, m_{\ell-1}\}$ which follows from Theorem 2.7. A basis for $D(\mathcal{A}^H)$ can be constructed a bit more flexibly, without the need to introduce the basic derivations. Namely, let $\{\varphi_1, \dots, \varphi_\ell\}$ be any basis for $D(\mathcal{A})$ with $\deg \varphi_j = m_j$ for $1 \leq j \leq \ell$. Note that m_ℓ appears exactly once in $\exp(\mathcal{A})$. Then, by Theorem 2.9, $\{\varphi_1^H, \dots, \varphi_{\ell-1}^H\}$ is a basis for $D(\mathcal{A}'')$.

A subset $\Gamma \subseteq \Phi$ is called a (root) subsystem if it is a root system in $\text{span}_{\mathbb{R}}(\Gamma) \subseteq V$. For any $J \subseteq \Delta$, set $\Phi(J) := \Phi \cap \text{span}(J)$. Let $W(J)$ be the group generated by $\{s_\delta \mid \delta \in J\}$. By [7, Proposition 2.5.1], $\Phi(J)$ is a subsystem of Φ , J is a base of $\Phi(J)$ and the Weyl group of $\Phi(J)$ is $W(J)$. For any subset $\Gamma \subseteq \Phi$ and $w \in W$, denote $w\Gamma := \{w(\alpha) \mid \alpha \in \Gamma\} \subseteq \Phi$. A parabolic subsystem is any subsystem of the form $w\Phi(J)$, likewise, a parabolic subgroup of W is any subgroup of the form $wW(J)w^{-1}$, where $J \subseteq \Delta$ and $w \in W$.

Theorem 2.16 For $\Gamma \subseteq \Phi$, $\Phi \cap \text{span}(\Gamma)$ is a parabolic subsystem of Φ and its Weyl group is a parabolic subgroup of W .

Proof See, e.g., [10, Proposition 2.6] and [14, Lemma 3.2.3]. □

Recall the notation of $\Phi_X, \Phi_X^+, \Delta_X$ for $X \in L(\mathcal{A})$ from Notation 1.1. Note that if $X \in L_p(\mathcal{A})$, then Φ_X is a parabolic subsystem of rank p (Theorem 2.16). For each $X \in L(\mathcal{A})$, define the *fixer* of X by

$$W_X := \{w \in W \mid w(x) = x \text{ for all } x \in X\}.$$

Proposition 2.17 If $X \in L(\mathcal{A})$, then

- (i) W_X is the Weyl group of Φ_X . Consequently, W_X is a parabolic subgroup of W .
- (ii) \mathcal{A}_X is the Weyl arrangement of Φ_X^+ .

Proof (i) The first statement follows from [7, Proposition 2.5.5]. The second statement follows from Theorem 2.16. (ii) follows from (i) and [6, Chapter V, §3.3, Proposition 2]. □

Definition 2.18 Two subsets $\Phi_1, \Phi_2 \subseteq \Phi$ (resp., two subspaces $X_1, X_2 \in L(\mathcal{A})$) lie in the same W -orbit if there exists $w \in W$ such that $\Phi_1 = w\Phi_2$ (resp., $X_1 = wX_2$). Two subgroups W_1, W_2 of W are W -conjugate if there exists $w \in W$ such that $W_1 = w^{-1}W_2w$.

Lemma 2.19 Let X_1, X_2 be subspaces in $L(\mathcal{A})$. The following statements are equivalent:

- (i) Φ_{X_1} and Φ_{X_2} lie in the same W -orbit.
- (ii) Δ_{X_1} and Δ_{X_2} lie in the same W -orbit.
- (iii) X_1 and X_2 lie in the same W -orbit.
- (iv) W_{X_1} and W_{X_2} are W -conjugate.

Consequently, if any one of the statements above holds, then $|\mathcal{A}^{X_1}| = |\mathcal{A}^{X_2}|$.

Proof The equivalence of the statements follows from [19, Lemmas (3.4), (3.5)] (see also [11, Chapter VIII, 27-3, Proposition B]). The consequence is straightforward. □

3 Enumerate the cardinalities of A_1^2 restrictions

In this section, we present the first step towards proving conceptually the exponent formula in Theorem 1.5, the most important result in our paper. When X is of type A_1^2 , we express the cardinality $|\mathcal{A}^X|$ in terms of the Coxeter number h and a certain sum of inner products of positive roots (Proposition 3.13).

Definition 3.1 A set $\{\beta_1, \beta_2\} \subseteq \Phi$ with $\beta_1 \neq \pm\beta_2$ is called an A_1^2 set if it spans a subsystem of type A_1^2 , i.e., $\Phi \cap \text{span}\{\beta_1, \beta_2\} = \{\pm\beta_1, \pm\beta_2\}$.

Thus, $X \in L(\mathcal{A})$ is of type A_1^2 if and only if Δ_X is an A_1^2 set (Notation 1.1).

- Lemma 3.2** (i) For any $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Phi$, the set of $\alpha \in \Delta$ such that $c_\alpha \neq 0$ forms a non-empty connected induced subgraph of the Dynkin graph of Φ .
 (ii) If G is a non-empty connected subgraph of the Dynkin graph, then $\sum_{\alpha \in G} \alpha \in \Phi$.
 (iii) If $\{\beta_1, \beta_2\} \subseteq \Delta$ and $(\beta_1, \beta_2) = 0$, then $\{\beta_1, \beta_2\}$ is an A_1^2 set.

Proof Proofs of (i) and (ii) can be found in [6, Chapter VI, §1.6, Corollary 3 of Proposition 19]. (iii) is an easy consequence of (i). □

Let $\mathcal{T}(A_1^2)$ (resp., $\mathcal{T}(RO)$) be the set consisting of A_1^2 (resp., RO) sets.

- Proposition 3.3** (i) If $\{\beta_1, \beta_2\}$ is A_1^2 (resp., RO), then $w\{\beta_1, \beta_2\}$ is A_1^2 (resp., RO) for all $w \in W$.
 (ii) $\mathcal{T}(A_1^2) = \mathcal{T}(\Delta)$, where $\mathcal{T}(\Delta) := \{w\{\alpha_i, \alpha_j\} \mid \{\alpha_i, \alpha_j\} \subseteq \Delta, (\alpha_i, \alpha_j) = 0, w \in W\}$.

Proof (i) is straightforward. (ii) follows from (i), Theorem 2.16 and Lemma 3.2(iii). □

Remark 3.4 By Proposition 3.3 and Definition 1.3, $\mathcal{T}(A_1^2) \neq \emptyset$ (resp., $\mathcal{T}(RO) \neq \emptyset$) only when $\dim(V) \geq 3$.

Remark 3.5 Only for giving additional information, we collect some numerical facts about $\mathcal{T}(A_1^2)$ and $\mathcal{T}(RO)$. These facts shall not be used in any of upcoming arguments that support the proof of Theorem 1.5 or Theorem 1.6(i). Some of these facts will be rementioned in the proof of Theorem 1.6(ii) (Example 5.2), which is the part we are unable to avoid the classification of all irreducible root systems. By a direct check, $\mathcal{T}(RO) \cap \mathcal{T}(A_1^2) \neq \emptyset$ if and only if $\Phi = D_\ell$ with $\ell \geq 3$ ($D_3 = A_3$). In general, $\mathcal{T}(RO) \setminus \mathcal{T}(A_1^2) \neq \emptyset$, for example when $\Phi = B_\ell$ ($\ell \geq 3$), $\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\}$ is RO but spans a subsystem of type B_2 (notation in [6]). There is only one orbit of A_1^2 sets, with the following exceptions:

- (i) when $\Phi = D_4$, $\mathcal{T}(RO) = \mathcal{T}(A_1^2)$, and there are three different orbits,
- (ii) when $\Phi = D_\ell$ ($\ell \geq 5$), $\mathcal{T}(RO) \subsetneq \mathcal{T}(A_1^2)$, and there are two different orbits: $\mathcal{T}(RO) = \{w\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\} \mid w \in W\}$ and $\mathcal{T}(A_1^2) \setminus \mathcal{T}(RO)$.
- (iii) when $\Phi \in \{B_\ell, C_\ell\}$ ($\ell \geq 4$), $\mathcal{T}(RO) \cap \mathcal{T}(A_1^2) = \emptyset$, and there are two different orbits: $\mathcal{T}(\Delta^\perp) := \{\{\alpha, \beta\} \in \mathcal{T}(A_1^2) \mid \|\alpha\| = \|\beta\|\}$, and $\mathcal{T}(A_1^2) \setminus \mathcal{T}(\Delta^\perp)$.

Recall the notation $\beta^\perp = \{\alpha \in \Phi \mid (\alpha, \beta) = 0\}$ for $\beta \in \Phi$.

Definition 3.6 For an A_1^2 set $\{\beta_1, \beta_2\} \subseteq \Phi$, define

$$\mathcal{N}_0 = \mathcal{N}_0(\{\beta_1, \beta_2\}) := \{\Psi \subseteq \Phi \mid \Psi \text{ is an irreducible subsystem of rank 3, } \{\beta_1, \beta_2\} \subseteq \Psi\},$$

and for each $i \in \{1, 2\}$,

$$\mathcal{M}_{\beta_{3-i}}(\beta_i) := \left\{ \Lambda \subseteq \beta_{3-i}^\perp \mid \begin{array}{l} \Lambda \text{ is an irreducible subsystem of rank 2, } \beta_i \in \Lambda, \\ \Phi \cap \text{span}(\{\beta_{3-i}\} \cup \Lambda) \text{ is a reducible subsystem of rank 3.} \end{array} \right\},$$

and

$$\mathcal{N}_{\beta_{3-i}}(\beta_i) := \{\Psi \subseteq \Phi \mid \Psi = \{\pm\beta_{3-i}\} \times \Lambda, \Lambda \in \mathcal{M}_{\beta_{3-i}}(\beta_i)\}.$$

Proposition 3.7 \mathcal{N}_0 is not empty.

Proof For $\beta_1, \beta_2 \in \Phi$, there exists $\delta \in \Phi$ such that $(\delta, \beta_1) \neq 0$ and $(\delta, \beta_2) \neq 0$, e.g., see [10, Lemma 2.10]. Thus, $\Phi \cap \text{span}\{\beta_1, \beta_2, \delta\} \in \mathcal{N}_0$. \square

In the remainder of this section, we assume that $X = H_{\beta_1} \cap H_{\beta_2}$ is an A_1^2 subspace with $\beta_1, \beta_2 \in \Phi^+$. If $Y \in \mathcal{A}^X$, then Φ_Y is a subsystem of rank 3 and contains $\Delta_X = \{\beta_1, \beta_2\}$.

Proposition 3.8 If $X \in L_2(\mathcal{A})$ is an A_1^2 subspace, then

$$\begin{aligned} \mathcal{N}_0 &= \{\Phi_Y \mid Y \in \mathcal{A}^X, \Phi_Y \text{ is irreducible}\}, \text{ and for each } i \in \{1, 2\}, \\ \mathcal{N}_{\beta_{3-i}}(\beta_i) &= \{\Phi_Y \mid Y \in \mathcal{A}^X, \Phi_Y = \{\pm\beta_{3-i}\} \times \Xi_Y \text{ and } \Xi_Y \text{ is irreducible of rank 2}\}. \end{aligned}$$

Proof We only give a proof for the first equality. The others follow by a similar method. Let $\Psi \in \mathcal{N}_0$. There exists $\delta \in \Phi^+$ such that $\Psi = \Phi \cap \text{span}\{\beta_1, \beta_2, \delta\}$. Thus, $\Psi = \Phi \cap Y^\perp$ where $Y := X \cap H_\delta$. Therefore, $\Psi = \Phi_Y$ with $Y \in \mathcal{A}^X$. \square

Lemma 3.9 Let $X \in L(\mathcal{A})$.

- (i) $\mathcal{A} = \bigcup_{Y \in \mathcal{A}^X} \mathcal{A}_Y$.
- (ii) If $Y, Y' \in \mathcal{A}^X$ and $Y \neq Y'$, then $\mathcal{A}_Y \cap \mathcal{A}_{Y'} = \mathcal{A}_X$.
- (iii) $\mathcal{A} \setminus \mathcal{A}_X = \bigsqcup_{Y \in \mathcal{A}^X} (\mathcal{A}_Y \setminus \mathcal{A}_X)$ (disjoint union).

Proof (i) is straightforward. For (ii), $\mathcal{A}_X \subseteq \mathcal{A}_Y \cap \mathcal{A}_{Y'}$, since $Y, Y' \subseteq X$. Arguing on the dimensions, and using the fact that $\dim(X) - \dim(Y) = 1$ yield $X = Y + Y'$. Thus, if $H \in \mathcal{A}_Y \cap \mathcal{A}_{Y'}$, then $X \subseteq H$, i.e., $H \in \mathcal{A}_X$. (iii) follows automatically from (i) and (ii). \square

Corollary 3.10 Set $\mathcal{N} := \mathcal{N}_0 \sqcup \mathcal{N}_{\beta_2}(\beta_1) \sqcup \mathcal{N}_{\beta_1}(\beta_2) \sqcup \mathcal{N}_3$, where

$$\mathcal{N}_3 = \mathcal{N}_3(\{\beta_1, \beta_2\}) := \{\Psi \subseteq \Phi \mid \Psi = \{\pm\beta_1\} \times \{\pm\beta_2\} \times \{\pm\gamma\}, \gamma \in \Phi^+\}.$$

For $\Psi \in \mathcal{N}$, set $\Psi^+ := \Phi^+ \cap \Psi$. Then,

$$\Phi^+ = \{\beta_1, \beta_2\} \sqcup \bigsqcup_{\Psi \in \mathcal{N}} (\Psi^+ \setminus \{\beta_1, \beta_2\}).$$

Proof It follows from Proposition 2.17(ii), Proposition 3.8 and Lemma 3.9. \square

Proposition 3.11 For each $i \in \{1, 2\}$, set

$$\mathcal{K}_0 := \sum_{\Psi \in \mathcal{N}_0} \sum_{\delta \in \Psi^+ \setminus \{\beta_i\}} (\widehat{\beta}_i, \widehat{\delta})^2,$$

$$\mathcal{K}_{\beta_{3-i}}(\beta_i) := \sum_{\Lambda \in \mathcal{M}_{\beta_{3-i}}(\beta_i)} \sum_{\delta \in \Lambda^+ \setminus \{\beta_i\}} (\widehat{\beta}_i, \widehat{\delta})^2.$$

Then, $2(\mathcal{K}_0 + \mathcal{K}_{\beta_{3-i}}(\beta_i) + 1) = h$. In particular, $\mathcal{K}_{\beta_2}(\beta_1) = \mathcal{K}_{\beta_1}(\beta_2)$.

Proof By Corollary 3.10,

$$2(\mathcal{K}_0 + \mathcal{K}_{\beta_{3-i}}(\beta_i) + 1) = 2 \sum_{\delta \in \Phi^+} (\widehat{\beta}_i, \widehat{\delta})^2,$$

which equals h by Theorem 2.11(vi). □

Proposition 3.12

$$|\Psi^+| = \begin{cases} 3 & \text{if } \Psi \in \mathcal{N}_3, \\ 1 + 2 \sum_{\delta \in \Psi^+} (\widehat{\beta}_i, \widehat{\delta})^2 & \text{if } \Psi \in \mathcal{N}_{\beta_{3-i}}(\beta_i), i \in \{1, 2\}, \\ 3 \sum_{\delta \in \Psi^+} (\widehat{\beta}_i, \widehat{\delta})^2 & \text{if } \Psi \in \mathcal{N}_0. \end{cases}$$

Proof It follows from items (iv) and (vi) of Theorem 2.11. □

Proposition 3.13 *If $X = H_{\beta_1} \cap H_{\beta_2}$ is of type A_1^2 , then for each $i \in \{1, 2\}$*

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^X| = h/2 + \mathcal{K}_{\beta_{3-i}}(\beta_i).$$

Proof The proof is similar in spirit to the proof of [20, Proposition 3.6]. By Corollary 3.10,

$$|\mathcal{A}| - 2 = \sum_{\Psi \in \mathcal{N}} (|\Psi^+| - 2).$$

It is not hard to see that $|\mathcal{A}^X| = |\mathcal{N}|$ (via the bijection $Y \mapsto \Phi_Y$). By Proposition 3.12,

$$\begin{aligned} |\mathcal{A}| - |\mathcal{A}^X| &= \sum_{\Psi \in \mathcal{N}} (|\Psi^+| - 3) + 2 \\ &= 3\mathcal{K}_0 + 2\mathcal{K}_{\beta_2}(\beta_1) + 2\mathcal{K}_{\beta_1}(\beta_2) + 2. \end{aligned}$$

By Theorem 2.14 and Proposition 3.11,

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^X| = |\mathcal{A}| - |\mathcal{A}^X| - m_\ell = h/2 + \mathcal{K}_{\beta_{3-i}}(\beta_i).$$

□

Remark 3.14 (i) The conclusion of Proposition 3.13 can also be written as

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^X| = m_\ell - \mathcal{K}_0.$$

(ii) For each $\Psi \in \mathcal{N}_0$, denote by $h(\Psi)$ the Coxeter number of Ψ and write $m_1(\Psi) \leq m_2(\Psi) \leq m_3(\Psi)$ for the exponents of Ψ . In fact, $h(\Psi) = 2m_2(\Psi)$ since $\text{rank}(\Psi) = 3$. Thus,

$$\mathcal{K}_0 = \sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1).$$

In particular, if $\ell = 3$, then $\mathcal{N}_0 = \{\Phi\}$ and $\mathcal{K}_0 = m_2 - 1$. In this case, $|\mathcal{A}^{H\beta_i}| - |\mathcal{A}^X| = m_2 = h/2$.

Remark 3.15 If $\text{codim}(X) > 2$, for example, X is of type A_1^k with $k > 2$, the calculation in Proposition 3.13 is expected to be more difficult (and harder to avoid classifications) as it involves the consideration on the rank $k + 1$ subsystems of Φ containing Φ_X .

4 Proof of Theorem 1.5

Theorem 1.5 is a combination of two theorems below:

- Theorem 4.4: the freeness part,
- Theorem 4.6: the exponent part.

Since we prove the freeness part as a consequence of the exponent part and the proof is much more simple, we give the proof for the freeness part first (provided that the exponent part is given). Let us recall some general facts on hyperplane arrangements.

Lemma 4.1 (i) If $X, Y \in L(\mathcal{A})$, then $(\mathcal{A}^X)^{X \cap Y} = \mathcal{A}^{X \cap Y}$. Similarly, $(\phi^X)^{X \cap Y} = \phi^{X \cap Y}$ for any $\phi \in D(\mathcal{A})$.
 (ii) If $H \in \mathcal{A}$ and $X \in L(\mathcal{A}^H)$, then $(\mathcal{A}^H)_X = (\mathcal{A}_X)^H$. We will use the notation \mathcal{A}_X^H to denote these arrangements.

Proof Straightforward. □

Lemma 4.2 Let (\mathcal{A}_i, V_i) be irreducible arrangements ($1 \leq i \leq n$). Let $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ and $V = \bigoplus_{i=1}^n V_i$. Then,

- (i) $\chi(\mathcal{A}, t) = \prod_{i=1}^n \chi(\mathcal{A}_i, t)$,
- (ii) for $H = V_1 \oplus \dots \oplus V_{k-1} \oplus H_k \oplus V_{k+1} \oplus \dots \oplus V_n \in \mathcal{A}$ ($H_k \in \mathcal{A}_k$),

$$\mathcal{A}^H = \mathcal{A}_1 \times \dots \times \mathcal{A}_{k-1} \times \mathcal{A}_k^{H_k} \times \mathcal{A}_{k+1} \times \dots \times \mathcal{A}_n.$$

Proof (i) is a well-known fact, e.g., [21, Lemma 2.50]. (ii) follows by the definition of restriction. □

The freeness of every restriction is settled by a case-by-case study in [22], and by a uniform method in [8]. In Theorem 4.4, we give a different and more direct proof for the freeness of A_1^2 restrictions by using the Addition–Deletion Theorem (Theorem 2.7) and the Combinatorial Deletion Theorem (Theorem 2.8). We also need the following immediate consequence of Theorems 2.14 and 4.6.

Proposition 4.3 *If $\{\alpha, \beta\}$ is an A_1^2 set, then $|\mathcal{A}^{H_\alpha}| - |\mathcal{A}^{H_\alpha \cap H_\beta}|$ is a root of $\chi(\mathcal{A}^{H_\alpha}, t) = \prod_{i=1}^{\ell-1} (t - m_i)$.*

Theorem 4.4 *If $\{\alpha, \beta\}$ is an A_1^2 set, then the following statements are equivalent:*

- (1) $\mathcal{A}^{H_\alpha \cap H_\beta}$ is free and $\exp(\mathcal{A}^{H_\alpha \cap H_\beta}) = \exp(\mathcal{A}) \setminus \{m_i, m_\ell\}$ for some i with $1 \leq i \leq \ell - 1$.
- (2) $|\mathcal{A}^{H_\alpha}| - |\mathcal{A}^{H_\alpha \cap H_\beta}| = m_i$.

Proof It is sufficient to prove (2) \Rightarrow (1). By Theorem 2.14, \mathcal{A}^{H_α} is free with $\exp(\mathcal{A}^{H_\alpha}) = \{m_1, \dots, m_{\ell-1}\}$. By Lemma 4.1, $\mathcal{A}^{H_\alpha \cap H_\beta} = (\mathcal{A}^{H_\alpha})^{H_\alpha \cap H_\beta}$. If we can prove that $\mathcal{A}^{H_\alpha} \setminus \{H_\alpha \cap H_\beta\}$ is free, then Theorem 2.7 and Condition (2) guarantee that $\mathcal{A}^{H_\alpha \cap H_\beta}$ is free with $\exp(\mathcal{A}^{H_\alpha \cap H_\beta}) = \exp(\mathcal{A}) \setminus \{m_i, m_\ell\}$ for some $1 \leq i \leq \ell - 1$.

To show the freeness of $\mathcal{A}^{H_\alpha} \setminus \{H_\alpha \cap H_\beta\}$, we use Theorem 2.8. We need to show that $|\mathcal{A}_X^{H_\alpha}| - |\mathcal{A}_X^{H_\alpha \cap H_\beta}|$ is a root of $\chi(\mathcal{A}_X^{H_\alpha}, t)$ for all $X \in L(\mathcal{A}^{H_\alpha \cap H_\beta})$. It is clearly true by Proposition 4.3 provided that \mathcal{A}_X is irreducible. If \mathcal{A}_X is reducible, write $\mathcal{A}_X = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ where each \mathcal{A}_i is irreducible. By Lemma 4.2, $|\mathcal{A}_X^{H_\alpha}| - |\mathcal{A}_X^{H_\alpha \cap H_\beta}|$ either equals $|\mathcal{A}_k^{H_k}| - |\mathcal{A}_k^{H_k \cap H'_k}|$ where $1 \leq k \leq n$, $H_k, H'_k \in \mathcal{A}_k$, $H_k \cap H'_k$ is A_1^2 with respect to \mathcal{A}_k , or equals $|\mathcal{A}_j| - |\mathcal{A}_j^{H_j}|$ for some $j \neq k$. In either case, $|\mathcal{A}_X^{H_\alpha}| - |\mathcal{A}_X^{H_\alpha \cap H_\beta}|$ is a root of $\chi(\mathcal{A}_X^{H_\alpha}, t) = \chi(\mathcal{A}_1, t) \dots \chi(\mathcal{A}_k^{H_k}, t) \dots \chi(\mathcal{A}_j, t) \dots \chi(\mathcal{A}_n, t)$. \square

Remark 4.5 If the subsystem $\Phi_{\{\alpha, \beta\}} \subseteq \Phi$ spanned by $\{\alpha, \beta\}$ is not of type A_1^2 , then Condition (2) in Theorem 4.4 may not occur, i.e., the number $|\mathcal{A}^{H_\alpha}| - |\mathcal{A}^{H_\alpha \cap H_\beta}|$ may not be an exponent of W . In this case, we are unable to use the Addition–Deletion Theorem (Theorem 2.7) to derive $\exp(\mathcal{A}^{H_\alpha \cap H_\beta})$. To see an example, let $\Phi = E_6$ and $\Phi_{\{\alpha, \beta\}}$ be of type A_2 , or let $\Phi = F_4$ and $\Phi_{\{\alpha, \beta\}}$ be of type B_2 (see, e.g., [19, Tables V and VI]).

Theorem 4.6 *If $\{\beta_1, \beta_2\} \subseteq \Phi^+$ is A_1^2 , then for each $i \in \{1, 2\}$*

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^{H_{\beta_1} \cap H_{\beta_2}}| = \begin{cases} h/2 & \text{if } \{\beta_1, \beta_2\} \text{ is RO,} \\ m_{\ell-1} & \text{if } \{\beta_1, \beta_2\} \text{ is not RO.} \end{cases}$$

Proof of Theorem 4.6 will be divided into two halves: Theorems 4.8 and 4.28.

4.1 First half of Proof of Theorem 4.6

Lemma 4.7 *Let $C = (c_{ij})$ with $c_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ be the Cartan matrix of Φ . The roots of the characteristic polynomial of C are $2 + 2 \cos(m_i \pi/h)$ ($1 \leq i \leq \ell$).*

Proof See, e.g., [5, Theorem 2]. \square

We denote by $\mathcal{D}(\Phi)$ the Dynkin graph and by $\widetilde{\mathcal{D}}(\Phi)$ the extended Dynkin graph of Φ .

Theorem 4.8 *Assume that there exists a set $\{\beta_1, \beta_2\} \subseteq \Phi^+$ such that $\{\beta_1, \beta_2\}$ is both A_1^2 and RO. Then, $h/2$ is an exponent of W . Moreover, for each $i \in \{1, 2\}$,*

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^{H_{\beta_1} \cap H_{\beta_2}}| = h/2.$$

Proof By Lemma 4.7, it suffices to prove that the characteristic polynomial of the Cartan matrix C admits 2 as a root. By Proposition 3.3, we may assume that $\{\beta_1, \beta_2\} \subseteq \Delta$. Since the Dynkin graph $\mathcal{D}(\Phi)$ of Φ is a tree, there is a unique path in $\mathcal{D}(\Phi)$ which admits β_1 and β_2 as endpoints. By the definition of RO sets, this path contains exactly one more vertex of $\mathcal{D}(\Phi)$, say β_3 with $(\beta_1, \beta_3) \neq 0$ and $(\beta_2, \beta_3) \neq 0$. Also, the other vertices of $\mathcal{D}(\Phi)$, if any, connect to the path only at β_3 . The Cartan matrix has the following form:

$$C = \begin{bmatrix} 2 & 0 & c_{13} & 0 & \cdots & 0 \\ 0 & 2 & c_{23} & 0 & \cdots & 0 \\ c_{13} & c_{23} & 2 & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & 2 \\ 0 & 0 & & & & 2 \end{bmatrix}.$$

By applying the Laplace’s formula, it is easily seen that $\det(xI_\ell - C)$ is divisible by $x - 2$.

Assume that $\mathcal{M}_{\beta_2}(\beta_1) \neq \emptyset$ and let $\Lambda \in \mathcal{M}_{\beta_2}(\beta_1)$ (notation in Definition 3.6). Since $\{\beta_1, \beta_2\}$ is RO, the fact that $\Lambda \subseteq \beta_2^\perp$ implies that $(\beta_1, \alpha) = 0$ for all $\alpha \in \Lambda \setminus \{\pm\beta_1\}$. This contradicts the irreducibility of Λ . Thus, $\mathcal{K}_{\beta_2}(\beta_1) = 0$. Proposition 3.13 completes the proof. \square

4.2 Root poset, Dynkin diagram and exponents

The second half of Proof of Theorem 4.6 is complicated but uses nothing rather than combinatorial properties of root systems. We need to recall and prove several (technical) properties of the root poset, Dynkin diagram and exponents. This section is devoted to doing so, and every statement will be provided in great detail.

First, we study the positive roots of height $\geq m_{\ell-1}$ in the root poset and a certain set of vertices in the extended Dynkin diagram. For $\alpha \in V$, $\beta \in V \setminus \{0\}$, denote $\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)}$. Let $\theta := \sum_{i=1}^\ell c_{\alpha_i} \alpha_i$ be the highest root of Φ , and we call $c_{\alpha_i} \in \mathbb{Z}_{>0}$ the coefficient of θ at the simple root α_i . Denote by $c_{\max} := \max\{c_{\alpha_i} \mid 1 \leq i \leq \ell\}$ the largest coefficient.

Proposition 4.9 *Let Φ be an irreducible root system in \mathbb{R}^ℓ . Let θ be the highest root of Φ , and denote $\lambda_0 := -\theta$, $c_{\lambda_0} := 1$. Suppose that the elements of a fixed base $\Delta = \{\lambda_1, \dots, \lambda_\ell\}$ are labeled so that $\Lambda := \{\lambda_0, \lambda_1, \dots, \lambda_q\}$ is a set of minimal cardinality such that $c_{\lambda_q} = c_{\max}$ and $(\lambda_s, \lambda_{s+1}) < 0$ for $0 \leq s \leq q - 1$.*

- (i) *Then, $c_{\lambda_s} = s + 1$ for $0 \leq s \leq q$ and $|\Lambda| = c_{\max}$.*

(ii) Assume that $c_{\max} \geq 2$. Then, $(\lambda_0, \lambda_1, \dots, \lambda_{q-1})$ is a simple chain of $\tilde{\mathcal{D}}(\Phi)$ connected to the other vertices only at λ_{q-1} .

Proof See, e.g., [18, Lemma B.27, Appendix B] and [28, Proposition 3.1]. □

Remark 4.10 Proposition 4.9 was first formulated and proved in terms of coroots in [25, Lemma 1.5] under the name *lemma of the string*. The proof of [18, Lemma B.27, Appendix B] contains a small error, which was resolved in [28, Proposition 3.1].

Corollary 4.11 (i) If $c_{\max} = 1$, then all roots of Φ have the same length. In addition, if $\ell \geq 2$, then $\mathcal{D}(\Phi)$ is a simple chain and $-\theta$ is connected only to two terminal vertices of $\mathcal{D}(\Phi)$.

(ii) If $c_{\max} \geq 2$, then $-\theta$ is connected only to one vertex λ of $\mathcal{D}(\Phi)$ with $\langle \theta, \lambda \rangle \in \{1, 2\}$ and $c_\lambda = 2$. In particular, if $\lambda_i \in \Delta$ with $c_{\lambda_i} = 1$ is connected to $\lambda_j \in \Delta$ with $c_{\lambda_j} \geq 2$, then λ_i must be a terminal vertex of $\mathcal{D}(\Phi)$ and $\langle \lambda_j, \lambda_i \rangle = -1$.

Proof The second statement of (ii) follows from the equation $\langle \theta, \lambda_i \rangle = 0$. The other statements can be found in [28, Proposition 3.1 and Remark 3.2]. □

Corollary 4.12 Assume that $c_{\max} \geq 2$. Either $\langle \lambda_{q-1}, \lambda_q \rangle \in \{-2, -3\}$ or λ_q is a ramification point of $\tilde{\mathcal{D}}(\Phi)$.

Proof See [28, Corollary 3.3]. □

Denote $\mathcal{U} := \{\theta_i \in \Phi^+ \mid \text{ht}(\theta_i) > m_{\ell-1}\}$, and set $m := |\mathcal{U}|$. By Theorems 2.12 and 2.11(i), (iii), we have $m = m_\ell - m_{\ell-1} = m_2 - 1 > 0$. Suppose that the elements of \mathcal{U} are labeled so that θ_1 denotes the highest root, and $\xi_i = \theta_i - \theta_{i+1} \in \Delta$ for $1 \leq i \leq m-1$. We also adopt a convention $\xi_0 := -\theta_1$. Set $\Xi := \{\xi_i \mid 0 \leq i \leq m-1\}$. Note that Ξ is a multiset, not necessarily a set. For a finite multiset $S = \{(a_1)^{b_1}, \dots, (a_n)^{b_n}\}$, write \bar{S} for the base set of S , i.e., $\bar{S} = \{a_1, \dots, a_n\}$. Let us call

Case 1: “there is an integer t such that $1 \leq t \leq m-1$ and $\langle \theta_t, \xi_t \rangle = 3$ ”.

Case 2: Negation of Case 1.

Proposition 4.13 (i) If Case 1 occurs, then $t = m-2$ and $\Xi = \{\xi_0, \xi_1, \dots, (\xi_{m-2})^2\}$ with $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m-2$. As a result, $|\bar{\Xi}| = m_2 - 2$.

(ii) If Case 2 occurs, then $\Xi = \{\xi_0, \xi_1, \dots, \xi_{m-1}\}$ with $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m-1$. As a result, $|\bar{\Xi}| = m_2 - 1$.

Proof See [28, Propositions 3.10 and 3.11]. □

Theorem 4.14 With the notations we have seen from Propositions 4.9 to 4.13, we have that $q = m-1$ and $\lambda_i = \xi_i$ for all $1 \leq i \leq q$. In particular, $\bar{\Xi} = \Lambda$ (as sets). Moreover, if Case 1 occurs, then $m_2 = c_{\max} + 2$; and if Case 2 occurs, then $m_2 = c_{\max} + 1$.

Proof See [28, Theorem 4.1]. □

Corollary 4.15 (i) If Case 1 occurs, then $\theta_i - \theta_j \in \Phi^+$ for $1 \leq i < j \leq m$, $\{i, j\} \neq \{m-2, m\}$, and $\theta_{m-2} - \theta_m \in 2\Delta$.

(ii) If Case 2 occurs, then $\theta_i - \theta_j \in \Phi^+$ for $1 \leq i < j \leq m$.

Proof See [28, Corollary 3.12]. □

Corollary 4.16 *The following statements are equivalent: (i) Case 1 occurs, (ii) $\text{rank}(\Phi) = 2$, (iii) $\Phi = G_2$, (iv) $c_{\max} = m_2 - 2$.*

Proof See [28, Theorem 4.2]. □

Recall the notation $\Theta^{(r)} = \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = r\}$.

Proposition 4.17 *There is always a long root in $\Theta^{(m_\ell-1)}$.*

Proof We may assume that $c_{\max} \geq 2$. If not, by Corollary 4.11, all roots of Φ have the same length. The assertion is trivial. If Case 1 occurs, then by Corollary 4.16, $\Phi = G_2$. The assertion is also trivial.

Now, we can assume that Case 2 occurs. By Proposition 4.13, $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m - 1$. If $\langle \xi_{m-2}, \xi_{m-1} \rangle = -2$, then $\langle \theta_{m-1}, \xi_{m-1} \rangle = 2$. Thus, $\theta_{m-1} - 2\xi_{m-1} \in \Theta^{(m_\ell-1)}$, that is a long root and we are done. We are left with the case $\langle \xi_{m-2}, \xi_{m-1} \rangle = -1$. By Corollary 4.12, ξ_{m-1} is connected to at least two vertices of $\mathcal{D}(\Phi)$ apart from ξ_{m-2} , say μ_1, \dots, μ_k ($k \geq 2$).

We claim that there exists μ_i such that $\langle \xi_{m-1}, \mu_i \rangle = -1$. Proof of the claim when $m = 2$ (i.e., $c_{\max} = 2$) and $m \geq 3$ uses very similar technique [the case $m = 2$ is actually Lemma 4.23(2)]. We only give a proof when $m \geq 3$. From $\langle \theta, \xi_{m-1} \rangle = 0$ (it equals 1 if $m = 2$), $c_{\xi_{m-1}} = c_{\max}$, $c_{\xi_{m-2}} = c_{\max} - 1$, and ξ_{m-1} is a long root, we have $c_{\max} + 1 - \sum_{i=1}^k c_{\mu_i} = 0$. Suppose to the contrary that $\langle \xi_{m-1}, \mu_i \rangle \leq -2$ for all $1 \leq i \leq k$. From $0 = \langle \theta, \mu_i \rangle \leq 2c_{\mu_i} + c_{\max} \langle \xi_{m-1}, \mu_i \rangle$, we obtain $c_{\mu_i} = c_{\max}$. Thus, $c_{\max} + 1 - kc_{\max} = 0$, a contradiction. So we can choose μ_i so that $\langle \xi_{m-1}, \mu_i \rangle = -1$. Therefore, $\langle \theta_m, \mu_i \rangle = -\langle \xi_{m-1}, \mu_i \rangle = 1$ and $\theta_m - \mu_i \in \Theta^{(m_\ell-1)}$, that is a long root. □

Lemma 4.18 *Suppose $\beta_1, \beta_2, \beta_3 \in \Phi$ with $\beta_1 + \beta_2 + \beta_3 \in \Phi$ and $\beta_i + \beta_j \neq 0$ for $i \neq j$. Then, at least two of the three partial sums $\beta_i + \beta_j$ belong to Φ .*

Proof See, e.g., [15, §11, Lemma 11.10]. □

Proposition 4.19 *If $\gamma \in \Theta^{(m_\ell-1)}$, then $\theta_i - \gamma \in k\Phi^+$ with $k \in \{1, 2, 3\}$ for every $1 \leq i \leq m$.*

Proof To avoid the triviality, we assume that Case 2 occurs and $1 \leq i \leq m - 1$. Denote $\mu := \theta_m - \gamma \in \Delta$.

Assume that $\mu = \xi_{m-1}$. Then, $\theta_{m-1} - \gamma = 2\xi_{m-1} \in 2\Delta$. We have $\langle \theta_{m-1}, \xi_{m-1} \rangle = \langle \gamma, \xi_{m-1} \rangle + 4$. It follows that $\langle \theta_{m-1}, \xi_{m-1} \rangle = 2$. Fix i with $1 \leq i \leq m - 2$, and set $\alpha := \theta_i - \theta_{m-1} \in \Phi^+$ (by Corollary 4.15). Since $\theta_i = \theta_1 - (\xi_1 + \dots + \xi_{i-1})$, we have $\langle \theta_i, \xi_{m-1} \rangle = 0$. Thus, $\langle \alpha, \xi_{m-1} \rangle = -\langle \theta_{m-1}, \xi_{m-1} \rangle = -2$. Then, $\theta_i - \gamma = \alpha + 2\xi_{m-1} \in \Phi^+$.

Assume that $\mu \neq \xi_{m-1}$. Then, $\theta_{m-1} = \gamma + \mu + \xi_{m-1}$. By Lemma 4.18, $\mu + \xi_{m-1} \in \Phi^+$ since $\gamma + \xi_{m-1} \notin \Phi^+$. Thus, μ and ξ_{m-1} are adjacent on $\mathcal{D}(\Phi)$. If $\mu \neq \xi_{m-2}$ (of course, $\mu \neq \xi_i$ for all $1 \leq i \leq m - 3$ since $\mathcal{D}(\Phi)$ is a tree), then by Lemma 3.2(ii), $\theta_i - \gamma = \xi_i + \dots + \xi_{m-1} + \mu \in \Phi^+$ for each $1 \leq i \leq m - 1$. If $\mu = \xi_{m-2}$, then $\theta_{m-2} = \gamma + \xi_{m-1} + 2\xi_{m-2}$. Thus, $\langle \theta_{m-2}, \xi_{m-2} \rangle = \langle \gamma, \xi_{m-2} \rangle + \langle \xi_{m-1}, \xi_{m-2} \rangle + 4$. Using the fact that ξ_{m-2} is a long root, we obtain a contradiction since the left-hand side is at most 1, while the right-hand side is at least 2. □

Corollary 4.20 *If $\ell \geq 5$, then $m_{\ell-2} < m_{\ell-1}$.*

Proof By Theorem 2.12, it suffices to prove that there are exactly two roots of height $m_{\ell-1}$, i.e., $|\Theta^{(m_{\ell-1})}| = 2$. To avoid the triviality, we assume that Case 2 occurs and $c_{\max} \geq 2$. We need to consider two cases: $\langle \xi_{m-2}, \xi_{m-1} \rangle = -2$ or $\langle \xi_{m-2}, \xi_{m-1} \rangle = -1$. Since the proofs are very similar, we only give a proof for the latter (slightly harder case).

Suppose to the contrary that $\Theta^{(m_{\ell-1})} = \{\gamma_1, \dots, \gamma_k\}$ with $k \geq 3$. By Proposition 4.19, $\theta_{m-1} - \gamma_i = \xi_{m-1} + \mu_i \in \Phi^+$, where $\mu_i \in \Delta$ ($1 \leq i \leq k$). Thus, μ_i is adjacent to ξ_{m-1} on $\tilde{\mathcal{D}}(\Phi)$. By the same argument as the one used in the end of Proof of Proposition 4.19, we obtain $\mu_i \neq \xi_{m-2}$ for all $1 \leq i \leq k$. If $m \geq 3$, then the same argument as in Proof of Proposition 4.17 gives $c_{\max} + 1 = \sum_{i=1}^k c_{\mu_i}$. From $0 = \langle \theta, \mu_i \rangle \leq 2c_{\mu_i} - c_{\max}$, we obtain $c_{\mu_i} \geq c_{\max}/2$ for all $1 \leq i \leq k$. This forces $k = 3$. But it implies that $c_{\max} \leq 2$, i.e., $m \leq 2$, a contradiction. Now, consider $m = 2$. A similar argument as above shows that $k = 3$ and $c_{\mu_1} = c_{\mu_2} = c_{\mu_3} = 1$. The second statement of Corollary 4.11(ii) implies that c_{μ_i} must be all terminal. Thus, $\ell = 4$, a contradiction. \square

From now on, we require the classification of root systems of rank ≤ 4 to make some arguments work. The classification of rank 3 or 4 root systems will be announced before use, while that of rank 2 root systems (has been and) will be used without announcing.

Lemma 4.21 *If $m_2 = h/2$, then $\ell \leq 4$. More specifically, when $\ell = 4$, $m_2 = h/2$ if and only if $\Phi = D_4$.*

Proof If $m_2 = c_{\max} + 2$, then by Corollary 4.16, $\Phi = G_2$. However, $m_2 > h/2$ by a direct check. Now, assume that $m_2 = c_{\max} + 1$. Recall from Proposition 4.9 that the coefficients of θ at elements of $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_q\}$ form an arithmetic progression, starting with $c_{\lambda_0} = 1$ and ending with $c_{\lambda_q} = c_{\max} = q + 1$. From $\sum_{\lambda \in \Lambda} c_{\lambda} + \sum_{\lambda \in \Delta \setminus \Lambda} c_{\lambda} = h$, we have

$$c_{\max}(c_{\max} + 1)/2 + \ell - (c_{\max} - 1) \leq 2c_{\max} + 2.$$

Thus, $\ell \leq (-c_{\max}^2 + 5c_{\max} + 2)/2$. Therefore, $\ell \leq 4$. The second statement is clear from the classification of irreducible root systems of rank 4. \square

Remark 4.22 The first statement of Lemma 4.21 is an easy consequence of the well-known fact that every exponent of Φ appears at most twice. A uniform proof of this fact is probably well-known among experts.

Now, we investigate the “local” picture of the extended Dynkin graph at the sub-graph induced by the negative $-\theta$ of the highest root and the simple root adjacent to it, and find a connection with RO properties.

Lemma 4.23 *Assume that $\ell \geq 2$ and $c_{\max} \geq 2$. Let λ be the unique simple root connected to $-\theta$ (Corollary 4.11). Denote by $\gamma_1, \dots, \gamma_k$ ($k \geq 1$) the simple roots connected to λ . Then, there are the following possibilities:*

- (1) If $\langle \theta, \lambda \rangle = 2$, then $k = 1, c_{\gamma_1} \in \{1, 2\}$.
- (2) If $\langle \theta, \lambda \rangle = 1$ (in particular, λ is long), then either (2a) $k = 3, c_{\gamma_1} = c_{\gamma_2} = c_{\gamma_3} = 1$ (i.e., $\Phi = D_4$), or (2b) $k = 2, c_{\gamma_1} = 2, c_{\gamma_2} = 1$ (γ_2 is terminal and long), or (2c) $k = 1, c_{\gamma_1} = 3$.

Proof We have $\langle \theta, \lambda \rangle = 2c_\lambda + \sum_{i=1}^k c_{\gamma_i} \langle \gamma_i, \lambda \rangle \leq 4 - \sum_{i=1}^k c_{\gamma_i}$. Since $\langle \theta, \lambda \rangle \geq 1$, we have $\sum_{i=1}^k c_{\gamma_i} \in \{1, 2, 3\}$. Then, we can list all possibilities and rule out impossibilities. For example, if $\langle \theta, \lambda \rangle = 1$ and $\sum_{i=1}^k c_{\gamma_i} = 2$, then either (i) $k = 1, c_{\gamma_1} = 2$, or (ii) $k = 2, c_{\gamma_1} = c_{\gamma_2} = 1$. For (i), it follows that $1 = \langle \theta, \lambda \rangle = 4 + 2\langle \gamma_1, \lambda \rangle$, which is a contradiction since λ is long. For (ii), the second statement of Corollary 4.11(ii) implies that γ_1, γ_2 must be all terminal. Thus, $\ell = 3$. However, such root system does not exist by the classification of irreducible root systems of rank 3. Similarly, to conclude that $\Phi = D_4$ in (2a), we need the classification of irreducible root systems of rank 4. □

It is known that θ^\perp is the standard parabolic subsystem of Φ generated by $\{\alpha \in \Delta \mid (\alpha, \theta) = 0\}$. Also, θ^\perp may be reducible and decomposed into irreducible, mutually orthogonal components.

Corollary 4.24 *If θ^\perp is reducible, then either Possibility (2a) or (2b) occurs.*

Proof It follows immediately from Lemma 4.23 (taking $k \geq 2$). □

Corollary 4.25 *When $\ell = 4$, there exists a set that is both A_1^2 and RO if and only if $\Phi = D_4$. Moreover, if $\Phi = D_4$, then every A_1^2 set is RO.*

Proof Use the classification of irreducible root systems of rank 4. □

Proposition 4.26 *Assume that $\ell \geq 4$. If θ^\perp is reducible and there exists an A_1^2 set that is not RO, then Possibility (2b) in Lemma 4.23 occurs. In particular, $\theta^\perp = \{\pm\gamma_2\} \times \Omega$ for a long simple root γ_2 and Ω is irreducible with $\text{rank}(\Omega) \geq 2$.*

Proof If $c_{\max} = 1$, then Corollary 4.11(i) implies that θ^\perp is irreducible and of rank at least 2, a contradiction. Now, consider $c_{\max} \geq 2$. By Corollary 4.24, either Possibility (2a) or (2b) occurs. However, Corollary 4.25 ensures that Possibility (2a) cannot occur because a non-RO A_1^2 set exists. Thus, Possibility (2b) must occur. Then, $\theta^\perp = \{\pm\gamma_2\} \times \Omega$ where Ω is irreducible with $\text{rank}(\Omega) \geq 2$. □

Proposition 4.27 *Assume that $\ell \geq 4$. Suppose that $\{\theta, \alpha\}$ is an A_1^2 set with $\alpha \in \Delta$. If $\{\pm\alpha\}$ is a component of θ^\perp , then $\{\theta, \alpha\}$ is RO.*

Proof We may assume that $c_{\max} \geq 2$. If not, Corollary 4.11(i) implies that θ^\perp is irreducible and of rank at least 2, a contradiction. Since $-\theta$ connects only to one vertex of $\mathcal{D}(\Phi)$, θ^\perp must be reducible; otherwise, $\theta^\perp = \{\pm\alpha\}$, a contradiction. Thus, $\{\beta, \alpha\}$ is A_1^2 for every $\beta \in \theta^\perp \setminus \{\pm\alpha\}$ because $\{\beta, \alpha\} = w\{\alpha', \alpha\}$ for some $w \in W$ and $\alpha' \in \Delta$ such that $(\alpha', \alpha) = 0$. With the notations in Definition 3.6 and Proposition 3.11, we have $\mathcal{M}_\theta(\alpha) = \emptyset$ and $\mathcal{K}_\theta(\alpha) = 0$.

Note that $\theta^\perp \setminus \{\pm\alpha\} \subseteq \alpha^\perp \setminus \{\pm\theta\}$ since $\{\pm\alpha\}$ is a component of θ^\perp . Suppose to the contrary that $\{\theta, \alpha\}$ is not RO. Then, there exists $\beta \in \alpha^\perp \setminus \{\pm\theta\}$ such that $(\beta, \theta) \neq 0$.

Note that $\Gamma := \Phi \cap \text{span}\{\beta, \theta, \alpha\}$ is a subsystem of rank 3 since it contains the A_1^2 set $\{\theta, \alpha\}$. Γ must be irreducible, otherwise, $\Gamma \in \mathcal{M}_\alpha(\theta)$ and $\mathcal{K}_\alpha(\theta) \neq 0$ which contradicts Proposition 3.11. Relying on the classification of rank-3 irreducible root systems, and two facts: (i) $\{\theta, \alpha\}$ is A_1^2 in Γ , (ii) $\alpha^\perp \cap \Gamma$ is an irreducible subsystem of rank 2 of Γ (as it contains β and θ), we conclude that $\Gamma = B_3$ and α is the unique short simple root of Γ . Since θ^\perp is reducible, either Possibility (2a) or (2b) occurs by Corollary 4.24. Thus, Possibility (2b) must occur since Φ contains the subsystem Γ of type B_3 . Since $\{\pm\alpha\}$ is a component of θ^\perp and α is short, with the notation in Possibility (2b), $\Delta = \{\gamma_2, \lambda, \alpha\}$. Thus, $\ell = 3$, a contradiction. \square

4.3 Second half of Proof of Theorem 4.6

In this subsection, we complete the Proof of Theorem 4.6 by proving its second half, Theorem 4.28.

Theorem 4.28 *If $\{\beta_1, \beta_2\} \subseteq \Phi^+$ is A_1^2 but not RO, then for each $i \in \{1, 2\}$*

$$|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^{H_{\beta_1} \cap H_{\beta_2}}| = m_{\ell-1}.$$

The first key ingredient is Theorem 4.30. It asserts that Theorem 4.28 is always true for a special class of A_1^2 sets, and we can describe it by the height function without requiring the RO condition. We were recently informed that Mücksch and Röhrle [17] study a similar property (to the non-RO case) of Weyl arrangement restrictions (which they called the accuracy) via MAT-free techniques of [1]. Their main result together with Corollary 4.20 indeed gives a proof of Theorem 4.30. We will use this proof here; however, we remark that our primary method gives a different and bijective proof, which we refer the interested reader to “Appendix” for more details.

Unless otherwise stated, we assume that $\ell \geq 3$ in the remainder of the paper.

Lemma 4.29 *If $\{\gamma_1, \gamma_2\} \subseteq \Theta^{(r)}$ with $\gamma_1 \neq \gamma_2$ and $r \geq \lfloor m_\ell/2 \rfloor + 1$ (floor function), then $\{\gamma_1, \gamma_2\}$ is A_1^2 . The assertion is true, in particular, if $r = m_{\ell-1}$.*

Proof We have $\gamma_1 + \gamma_2 \notin \Phi$ and $\gamma_1 - \gamma_2 \notin \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\alpha$. Use the classification of irreducible root systems of rank 2. \square

Theorem 4.30 *If $\gamma_1, \gamma_2 \in \Theta^{(m_{\ell-1})}$ with $\gamma_1 \neq \gamma_2$, then for each $i \in \{1, 2\}$*

$$|\mathcal{A}^{H_{\gamma_i}}| - |\mathcal{A}^{H_{\gamma_1} \cap H_{\gamma_2}}| = m_{\ell-1}.$$

Proof The formula holds true trivially for $\ell = 3$, and we use the classification of irreducible root systems for $\ell = 4$. Assume that $\ell \geq 5$. Then, by Corollary 4.20, there are exactly two roots of height $m_{\ell-1}$. The rest follows from [17, Theorem 4.3]. \square

The second key ingredient is Proposition 4.31. In fact, the A_1^2 sets described in Theorem 4.30 are not enough to generate all possible W -orbits of the A_1^2 sets (cf. Remark 3.5). Notice that the focus of Theorem 4.28 is non-RO A_1^2 sets. Although, Theorem 4.30 alone is not enough to prove Theorem 4.28, it guarantees that the

problem is solved if the involving non-RO A_1^2 set lies in the same W -orbit with a pair of roots of height $m_{\ell-1}$.

Proposition 4.31 *If there exists an A_1^2 set that is not RO, then the set $\mathcal{S} := \{\{\gamma_1, \gamma_2\} \subseteq \Theta^{(m_{\ell-1})} \mid \text{at least one of } \gamma_1, \gamma_2 \text{ is a long root}\}$ contains a non-RO set.*

Proof Note that $\mathcal{S} \neq \emptyset$ since $\Theta^{(m_{\ell-1})}$ always contains a long root (Proposition 4.17). The case $\ell = 3$ is checked directly by the classification. Assume that $\ell \geq 4$ and suppose to the contrary that every element in \mathcal{S} is RO. We can take $\{\gamma_1, \gamma_2\} \in \mathcal{S}$ and assume that γ_1 is long. By Theorems 4.8 and 4.30, we have $m_{\ell-1} = h/2$. By Lemma 4.21, $\ell = 4$. By Corollary 4.25, $\Phi = D_4$, and all A_1^2 sets must be RO. This contradicts the Proposition’s assumption. \square

Of course, if $\ell \geq 5$, then Corollary 4.20 implies that the set \mathcal{S} contains only one element. However, the present statement is enough for us.

The third (and the final) key ingredient is Corollary 4.32. From the previous discussions, we will be in the situation that there exist two non-RO A_1^2 sets which may form different orbits. So we want to find a relation between them.

Corollary 4.32 *Assume that $\ell \geq 4$. If there are two A_1^2 sets $\{\theta, \lambda_1\}, \{\theta, \lambda_2\}$ that are both non-RO, then both λ_1 and λ_2 lie in the unique irreducible component Ω of θ^\perp with $\text{rank}(\Omega) \geq 2$.*

Proof The statement is trivial if θ^\perp is irreducible. If θ^\perp is reducible, then the statement follows from Propositions 4.26 and 4.27. \square

We need one more simple lemma.

Lemma 4.33 *If $\{\beta_1, \beta_2\} \subseteq \Phi^+$ contains a long root and $(\beta_1, \beta_2) = 0$, then $\{\beta_1, \beta_2\}$ lies in the same W -orbit with $\{\theta, \mu\}$ for some $\mu \in \Delta$.*

Proof This is well-known. There is an irreducible component $\Psi \subseteq \theta^\perp$ such that $\{\beta_1, \beta_2\}$ lies in the same W -orbit with $\{\theta, \gamma\}$ for some $\gamma \in \Psi$. There exist $\mu \in \Delta \cap \Psi$ and $w \in W(\Psi)$ such that $\gamma = w(\mu)$ and this w fixes θ , i.e., $\theta = w(\theta)$. Thus, $\{\beta_1, \beta_2\}$ lies in the same W -orbit with $\{\theta, \mu\}$. \square

Now, we are ready to prove Theorem 4.28.

Proof of Theorem 4.28 It suffices to prove the theorem under the condition that the A_1^2 set $\{\beta_1, \beta_2\}$ contains a long root. Otherwise, we would consider the dual root system Φ^\vee where short roots become long roots. By Lemma 4.33, $\{\beta_1, \beta_2\}$ lies in the same W -orbit with $\{\theta, \lambda_1\}$ for some $\lambda_1 \in \Delta$. We may also assume that $\ell \geq 4$ since the case $\ell = 3$ is done in Remark 3.14(ii). We note that by Remark 3.14(i), proving Theorem 4.28 is equivalent to showing that $\mathcal{K}_\theta(\lambda_1) = m_{\ell-1} - h/2$ (notation in Proposition 3.11).

By Proposition 4.31, we can find a non-RO set $\{\gamma_1, \gamma_2\} \subseteq \Theta^{(m_{\ell-1})}$ where γ_1 is a long root. Again by Lemma 4.33, $\{\gamma_1, \gamma_2\}$ lies in the same W -orbit with $\{\theta, \lambda_2\}$ for some $\lambda_2 \in \Delta$. By Proposition 3.3(i), $\{\theta, \lambda_1\}$ and $\{\theta, \lambda_2\}$ are A_1^2 and non-RO. Corollary 4.32 implies that λ_1 and λ_2 lie in the unique irreducible component Ω of

θ^\perp with $\text{rank}(\Omega) \geq 2$. We already know from Theorem 4.30 that Theorem 4.28 is automatically proved for $\{\theta, \lambda_2\}$, i.e., $\mathcal{K}_\theta(\lambda_2) = m_{\ell-1} - h/2$. So, we want to prove that

$$\mathcal{K}_\theta(\lambda_1) = \mathcal{K}_\theta(\lambda_2). \tag{1}$$

If $\|\lambda_1\| = \|\lambda_2\|$, then $\{\theta, \lambda_1\}$ and $\{\theta, \lambda_2\}$ lie in the same W -orbit. So, Formula (1) follows. Now, consider $\|\lambda_1\| \neq \|\lambda_2\|$. Note that at most two root lengths occur in Ω , they are $\|\lambda_1\|$ and $\|\lambda_2\|$. Then, the fact that $\{\theta, \lambda_1\}$ and $\{\theta, \lambda_2\}$ are both A_1^2 implies that $\{\theta, \beta\}$ is A_1^2 for all $\beta \in \Omega$. For $\beta \in \Omega$, with the notations in Definition 3.6 applied to the A_1^2 set $\{\theta, \beta\}$, we have

$$\mathcal{M}_\theta(\beta) = \left\{ \Lambda \subseteq \Omega \mid \begin{array}{l} \Lambda \text{ is an irreducible subsystem of rank 2, } \beta \in \Lambda, \\ \Phi \cap \text{span}(\{\theta\} \cup \Lambda) \text{ is a reducible subsystem of rank 3.} \end{array} \right\}.$$

Indeed, if $\Lambda \subseteq \theta^\perp$ and Λ is irreducible, then $\Lambda \subseteq \Omega$. We further make the following definition

$$\mathcal{M}'_\theta(\beta) := \left\{ \Lambda \subseteq \Omega \mid \begin{array}{l} \Lambda \text{ is an irreducible subsystem of rank 2, } \beta \in \Lambda, \\ \Phi \cap \text{span}(\{\theta\} \cup \Lambda) \text{ is an irreducible subsystem of rank 3.} \end{array} \right\}.$$

Using Proposition 3.11 and Theorem 2.11(vi), we compute

$$\begin{aligned} 2\mathcal{K}_\theta(\beta) &= 2 \sum_{\Lambda \in \mathcal{M}_\theta(\beta)} \sum_{\delta \in \Lambda^+ \setminus \{\beta\}} (\widehat{\beta}, \widehat{\delta})^2 \\ &= h(\Omega) - 2 - 2 \sum_{\Lambda \in \mathcal{M}'_\theta(\beta)} \sum_{\delta \in \Lambda^+ \setminus \{\beta\}} (\widehat{\beta}, \widehat{\delta})^2. \end{aligned}$$

We claim that $\mathcal{M}'_\theta(\beta) = \emptyset$ for every $\beta \in \Omega$. Suppose not and let $\Lambda \in \mathcal{M}'_\theta(\beta)$. Note that $\Gamma := \Phi \cap \text{span}(\{\theta\} \cup \Lambda)$ admits θ as the highest root in its positive system $\Phi^+ \cap \text{span}(\{\theta\} \cup \Lambda)$. By a direct check on all rank-3 irreducible root systems and using the fact that $\{\theta, \beta\}$ is A_1^2 for all $\beta \in \Lambda \subseteq \Gamma \cap \Omega$, we obtain a contradiction. Thus, the claim is proved and we have $\mathcal{M}'_\theta(\lambda_1) = \mathcal{M}'_\theta(\lambda_2) = \emptyset$. By the computation above, $\mathcal{K}_\theta(\lambda_1) = \mathcal{K}_\theta(\lambda_2) = h(\Omega)/2 - 1$. This completes the proof. \square

We close this section (Sect. 4) by giving two corollaries.

Corollary 4.34 (Local–global inequalities) *Assume that a set $\{\beta_1, \beta_2\} \subseteq \Phi$ is A_1^2 . Recall the notation $\mathcal{N}_0 = \mathcal{N}_0(\{\beta_1, \beta_2\})$ in Definition 3.6. For each $\Psi \in \mathcal{N}_0$, denote by $c_{\max}(\Psi)$ the largest coefficient of the highest root of the subsystem Ψ . Then,*

- (a) $\sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1) \geq m_2 - 1$,
- (b) $\sum_{\Psi \in \mathcal{N}_0} c_{\max}(\Psi) \geq c_{\max}$.

The equality in (a) [resp., (b)] occurs if and only if either (i) $\ell \leq 4$, or (ii) $\ell \geq 5$ and $\{\beta_1, \beta_2\}$ is not RO.

Proof Since $\ell \geq 3$, $m_2 = c_{\max} + 1$ by Theorem 4.14 and Corollary 4.16. Thus, (a) and (b) are essentially equivalent. The left-hand sides of these inequalities are equal to \mathcal{K}_0 by Remark 3.14. By Theorem 4.6 and Remark 3.14,

$$\mathcal{K}_0 = \begin{cases} h/2 - 1 & \text{if } \{\beta_1, \beta_2\} \text{ is RO,} \\ m_2 - 1 & \text{if } \{\beta_1, \beta_2\} \text{ is not RO.} \end{cases}$$

Thus, the inequalities follow. If $\ell = 3$, the equalities always occur since $h/2 = m_2$. If $\ell = 4$, we need only care about the case $\{\beta_1, \beta_2\}$ is both A_1^2 and RO. This condition forces $\Phi = D_4$ by Corollary 4.25. Again, we have $h/2 = m_2$. So, the equalities always occur if $\ell \leq 4$. If $\ell \geq 5$, by Lemma 4.21, $h/2 > m_2$. Thus, the equalities occur if $\{\beta_1, \beta_2\}$ is not RO. \square

Corollary 4.35 *Let Ω be an irreducible component of θ^\perp . If Φ is simply-laced, then the Coxeter number of Ω is given by*

$$h(\Omega) = \begin{cases} 2 & \text{if } \text{rank}(\Omega) = 1, \\ h - 2m_2 + 2 & \text{if } \text{rank}(\Omega) \geq 2. \end{cases}$$

Proof We need only to give a proof for the second line. Note that by Lemma 4.23, there exists at most one irreducible component Ω of θ^\perp satisfying $\text{rank}(\Omega) \geq 2$. For every $\beta \in \Omega$, $\{\theta, \beta\}$ is A_1^2 since Φ is simply laced. Moreover, $\{\theta, \beta\}$ is not RO by the reason of rank. With the notations in Proof of Theorem 4.28, $\mathcal{M}'_\theta(\beta) = \emptyset$. It completes the proof. \square

5 Proof of Theorem 1.6

Theorem 1.6 follows from Theorem 5.1 and Example 5.2.

Theorem 5.1 *Assume that $X = H_1 \cap X_2$ is A_1^2 but not RO. Let $\{\varphi_1, \dots, \varphi_\ell\}$ be a basis for $D(\mathcal{A})$ with $\deg \varphi_j = m_j$ ($1 \leq j \leq \ell$). Then, $\{\varphi_1^X, \dots, \varphi_{\ell-2}^X\}$ is a basis for $D(\mathcal{A}^X)$.*

Proof The statement is checked by a case-by-case method when $\ell \leq 4$. If $\ell \geq 5$, then $m_{\ell-1}$ appears exactly once in $\exp(\mathcal{A})$ (Corollary 4.20). By Remark 2.15, \mathcal{A}^{H_1} is free and $\{\varphi_1^{H_1}, \dots, \varphi_{\ell-1}^{H_1}\}$ is a basis for $D(\mathcal{A}^{H_1})$. By Theorem 4.6, $|\mathcal{A}^{H_1}| - |\mathcal{A}^X| = m_{\ell-1}$. By Proof of Theorem 4.4, $\mathcal{A}^{H_1} \setminus \{X\}$ is also free. Theorem 2.9 completes the proof. \square

Example 5.2 *Assume that $X \in L(\mathcal{A})$ is both A_1^2 and RO. By Remark 3.5, $\Phi = D_\ell$ with $\ell \geq 3$ ($D_3 = A_3$). Suppose that*

$$Q = \prod_{1 \leq i < j \leq \ell} (x_i - x_j) \prod_{1 \leq i < j \leq \ell} (x_i + x_j),$$

where $\{x_1, \dots, x_\ell\}$ is an orthonormal basis for V^* . Let $H_1 = \ker(x_1 + x_2)$, $H_2 = \ker(x_1 - x_2)$, and $X = H_1 \cap X_2$. Then, again by Remark 3.5, X is A_1^2 and RO. Define

$$\tau_i := \sum_{k=1}^{\ell} x_k^{2i-1} \partial_k \quad (1 \leq i \leq \ell - 1),$$

$$\eta := \sum_{k=1}^{\ell} \frac{x_1 \dots x_\ell}{x_k} \partial_k.$$

Then, it is known that $\tau_1, \dots, \tau_{\ell-1}, \eta$ form a basis for $D(\mathcal{A})$. Let

$$\varphi := \left(\prod_{k=3}^{\ell} (x_1^2 - x_k^2) \right) \partial_1 + \left(\prod_{k=3}^{\ell} (x_2^2 - x_k^2) \right) \partial_2.$$

Then, it is not hard to verify that $\varphi \in D(\mathcal{A} \setminus \{H_1\})$ and thus $(x_1 + x_2)\varphi \in D(\mathcal{A})$. By Saito’s criterion, we may show that $\tau_1, \dots, \tau_{\ell-2}, \eta, (x_1 + x_2)\varphi$ also form a basis for $D(\mathcal{A})$, and $\tau_1, \dots, \tau_{\ell-2}, \eta, \varphi$ form a basis for $D(\mathcal{A} \setminus \{H_1\})$. Therefore, $\{\tau_1^{H_1}, \dots, \tau_{\ell-2}^{H_1}, \eta^{H_1}\}$ is a basis for $D(\mathcal{A}^{H_1})$. This basis may have two elements having the same degree, for example, $\deg \eta = \deg \tau_{\ell/2} = \ell - 1$ when ℓ is an even number. However, it is easy to check that for every case $\eta^X = 0$. Hence, $\{\tau_1^X, \dots, \tau_{\ell-2}^X\}$ is a basis for $D(\mathcal{A}^X)$ and $\exp(\mathcal{A}^X) = \{1, 3, 5, \dots, 2\ell - 5\}$ as predicted in Theorem 2.9. This is also consistent with the fact that the \mathcal{A}^X above is exactly the Weyl arrangement of type $B_{\ell-2}$.

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6 Appendix: A bijective proof of Theorem 4.30

In this section, we give an alternative and bijective proof for Theorem 4.30. In comparison with the proof used in Sect. 4.3, we do not use here the classification of rank-4 irreducible root systems.

Definition 6.1 Let $\{\gamma_1, \gamma_2\}$ be an A_1^2 set. Let $\Psi \in \mathcal{N}_0 = \mathcal{N}_0(\{\gamma_1, \gamma_2\})$ (see notation in Definition 3.6). For any $\alpha = \sum_{\mu \in \Delta(\Psi)} c_\mu \mu \in \Psi^+ = \Phi^+ \cap \Psi$ ($\Delta(\Psi)$ is the base of Ψ associated with Ψ^+), its local height in Ψ is defined by $\text{ht}_\Psi(\alpha) := \sum_{\mu \in \Delta(\Psi)} c_\mu$.

Lemma 6.2 With the notations and assumptions in Lemma 4.29, Definition 6.1 and Remark 3.14(ii), we have $\text{ht}_\Psi(\gamma_i) = m_2(\Psi)$ for $i \in \{1, 2\}$.

Proof This follows from a direct verification on all rank-3 irreducible root systems. We will check only a (most) non-obvious case when $\Psi = C_3$, and $\gamma_1 = \mu_1 + \mu_2, \gamma_2 =$

Table 1 Ψ^+ when $\Psi = C_3$

		Height	
5	$2\mu_1 + 2\mu_2 + \mu_3$		
4	$\mu_1 + 2\mu_2 + \mu_3$		
3	$\mu_1 + \mu_2 + \mu_3$	$2\mu_2 + \mu_3$	
2	$\mu_1 + \mu_2$	$\mu_2 + \mu_3$	
1	μ_1	μ_2	μ_3

$2\mu_2 + \mu_3$ (see Table 1). Set $\alpha := 2\mu_1 + 2\mu_2 + \mu_3$. Since $\alpha = 2\gamma_1 + \mu_3 \in \Psi^+ \subseteq \Phi^+$, we have $\text{ht}(\alpha) > 2\text{ht}(\gamma_1) > m_\ell$, which is a contradiction. \square

A bijective proof of Theorem 4.30 By Remark 3.14, Theorem 4.30 is proved once we prove

$$\sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1) = m_2 - 1. \tag{6.1}$$

Recall the notation $\mathcal{U} = \{\theta_j \in \Phi^+ \mid \text{ht}(\theta_j) > m_{\ell-1}\}$, and $|\mathcal{U}| = m_2 - 1$. For each $\Psi \in \mathcal{N}_0$, set $\mathcal{U}_\Psi := \{\mu \in \Psi^+ \mid \text{ht}_\Psi(\mu) > \text{ht}_\Psi(\gamma_i)\}$ (by Lemma 6.2, this definition does not depend on the index i). Moreover, $|\mathcal{U}_\Psi| = m_2(\Psi) - 1$. Since $\mathcal{U}_\Psi \cap \mathcal{U}_{\Psi'} = \emptyset$ for $\Psi \neq \Psi'$, we have

$$\left| \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi \right| = \sum_{\Psi \in \mathcal{N}_0} |\mathcal{U}_\Psi| = \sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1).$$

Equality (6.1) will be proved once we prove $\mathcal{U} = \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi$. For any $\mu \in \mathcal{U}_\Psi$,

$$\mu - \gamma_i = \sum_{\mu \in \Delta(\Psi)} \mathbb{Z}_{\geq 0} \mu \subseteq \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha \quad (i \in \{1, 2\}).$$

Thus, $\text{ht}(\mu) > \text{ht}(\gamma_i) = m_{\ell-1}$ hence $\mu \in \mathcal{U}$. Therefore, $\mathcal{U} \supseteq \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi$. To prove the inclusion, it suffices to prove that for every $\theta_j \in \mathcal{U}$, the subsystem $\Gamma := \Phi \cap \text{span}\{\theta_j, \gamma_1, \gamma_2\}$ is an element of \mathcal{N}_0 . Obviously, Γ is of rank 3 since it contains the A_1^2 set $\{\gamma_1, \gamma_2\}$. The irreducibility of Γ follows from Proposition 4.19. \square

References

1. Abe, T., Barakat, M., Cuntz, M., Hoge, T., Terao, H.: The freeness of ideal subarrangements of Weyl arrangements. *J. Eur. Math. Soc.* **18**, 1339–1348 (2016)
2. Abe, T.: Plus-one generated and next to free arrangements of hyperplanes. *Int. Math. Res. Not.* (2019). <https://doi.org/10.1093/imrn/rnz099>
3. Abe, T.: Addition-deletion theorem for free hyperplane arrangements and combinatorics. arXiv preprint (2018). [arxiv:1811.03780](https://arxiv.org/abs/1811.03780)

4. Abe, T.: Deletion theorem and combinatorics of hyperplane arrangements. *Math. Ann.* **373**(1–2), 581–595 (2019)
5. Berman, S., Lee, Y.S., Moody, R.V.: The spectrum of a Coxeter transformation, affine Coxeter transformations, and the defect map. *J. Algebra* **121**, 339–357 (1989)
6. Bourbaki, N.: *Groupes et Algèbres de Lie*. Chapitres 4,5 et 6, Hermann, Paris (1968)
7. Carter, R.W.: *Simple Groups of Lie Type*. Wiley-Interscience, New York (1972)
8. Douglass, J.M.: The adjoint representation of a reductive group and hyperplane arrangements. *Represent. Theory* **3**, 444–456 (1999)
9. Green, R.M.: *Combinatorics of Minuscule Representations*. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2013)
10. Howlett, R.B., Rowley, P.J., Taylor, D.E.: On outer automorphism groups of Coxeter groups. *Manuscripta Math.* **93**(1), 499–513 (1997)
11. Kane, R.: *Reflection Groups and Invariant Theory*, 1st edn. Springer, New York (2001)
12. Kostant, B.: On the conjugacy of real Cartan subalgebras. I. *Proc. Nat. Acad. Sci. USA* **41**, 967–970 (1955)
13. Kostant, B.: The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.* **81**, 973–1032 (1959)
14. Krammer, D.: *The conjugacy problem for Coxeter groups*. Dissertation, Universiteit Utrecht (1994)
15. Loos, O., Neher, E.: *Locally Finite Root Systems*. *Memoirs of the American Mathematical Society* (2004)
16. Macdonald, I.G.: The Poincaré series of a Coxeter group. *Math. Ann.* **199**, 161–174 (1972)
17. Mücksch, P., Röhrle, G.: Accurate arrangements. arXiv preprint (2020). <https://arxiv.org/abs/2003.01435>
18. Malle, G., Testerman, D.: *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge Studies in Advanced Mathematics, vol. 133. Cambridge University Press, Cambridge (2011)
19. Orlik, P., Solomon, L.: Coxeter arrangements. *Proc. Symp. Pure Math. Amer. Math. Soc.* **40**(2), 269–292 (1983)
20. Orlik, P., Solomon, L., Terao, H.: On Coxeter arrangements and the Coxeter number. *Adv. Stud. Pure Math.* **8**, 461–77 (1986)
21. Orlik, P., Terao, H.: *Arrangements of Hyperplanes*. Grundlehren der Mathematischen Wissenschaften 300. Springer, Berlin (1992)
22. Orlik, P., Terao, H.: Coxeter arrangements are hereditarily free. *Tôhoku Math. J.* **52**, 369–383 (1993)
23. Saito, K.: On a linear structure of the quotient variety by a finite reflexion group. *Publ. Res. Inst. Math. Sci.* **29**, 535–579 (1993)
24. Steinberg, R.: Finite reflection groups. *Trans. Amer. Math. Soc.* **91**, 493–504 (1959)
25. Steinberg, R.: Torsion in reductive groups. *Adv. Math.* **15**, 63–92 (1975)
26. Terao, H.: Arrangements of hyperplanes and their freeness I, II. *J. Fac. Sci. Univ. Tokyo* **27**, 293–320 (1980)
27. Terao, H.: Generalized exponents of a free arrangement of hyperplanes and Shepherd–Todd–Brieskorn formula. *Invent. Math.* **63**(1), 159–179 (1981)
28. Tran, T.N.: The largest coefficient of the highest root and the second smallest exponent. *Graphs Comb.* (2020). [arxiv:1704.03775v2](https://arxiv.org/abs/1704.03775v2)