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A generalization of the quadratic cone of PG(3, q^n) and its relation with the affine set of the Lüneburg spread

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Abstract Using a variation of Seydewitz's method of projective generation of quadratic cones, we define an algebraic surface of PG$(3, q^n)$, called σ-cone, whose $\mathbb{F}_{q^n}$-rational points are the union of a line with a set $\mathcal{A}$ of q^{2n} points. If $q^n = 2^{2h+1}$, $h \geq 1$, and σ is the automorphism of $\mathbb{F}_{q^n}$ given by $x \mapsto x^{2^h}$, then the set $\mathcal{A}$ is the affine set of the Lüneburg spread of PG$(3, q^n)$. If $n = 2$ and σ is the involutory automorphism of $\mathbb{F}_{q^2}$, then a σ-cone is a subset of a Hermitian cone and the set $\mathcal{A}$ is the union of q non-degenerate Hermitian curves.

Keywords Luneburg spread · Quadratic cone · Hermitian curve

1 Introduction

Let A and B be two distinct points of a three-dimensional projective space. Let $\mathcal{S}_A$ be the star of lines through A, let $\mathcal{S}_B^*$ be the star of planes through B, and let Φ be a projectivity between $\mathcal{S}_A$ and $\mathcal{S}_B^*$. In 1848 F. Seydewitz proved that quadrics may be generated as the set of points of intersection of corresponding elements under Φ (see e.g., [10]).

If the line AB is mapped under Φ into a plane, say π_{AB}, containing AB and the lines through A of the plane π_{AB} are mapped into the planes through the line AB, then the set of points of intersection of corresponding elements under Φ is a quadratic cone. In this

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paper we define an algebraic surface of $\mathrm{PG}(3, q^n)$ by using a variation of Seydewitz's projective generation of quadratic cones by means of a suitable collineation instead of a projectivity.

Let A and B be two distinct points of $\mathrm{PG}(3, q^n)$, and let σ be an automorphism of $\mathbb{F}_{q^n}$ such that $Fix(\sigma) = \mathbb{F}_q$. Let Φ be a σ-collineation between the star of lines with center A and the star of planes with center B. Suppose that the line AB is mapped under Φ into a plane containing AB and that the lines through A of the plane π_{AB} are mapped into the planes through the line AB, then the set of points of intersection of corresponding elements under Φ is a σ*-cone* of $\mathrm{PG}(3, q^n)$ with *vertices* A and B. We will prove the following results:

Theorem 1.1 *Every* σ*-cone* $\mathcal{K}$ *of* $\mathrm{PG}(3, q^n)$ *is projectively equivalent to the set of* $\mathbb{F}_{q^n}$*-rational points of the algebraic surface with equation*

$$x_1{}^{\sigma+1} + x_2 x_3^{\sigma} + x_3 x_2^{\sigma} - x_4 x_2{}^{\sigma} = 0.$$

It has size $q^{2n} + q^n + 1$*, it is of type* $(0, 1, 2, q+1, q^n+1)_1$*, every* $(q+1)$*-secant line meets* $\mathcal{K}$ *in a subline over* $\mathbb{F}_q$*. Moreover* AB *is the unique line contained in* $\mathcal{K}$ *and* π_{AB} *is the unique plane that meets* $\mathcal{K}$ *exactly in* AB*.*

Theorem 1.2 *A* σ*-cone and a* σ'*-cone of* $\mathrm{PG}(3, q^n)$ *are* $P\Gamma L$*-equivalent if, and only if, either* $\sigma' = \sigma$ *or* $\sigma' = \sigma^{-1}$*.*

Theorem 1.3 *Let* $\mathcal{K}$ *be a* σ*-cone of* $\mathrm{PG}(3, 2^n)$ *with vertices A and B. The set* $\mathcal{K} \setminus AB$ *is a subset of* $\mathrm{PG}(3, 2^n) \setminus \pi_{AB}$ *of size* 2^{2n}*. The lines joining any two points of* $\mathcal{K} \setminus AB$ *are disjoint from a translation hyperoval* $\mathcal{O}_\infty$ *of the plane* π_{AB}*, projectively equivalent to the set* $\{(0, t, t^{\sigma^{-2}}, 1) : t \in \mathbb{F}_{2^n}\} \cup \{A, B\}$*, .*

Corollary 1.4 *If* $n = 2h+1$ *and* σ *is the automorphism of* $\mathbb{F}_{2^n}$ *given by* $\sigma : x \mapsto x^{2^h}$*, then* $\mathcal{O}_\infty$ *is the union of a non-degenerate conic with its nucleus. Hence the set* $\mathcal{K} \setminus AB$ *is the affine set of the Lüneburg spread of* $\mathrm{PG}(3, q^n)$*.*

2 Preliminary results

Let τ be a symplectic polarity of $\mathrm{PG}(3, q^n)$. A line ℓ is called *self-polar* if $\tau(\ell) = \ell$. The *symplectic polar space* $W_3(q^n)$ is the classical generalized quadrangle whose pointset is the set of points of $\mathrm{PG}(3, q^n)$ and whose lineset is the set of the self-polar lines with respect to τ. A *spread* $\mathcal{S}$ of $\mathrm{PG}(3, q^n)$, i.e., a set of $q^{2n} + 1$ pairwise skew lines, is called *symplectic* if all lines of $\mathcal{S}$ belong to a symplectic polar space $W_3(q^n)$. Via the Plücker embedding, to a spread $\mathcal{S}$ of $\mathrm{PG}(3, q^n)$ there corresponds an *ovoid* $\mathcal{O}(\mathcal{S})$ of $Q^+(5, q^n)$, i.e., a set of $q^{2n} + 1$ points pairwise non-collinear on $Q^+(5, q^n)$. The spread $\mathcal{S}$ is symplectic if and only if the ovoid $\mathcal{O}(\mathcal{S})$ is contained in a $Q(4, q^n)$. In the what follows we will denote by $\perp$ the polarity of $Q^+(5, q^n)$.

A *polarity* of $W_3(q^n)$ is a bijection of order 2 interchanging points and lines. An *absolute* line of a polarity of $W_3(q^n)$ is a line of $W_3(q^n)$ containing its polar point.

In 1965, H. Lüneburg in [9] proved that if $q^n = 2^{2h+1}$, $h \geq 1$, then the set of absolute lines of a polarity of $W_3(q^n)$ is a symplectic spread, now called the *Lüneburg spread* of $\mathrm{PG}(3, q^n)$.

Let Σ_∞ be a hyperplane of $\mathrm{PG}(4, q^n)$ and let $Q^+(3, q^n)$ be a hyperbolic quadric of Σ_∞. A set $\mathcal{A}$ of q^{2n} points of $\mathrm{PG}(4, q^n) \setminus \Sigma_\infty$ s. t. that the line joining any two of them is disjoint from $Q^+(3, q^n)$ called an *affine set* of $\mathrm{PG}(4, q^n) \setminus \Sigma_\infty$.

In [6] and also in [7] the following result has been proved.

Theorem 2.1 *Let $\mathcal{O}$ be an ovoid of $Q^+(5, q^n)$, let x be a point of $\mathcal{O}$ and let Ω be a hyperplane of* $\mathrm{PG}(5, q^n)$ *not containing x. The set $\mathcal{A}_x(\mathcal{O})$ obtained by projecting $\mathcal{O}$ from the point x onto Ω is an affine set of $\Omega \setminus x^\perp$. Conversely, if $\mathcal{A}$ is an affine set of $\Omega \setminus x^\perp$, then the set $\mathcal{O} = \{xy \cap Q^+(5, q^n) : y \in \mathcal{A}\}$ is an ovoid of $Q^+(5, q^n)$.*

If $\mathcal{S}$ is a spread of $\mathrm{PG}(3, q^n)$ and ℓ is a line of $\mathcal{S}$, then we will denote by $\mathcal{A}_\ell(\mathcal{S})$ the affine set arising from $\mathcal{S}$ with respect to ℓ.

If $\mathcal{S}$ is a symplectic spread, then $\mathcal{A}_\ell(\mathcal{S})$ is a set of q^{2n} points of an affine space $\mathrm{PG}(3, q^n) \setminus \pi_\infty$ such that the line joining any two of them is disjoint from a given non-degenerate conic $\mathcal{C}$ of π_∞.

The affine set arising from the Lüneburg spread has been studied by A. Cossidente, G. Marino and O. Polverino in [1], where the following result has been obtained.

Theorem 2.2 *The affine set $\mathcal{A}$ of the Lüneburg spread of* $\mathrm{PG}(3, q^n)$, $q^n = 2^{2h+1}$, *is the union of q^n q^n-arcs, each of which can be completed to a translation hyperoval. The directions of $\mathcal{A}$ on π_∞ are the complement of a conic and its nucleus. Moreover, the planes containing the q^n translation hyperovals, together with π_∞, form a pencil of planes with axis a line that is also the axis of all the translation hyperovals.*

In [5] a C_F^m*-set* (*degenerate* C_F^m*-set*) of $\mathrm{PG}(2, q^n)$ is defined as the set of points of intersection of corresponding lines under a collineation between two pencils of lines with vertices two distinct points A and B mapping the line AB not into itself (into itself). Every C_F^m-set of $\mathrm{PG}(2, q^n)$ is the union of $\{A, B\}$ with $q - 1$ scattered $\mathbb{F}_q$-linear sets of rank n. Every degenerate C_F^m-set of $\mathrm{PG}(2, q^n)$ is the union of the line AB with a scattered $\mathbb{F}_q$-linear set of rank $n + 1$ meeting the line AB in a $\mathbb{F}_q$-linear set of pseudoregulus type with transversal points A and B and vice versa.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two Baer subpencils of lines of $\mathrm{PG}(2, q^2)$ with vertices two distinct points V and V', respectively. If the line VV' belongs to $\mathcal{P}$ and does not belong to $\mathcal{P}'$ (or vice versa), then the set of points of intersection between the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$ is called in a *K-set* of $\mathrm{PG}(2, q^2)$ (see [3]). If the line VV' belongs neither to $\mathcal{P}$ nor to $\mathcal{P}'$, then the set of points of intersection between the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$ is called an *H-set* of $\mathrm{PG}(2, q^2)$ (see [3]). Finally let $\mathcal{H}$ be a non-degenerate Hermitian curve of $\mathrm{PG}(2, q^2)$ containing the point V, with the tangent line to $\mathcal{H}$ at V being a line of $\mathcal{P}$. The set of points of intersection between $\mathcal{H}$ and the lines of $\mathcal{P}$ is called a *Γ-set* (see [4]).

3 σ-cones

We start by proving Theorem 1.1. Let $\mathcal{K}$ be a σ-cone of $\mathrm{PG}(3, q^n)$ with vertices A and B. Wlog we may assume that $A = (0, 0, 0, 1)$, $B = (0, 0, 1, 1)$ and that Φ

maps the line joining A and $(y_1, y_1, y_3, 0)$ onto the plane through B with equation $b_1x_1 + b_2x_2 + b_3x_3 - b_3x_4 = 0$, where

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} {y_1}^\sigma \\ {y_2}^\sigma \\ {y_3}^\sigma \end{pmatrix},$$

and (a_{ij}) is a non-singular matrix over $\mathbb{F}_{q^n}$.

We may assume that the line joining A and $(1, 0, 0, 0)$ is mapped onto the plane with equation $x_1 = 0$, that the line joining A and $(0, 1, 0, 0)$ is mapped onto the plane with equation $x_3 - x_4 = 0$ and that the line joining A and $(0, 0, 1, 0)$ is mapped onto the plane π_{AB} with equation $x_2 = 0$. Under these assumptions it follows readily that $a_{21} = a_{31} = a_{12} = a_{22} = a_{13} = a_{33} = 0$, and that $a_{11}a_{23}a_{32} \neq 0$.

A line through A may be assumed as the line joining A with a point $(y_1, y_2, y_3, 0)$. The unique line ℓ through A contained in the plane $\Phi(\ell)$ is the line AB. If ℓ is different from AB, then $\ell \cap \Phi(\ell)$ is a point with homogeneous coordinates

$$(a_{32}y_2^\sigma y_1, a_{32}y_2^{\sigma+1}, a_{32}y_2^\sigma y_3, a_{11}y_1^{\sigma+1} + a_{23}y_2y_3^\sigma + a_{32}y_3y_2^\sigma),$$

and this gives a parametric representation of the set $\mathcal{K}$. Therefore $\mathcal{K}$ has an equation of the form

$$ax_1^{\sigma+1} + bx_2x_3^\sigma + x_3x_2^\sigma - x_4x_2^\sigma = 0,$$

with $a = \frac{a_{11}}{a_{32}} \neq 0$ and $b = \frac{a_{23}}{a_{32}} \neq 0$.

Assuming that the point $(0, 1, 1, 2)$ belongs to $\mathcal{K}$, it follows that $b = 1$, and assuming that the point $(1, 1, 1, 3)$ belongs to $\mathcal{K}$, it follows that $a = 1$. Hence the equation of a σ-cone assumes the canonical form

$$x_1^{\sigma+1} + x_2x_3^\sigma + x_3x_2^\sigma - x_4x_2^\sigma = 0.$$

Let $\mathcal{P}_A$ be the pencil of lines mapped under Φ onto the pencil of planes with axis the line AB. Every line of $\mathcal{P}_A$, different from AB, intersects $\mathcal{K}$ exactly in A. Every line ℓ through A, not in π_{AB}, intersects $\mathcal{K}$ in two points, namely A and $\ell \cap \Phi(\ell)$. Similarly, every line through B, different from AB, either intersects $\mathcal{K}$ exactly in B or intersects $\mathcal{K}$ in two distinct points. It follows that $\mathcal{K}$ has $q^{2n} + q^n + 1$ points. Let ℓ be a line of $\mathrm{PG}(3, q^n)$ neither through A nor through B. If there exist a point R on ℓ such that $\ell \subseteq \Phi(AR)$, then the axis of the pencil of planes $\{\Phi(AP) : P \in \ell\}$ intersects ℓ in a point R' possibly coincident to R. It follows that for every point $P \in \ell$, distinct from R and distinct from R', the plane $\Phi(AP)$ cannot contain P, so $\mathcal{K} \cap \ell = \{R, R'\}$. If the line ℓ is not contained in any plane of the pencil $\{\Phi(AP) : P \in \ell\}$, then Φ induces a σ-collineation of the line ℓ into itself defined by

$$\phi_\ell : P \in \ell \longrightarrow \Phi(AP) \cap \ell \in \ell.$$

The points of the line ℓ which belong to $\mathcal{K}$ are exactly all the fixed points of ϕ_ℓ. The system of fixed points of ϕ_ℓ is one of the following (see [2]): the empty set, a single point, a pair of distinct points or a subline formed by all the points of ℓ coordinatized over the subfield $Fix(\sigma) = \mathbb{F}_q$, with respect to a suitable basis of ℓ. In the last case this set is an $\mathbb{F}_q$-subline of the line ℓ. From these arguments it follows that every line of $\mathrm{PG}(3, q^n)$, neither through A nor through B, intersects $\mathcal{K}$ in 0, 1, 2 or $q+1$ points.

The Proof of Theorem 1.1 is complete.

Suppose that σ is the automorphism of $\mathbb{F}_{q^n}$ mapping x into x^{q^m}, $(m, n) = 1$.

Theorem 3.1 *Let $\mathcal{K}$ be a σ-cone of* $\mathrm{PG}(3, q^n)$ *with vertices A and B. Every plane π through the line AB, different from π_{AB}, intersects $\mathcal{K}$ in a degenerate C_F^m-set, say $\mathcal{K}_\pi$, with vertices A and B. Moreover the set of the directions of $\mathcal{K}_\pi \setminus AB$ on the line AB is independent of π.*

Proof Suppose that $\mathcal{K}$ is defined by a collineation Φ between $\mathcal{S}_A$ and $\mathcal{S}_B^*$ and suppose that $\mathcal{K}$ has canonical equation $x_1^{\sigma+1} + x_2 x_3^\sigma + x_3 x_2^\sigma - x_4 x_2^\sigma = 0$. Let $\mathcal{P}_A^\pi$ and $\mathcal{P}_B^\pi$ be the pencils of lines contained in π with vertices A and B, respectively. The collineation Φ induces a collineation Φ_π given by

$$\Phi_\pi : \ell \in \mathcal{P}_A^\pi \longrightarrow \Phi(\ell) \cap \pi \in \mathcal{P}_B^\pi.$$

The σ-cone $\mathcal{K}$ meets the plane π exactly in the set of points of intersection of corresponding lines under Φ_π. Since Φ_π maps the line AB into itself, it follows that $\mathcal{K} \cap \pi$ is a degenerate C_F^m-set of the plane π. Consider the plane through the line AB with equation $ax_1 = bx_2$, with $a \neq 0$, and consider in this plane the points of $\mathcal{K}$ given by $L = (b, a, 0, b(a^\sigma)^{-1})$, $M = (b, a, a, 2a + b(a^\sigma)^{-1})$. The direction of the line LM on the line AB is independent of (a, b) and it is given by $D = (0, 0, 1, 2)$. The set of the directions of $\mathcal{K}_\pi \setminus AB$ on the line AB is a linear set of pseudoregulus type with transversal points A and B (see [5,8]). Since the linear sets of pseudoregulus type with transversal points A and B partition $AB \setminus \{A, B\}$, it follows that the set of the directions of $\mathcal{C}_\pi \setminus AB$ is independent of π. □

Remark 3.2 Let $\mathcal{K}$ be a σ-cone of $\mathrm{PG}(3, q^n)$ with vertices A and B. From the previous theorem it follows that A and B are the unique points of $\mathcal{K}$ not incident with $(q+1)$-secant lines to $\mathcal{K}$.

Let σ' be the automorphism of $\mathbb{F}_{q^n}$ mapping x into $x^{q^{m'}}$, with $(m', n) = 1$.

Lemma 3.3 *Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be a degenerate C_F^m-set (resp. $C_F^{m'}$-set) of* $\mathrm{PG}(2, q^n)$. *The sets $\mathcal{S}$ and $\mathcal{S}'$ are $P\Gamma L$-equivalent if, and only if, either $\sigma' = \sigma$ or $\sigma' = \sigma^{-1}$*

Proof Let Φ (resp. Φ') be a σ-collineation (resp. σ'-collineation) defining $\mathcal{S}$ (resp. $\mathcal{S}'$). We may assume that both $\mathcal{S}$ and $\mathcal{S}'$ have the same vertices A and B. Observe that if $\mathcal{S}$ is a degenerate C_F^m-set of $\mathrm{PG}(2, q^n)$ with vertices A and B defined by a σ-collineation Φ, then $\mathcal{S}$ is also a C_F^{n-m}-set with vertices B and A, generated by the σ^{-1}-collineation Φ^{-1}.

Let f be a collineation of $\mathrm{PG}(2, q^n)$ mapping $\mathcal{S}$ into $\mathcal{S}'$. Since A and B are the unique points of both $\mathcal{S}$ and $\mathcal{S}'$ not incident with $(q+1)$-secant lines, it follows that

f stabilizes the set $\{A, B\}$. First assume that $f(A) = B$. For every line ℓ through the point A we have that $f(\Phi(\ell)) = (\Phi')^{-1}(f(\ell))$. As Φ and $(\Phi')^{-1}$ are collineations with accompanying automorphism $x \mapsto x^{q^m}$ and $x \mapsto x^{q^{n-m'}}$, respectively, we have that $m = n - m'$, and so $m = m' = \frac{n}{2}$. Since $(m, n) = 1$ it follows that $n = 2, m = 1$. Next suppose that $f(A) = A$, $f(B) = B$. For every line ℓ through A we have that $f(\Phi(\ell)) = \Phi'(f(\ell))$, hence $\sigma = \sigma'$. □

Proof of Theorem 1.2 Let Φ (resp. Φ') be a σ-collineation (resp. σ'-collineation) defining $\mathcal{K}$ (resp. $\mathcal{K}'$). We may assume that both $\mathcal{K}$ and $\mathcal{K}'$ have the same vertices A and B and the same plane π_{AB}. Observe that if $\mathcal{K}$ is defined by a σ-collineation Φ, then $\mathcal{K}$ is also a σ^{-1}-cone with vertices B and A generated by the σ^{-1}-collineation Φ^{-1}.

Let f be a collineation of $\mathrm{PG}(3, q^n)$ mapping $\mathcal{K}$ into $\mathcal{K}'$. Since A and B are the unique points of both $\mathcal{K}$ and $\mathcal{K}'$ through which do not pass $(q+1)$-secant lines, it follows that f stabilizes both the set $\{A, B\}$ and the pencil of planes with axes the line AB. Let π be a plane through the line AB, different from π_{AB}. The set $\pi \cap \mathcal{K}$ is a degenerate C_F^m-set of π and it is mapped under f into the set $f(\pi) \cap \mathcal{K}'$, that is a degenerate $C_F^{m'}$-set if $f(\pi)$. From Lemma 3.3 we have that either $\sigma' = \sigma$ or $\sigma' = \sigma^{-1}$. □

4 Affine sets of the Lüneburg spread

In this section we will prove Theorem 1.3 and Corollary 1.4. Let $\mathcal{K}$ be the σ-cone with vertices $A = (0, 0, 0, 1)$ and $B = (0, 0, 1, 1)$ with canonical equation

$$x_1{}^{\sigma+1} + x_2 x_3^{\sigma} + x_3 x_2^{\sigma} - x_4 x_2{}^{\sigma} = 0.$$

Consider the projectivity given by

$$\begin{pmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ x_4{}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

that maps $\mathcal{K}$ into the σ-cone $\mathcal{K}'$ with vertices $A' = (0, 0, 1, 0)$ and $B' = (0, 1, 0, 0)$ and with equation

$$x_4{}^{\sigma+1} + x_1 x_2^{\sigma} - x_3 x_1{}^{\sigma} = 0.$$

This projectivity maps the plane π_{AB} with equation $x_2 = 0$ onto the plane $\pi_{A'B'}$ with equation $x_1 = 0$. The set $\mathcal{A}' = \mathcal{K}' \setminus \pi_{A'B'}$ is given by $\{(1, x, x^{\sigma} + y^{\sigma+1}, y) : x, y \in \mathbb{F}_{q^n}\}$. Arguing as in Proposition 5.2 in [1], we obtain that the set of directions determined by $\mathcal{A}'$ onto the plane $\pi_{A'B'}$ cover all the points of $\pi_{A'B'}$ but the points of a hyperoval $\mathcal{C}$ given by $\mathcal{C} = \{(0, x, x^{\sigma^{-2}}, 1) : x \in \mathbb{F}_{q^n}^*\}$. Note that if $q = 2^{2h+1}$ and

$\sigma : x \mapsto x^{2^h}$, then the hyperoval $\mathcal{C}$ is a hyperconic and hence both Theorem 1.3 and Corollary 1.4 are proved.

5 σ-cones in PG(3, q^2)

In this section we will study a σ-cone in PG$(3, q^2)$ with $\sigma : x \mapsto x^q$, that will be called a *q-cone*.

Theorem 5.1 *Every q-cone of* PG$(3, q^2)$ *is the union of q non-degenerate Hermitian curves with a common point with the line AB, such that the planes containing the q Hermitian curves, together with the plane π_{AB}, form a Baer subpencil of planes.*

Proof Let $\mathcal{K}$ be a q-cone. Wlog we may assume that $\mathcal{K}$ has canonical equation $x_1^{q+1} + x_2x_3^q + x_3x_2^q - x_4x_2^q = 0$. Let $\mathcal{F}$ be the pencil formed by all the planes with equation $x_4 = hx_2, h \in \mathbb{F}_{q^2}$, together with the plane π_{AB} with equation $x_2 = 0$. The intersection between π_{AB} and $\mathcal{K}$ is the line AB. The intersection between the plane with equation $x_4 = hx_2$ with $\mathcal{K}$ is a set $\mathcal{K}_h$ with equations $x_4 = hx_2, x_1^{q+1} - hx_2^{q+1} + x_2x_3^q + x_3x_2^q = 0$. The set $\mathcal{K}_h$ is a non-degenerate Hermitian curve of the plane with equation $x_4 = hx_2$ if, and only if, $h \in \mathbb{F}_q$. Therefore $\mathcal{K}$ intersects each of the q planes of $\mathcal{F}$, different from π_{AB}, in a non-degenerate Hermitian curve and intersects π_{AB} in the line AB. Observe that the set of planes with equations $x_4 = hx_2$, $h \in \mathbb{F}_q$, together with the plane π_{AB}, form a Baer subpencil of planes with axis the line ℓ with equations $x_2 = x_4 = 0$, and observe that every non-degenerate Hermitian curve $\mathcal{K}_h$, $h \in \mathbb{F}_q$, contains the point $V = (0, 0, 1, 0)$. Finally, the set $\mathcal{K}$ contains the set

$$\mathcal{K}' = \bigcup_{h \in \mathbb{F}_q} \mathcal{K}_h \cup (\mathcal{K} \cap \pi_{AB}),$$

and since $|\mathcal{K}'| = q^4 + q^2 + 1 = |\mathcal{K}|$, it follows that $\mathcal{K} = \mathcal{K}'$. □

Theorem 5.2 *Every q-cone of* PG$(3, q^2)$ *is contained in a Hermitian cone.*

Proof Let $\mathcal{K}$ be a q-cone of PG$(3, q^2)$. Wlog we may assume that $\mathcal{K}$ has canonical equation $x_1^{q+1} + x_2x_3^q + x_3x_2^q - x_4x_2^q = 0$. It is known that the polynomial $x^q + x + 1$ has q roots over $\mathbb{F}_{q^2}$. Hence there exists an element γ of $\mathbb{F}_{q^2}$ satisfying the condition $\gamma^q + \gamma + 1 = 0$. It follows that $\mathcal{K}$ is contained in the Hermitian cone with equation $x_1^{q+1} + x_2x_3^q + x_3x_2^q + \sigma x_2x_4^q + \sigma^q x_4x_2^q = 0$ whose vertex is the point $V' = (0, 0, -\sigma, 1)$. □

Theorem 5.3 *Every plane of* PG$(3, q^2)$ *intersects a q-cone in one of the following: a point, a Baer subline, a non-degenerate Hermitian curve, a line, a (possibly degenerate) $\mathcal{C}_F^1$-set, a K-set, a Γ-set.*

Proof Let π be a plane of PG$(3, q^2)$ and let $\mathcal{K}$ be a q-cone with vertices A and B, contained in a Hermitian cone $\mathcal{C}$. The set $\mathcal{K}$ is contained in a Baer subpencil of planes $\mathcal{P}$ with axis a line ℓ. If π is a plane of $\mathcal{P}$, then π intersects $\mathcal{K}$ either in a non-degenerate Hermitian curve or in the line AB. If π belongs to the pencil of planes with axis ℓ and does not belong to $\mathcal{P}$, then $\pi \cap \mathcal{K}$ is a point. If π is a plane not containing the line ℓ, then $\pi \cap \mathcal{K} = \mathcal{P} \cap \pi \cap \mathcal{C}$ and so $\pi \cap \mathcal{K}$ is the intersection between the Hermitian curve $\pi \cap \mathcal{C}$ (possibly degenerate) with the degenerate Hermitian curve of $\pi \cap \mathcal{P}$. In [4] the intersection between two distinct, possibly degenerate, Hermitian curves has been studied and it has been proved that, if one of the two Hermitian curves is degenerate, then their intersection is one of the following: a point, a Baer subline, a (possibly degenerate) $\mathcal{C}_F^1$-set, an H-set, a K-set, a Γ-set, a line, a pair of distinct lines. It follows that if $\pi \cap \mathcal{C} \neq \pi \cap \mathcal{P}$, then $\pi \cap \mathcal{K}$ is one of the intersection configurations described above. A pair of distinct lines cannot occur since $\mathcal{K}$ contains a unique line. Moreover there are no planes meeting $\mathcal{K}$ in an H-set. Indeed a plane of the pencil $\mathcal{F}$ in Theorem 5.1 meets $\mathcal{K}$ either in a point or a line or a non-degenerate Hermitian curve. A plane π not in $\mathcal{F}$ meets each of the Hermitian curves $\mathcal{K}_h$, $h \in \mathbb{F}_q$ either in a point or in a Baer subline and meets the line AB in a point. It follows that $|\mathcal{K} \cap \pi| \leq q(q+1)+1$, so $\pi \cap \mathcal{K}$ cannot be an H-set.

Finally, if $\pi \cap \mathcal{C} = \pi \cap \mathcal{P}$, then $\pi \cap \mathcal{K}$ is a Baer subpencil of lines, which is not possible since $\mathcal{K}$ does not contain Baer subpencils of lines. □

Theorem 5.4 *Let π be a plane of* $\mathrm{PG}(3, q^2)$ *and let $\mathcal{H}$ be a non-degenerate Hermitian curve contained in π. Let A and B be two distinct points not in π such that $AB \cap \pi$ is a point P of $\mathcal{H}$. There exists a unique q-cone containing $\mathcal{H}$.*

Proof Let u be the unitary polarity associated with $\mathcal{H}$ and let Φ be the map sending every line ℓ of $\mathrm{PG}(3, q^2)$ through A onto the plane $\langle u(\ell \cap \pi), B\rangle$ spanned by the line $u(\ell \cap \pi)$ and the point B. The map Φ is a σ-collineation mapping the line AB onto the plane $\langle B, u(P)\rangle$. Let $\mathcal{K}$ be the set of points of intersection of corresponding elements under Φ. Since the points of the line $u(P)$ are mapped under u onto the lines through P contained in π, it follows that the lines on A contained in $\langle A, u(P)\rangle$ are mapped under Φ onto the planes containing AB. Hence $\pi_{AB} = \langle A, u(P)\rangle = \Phi(AB)$ and so $\mathcal{K}$ is a q-cone of $\mathrm{PG}(3, q^2)$. It is clear that $\mathcal{K} \cap \pi = \mathcal{H}$. We will now prove the uniqueness of $\mathcal{K}$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two q-cones of $\mathrm{PG}(3, q^2)$ both with vertices A and B generated by two σ-collineation Φ_1 and Φ_2 such that $\mathcal{K}_1 \cap \pi = \mathcal{K}_2 \cap \pi = \mathcal{H}$. The maps Φ_1 and Φ_2 induce on π two correlations u_1 and u_2 both with $\mathcal{H}$ as set of absolute points. Hence $u_1 = u_2 = u$. So $\mathcal{K}_1 = \mathcal{K}_2$. □

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