A generalization of the alcove model and its applications

Cristian Lenart · Arthur Lubovsky

Abstract The alcove model of the first author and Postnikov uniformly describes highest weight crystals of semisimple Lie algebras. We construct a generalization, called the quantum alcove model. In joint work of the first author with Naito, Sagaki, Schilling, and Shimozono, this was shown to uniformly describe tensor products of column shape Kirillov–Reshetikhin crystals in all untwisted affine types; moreover, an efficient formula for the corresponding energy function is available. In the second part of this paper, we specialize the quantum alcove model to types A and C. We give explicit affine crystal isomorphisms from the specialized quantum alcove model to the corresponding tensor products of column shape Kirillov–Reshetikhin crystals, which are realized in terms of Kashiwara–Nakashima columns.

Keywords Kirillov–Reshetikhin crystals · Energy function · Alcove model · Quantum Bruhat graph · Kashiwara–Nakashima columns

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1 Introduction

Kashiwara's *crystals* [11] are colored directed graphs encoding the structure of certain bases (called crystal bases) for certain representations of quantum groups $U_q(\mathfrak{g})$ as q goes to zero. The first author and Postnikov [23,24] defined the so-called *alcove*

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model for highest weight crystals associated to a semisimple Lie algebra 𝔤 (in fact, the model was defined more generally, for symmetrizable Kac–Moody algebras 𝔤). A related model is the one of Gaussent–Littelmann, based on *LS-galleries* [7]. Both models are discrete counterparts of the celebrated *Littelmann path model* [26,27].

In this paper, we construct a generalization of the alcove model, which we call the *quantum alcove model*, as it is based on enumerating paths in the so-called *quantum Bruhat graph* of the corresponding finite Weyl group. This graph originates in the quantum cohomology theory for flag varieties [6], and was first studied in [2]. The path enumeration is determined by the choice of a certain sequence of alcoves (an alcove path or, equivalently, a λ-chain of roots), like in the classical alcove model. If we restrict to paths in the Hasse diagram of the Bruhat order, we recover the classical alcove model. The mentioned paths in the quantum Bruhat graph first appeared in [18], where they index the terms in the specialization $t = 0$ of the Ram–Yip formula [30] for *Macdonald polynomials* $P_\lambda(X; q, t)$. We construct combinatorial crystal operators on the mentioned paths, and prove various properties of them.

The main application [21,22] is that the new model uniformly describes tensor products of column shape *Kirillov–Reshetikhin (KR) crystals* [13], for all untwisted affine types. (KR crystals correspond to certain finite-dimensional representations of affine algebras.) More precisely, the model realizes the crystal operators on the mentioned tensor product, and also gives an efficient formula (based on the so-called height statistic) for the corresponding *energy function* [8]. (The energy can be viewed as an affine grading on a tensor product of KR crystals [28,32].) This result, combined with the Ram–Yip formula for Macdonald polynomials [30], implies that the graded character of a tensor product of column shape KR modules (the grading being by the energy function) concides with the corresponding Macdonald polynomial specialized at $t = 0$ [22].

In the second part of this paper, we specialize the quantum alcove model to types *A* and *C*, and prove that the bijections constructed in [18], from the objects of the specialized quantum alcove model to the tensor products of the corresponding *Kashiwara–Nakashima (KN) columns* [12], are affine crystal isomorphisms. (A column shape KR crystal is realized by a KN column in these cases.) Note that this result has no overlap with the type-independent result in [21,22], because the λ-chains on which the quantum alcove model is based in the two cases are different. Moreover, note that having such explicit bijections to models based on diagram fillings, which are known to be crystal isomorphisms, is important for the following reason. It is often easier to define extra structure on the quantum alcove model, which describes additional structure of the crystal, and only then translate it to the models based on fillings via the bijection. One such example was mentioned above, in connection with the energy function; the height statistic in the quantum alcove model has been translated to the so-called *charge* statistic for fillings in types *A* and *C* in [18], while type *B* is currently under investigation in [1]. Another example is related to the combinatorial *R*-matrix mentioned below.

We also conjecture that, like the alcove model, its generalization given here is independent of the λ-chain of roots on which the whole construction is based, cf. [15]. This conjecture is currently under investigation in [19]. We intend to realize an affine crystal isomorphism between the models based on two λ-chains by extending to the

quantum alcove model the alcove model version of Schützenberger's *jeu de taquin* [5] on Young tableaux in [15]; the latter is based on so-called *Yang–Baxter moves*. Another application of this construction would be a uniform realization of the *combinatorial R-matrix* (i.e., the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).

2 Background

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\Phi^- := \Phi \backslash \Phi^+$. For $\alpha \in \Phi$, we will say that $\alpha > 0$ if $\alpha \in \Phi^+$, and $\alpha < 0$ if $\alpha \in \Phi^-$. The sign of the root α, denoted $\mathrm{sgn}(\alpha)$, is defined to be 1 if $\alpha \in \Phi^+$, and -1 otherwise. Let $|\alpha| = \mathrm{sgn}(\alpha)\alpha$. Let $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha)$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*, and $s_i := s_{\alpha_i}$ the corresponding simple reflections. We denote $\langle \cdot, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ and reflection s_α. If $\alpha = \sum_i c_i \alpha_i$, then the *height* of α, denoted $\mathrm{ht}(\alpha)$, is given by $\mathrm{ht}(\alpha) := \sum_i c_i$. We denote by $\widetilde{\alpha}$ the highest root in Φ^+; we let $\theta = \alpha_0 := -\widetilde{\alpha}$ and $s_0 := s_{\widetilde{\alpha}}$.

Let W be the corresponding *Weyl group*. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) + 1$. Define $w \lhd ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1$. The *quantum Bruhat graph* [6] is the directed graph on W with edges labeled by positive roots

$$w \xrightarrow{\alpha} ws_\alpha \quad \text{for } w \lessdot ws_\alpha \text{ or } w \lhd ws_\alpha; \tag{1}$$

see Example 4.2.

The *weight lattice* Λ is given by

$$\Lambda := \left\{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \; : \; \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi \right\}. \tag{2}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots \omega_r$, which form the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \left\{ \lambda \in \Lambda \; : \; \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+ \right\}. \tag{3}$$

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \left\{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \; : \; \langle \lambda, \alpha^\vee \rangle = k \right\}. \tag{4}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \left\{ \alpha^\vee \mid \alpha \in \Phi \right\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves*. The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \left\{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+ \right\}. \tag{5}$$

We will need the following properties of the quantum Bruhat graph which were proved in [21]; more precisely, Lemma 2.1 below is a simplified version of Proposition 5.4.2 in the mentioned paper, whereas Lemmas 2.2 and 2.3 below are simplified versions of different parts of the Diamond Lemma 5.5.2.

Lemma 2.1 *Let $w \in W$. We have $w^{-1}(\theta) > 0$ if and only if $w \lhd s_\theta w$. We also have $w^{-1}(\theta) < 0$ if and only if $s_\theta w \lhd w$.*

Lemma 2.2 *Let $w \in W$, let α be a simple root, $\beta \in \Phi^+$, and assume $s_\alpha w \neq w s_\beta$. Then $w \lessdot s_\alpha w$ and $w \longrightarrow w s_\beta$ if and only if $w s_\beta \lessdot s_\alpha w s_\beta$ and $s_\alpha w \longrightarrow s_\alpha w s_\beta$, cf. the diagram below. Furthermore, in this context we have $w \lessdot w s_\beta$ if and only if $s_\alpha w \lessdot s_\alpha w s_\beta$.*
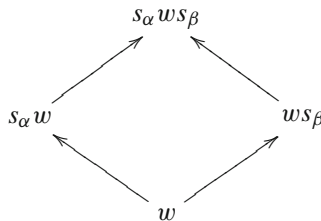
Lemma 2.3 *Let $w \in W$, $\beta \in \Phi^+$, and assume $s_\theta w \neq w s_\beta$. Then $w \lhd s_\theta w$ and $w \longrightarrow w s_\beta$ if and only if $w s_\beta \lhd s_\theta w s_\beta$ and $s_\theta w \longrightarrow s_\theta w s_\beta$.*

2.1 Kirillov–Reshetikhin (KR) crystals

A $\mathfrak{g}$-*crystal* (for a symmetrizable Kac–Moody $\mathfrak{g}$) is a nonempty set B together with maps $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for $i \in I$ (I indexes the simple roots, as usual, and $\mathbf{0} \notin B$), and $\mathrm{wt} : B \to \Lambda$. We require $b' = f_i(b)$ if and only if $b = e_i(b')$, and $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$. The maps e_i and f_i are called *crystal operators* and are represented as arrows $b \to b' = f_i(b)$ colored i; thus they endow B with the structure of a colored directed graph. For $b \in B$, we set $\varepsilon_i(b) := \max\{k \mid e_i^k(b) \neq \mathbf{0}\}$, and $\varphi_i(b) := \max\{k \mid f_i^k(b) \neq \mathbf{0}\}$. Given two $\mathfrak{g}$-crystals B_1 and B_2, we define their tensor product $B_1 \otimes B_2$ as follows. As a set, $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\mathrm{wt}(b) := \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$. The crystal operators are given by

$$f_i(b_1 \otimes b_2) := \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2) \\ b_1 \otimes f_i(b_2) & \text{otherwise,} \end{cases} \tag{6}$$

and similarly for e_i. The *highest weight crystal* $B(\lambda)$ of highest weight $\lambda \in \Lambda^+$ is a certain crystal with a unique element u_λ such that $e_i(u_\lambda) = \mathbf{0}$ for all $i \in I$ and $\mathrm{wt}(u_\lambda) = \lambda$. It encodes the structure of the crystal basis of the $U_q(\mathfrak{g})$-irreducible representation with highest weight λ as q goes to 0.

A *Kirillov–Reshetikhin (KR) crystal* [13] is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height r and width s, where $r \in I \setminus \{0\}$ and s is any positive integer. We refer, throughout, to the untwisted affine types $A_{n-1}^{(1)} - G_2^{(1)}$, and only consider column shape KR crystals $B^{r,1}$.

We now describe the models based on diagram fillings for KR crystals $B^{r,1}$ of type $A_{n-1}^{(1)}$ and $C_n^{(1)}$, where $r \in \{1, 2, \ldots, n-1\}$ and $r \in \{1, 2, \ldots, n\}$, respectively. As a classical type A_{n-1} (resp. C_n) crystal, the KR crystal $B^{r,1}$ is isomorphic to the corresponding $B(\omega_r)$. Therefore, we can use the corresponding models in terms of fillings, as mentioned below.

In type $A_{n-1}^{(1)}$, an element $b \in B^{r,1}$ is represented by a strictly increasing filling of a height r column, with entries in $[n] := \{1, \ldots, n\}$. We will now describe the crystal operators on a tensor product of type $A_{n-1}^{(1)}$ KR crystals $B^{r,1}$ in terms of the so-called signature rule, which is just a translation of the tensor product rule (6). To apply f_i (or e_i) on $b := b_1 \otimes \cdots \otimes b_k$ in $B^{i_1,1} \otimes \cdots \otimes B^{i_k,1}$, consider the word with letters i and $i + 1$, if $1 \le i \le n - 1$ (resp., the letters n and 1, if $i = 0$) formed by recording these letters in $b_1, \ldots, b_k$, which are scanned from left to right and bottom to top; we make the convention that if $i = 0$ and a column contains both 1 and n, then we discard this column. We replace the letter i with the symbol $+$ and the letter $i + 1$ with $-$ (resp., n with $+$ and 1 with $-$, if $i = 0$). Then, we remove from our binary word adjacent pairs $-+$, as long as this is possible. At the end of this process, we are left with a word

$$\rho_i(b) = \underbrace{+ + \ldots +}_{x} \underbrace{- - \ldots -}_{y}, \tag{7}$$

called the i-signature of b.

Definition 2.4 (1) If $y > 0$, then $e_i(b)$ is obtained by replacing in b the letter $i + 1$ which corresponds to the leftmost $-$ in $\rho_i(b)$ with the letter i (resp., the letter 1 with n, after which we sort the column, if $i = 0$). If $y = 0$, then $e_i(b) = \mathbf{0}$.

(2) If $x > 0$, then $f_i(b)$ is obtained by replacing in b the letter i which corresponds to the rightmost $+$ in $\rho_i(b)$ with the letter $i + 1$ (resp., the letter n with 1, after which we sort the column, if $i = 0$). If $x = 0$, then $f_i(b) = \mathbf{0}$.

Example 2.5 Let $n = 3$, $b = \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 3 & 2 & \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \boxed{1}$, and has $+ - -$ as its

0-signature. So we have $f_0 \left(\begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 3 & 2 & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline \end{array}$.

In type $C_n^{(1)}$, the elements of $B^{r,1}$ are represented by *Kashiwara–Nakashima (KN) columns* [12] of height r, with entries in the set $[\overline{n}] := \{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$, which we will now describe.

Definition 2.6 A column-strict filling $C = x_1 \ldots x_r$ with entries in $[\overline{n}]$ is a KN column if there is no pair $(z, \overline{z})$ of letters in C such that

$$z = x_p, \quad \overline{z} = x_q, \quad q - p \le r - z.$$

Crystal operators f_i and e_i are defined on tensor products of KN columns in a similar way to type $A_{n-1}^{(1)}$. To apply f_i (or e_i) on $b := b_1 \otimes \cdots \otimes b_k$ in $B^{i_1,1} \otimes \cdots \otimes B^{i_k,1}$, consider the word with letters $i, i+1, \bar{i}, \overline{i+1}$, if $1 \le i \le n-1$ (resp., the letters n and $\bar{n}$, if $i = n$, or $\bar{1}$ and 1, if $i = 0$) formed by recording these letters in $b_1, \dots, b_k$, which are scanned from from left to right and bottom to top; note that the letters 1 and $\bar{1}$ cannot simultaneously appear in a column, so we do not need an exception like in type $A_{n-1}^{(1)}$ if $i = 0$. We replace the letters $i, \overline{i+1}$ with the symbol $+$ and the letters $i+1, \bar{i}$ with $-$, if $1 \le i \le n-1$ (if $i = n$ we replace n with $+$ and $\bar{n}$ with $-$, and if $i = 0$ we replace $\bar{1}$ with $+$ and 1 with $-$). We proceed like in type $A_{n-1}^{(1)}$ by canceling adjacent pairs $-+$ as long as possible, and we obtain the i-signature $\rho_i(b)$. The crystal operators f_i and e_i are again given in terms $\rho_i(b)$, by a similar procedure to the one in Definition 2.4. Namely, if $1 \le i \le n-1$, changing $+$ to $-$ means changing i to $i+1$, if $+$ corresponds to i, and changing $\overline{i+1}$ to $\bar{i}$, if $+$ corresponds to $\overline{i+1}$; similarly changing $-$ to $+$ means changing $i+1$ to i or $\bar{i}$ to $\overline{i+1}$. On another hand, changing $+$ to $-$ means changing n to $\bar{n}$ if $i = n$, and changing $\bar{1}$ to 1 if $i = 0$.

We will need a different definition of KN columns which was proved to be equivalent to the one above in [31].

Definition 2.7 Let C be column and $I = \{z_1 > \dots > z_k\}$ the set of unbarred letters z such that the pair $(z, \bar{z})$ occurs in C. The column C can be *split* when there exists a set of k unbarred letters $J = \{t_1 > \dots > t_k\} \subset [n]$ such that

- t_1 is the greatest letter in $[n]$ satisfying: $t_1 < z_1$, $t_1 \notin C$, and $\overline{t_1} \notin C$,
- for $i = 2, \dots, k$, the letter t_i is the greatest one in $[n]$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$, and $\overline{t_i} \notin C$.

In this case we write

- rC for the column obtained by changing $\overline{z_i}$ into $\overline{t_i}$ in C for each letter $z_i \in I$, and by reordering if necessary,
- lC for the column obtained by changing z_i into t_i in C for each letter $z_i \in I$, and by reordering if necessary.

The pair (lC, rC) will be called a *split column*, which we well sometimes denote by $lCrC$.

Example 2.8 The following is a KN column of height 5 in type C_n for $n \ge 5$, together with the corresponding split column:

$$
C = \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline \end{array}, \quad
(lC, rC) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline \bar{5} & \bar{3} \\ \hline \bar{4} & \bar{2} \\ \hline \bar{3} & \bar{1} \\ \hline \end{array}
$$

We used the fact that $\{z_1 > z_2\} = \{5 > 4\}$, so $\{t_1 > t_2\} = \{2 > 1\}$.

Proposition 2.9   (1) *[31] A column with entries in $[\overline{n}]$ is a KN column if and only if it can be split.*

(2) *[10, Theorem 5.1] The splitting is compatible with the action of the crystal operators, i.e., if $f_i(C) = C'$ then $lC'rC' = f_i^2(lCrC)$. This holds more generally, for tensor products of columns, i.e., if $f_i(C_1 \ldots C_n) = C'_1 \ldots C'_n$, then $lC'_1 rC'_1 \ldots lC'_n rC'_n = f_i^2(lC_1 rC_1 \ldots lC_n rC_n)$.*

In what follows we will use Definition 2.7 as the definition of KN columns.

We refer again to (column shape) KR crystals of arbitrary (untwisted) type. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ be a partition, which encodes a dominant weight in classical types; let λ' be the conjugate partition. We define

$$B^{\otimes \lambda} := \bigotimes_{i=1}^{\lambda_1} B^{\lambda'_i, 1}, \tag{8}$$

assuming that the corresponding column shape KR crystals exist. We denote such a tensor product generically by B.

Remarks 2.10 (1) It is known that B is connected as an affine crystal, but disconnected as a classical crystal (i.e., with the 0-arrows removed).

(2) Let $\mathbf{p}'$ be a composition obtained from $\mathbf{p}$ by permuting its parts. There is an affine crystal isomorphism betwee $B^{\otimes \mathbf{p}}$ and $B^{\otimes \mathbf{p}'}$, which is unique by the previous remark. This isomorphism is called the *combinatorial R-matrix*.

We need to distinguish certain arrows in B, which are related to affine Demazure crystals, as we shall explain.

Definition 2.11 An arrow $b \to f_i(b)$ in B is called a *Demazure arrow* if $i \neq 0$, or $i = 0$ and $\varepsilon_0(b) \geq 1$. An arrow $b \to f_i(b)$ in B is called a *dual Demazure arrow* if $i \neq 0$, or $i = 0$ and $\varphi_i(b) \geq 2$.

Remarks 2.12 (1) By Fourier–Littelmann [3], in simply laced types, the tensor product of KR crystals B is isomorphic, as a classical crystal (discard the affine 0-arrows) with a certain *Demazure crystal* for the corresponding affine algebra. (Demazure modules are submodules of highest weight ones determined by a Borel subalgebra acting on an extremal weight vector.) Moreover, by [4], the 0-arrows in the latter correspond precisely to the Demazure arrows in B.

(2) In the case when all of the tensor factors in B are *perfect* crystals [9], B remains connected upon removal of the non-Demazure (resp. non-dual Demazure) 0-arrows.

(3) In classical types, $B^{k,1}$ is perfect as follows: in types $A_{n-1}^{(1)}$ and $D_n^{(1)}$ for all k, in type $B_n^{(1)}$ only for $k \neq n$, and in type $C_n^{(1)}$ only for $k = n$ (using the standard indexing of the Dynkin diagram); in other words, for all the Dynkin nodes in simply laced types, and only for the nodes corresponding to the long roots in non-simply-laced types. It was conjectured in [8] that the same is true in the exceptional types. In type $G_2^{(1)}$ this was confirmed in [33], while for types $E_{6,7}^{(1)}$

and $F_4^{(1)}$ it was checked by computer, based on a model closely related to the quantum alcove model, see Sect. 3 and [22].

The *energy function* $D = D_B$ is a function from B to the integers, defined by summing the so-called local energies of all pairs of tensor factors [8]. We will only refer here to the so-called tail energy [22], so we will not make this specification. (There are two conventions in defining the local energy of a pair of tensor factors: commuting the right one toward the head of the tensor product, or the left one toward the tail; the tail energy corresponds to the second choice.) We will only need the following property of the energy function, which defines it as an affine grading on B.

Theorem 2.13 *[28,32] The energy is preserved by the classical crystal operators f_i, i.e., $i \neq 0$. If $b \rightarrow f_0(b)$ is a dual Demazure arrow, then $D(f_0(b)) = D(b) - 1$.*

Remarks 2.14 Theorem 2.13 shows that the energy is determined up to a constant on the connected components of the subgraph of the affine crystal B containing only the dual Demazure arrows. See also Remark 2.12 (2).

One can define a statistic called *charge* on the model based on KN columns for $B^{\otimes \lambda}$ in types A and C. This was done in [18], by translating a certain statistic in the Ram–Yip formula for Macdonald polynomials (i.e., the height statistic in (11)) to the model based on KN columns, via certain bijections recalled in Sect. 4, cf. Remarks 4.8 (2) and 4.23 (2). In type A, this procedure leads to the same statistic that was originally defined by Lascoux and Schützenberger [14]. A similar procedure to the one in [18] is under investigation in type B in [1]. The charge statistic is related to the energy function by the following theorem.

Theorem 2.15 *[25,29] Let $B^{\otimes \lambda}$ be a tensor product of KR crystals in type $A_{n-1}^{(1)}$ or type $C_n^{(1)}$. For all $b \in B^{\otimes \lambda}$, we have $D(b) = -\text{charge}(b)$.*

The charge gives a much easier method to compute the energy than the recursive one based on Theorem 2.13. See also Theorem 3.24 (2) for a much more general result.

3 The quantum alcove model

In this section we construct the quantum alcove model and study its main properties.

3.1 λ-Chains and admissible subsets

We say that two alcoves are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 3.1 [23] An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \ldots m$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_m.
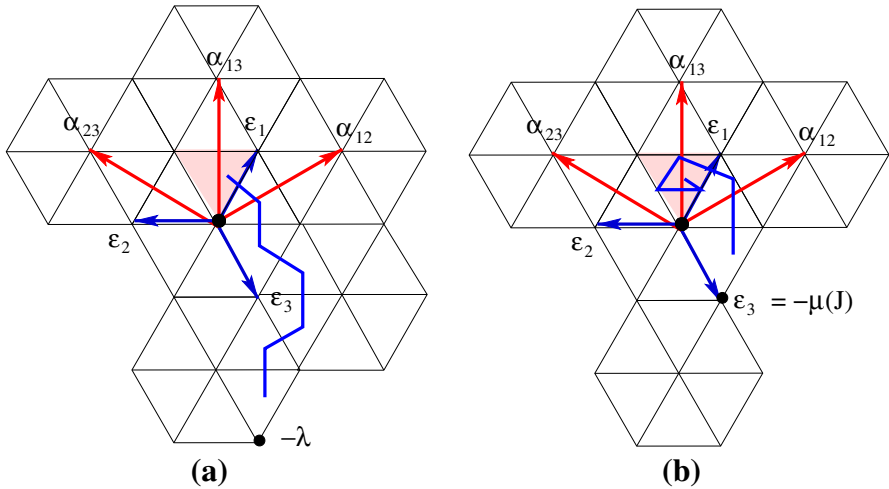
Fig. 1 Unfolded and folded λ-chain. **a** Γ for $\lambda = 3\varepsilon_1 + 2\varepsilon_2$. **b** $\Gamma(J)$ for $J = \{1, 2, 3, 5\}$

Let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by the weight λ.

Definition 3.2 [23] The sequence of roots $(\beta_1, \beta_2, \ldots, \beta_m)$ is called a λ-*chain* if

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

We now fix a dominant weight λ and an alcove path $\Pi = (A_0, \ldots, A_m)$ from $A_0 = A_\circ$ to $A_m = A_{-\lambda}$. Note that Π is determined by the corresponding λ-chain $\Gamma := (\beta_1, \ldots, \beta_m)$, which consists of positive roots. A specific choice of a λ-chain, called a lex λ-chain and denoted Γ_{lex}, is given in [24] [Proposition 4.2]; this choice depends on a total order on the simple roots. We let $r_i := s_{\beta_i}$, and let $\widehat{r_i}$ be the affine reflection in the hyperplane containing the common face of A_{i-1} and A_i, for $i = 1, \ldots, m$; in other words, $\widehat{r_i} := s_{\beta_i, -l_i}$, where $l_i := |\{j < i \,;\, \beta_j = \beta_i\}|$. We define $\widetilde{l_i} := \langle \lambda, \beta_i^\vee \rangle - l_i = |\{j \geq i \,;\, \beta_j = \beta_i\}|$.

Example 3.3 Consider the dominant weight $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ in the root system A_2 (cf. Sect. 4.1 and the notation therein). A λ-chain is $\Gamma = (\alpha_{23}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$. The corresponding l_i are $(0, 0, 1, 1, 0, 2)$ and $\widetilde{l_i}$ are $\{2, 3, 1, 2, 1, 1\}$. The alcove path is shown in Fig. 1a; here $A_\circ$ is shaded, and $A_{-\lambda}$ is the alcove at the end of the path.

Let $J = \{j_1 < j_2 < \cdots < j_s\}$ be a subset of $[m]$. The elements of J are called *folding positions*. We fold Π in the hyperplanes corresponding to these positions and obtain a folded path, see Example 3.6 and Fig. 1b. Like Π, the folded path can be recorded by a sequence of roots, namely $\Gamma(J) = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, where

$$\gamma_k := r_{j_1} r_{j_2} \ldots r_{j_p}(\beta_k), \tag{9}$$

with j_p the largest folding position less than k. We define $\gamma_\infty := r_{j_1} r_{j_2} \ldots r_{j_s}(\rho)$.
Upon folding, the hyperplane separating the alcoves A_{k-1} and A_k in Π is mapped to

$$H_{|\gamma_k|, -l_k^J} = \widehat{r}_{j_1} \widehat{r}_{j_2} \ldots \widehat{r}_{j_p} (H_{\beta_k, -l_k}), \tag{10}$$

for some l_k^J, which is defined by this relation.

Given $i \in J$, we say that i is a *positive folding position* if $\gamma_i > 0$, and a *negative folding position* if $\gamma_i < 0$. We denote the positive folding positions by J^+, and the negative ones by J^-. We call $\mu = \mu(J) := -\widehat{r}_{j_1} \widehat{r}_{j_2} \ldots \widehat{r}_{j_s}(-\lambda)$ the *weight* of J. We define

$$\text{height}(J) := \sum_{j \in J^-} \widetilde{l}_j. \tag{11}$$

Definition 3.4 A subset $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ (possibly empty) is an *admissible subset* if we have the following path in the quantum Bruhat graph on W:

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \cdots r_{j_s}. \tag{12}$$

We call $\Gamma(J)$ an *admissible folding*. We let $\mathcal{A}(\Gamma)$ be the collection of admissible subsets.

Remarks 3.5 If we restrict to admissible subsets for which the path (12) has no down steps, we recover the classical alcove model in [23,24].

Example 3.6 We continue Example 3.3. Let $J = \{1, 2, 3, 5\}$, then $\Gamma(J) = (\alpha_{23}, \alpha_{12}, \alpha_{31}, \alpha_{23}, \alpha_{21}, \alpha_{13})$. The folded path is shown in Fig. 1b. We have $J^+ = \{1, 2\}$, $J^- = \{3, 5\}$, $\mu(J) = -\varepsilon_3$, and height$(J) = 2$. In Sect. 4.1 we will describe an easy way to verify that J is admissible.

3.2 Crystal operators

In this section we define the crystal operators on $\mathcal{A}(\Gamma)$. Given $J \subseteq [m]$ and $\alpha \in \Phi$, we will use the following notation:

$$I_\alpha = I_\alpha(J) := \{i \in [m] \mid \gamma_i = \pm\alpha\}, \qquad \widehat{I}_\alpha = \widehat{I}_\alpha(J) := I_\alpha \cup \{\infty\},$$

and $l_\alpha^\infty := \langle \mu(J), \text{sgn}(\alpha)\alpha^\vee \rangle$. The following graphical representation of the heights l_i^J for $i \in I_\alpha$ and l_α^∞ is useful for defining the crystal operators. Let

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n < i_{n+1} = \infty\} \text{ and } \varepsilon_i := \begin{cases} 1 & \text{if } i \notin J \\ -1 & \text{if } i \in J \end{cases}.$$
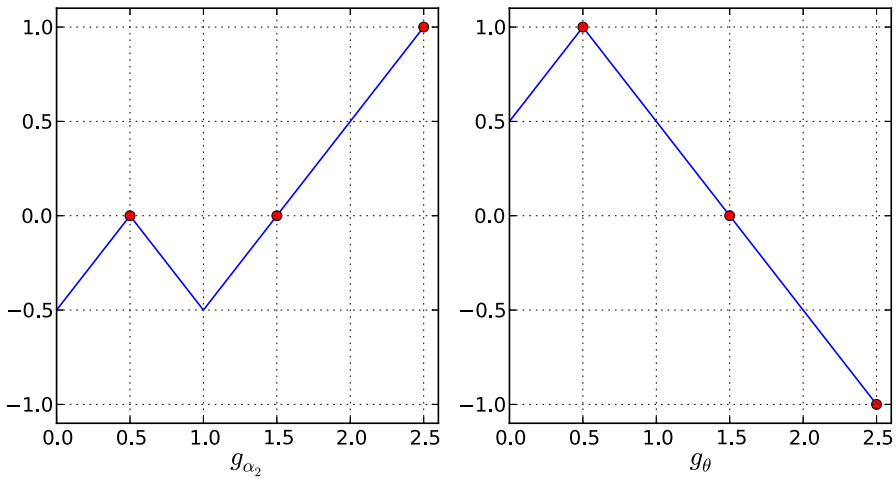
Fig. 2 Examples of g_α

If $\alpha > 0$, we define the continuous piecewise linear function $g_\alpha : [0, n + \frac{1}{2}] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}), \ k = 1, \ldots, n \\ \varepsilon_{i_k} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x \in (k-\frac{1}{2}, k), \ k = 1, \ldots, n \\ \operatorname{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle) & \text{if } x \in (n, n+\frac{1}{2}). \end{cases}$$
(13)

If $\alpha < 0$, we define g_α to be the graph obtained by reflecting $g_{-\alpha}$ in the x-axis. By [24][Propositions 5.3 and 5.5], for any α we have

$$\operatorname{sgn}(\alpha) l^J_{i_k} = g_\alpha \left(k - \frac{1}{2} \right), k = 1, \ldots, n, \text{ and } \operatorname{sgn}(\alpha) l^\infty_\alpha := \langle \mu(J), \alpha^\vee \rangle = g_\alpha \left(n + \frac{1}{2} \right).$$
(14)

Example 3.7 We continue Example 3.6. The graphs of g_{α_2} and g_θ are given in Fig. 2.

Let J be an admissible subset. Let $\delta_{i,j}$ be the Kronecker delta function. Fix p in $\{0, \ldots, r\}$, so α_p is a simple root if $p > 0$, or θ if $p = 0$. Let M be the maximum of g_{α_p}. Let m be the minimum index i in $\widehat{I}_{\alpha_p}$ for which we have $\operatorname{sgn}(\alpha_p) l^J_i = M$. By Proposition 3.22, if $M \geq \delta_{p,0}$, then we have either $m \in J$ or $m = \infty$; furthermore, if $M > \delta_{p,0}$, then m has a predecessor k in $\widehat{I}_{\alpha_p}$, and we have $k \notin J$. We define

$$f_p(J) := \begin{cases} (J \setminus \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\ \mathbf{0} & \text{otherwise .} \end{cases}$$
(15)

Now we define e_p. Again let $M := \max g_{\alpha_p}$. Assuming that $M > \langle \mu(J), \alpha_p^\vee \rangle$, let k be the maximum index i in I_{α_p} for which we have $\operatorname{sgn}(\alpha_p) l^J_i = M$, and let m be the successor of k in $\widehat{I}_{\alpha_p}$. Assuming also that $M \geq \delta_{p,0}$, by Proposition 3.23 we have

$k \in J$, and either $m \notin J$ or $m = \infty$. Define

$$e_p(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mu(J), \alpha_p^\vee \rangle \text{ and } M \geq \delta_{p,0} \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (16)$$

In the above definitions, we use the convention that $J \setminus \{\infty\} = J \cup \{\infty\} = J$.

Example 3.8 We continue Example 3.7. We find $f_2(J)$ by noting that $\widehat{I}_{\alpha_2} = \{1, 4, \infty\}$. From g_{α_2} in Fig. 2 we can see that the heights l_i^J and $l_{\alpha_2}^\infty$ corresponding to these positions are 0, 0, 1, so $k = 4$, $m = \infty$, and $f_2(J) = J \cup \{4\} = \{1, 2, 3, 4, 5\}$. We can see from Fig. 2 that the maximum of $g_\theta = 1$, hence $f_0(J) = \mathbf{0}$. To compute $e_0(J)$ observe that $\widehat{I}_\theta = \{3, 6\}$ with $k = 3$ and $m = 6$. So $e_0(J) = (J \setminus \{k\}) \cup \{m\} = \{1, 2, 5, 6\}$.

The following theorem is one of our main results, and will be proved in Sect. 3.3.

Theorem 3.9   (1) *If J is an admissible subset and if $f_p(J) \neq \mathbf{0}$, then $f_p(J)$ is also an admissible subset. Similarly for $e_p(J)$. Moreover, $f_p(J) = J'$ if and only if $e_p(J') = J$.*
(2) *We have $\mu(f_p(J)) = \mu(J) - \alpha_p$. Moreover, if $M \geq \delta_{p,0}$, then*

$$\varphi_p(J) = M - \delta_{p,0}, \quad \varepsilon_p(J) = M - \langle \mu(J), \alpha_p^\vee \rangle,$$

while otherwise $\varphi_p(J) = \varepsilon_p(J) = 0$.

3.3 Proofs

In this section we collect necessary results for the proof of Theorem 3.9. The techniques are similar to those in [24]; we go in detail over the parts of the proofs where there are notable differences, and we refer to the mentioned paper for the remaining parts.

Lemma 3.10 *Let $w \in W$, α a simple root or θ, and β a positive root. If we have $w \longrightarrow ws_\beta$, as well as $w^{-1}(\alpha) > 0$ and $s_\beta w^{-1}(\alpha) < 0$, then $w^{-1}(\alpha) = \beta$.*

Proof If $s_\alpha w = ws_\beta$, then $w^{-1}(\alpha) = \pm\beta$, and $w^{-1}(\alpha) > 0$ implies $w^{-1}(\alpha) = \beta$. Suppose by way of contradiction that $s_\alpha w \neq ws_\beta$. First suppose α is a simple root. Since $w^{-1}(\alpha) > 0$, then $w \lessdot s_\alpha w$. By assumption we have $w \longrightarrow ws_\beta$, hence by Lemma 2.2 we have $ws_\beta \lessdot s_\alpha ws_\beta$. But $s_\beta w^{-1}(\alpha) < 0$ implies $s_\alpha ws_\beta \lessdot ws_\beta$, which is a contradiction.

Suppose $\alpha = \theta$. Since $w^{-1}(\theta) > 0$, we deduce $w \vartriangleleft s_\theta w$, by Lemma 2.1, and then $ws_\beta \vartriangleleft s_\theta ws_\beta$, by Lemma 2.3. On another hand, since $s_\beta w^{-1}(\theta) < 0$, Lemma 2.1 implies that $s_\theta ws_\beta \vartriangleleft ws_\beta$, which is a contradiction. $\square$

Lemma 3.11 *Let $J = \{j_1 < j_2 < \ldots < j_s\}$ be an admissible subset. Assume that $r_{j_a} \ldots r_{j_1}(\alpha) > 0$ and $r_{j_b} \ldots r_{j_1}(\alpha) < 0$, where α is a simple root or θ, and $0 \leq a < b$ (if $a = 0$, then the first condition is void). Then there exists i with $a \leq i < b$ such that $\gamma_{j_{i+1}} = \alpha$.*

Proof Find i with $a \leq i < b$ such that $r_{j_i} \dots r_{j_1}(\alpha) > 0$ and $r_{j_{i+1}} \dots r_{j_1}(\alpha) < 0$. By Lemma 3.10, we have $\beta_{j_{i+1}} = r_{j_i} \dots r_{j_1}(\alpha)$. This means that $\gamma_{j_{i+1}} = r_{j_1} \dots r_{j_i}(\beta_{j_{i+1}}) = \alpha$.   □

Proposition 3.12 *Let* $J = \{j_1 < j_2 < \cdots < j_s\}$ *be an admissible subset. Assume that* α *is a simple root or* θ, *with* $I_\alpha \neq \emptyset$. *Let* $m \in I_\alpha$ *be an element for which its predecessor* k *(in* I_α*) satisfies* $(\gamma_k, \varepsilon_k) \in \{(\alpha, 1), (-\alpha, -1)\}$. *Then we have* $\gamma_m = \alpha$.

Proof First suppose that $(\gamma_k, \varepsilon_k) = (\alpha, 1)$. Assume that $\gamma_m = -\alpha$. Let us define the index b by the condition $j_b < m \leq j_{b+1}$ (possibly $b = s$, in which case the second inequality is dropped). We define the index a by the condition $j_a < k < j_{a+1}$ (possibly $a = 0$, in which case the first inequality is dropped). We clearly have $r_{j_1} \dots r_{j_b}(\beta_m) = -\alpha$, which implies $r_{j_b} \dots r_{j_1}(\alpha) < 0$. We also have $r_{j_1} \dots r_{j_a}(\beta_k) = \alpha$, so $r_{j_a} \dots r_{j_1}(\alpha) > 0$ (hence $a < b$). Note that if $\alpha = \theta$, then $a > 0$. We can now apply Lemma 3.11 to conclude that $\gamma_{j_i} = \alpha$ for some $i \in [a+1, b]$. Since $k < j_{a+1} \leq j_b < m$, we contradicted the assumption that γ_k is the predecessor of γ_m in I_α.

Now suppose that $(\gamma_k, \varepsilon_k) = (-\alpha, -1)$. Assume that $\gamma_m = -\alpha$ and define b as in the previous case. Again we have $r_{j_b} \dots r_{j_1}(\alpha) < 0$. Define a by the condition $j_a = k < j_{a+1}$. Hence $r_{j_1} \dots r_{j_{a-1}}(\beta_{j_a}) = -\alpha$, so $r_{j_a} \dots r_{j_1}(\alpha) > 0$. This leads to a contradiction by a similar reasoning to the one above.   □

Proposition 3.13 *Let* J *be an admissible subset. Assume that* α *is a simple root for which* $I_\alpha \neq \emptyset$. *Let* $m \in I_\alpha$ *be the minimum of* I_α. *Then we have* $\gamma_m = \alpha$.

Proof The proof of Proposition 3.12 carries through with $a = 0$.   □

Proposition 3.14 *Let* $J = \{j_1 < j_2 < \cdots < j_s\}$ *be an admissible subset. Assume that* α *is a simple root or* θ. *Suppose that* $I_\alpha \neq \emptyset$, *and* $(\gamma_m, \varepsilon_m) \in \{(\alpha, 1), (-\alpha, -1)\}$ *for* $m = \max I_\alpha$. *Then we have* $\langle \gamma_\infty, \alpha^\vee \rangle > 0$.

Proof Assume that the conclusion fails, which means that $r_{j_s} \dots r_{j_1}(\alpha) < 0$. First suppose that $(\gamma_m, \varepsilon_m) = (\alpha, 1)$. Define the index a by the condition $j_a < m < j_{a+1}$. (If $a = 0$ or $a = s$ one of the two inequalities is dropped). We have $r_{j_1} \dots r_{j_a}(\beta_m) = \alpha$, so $r_{j_a} \dots r_{j_1}(\alpha) > 0$ (hence $a \neq s$). Note that if $\alpha = \theta$, then $a > 0$. We now apply Lemma 3.11 to conclude that $\gamma_{j_i} = \alpha$ for $i \in [a+1, s]$. Since $m < j_{a+1} \leq j_s$, this contradicts that $m = \max I_\alpha$.

Now suppose that $(\gamma_m, \varepsilon_m) = (-\alpha, -1)$. In this case we define the index a by $j_a = m < j_{a+1}$. We have $r_{j_1} \dots r_{j_{a-1}}(\beta_{j_a}) = -\alpha$, so $r_{j_a} \dots r_{j_1}(\alpha) > 0$. This leads to a contradiction by a similar reasoning to the one above.   □

Proposition 3.15 *Let* J *be an admissible subset. Assume that, for some simple root* α, *we have* $I_\alpha = \emptyset$. *Then* $\langle \gamma_\infty, \alpha^\vee \rangle > 0$.

Proof The proof of Proposition 3.14 carries through with $a = 0$.

Let us now fix a simple root α. We will rephrase some of the above results in a simple way in terms of g_α, and we will deduce some consequences. Assume that $I_\alpha = $

$\{i_1 < i_2 < \cdots < i_n\}$, so that g_α is defined on $[0, n + \frac{1}{2}]$, and let M be the maximum of g_α. Note first that the function g_α is determined by the sequence $(\sigma_1, \ldots, \sigma_{n+1})$, where $\sigma_j = (\sigma_{j,1}, \sigma_{j,2}) := (\mathrm{sgn}(\gamma_{i_j}), \varepsilon_{i_j} \mathrm{sgn}(\gamma_{i_j}))$ for $1 \leq j \leq n$, and $\sigma_{n+1} = \sigma_{n+1,1} :=$ $\mathrm{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle)$. From Propositions 3.12, 3.13, 3.14 and 3.15 we have the following restrictions.

(C1) $\sigma_{1,1} = 1$.
(C2) $\sigma_{j,2} = 1 \Rightarrow \sigma_{j+1,1} = 1$.

Proposition 3.16 *If $g_\alpha(x) = M$, then $M \in \mathbb{Z}_{\geq 0}$, $x = m + \frac{1}{2}$ for $0 \leq m \leq n$, and $\sigma_{m+1} \in \{(1, -1), 1\}$.*

Proof By (C1), we have $M \geq 0$, therefore $g_\alpha(0) = -\frac{1}{2} \neq M$. For $m \in \{1, \ldots, n\}$, if $g_\alpha(m) = M$ then $\sigma_{m,2} = 1$, and (C2) leads to a contradiction. The last statement is obvious. $\square$

We use Proposition 3.16 implicitly in the proof of Proposition 3.17 and Proposition 3.18.

Proposition 3.17 *Assume that $M > 0$, and let m be such that $m + \frac{1}{2} = \min g_\alpha^{-1}(M)$. We have $m > 0$, $\sigma_m = (1, 1)$, and $g_\alpha(m - \frac{1}{2}) = M - 1$. Moreover, we have $g_\alpha(x) \leq M - 1$ for $0 \leq x \leq m - \frac{1}{2}$.*

Proof By (C1) we have $g_\alpha(\frac{1}{2}) = 0$, so $m > 0$. If $\sigma_m \in \{(-1, -1), (1, -1)\}$, then we have $g_\alpha(m - \frac{1}{2}) = M$, which contradicts the definition of m. If $\sigma_m = (-1, 1)$, then $g_\alpha(m - 1) = M - \frac{1}{2} > -\frac{1}{2}$. By (C1) we have $m \geq 2$, and by (C2) we have $\sigma_{m-1,2} = -1$. This implies that $g_\alpha(m - \frac{3}{2}) = M$, contradicting the definition of m. Hence $\sigma_m = (1, 1)$.

Suppose by way of contradiction that the last statement in the corollary fails. Then there exists a k with $1 \leq k \leq m - 1$ such that $g_\alpha(k-1) = M - \frac{1}{2} > -\frac{1}{2}$ and $\sigma_{k,1} = -1$. Condition (C1) implies that $k \geq 2$ and Condition (C2) implies $\sigma_{k-1,2} = -1$. This implies $g_\alpha(k - \frac{3}{2}) = M$, contradicting the definition of m. $\square$

Proposition 3.18 *Assume that $M > g_\alpha(n + \frac{1}{2})$, and let k be such that $k - \frac{1}{2} = \max g_\alpha^{-1}(M)$. We have $k \leq n$, $\sigma_{k+1} \in \{(-1, -1), -1\}$, and $g_\alpha(k + \frac{1}{2}) = M - 1$. Moreover, we have $g_\alpha(x) \leq M - 1$ for $k + \frac{1}{2} \leq x \leq n + \frac{1}{2}$.*

Proof Since $M > g_\alpha(n + \frac{1}{2})$, it follows that $k \leq n$. If $\sigma_{k+1} \in \{(1, 1), (1, -1), 1\}$ then $g_\alpha(k + \frac{1}{2}) = M$, contradicting the choice of k. If $\sigma_{k+1} = (-1, 1)$, then by (C2) we have $\sigma_{k+2,1} = 1$, and $g_\alpha(k + \frac{3}{2}) = M$, contradicting the choice of k. Hence $\sigma_{k+1} \in \{(-1, -1), -1\}$.

Suppose by way of contradiction the last statement in the corollary fails. Then there exists an m with $k + 2 \leq m \leq n$ such that $g_\alpha(m) = M - \frac{1}{2}$ and $\sigma_{m,2} = 1$. Condition (C2) implies that $\sigma_{m+1,1} = 1$, so $g_\alpha(m + \frac{1}{2}) = M$, contradicting the choice of k. $\square$

We now consider g_θ. Since $\theta < 0$, the definition of the piecewise linear function g_θ requires us to define its linear steps by $\sigma_j = (\sigma_{j,1}, \sigma_{j,2}) := (-\mathrm{sgn}(\gamma_{i_j}), -\varepsilon_{i_j} \mathrm{sgn}(\gamma_{i_j}))$

for $1 \le j \le n$, and $\sigma_{n+1} = \sigma_{n+1,1} := \mathrm{sgn}(\langle \gamma_\infty, \theta^\vee \rangle)$. From Propositions 3.12 and 3.14 we conclude that condition (C2) holds for g_θ. We can replace condition (C1) by restricting to admissible subsets J where M is large enough, as we will now explain. In the proof of Proposition 3.16, condition (C1) is needed to conclude that $g_\alpha(0) \ne M$. It is possible that $g_\theta(0) = M$, but if we restrict to g_θ where $M \ge 1$ we can conclude that $g_\theta(0) = \frac{1}{2} \ne M$, and the rest of the proof follows through. In the proof of Proposition 3.17, condition (C1) first implies $m \ge 1$; we can conclude the same thing if we assume $M \ge 2$, since $g_\theta(\frac{1}{2}) \le 1$. Then we need to derive $m \ge 2$ from $g_\theta(m-1) = M - \frac{1}{2}$; again, if $M \ge 2$, then $M - \frac{1}{2} > g_\theta(0) = \frac{1}{2}$, so $m - 1 > 0$. Note that Proposition 3.18 depends on Proposition 3.16 so we need to assume $M \ge 1$ here too. We have therefore proved the following propositions.

Proposition 3.19 *Suppose $M \ge 1$. If $g_\theta(x) = M$, then $M \in \mathbb{Z}_{\ge 1}$, $x = m + \frac{1}{2}$ for $0 \le m \le n$, and $\sigma_{m+1} \in \{(1, -1), 1\}$.*

Proposition 3.20 *Assume that $M \ge 2$, and let m be such that $m + \frac{1}{2} = \min g_\theta^{-1}(M)$. We have $m > 0$, $\sigma_m = (1, 1)$, and $g_\theta(m - \frac{1}{2}) = M - 1$. Moreover, we have $g_\theta(x) \le M - 1$ for $0 \le x \le m - \frac{1}{2}$.*

Proposition 3.21 *Assume $M \ge 1$, and also that $M > g_\theta(n + \frac{1}{2})$. Let k be such that $k - \frac{1}{2} = \max g_\theta^{-1}(M)$. We have $k \le n$, $\sigma_{k+1} \in \{(-1, -1), -1\}$, and $g_\theta(k + \frac{1}{2}) = M - 1$. Moreover, we have $g_\theta(x) \le M - 1$ for $k + \frac{1}{2} \le x \le n + \frac{1}{2}$.*

Recall from from Sect. 3.2 the definitions of the finite sequences $I_\alpha(J)$ and $\widehat{I}_\alpha(J)$, where α is a root, of g_α, as well as the related notation.

Fix p, so α_p is a simple root if $p > 0$, or θ if $p = 0$. Let M be the maximum of g_{α_p}, and suppose that $M \ge \delta_{p,0}$. Note this is always true for $p \ne 0$ by Proposition 3.16. Let m be the minimum index i in $\widehat{I}_{\alpha_p}(J)$ for which we have $\mathrm{sgn}(\alpha_p) l_i^J = M$. The following proposition is an immediate consequence of Propositions 3.16, 3.17, 3.19, 3.20.

Proposition 3.22 *Given the above setup, the following hold.*

(1) *If $m \ne \infty$, then $\gamma_m = \alpha_p$ and $m \in J$.*
(2) *If $M > \delta_{p,0}$, then m has a predecessor k in $\widehat{I}_{\alpha_p}(J)$ such that*

$$\gamma_k = \alpha_p, \ k \notin J, \ and \ \mathrm{sgn}(\alpha_p) l_k^J = M - 1.$$

Now assume that $M > \langle \mu(J), \alpha_p^\vee \rangle$. Let k be the maximum index i in $I_{\alpha_p}(J)$ for which we have $\mathrm{sgn}(\alpha_p) l_i^J = M$, and let m be the successor of k in $\widehat{I}_{\alpha_p}(J)$. The following analog of Proposition 3.22 is proved in a similar way, based on Propositions 3.16, 3.18, 3.19, 3.21.

Proposition 3.23 *Given the above setup, and assuming also that $M \ge \delta_{p,0}$, the following hold.*

(1) *We have $\gamma_k = \alpha_p$ and $k \in J$.*
(2) *If $m \neq \infty$, then*

$$\gamma_m = -\alpha_p, \ m \notin J, \ \text{and} \ \text{sgn}(\alpha_p)l_m^J = M - 1.$$

Proof of Theorem 3.9 Suppose $p \neq 0$. We consider f_p first. The cases corresponding to $m \neq \infty$ and $m = \infty$ can be proved in similar ways, so we only consider the first case. Let $J = \{j_1 < j_2 < \ldots < j_s\}$, and let $w_i := r_{j_1} r_{j_2} \ldots r_{j_i}$. Based on Proposition 3.22, let $a < b$ be such that

$$j_a < k < j_{a+1} < \cdots < j_b = m < j_{b+1} ;$$

if $a = 0$ or $b + 1 > s$, then the corresponding indices j_a, respectively j_{b+1}, are missing. To show that $(J \setminus \{m\}) \cup \{k\}$ is an admissible subset, it is enough to prove that we have the path in the quantum Bruhat graph

$$w_a \longrightarrow w_a r_k \longrightarrow w_a r_k r_{j_{a+1}} \longrightarrow \cdots \longrightarrow w_a r_k r_{j_{a+1}} \ldots r_{j_{b-1}} = w_b . \qquad (17)$$

By our choice of k, we have

$$w_a(\beta_k) = \alpha_p \iff w_a^{-1}(\alpha_p) = \beta_k > 0 \iff w_a \lessdot s_p w_a = w_a r_k . \qquad (18)$$

So we can rewrite (17) as

$$w_a \longrightarrow s_p w_a \longrightarrow s_p w_{a+1} \longrightarrow \cdots \longrightarrow s_p w_{b-1} = w_b . \qquad (19)$$

We will now prove that (19) is a path in the quantum Bruhat graph. Observe that, for $a < i \leq b$, we have

$$s_p w_{i-1} = w_i = w_{i-1} r_{j_i} \iff w_{i-1}(\beta_{j_i}) = \pm \alpha_p \iff j_i \in I_\alpha .$$

Our choice of k and b implies that we have

$$s_p w_{i-1} \neq w_i \ \text{for} \ a < i < b, \quad \text{and} \ s_p w_{b-1} = w_b . \qquad (20)$$

Since J is admissible, we have

$$w_{i-1} \longrightarrow w_i . \qquad (21)$$

Starting from (18), and then using (20)–(21), we can apply Lemma 2.2 repeatedly to conclude that

$$s_p w_{i-1} \longrightarrow s_p w_i \ \text{and} \ w_i \lessdot s_p w_i , \quad \text{for} \ a < i < b . \qquad (22)$$

The proof for $e_p(J)$ is similar. The main difference is that we need the "if" part of Lemma 2.2, whereas above we used the "only if" part.

The above proof follows through for $p = 0$, based on Lemma 2.1, which is used to derive the analog of (18), and Lemma 2.3, which replaces Lemma 2.2.

We can prove that $f_p(J) = J'$ if and only if $e_p(J') = J$ based on [24] [Proposition 7.4 (1)]; this still holds in the above setup (for any p, including $p = 0$), based on Propositions 3.16–3.21. The same result can be invoked to derive the formulas for $\varphi_p(J)$ and $\varepsilon_p(J)$.

In order to show that f_p changes weights by $-\alpha_p$, the proof of [24][Proposition 7.1 (3)] can be applied in our context. In essence, we note that $\mu(f_p(J))$ is $-\widehat{t_k}\widehat{t_m}(-\mu)$ if $m \neq \infty$, and $-\widehat{t_k}(-\mu)$ otherwise, where $\widehat{t_j} := s_{|\gamma_j|, -l_j^J} = s_{\gamma_j, -\mathrm{sgn}(\gamma_j)l_j^J}$. By Proposition 3.22 and (14), we have $\widehat{t_k} = s_{\alpha_p, -M}$ and $\widehat{t_m} = s_{\alpha_p, -(M-1)}$. The rest of the calculation is the same as in the proof mentioned above. □

3.4 Main application

We summarize the main results in [22], cf. also [20,21]. The setup is that of untwisted affine root systems.

Theorem 3.24 *[22] Consider a composition* $\mathbf{p} = (p_1, \ldots, p_k)$ *and the corresponding crystal* $B := \bigotimes_{i=1}^{k} B^{p_i,1}$. *Let* $\lambda := \omega_{p_1} + \ldots + \omega_{p_k}$, *and let* Γ_{lex} *be a corresponding lex* λ-*chain (see above).*

(1) *The (combinatorial) crystal* $\mathcal{A}(\Gamma_{\mathrm{lex}})$ *is isomorphic to the subgraph of* B *consisting of the dual Demazure arrows, via a specific bijection which preserves the weights of the vertices.*
(2) *If the vertex* b *of* B *corresponds to* J *under the isomorphism in part* (1)*, then the energy is given by* $D_B(b) - C = -\mathrm{height}(J)$, *where* C *is a global constant.*

Remarks 3.25 (1) The entire crystal B is realized in [22] in terms of the so-called *quantum LS path model*. If we identify the two, the bijection in Theorem 3.24 (1) is the "forgetful map" from the quantum alcove model to the quantum LS path model, so it is a very natural map. Therefore, we think of the former model as a mirror image of the latter, via this bijection. However, if we use this identification to construct the non-dual Demazure arrows in the quantum alcove model, we quickly realize that, in general, the constructions are considerably more involved than (15)–(16), cf. Remark 4.8 (1) and Example 4.9.

(2) Although the quantum alcove model so far misses the non-dual Demazure arrows, it has the advantage of being a discrete model. Therefore, combinatorial methods are applicable, for instance in proving the independence of the model from the choice of an initial alcove path (or λ-chain of roots), see below, including the application in Remark 3.27 (2). This should be compared with the continuous arguments used for the similar purpose in the Littelmann path model [27].

(3) Theorem 3.24, combined with the Ram–Yip formula for Macdonald polynomials [30], implies that the graded character of a tensor product of column shape KR modules (the grading being by the energy function) concides with the corresponding Macdonald polynomial specialized at $t = 0$ [22].

Based on Theorem 3.24 (1), as well as on the realization of the same subgraph of B in types A and C in terms of a different λ-chain (see Theorems 4.7 and 4.22), we make the following conjecture.

Conjecture 3.26 *Theorem 3.24 holds for any choice of a λ-chain (instead of a lex λ-chain).*

We plan to prove this conjecture in [19] by using Theorem 3.24 as the starting point. Then, given two λ-chains Γ and Γ', we would construct a bijection between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ preserving the dual Demazure arrows, the weights of the vertices, and the height statistic; this would mean that the quantum alcove model does not depend on the choice of a λ-chain. This construction will be based on generalizing to the quantum alcove model the so-called Yang–Baxter moves in [15]. As a result, we would obtain a collection of a priori different affine crystal isomorphisms between B and $\mathcal{A}(\Gamma)$.

Remarks 3.27 (1) We believe that the bijections mentioned above would be identical. In fact, this would clearly be the case if all the tensor factors of B are perfect crystals, see Sect. 2.1. Indeed, then the subgraph of B consisting of the dual Demazure arrows is connected, so there is no more than one isomorphism between it and $\mathcal{A}(\Gamma)$.

(2) In the case when all the tensor factors of B are perfect crystals, a corollary of the work in [19] would be the following application of the quantum alcove model, cf. Remark 3.27 (1). By making specific choices for the λ-chains Γ and Γ', the bijection between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ mentioned above would give a uniform realization of the combinatorial R-matrix (see Remark 2.10 (2)); the details will appear in the forthcoming paper [19]. In fact, we believe that the previous statement would hold in full generality, rather than just the perfect case.

4 The quantum alcove model in types A and C

In this section we specialize the quantum alcove model to types A and C, and prove that the bijections constructed in [18], from the objects of the specialized quantum alcove model to the tensor products of the corresponding KN columns (see Sect. 2.1), are affine crystal isomorphisms.

4.1 Type A

We start with the basic facts about the root system of type A_{n-1}. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient $V := \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$, where $\mathbb{R}(1, \ldots, 1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1, \ldots, 1)$. Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j : i \neq j, \ 1 \leq i, j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1, \ldots, n-1$. The highest root $\widetilde{\alpha} = \alpha_{1n}$. We let $\alpha_0 = \theta = \alpha_{n1}$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n-1$. A dominant weight $\lambda = \lambda_1 \varepsilon_1 + \ldots + \lambda_{n-1} \varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ having at most $n-1$ parts. Note that $\rho = (n-1, n-2, \ldots, 0)$. Considering the Young diagram of the dominant weight λ as a concatenation of columns, whose heights are $\lambda'_1, \lambda'_2, \ldots$, corresponds to expressing λ as $\omega_{\lambda'_1} + \omega_{\lambda'_2} + \ldots$ (as usual, λ' is the conjugate partition to λ).

Fig. 3 The quantum Bruhat graph on S_3

The Weyl group W is the symmetric group S_n, which acts on V by permuting the coordinate vectors $\varepsilon_1, \ldots, \varepsilon_n$. Permutations $w \in S_n$ are written in one-line notation $w = w(1) \ldots w(n)$. For simplicity, we use the same notation (i, j) with $1 \leq i < j \leq n$ for the root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition t_{ij} of i and j. We recall a criterion for the edges of the type A quantum Bruhat graph. We need the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i + 1 \prec_i \ldots \prec_i n \prec_i 1 \prec_i \ldots \prec_i i - 1$. It is convenient to think of this order in terms of the numbers $1, \ldots, n$ arranged on a circle clockwise. We make the convention that, whenever we write $a \prec b \prec c \prec \ldots$, we refer to the circular order $\prec = \prec_a$.

Proposition 4.1 *[18] For $1 \leq i < j \leq n$, we have an edge $w \xrightarrow{(i,j)} w(i, j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

Example 4.2 The quantum Bruhat graph of type A_2, i.e., on the symmetric group S_3, is indicated in Fig. 3.

We now consider the specialization of the quantum alcove model to type A. For any $k = 1, \ldots, n - 1$, we have the following ω_k-chain, from $A_\circ$ to $A_{-\omega_k}$, denoted by $\Gamma(k)$ [23]:

$$
\begin{aligned}
&((k, k+1), \quad (k, k+2) \quad , \ldots, (k, n), \\
&(k-1, k+1), (k-1, k+2), \ldots, (k-1, n), \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\
&(1, k+1), \quad (1, k+2) \quad , \ldots, (1, n)) .
\end{aligned}
\tag{23}
$$

Example 4.3 We specialize (23) to $n = 4$ and $k = 1, 2, 3$. It is best to visualize $\Gamma(k)$ based on a column of height n broken into two pieces, with the top part of height k and the bottom one of height $n - k$; then $\Gamma(k)$ is obtained by pairing row numbers in the top and bottom parts, in the prescribed order.

$$\boxed{1}$$

$$\boxed{2}, \quad \Gamma(1) = ((1,2),(1,3),(1,4)); \qquad \boxed{\begin{matrix}1\\2\\3\end{matrix}}, \quad \Gamma(3) = ((3,4),(2,4),(1,4));$$

$$\boxed{\begin{matrix}1\\2\\3\\4\end{matrix}}, \quad \Gamma(2) = ((2,3),(2,4),(1,3),(1,4)).$$

Fix a dominant weight/partition λ for the remainder of this section. We construct a λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ as the concatenation $\Gamma := \Gamma^1 \ldots \Gamma^{\lambda_1}$, where $\Gamma^j = \Gamma(\lambda'_j)$. Let $J = \{j_1 < \cdots < j_s\}$ be a set of folding positions in Γ, not necessarily admissible, and let T be the corresponding list of roots of Γ, also viewed as transpositions. The factorization of Γ induces a factorization of T as $T = T^1 T^2 \ldots T^{\lambda_1}$, and of $\Delta = \Gamma(J)$ as $\Delta = \Delta^1 \ldots \Delta^{\lambda_1}$. Recalling that the roots in Δ were denoted γ_k, we use the notation $\gamma_k \in \Delta^q$ to indicate that the kth root in Δ falls in the segment Δ^q (rather than the fact that Δ^q contains a root equal to γ_k). We denote by $T^1 \ldots T^j$ the permutation obtained by composing the transpositions in $T^1, \ldots, T^j$ left to right. For $w \in W$, let $w_i = w(i)$. For w written in one-line notation as $w = w_1 w_2 \ldots w_n$, let $w[i, j] = w_i \ldots w_j$.

We now recall from [18] the construction of the correspondence between the type A quantum alcove model and model based on diagram fillings.

Definition 4.4 Let $\pi_j = \pi_j(T) := T^1 \ldots T^j$. We define the *filling map*, which associates with each $J \subseteq [m]$ a filling of the Young diagram λ, by

$$\text{fill}(J) = \text{fill}(T) := C_1 \ldots C_{\lambda_1}, \quad \text{where } C_i := \pi_i[1, \lambda'_i], \tag{24}$$

see the notation above. We define the *sorted filling map* sfill(J) by sorting ascendingly the columns of fill(J).

In other words, the ith column C_i of fill(J) consists of the first λ'_i entries of the permutation π_i, written in one-line notation; see Example 4.5.

Example 4.5 Let $n = 3$ and $\lambda = (4, 3, 0)$, which is identified with $4\varepsilon_1 + 3\varepsilon_2 = 3\omega_2 + \omega_1$, and corresponds to the Young diagram ⬚⬚⬚⬚. We have

$$\Gamma = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = \Gamma(2)\Gamma(2)\Gamma(2)\Gamma(1)$$
$$= ((2,3), \underline{(1,3)} \mid (2,3), (1,3) \mid (2,3), (1,3) \mid (1,2), (1,3)),$$

where we underlined the roots in positions $J = \{1, 2, 3, 5, 7\}$. Then

$$T = ((2, 3), (1, 3) \mid (2, 3) \mid (2, 3) \mid (1, 2)), \quad \text{and}$$

$$
\begin{aligned}
\Gamma(J) = \Delta \\
= \Delta^1 \Delta^2 \Delta^3 \Delta^4 = ((\underline{2, 3)}, \underline{(1, 2)} \mid \underline{(3, 1)}, (2, 3) \mid \underline{(1, 3)}, (2, 1) \mid \underline{(2, 3)}, (3, 1)),
\end{aligned}
\tag{25}
$$

where we again underlined the folding positions, and indicated the factorizations of T and Δ by vertical lines. It is easy to check that J is admissible; indeed, the sequence of permutations (12) corresponding to J is a path in the quantum Bruaht graph, cf. Proposition 4.1 and Example 4.2:

$$\tag{26}$$

Here each permutation in (12) is written vertically in one-line notation, with the entries in bold to be transposed; moreover, if the transposition to be applied lies in Γ^i, then the corresponding permutation is represented as a broken column with the top part of height λ_i'; see the structure of Γ^i in (23) and Example 4.3. By considering the top part of the last column in each segment and by concatenating these columns left to right, we obtain fill(J), i.e., fill(J) = $\begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\\hline 3 & 1 & 3 \\\cline{1-3}\end{array}$.

Theorem 4.6 *[18] [Theorem 4.1] The map* sfill *is a bijection between $\mathcal{A}(\Gamma)$ and $B^{\otimes \lambda}$, see* (8).

We now state the main result of this section.

Theorem 4.7 *The map* sfill *is an affine crystal isomorphism between $\mathcal{A}(\Gamma)$ and the subgraph of $B^{\otimes \lambda}$ consisting of the dual Demazure arrows. In other words, given* sfill$(J) = b$, *there is a dual Demazure arrow $b \to f_p(b)$ if and only if $f_p(J) \neq \mathbf{0}$, and we have $f_i(b) = $ sfill$(f_i(J))$.*

Remarks 4.8 (1) The affine crystal isomorphism in Theorem 4.7 is unique, cf. Remark 3.27 (1). Therefore, this isomorphism gives the unique way to realize the non-dual Demazure arrows in $\mathcal{A}(\Gamma)$, cf. Remark 3.25 (1) and Example 4.9 below.

(2) In [17] it was proved that the map sfill preserves weights, cf. Lemma 4.10. Furthermore, in [18] it was shown this map translates the height statistic to the charge statistic, which is known to express the energy function in the model based on diagram fillings, cf. Theorem 2.15. This should be compared with Theorem 3.24 (2), where the constant C is 0 in this case.

Example 4.9 In type A_2, consider $\lambda = (3, 2, 0)$, the λ-chain in Example 3.3, and the admissible subset $J = \{1, 2, 3, 5\}$, cf. Examples 3.6, 3.7, and 3.8. We have

$b = \text{sfill}(J) = \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 3 & 2 \\ \hline \end{array}$, and $f_0(b) = \text{sfill}(\emptyset) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$, cf. Example 2.5.
However, $b \to f_0(b)$ is not a dual Demazure arrow, and indeed $f_0(J) = \mathbf{0}$, cf.
Example 3.8. In order to realize this arrow in the quantum alcove model, we would
have to define $f_0(J) = \emptyset$. This shows that, in general, the changes in an admissible
subset corresponding to non-dual Demazure arrows are hard to control. Neverthe-
less, such arrows are sometimes still realized by our construction (15), assuming
that we drop the corresponding condition $M > 1$. For an example, still in type A_2,
consider $\lambda = (3, 0, 0)$, the λ-chain $(\alpha_{12}, \alpha_{13}, \alpha_{12}, \alpha_{13}, \alpha_{12}, \alpha_{13})$, and $J = \{3, 4\}$. We
have $b = \text{sfill}(J) = \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline \end{array}$, and $b \to f_0(b) = \begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline \end{array}$ is not a dual Demazure
arrow. Now note that the corresponding arrow in the quantum alcove model is
$J \mapsto J \cup \{5\}$, which is given by the mentioned relaxed version of (15).

The main idea of the proof of Theorem 4.7 is the following. The signature of a
filling, used to define the crystal operator f_p, can be interpreted as a graph similar to
the graph of g_{α_p}, which is used to define the crystal operator on the corresponding
admissible subsequence. The link between the two graphs is given by Lemma 4.12
below, called the height counting lemma, which we now explain.

Let $N_c(\sigma)$ denote the number of entries c in a filling σ. Let $\text{ct}(\sigma) = (N_1(\sigma), \ldots, N_n(\sigma))$ be the content of σ, which is identified with a type A_{n-1} weight.
Let $\sigma[q]$ be the filling consisting of the columns $1, 2, \ldots, q$ of σ. Given a λ-chain
and a corresponding sequence J (not necessarily admissible), recall the related nota-
tion, including the heights l_k^J in (10), the sequence of roots Δ, and its factorization
illustrated in (25).

Lemma 4.10 *[17] [Proposition 3.6] Let $J \subseteq [m]$, and $\sigma = \text{sfill}(J)$. Then we have*
$\mu(J) = \text{ct}(\sigma)$.

Corollary 4.11 *Let $J \subseteq [m]$, $\sigma = \text{fill}(J)$, and $\alpha \in \Phi$. Then $\text{sgn}(\alpha)l_\alpha^\infty = \langle \text{ct}(\sigma), \alpha^\vee \rangle$.*

The height counting lemma can be viewed as an extension of Corollary 4.11.

Lemma 4.12 *[17] [Proposition 4.1] Let $J \subseteq [m]$, and $\sigma = \text{fill}(J)$. For a fixed k, let
$\gamma_k = (c, d)$ be a root in Δ^{q+1}. We have*

$$\text{sgn}(\gamma_k)\, l_k^J = \langle \text{ct}(\sigma[q]), \gamma_k^\vee \rangle = N_c(\sigma[q]) - N_d(\sigma[q]).$$

We now introduce notation to be used for the remainder of this section. Let $p \in \{1, \ldots, n-1\}$. Let J be an admissible subset and let $\sigma = \text{sfill}(J) = C_1 \ldots C_{\lambda_1}$. For
$i = 1, \ldots, \lambda_1$, let $a_i := \langle \text{ct}(C_i), \alpha_p^\vee \rangle$, and note that $a_i \in \{1, -1, 0\}$; here $a_i = 1$ (resp.
$a_i = -1$) corresponds to C_i containing p but not $p + 1$ (resp. $p + 1$ but not p), while
$a_i = 0$ corresponds to C_i containing both p and $p + 1$, or neither of them.

The sequence a_i corresponds to the p-signature from Sect. 2.1, as we now explain.
For $j = 0, \ldots, \lambda_1$, let $h_j := \langle \text{ct}(\sigma[j]), \alpha_p^\vee \rangle = \sum_{i=0}^{j} a_i$, where we set $a_0 = h_0 := 0$.
It is useful to think of this sequence as a piecewise linear function, by analogy with
the function g_{α_p} used to define the crystal operator f_p in the quantum alcove model.
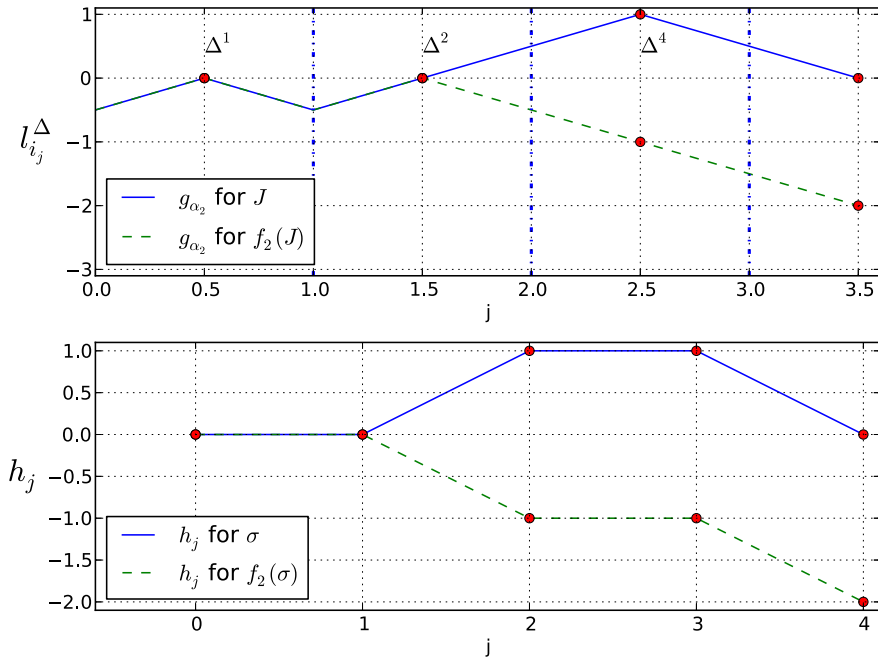
Fig. 4 Effects of f_2 on g_α and h_j

Let M' be the maximum of h_j, and let m' be minimal with the property $h_{m'} = M'$. We clearly have $M' \geq 0$. If $M' > 0$ then $a_{m'} = 1$, and m' is the position of the rightmost $+$ in the p-signature of σ. It follows that f_p changes the p in column m' of σ to $p+1$. The previous observations hold if we replace α_p, f_p, and the p-signature with α_0, f_0, and the 0-signature, while at the same time we replace the entries p and $p+1$ in a filling with n and 1, respectively. Therefore, from now on we assume that the index p is in $\{0, 1, \ldots, n-1\}$.

Example 4.13 We continue Example 4.5. Let $\sigma = \mathrm{sfill}(J) = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 2 & 3 \\ \hline 3 & 2 & 3 \\ \cline{1-3} \end{array}$, then

$f_2(\sigma) = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 2 & 3 \\ \hline 3 & 3 & 3 \\ \cline{1-3} \end{array}$. Let $p = 2$ and refer to Fig. 4. From the graph g_{α_2} for J,

we can see that $M = 1$. We note that $m = 7$, with $\gamma_7 \in \Delta^4$, and $k = 4$ with $\gamma_k \in \Delta^2$. So $f_2(J) = (J\backslash\{7\}) \cup \{4\} = \{1, 2, 3, 4, 5\}$, and

$$\Gamma(f_2(J)) = (\underline{(2,3)}, \underline{(1,2)} \mid \underline{(3,1)}, \underline{(2,3)} \mid \underline{(1,2)}, (3,1) \mid (3,2), (3,1)),$$

where we underlined roots in positions $f_2(J)$. From the graph corresponding to h_j for J, we can see that $M' = 1$ and $m' = 2$.

Lemma 4.14 *If $\alpha_p = \gamma_k \in \Delta^q$ with $k \notin J$ then $a_q = 1$.*

Proof Recall that $\alpha_p = \gamma_k = w(\beta_k)$ for the corresponding w defined in (9), and let $\beta_k = (b, c)$. The result follows from the claim that $w(b) = \pi_q(b)$ and $w(c) = \pi_q(c)$ (cf. Definition 4.4), which is a consequence of the structure of Γ^q (cf. (23)), as we now explain. The only reflections in Γ^q to the right of β_k that affect values in positions b or c are (b, c') for $c' > c$, and (b', c) for $b' < b$. Applying these reflections (on the right) to any w' satisfying $w'(b) = w(b)$ and $w'(c) = w(c)$ does not give an edge in the quantum Bruhat graph, by the corresponding criterion in Proposition 4.1.

Fix an admissible subset J, and recall Proposition 3.22, including the notation therein. In particular, M is the maximum of g_{α_p}. Moreover, if $M > \delta_{p,0}$ we defined $k \notin J$ and $m \in J \cup \{\infty\}$ with $\gamma_k = \alpha_p$, $\mathrm{sgn}(\alpha_p)l_k^J = M - 1$, and $\gamma_m = \alpha_p$ when $m \neq \infty$. We will implicitly use the following observation when applying Lemma 3.11 in the next two proofs: if $a_i \neq 0$ then $\mathrm{sgn}(a_i) = \mathrm{sgn}(\pi_i^{-1}(\alpha_p))$, where π_i is given in Definition 4.4.

Proposition 4.15 *We have* $M \geq M'$. *If* $M \geq \delta_{p,0}$ *then* $M = M'$.

Proof We first prove that $M \geq M'$. By Corollary 4.11 we have $h_{\lambda_1} = \mathrm{sgn}(\alpha)l_{\alpha_p}^\infty$, so the case $M' = h_{\lambda_1}$ is clear. The case $M' = 0$ is trivial, since $M \geq 0$. Therefore, we can assume that the maximum of the sequence h_i does not occur at its endpoints $i = 0$ and $i = \lambda_1$. Then we can find $i < j$ such that $h_i = M'$, $a_i > 0$, $a_j < 0$, and $a_t = 0$ for $t \in (i, j)$. By Lemma 3.11 there exists $\gamma_{k'} = \alpha_p \in \Delta^q$ with $q \in (i, j]$, and by Lemma 4.12 we have $\mathrm{sgn}(\alpha_p)l_{k'}^J = h_{q-1} = h_i = M'$. Hence $M \geq M'$.

By (14), Lemma 4.12, and Corollary 4.11, all the values of g_{α_p} at points in $\mathbb{Z} + \frac{1}{2}$ are among the values h_i. If $M \geq \delta_{p,0}$ then, by Propositions 3.16 and 3.19, the maximum M is attained at a point in $\mathbb{Z} + \frac{1}{2}$. It follows that $M \leq M'$, hence $M = M'$. $\square$

The previous proposition states that $M = M'$ except in a few corner cases that occur when $p = 0$. We will sometimes use one symbol in favor of the other to allude to the corresponding graph.

Proposition 4.16 *Assume that* $M > \delta_{p,0}$, *so* $M = M'$ *(by Proposition 4.15) and* $f_p(J) \neq \mathbf{0}$. *Then* $\gamma_k \in \Delta^{m'}$. *If* $m \neq \infty$, *so* $\gamma_m \in \Delta^{m''}$ *for some* $m'' \geq m'$, *then* $a_i = 0$ *for* $i \in (m', m'')$; *if* $m = \infty$ *then* $a_i = 0$ *for* $i > m'$.

Proof Assuming $\gamma_k \in \Delta^j$, by Proposition 3.22 (2), Lemma 4.14, and Lemma 4.12, we have $a_j = 1$ and $\mathrm{sgn}(\alpha_p)l_k^J = M - 1 = h_{j-1}$. It follows that $h_j = M = M'$. By the definition of m' and the fact that $M' > 0$, we have $1 \leq m' \leq j$ and $a_{m'} = 1$. By way of contradiction suppose that $m' < j$. It follows that the set $\{i \in (m', j] \mid a_i \neq 0\}$ is not empty, so let t be its minimum. We have $a_t = -1$ and $t < j$, because $a_t = 1$ would imply $h_t > h_{m'} = M'$. We can now apply Lemma 3.11 to show that there exists $\gamma_{k'} = \alpha_p \in \Delta^q$ with $q \in (m', t]$ and $k' \in J$. By Lemma 4.12, we have $\mathrm{sgn}(\alpha_p)l_{k'}^J = h_{q-1} = h_{m'} = M' = M$. Thus, since $k' < k < m$, the minimality of m is contradicted. We conclude that $j = m'$. If $m \neq \infty$, we can use a similar proof to conclude that the set $\{i \in (m', m'') \mid a_i \neq 0\}$ is empty. The case $m = \infty$ is done similarly. $\square$

Proof of Theorem 4.7 We continue to use the notation from the above setup. Recall that $b = \mathrm{sfill}(J)$. The statement that there is a dual Demazure arrow $b \to f_p(b)$ if and only if $f_p(J) \neq \mathbf{0}$ follows from Proposition 4.15; indeed, it is clear that $\varphi_p(b) = M'$, whereas $\varphi_p(J) = M - \delta_{p,0}$ in the quantum alcove model if $M \geq \delta_{p,0}$, by Theorem 3.9 (2).

We next show that $f_p(b) = \mathrm{sfill}(f_p(J))$, when $f_p(J) \neq \mathbf{0}$. Since $f_p(b) \neq \mathbf{0}$, we have $M' > 0$, and f_p changes the p in column m' to $p + 1$ (f_0 changes n to 1 and sorts the column). Now let us turn to $f_p(J)$, where we write the admissible subset J as $\{j_1 < \cdots < j_s\}$. Let $w_i := r_{j_1} \ldots r_{j_i}$ be the corresponding sequence of permutations. (Recall that the filling $\mathrm{fill}(J)$ is constructed from a subsequence of w_i, see Definition 4.4.) We assume $m \neq \infty$, as the case $m = \infty$ is proved similarly. There exist $a < b$ such that

$$j_a < k < j_{a+1} < \cdots < j_b = m < j_{b+1} \, ;$$

if $a = 0$ or $b = s$, then the corresponding indices j_a, respectively j_{b+1} are missing. The sequence of permutations associated to $f_p(J)$ is

$$1, \, w_1, \, \ldots, \, w_a, \, s_p w_a, \, s_p w_{a+1}, \, \ldots, \, s_p w_{b-1} = w_b, \, w_{b+1}, \, \ldots, \, w_s$$

(see (19)). By the first part of Proposition 4.16 and by using the notation therein, we conclude that $\mathrm{sfill}(f_p(J))$ is obtained from $\mathrm{sfill}(J)$ by interchanging p and $p + 1$ in columns i for $i \in [m', m'')$ (interchange n with 1 if $p = 0$). By the second part of Proposition 4.16, this amounts to changing the p in column m' to $p + 1$ (n to 1 if $p = 0$). $\qquad\square$

4.2 Type C

We start with the basic facts about the root system of type C_n. We can identify the space $\mathfrak{h}_{\mathbb{R}}^*$ with $V := \mathbb{R}^n$, the coordinate vectors being $\varepsilon_1, \ldots, \varepsilon_n$. The root system is $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n - 1$, and $\alpha_n = 2\varepsilon_n$. The highest root $\widetilde{\alpha} = 2\varepsilon_1$. We let $\alpha_0 = \theta = -2\varepsilon_1$. The weight lattice is $\Lambda = \mathbb{Z}^n$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \ldots, n$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0)$ of length at most n. Note that $\rho = (n, n - 1, \ldots, 1)$. Like in type A, writing the dominant weight λ as a sum of fundamental weights corresponds to considering the Young diagram of λ as a concatenation of columns. We fix a dominant weight λ throughout this section.

The Weyl group W is the group of signed permutations B_n, which acts on V by permuting the coordinates and changing their signs. A signed permutation is a bijection w from $[\overline{n}] := \{1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{1}\}$ to $[\overline{n}]$ satisfying $w(\overline{i}) = \overline{w(i)}$. Here $\overline{i}$ is viewed as $-i$, so $\overline{\overline{i}} = i$, $|\overline{i}| = i$, and $\mathrm{sign}(\overline{i}) = -1$. We use both the window notation $w = w_1 \ldots w_n$ and the full one-line notation $w = w(1) \ldots w(n) w(\overline{n}) \ldots w(\overline{1})$ for signed permutations. For simplicity, given $1 \leq i < j \leq n$, we denote by (i, j) the root $\varepsilon_i - \varepsilon_j$ and the corresponding reflection,

which is identified with the composition of transpositions $t_{ij}t_{\bar{j}\bar{\imath}}$. Similarly, we denote by $(i,\bar{\jmath}) = (j,\bar{\imath})$, for $1 \leq i < j \leq n$, the root $\varepsilon_i + \varepsilon_j$ and the corresponding reflection, which is identified with the composition of transpositions $t_{i\bar{\jmath}}t_{j\bar{\imath}}$. Finally, we denote by $(i,\bar{\imath})$ the root $2\varepsilon_i$ and the corresponding reflection, which is identified with the transposition $t_{i\bar{\imath}}$.

We recall a criterion for the edges of the type C quantum Bruhat graph. We need the circular order $\prec_i$ on $[\bar{n}]$ starting at i, which is defined in the obvious way, cf. Sect. 4.1. It is convenient to think of this order in terms of the numbers $1, \ldots, n, \bar{n}, \ldots, \bar{1}$ arranged on a circle clockwise. We make the same convention as in Sect. 4.1 that, whenever we write $a \prec b \prec c \prec \ldots$, we refer to the circular order $\prec = \prec_a$.

Proposition 4.17 *[18]*

(1) *Given $1 \leq i < j \leq n$, we have an edge $w \xrightarrow{(i,j)} w(i,j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

(2) *Given $1 \leq i < j \leq n$, we have an edge $w \xrightarrow{(i,\bar{\jmath})}$ if and only if $w(i) < w(\bar{\jmath})$, $\mathrm{sign}(w(i)) = \mathrm{sign}(w(\bar{\jmath}))$, and there is no k such that $i < k < \bar{\jmath}$ and $w(i) < w(k) < w(\bar{\jmath})$.*

(3) *Given $1 \leq i \leq n$, we have an edge $w \xrightarrow{(i,\bar{\imath})} w(i,\bar{\imath})$ if and only if there is no k such that $i < k < \bar{\imath}$ (or, equivalently, $i < k \leq n$) and $w(i) \prec w(k) \prec w(\bar{\imath})$.*

We now consider the specialization of the quantum alcove model to type C. For any $k = 1, \ldots, n$, we have the following ω_k-chain, from $A_\circ$ to $A_{-\omega_k}$, denoted by $\Gamma(k)$ [16]:

$$
\begin{aligned}
&\Gamma(k) := \Gamma_l(k)\Gamma_r(k) \text{ where} \\
&\Gamma_l(k) := \Gamma_{kk} \ldots \Gamma_{k1}, \quad \Gamma_r(k) := \Gamma_k \ldots \Gamma_2, \qquad\qquad (27)\\
&\Gamma_i := ((i,\overline{i-1}), (i,\overline{i-2}), \ldots, (i,\bar{1})), \\
&\Gamma_{ki} := ((i,k+1), (i,k+2), \ldots, (i,n), \\
&\qquad\quad (i,\bar{\imath}), \\
&\qquad\quad (i,\bar{n}), \quad (i,\overline{n-1}), \ldots, (i,\overline{k+1}), \\
&\qquad\quad (i,\overline{i-1}), (i,\overline{i-2}), \ldots, (i,\bar{1})) \, .
\end{aligned}
$$

Example 4.18 We specialize (27) to $n = 3$ and $k = 2$. In this case

$$\Gamma(2) = \Gamma_l(2)\Gamma_r(2) = ((2,3), (2,\bar{2}), (2,\bar{3}), (2,\bar{1}), (1,3), (1,\bar{1}), (1,\bar{3}) \mid (2,\bar{1})).$$

On the right hand side we use a vertical line to denote the end of $\Gamma_l(2)$. For a graphical interpretation similar to the one in Example 4.3, based on broken columns, we refer to [18][eq. (7.1)].

Fix a dominant weight/partition λ for the remainder of this section. We construct a λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ as a concatenation $\Gamma := \Gamma^1 \ldots \Gamma^{\lambda_1}$, where $\Gamma^j = \Gamma(\lambda'_j)$; we also let $\Gamma^j_l := \Gamma_l(\lambda'_j)$ and $\Gamma^j_r := \Gamma_r(\lambda'_j)$. Like in type A, given a set $J = \{j_1 < \cdots < j_s\}$ of folding positions in Γ, not necessarily admissible, we let T be

the corresponding list of roots of Γ, also viewed as type C transpositions. We factor Γ as $\Gamma = \widetilde{\Gamma}^1 \ldots \widetilde{\Gamma}^{2\lambda_1}$, where $\widetilde{\Gamma}^{2i-1} = \Gamma_l^i$ and $\widetilde{\Gamma}^{2i} = \Gamma_r^i$, for $1 \le i \le \lambda_1$. This factorization of Γ induces a factorization of T as $T^1 T^2 \ldots T^{2\lambda_1}$, and of $\Delta = \Gamma(J)$ as $\Delta = \Delta^1 \ldots \Delta^{2\lambda_1}$. Like in type A, we use the notation $\gamma_k \in \Delta^q$ to indicate that the kth root in Δ falls in the segment Δ^q. We denote by $T^1 T^2 \ldots T^j$ the signed permutation obtained by composing the type C transpositions in $T^1, \ldots, T^j$ left to right. For $w \in W$, let $w_i = w(i)$. For w written in the window notation as $w = w_1 w_2 \ldots w_n$, let $w[i, j] = w_i \ldots w_j$.

We now recall from [18] the construction of the correspondence between the type C quantum alcove model and model based on diagram fillings; it is useful to compare with Definition 4.4 in type A.

Definition 4.19 Let $\pi_j = \pi_j(T) := T^1 \ldots T^j$. We define the *filling map*, which associates with each $J \subseteq [m]$ a filling of the Young diagram 2λ, by

$$\text{fill}(J) = \text{fill}(T) := C_1 \ldots C_{2\lambda_1}, \quad \text{where } C_i := \pi_i[1, \lambda'_{\lceil \frac{i}{2} \rceil}], \qquad (28)$$

see the notation above. We define the *sorted filling map* sfill(J) by sorting ascendingly the columns of the filling fill(J).

Example 4.20 Let $n = 3$ and $\lambda = (2, 1, 0)$, which is identified with $2\varepsilon_1 + \varepsilon_2 = \omega_2 + \omega_1$ and corresponds to the Young diagram ⊟. We have the following λ-chain, see Example 4.18:

$$\Gamma = \Gamma(2)\Gamma(1) = \Gamma_l(2)\Gamma_r(2)\Gamma_l(1)\Gamma_r(1)$$
$$= ((2, 3), \underline{(2, \bar{2})}, \underline{(2, \bar{3})}, (2, \bar{1}), (1, 3), (1, \bar{1}), (1, \bar{3}) \mid \underline{(2, \bar{1})} \mid\mid (1, 2), (1, 3), \underline{(1, \bar{1})}, (1, \bar{3}), \underline{(1, \bar{2})} \mid),$$

where we underlined the roots in positions $J = \{1, 2, 3, 8, 9, 11\}$. On the right hand side we used double vertical lines to indicate the end of $\Gamma(2)$, and single vertical lines indicate the end of $\Gamma_l(k)$. Then

$$T = ((2, 3), (2, \bar{2}), (2, \bar{3}) \mid (2, \bar{1}) \mid\mid (1, 2), (1, \bar{1}) \mid),$$

and

$$\Gamma(J) = \Delta = \Delta^1 \Delta^2 \Delta^3 \Delta^4$$
$$= (\underline{(2, 3)}, \underline{(3, \bar{3})}, \underline{(\bar{3}, \bar{2})}, (\bar{2}, \bar{1}), (1, 3), (1, \bar{1}), (1, \bar{3}) \mid \underline{(\bar{2}, \bar{1})} \mid\mid \underline{(2, \bar{1})}, (\bar{1}, 3), \underline{(\bar{1}, 1)}, (1, \bar{3}), (1, \bar{2}) \mid);$$

here we again underlined the folding positions, and indicated the factorizations of T and Δ by vertical lines.

It is easy to check that J is admissible using Proposition 4.17; indeed, the sequence of signed permutations (12) corresponding to J is a path in the quantum Bruhat graph:

$$\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} <\boxed{\begin{smallmatrix}1\\3\\2\end{smallmatrix}} <\boxed{\begin{smallmatrix}1\\\overline{3}\\2\end{smallmatrix}} |\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} <\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} <\boxed{\begin{smallmatrix}2\\1\\3\end{smallmatrix}} ||\boxed{\begin{smallmatrix}2\\\overline{1}\\3\end{smallmatrix}} <\boxed{\begin{smallmatrix}\overline{1}\\2\\3\end{smallmatrix}} \lhd\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} |\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} || \, .$$

Here, like in Example 4.5 in type A, each signed permutation in (12) is written vertically in the window notation, with the entries in bold to be changed by the corresponding reflection; moreover, if this reflection lies in Γ_l^i or Γ_r^i, then the corresponding signed permutation is represented as a broken column with the top part of height λ_i'; see the structure of Γ_l^i and Γ_r^i in (27) and Example 4.18. By considering the top part of the last column in each segment and by concatenating these columns left to right, we obtain fill(J), i.e., sfill(J) = fill(J) = $\begin{smallmatrix}\boxed{1\,2\,1\,1}\\\boxed{2\,1}\end{smallmatrix}$.

Recall from (8) the definition of $B^{\otimes\lambda}$, which is now realized with split KN columns, see Sect. 2.1. As such, its arrows are given by f_p^2, according to Proposition 2.9 (2).

Theorem 4.21 *[18] [Theorem 6.1] The map* sfill *is a bijection between* $\mathcal{A}(\Gamma)$ *and* $B^{\otimes\lambda}$.

We now state the main result of this section, cf. Theorem 4.7 in type A.

Theorem 4.22 *The map* sfill *is an affine crystal isomorphism between* $\mathcal{A}(\Gamma)$ *and the subgraph of* $B^{\otimes\lambda}$ *consisting of the dual Demazure arrows.*

Remarks 4.23 (1) The affine crystal isomorphism in Theorem 4.22 is not guaranteed to be unique in general. However, we believe that it coincides with the bijection that we plan to construct (in a type-independent setup) in order to prove Conjecture 3.26. See Remark 3.27 (1).

(2) In [16] it was proved that the map sfill preserves weights, cf. Lemma 4.24. Furthermore, in [18] it was shown that this map translates the height statistic to the charge statistic, which is known to express the energy function in the model based on KN columns, cf. Theorem 2.15. This should be compared with Theorem 3.24 (2), where the constant C is 0 in this case.

The proof of Theorem 4.21 is parallel to the proof of Theorem 4.7. In this case, we use the height counting lemma in type C_n, namely Lemma 4.26. As before let $N_i(\sigma)$ denote the number of entries i in a filling σ. Let $c_i = c_i(\sigma) := \frac{1}{2}(N_i(\sigma) - N_{\bar{i}}(\sigma))$ and define the content of a filling σ as $\mathrm{ct}(\sigma) := (c_1, c_2, \ldots, c_n)$, which is identified with a type C_n weight. Let $\sigma[q]$ be the filling consisting of the columns $1, 2, \ldots, q$ of σ. Given a λ-chain and a corresponding sequence J (not necessarily admissible), recall the related notation, including the heights l_k^J in (10), the sequence of roots Δ, and its factorization.

Lemma 4.24 *[16][Proposition 4.6 (2)] Let $J \subseteq [m]$, and $\sigma = \mathrm{sfill}(J)$. Then we have $\mu(J) = \mathrm{ct}(\sigma)$.*

Corollary 4.25 *Let $J \subseteq [m]$, $\sigma = \mathrm{fill}(J)$, and $\alpha \in \Phi$. Then $\mathrm{sgn}(\alpha)l_\alpha^\infty = \langle \mathrm{ct}(\sigma), \alpha^\vee \rangle$.*

Lemma 4.26 *[16][Proposition 6.1] Let $J \subseteq [m]$, and $\sigma = \mathrm{fill}(J)$. For a fixed k, let γ_k be a root in Δ^{q+1}. We have*

$$\mathrm{sgn}(\gamma_k)\, l_k^J = \langle \mathrm{ct}(\sigma[q]), \gamma_k^\vee \rangle.$$

We now introduce notation to be used for the remainder of this section. Let $p \in \{0, 1, \ldots, n\}$. Let J be an admissible subset, and $\sigma = \mathrm{sfill}(J) = C_1 \ldots C_{2\lambda_1}$, which is guaranteed to be in $B^{\otimes \lambda}$ by Theorem 4.21. Let $a_i := \langle \mathrm{ct}(C_i), \alpha_p^\vee \rangle$. We have $a_i \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$, for $1 \leq p \leq n-1$, and $a_i \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ for $p \in \{0, n\}$.

Remarks 4.27 (1) The value of a_i indicates which entries related to the action of f_p are contained in column C_i, as we now explain. Assuming first that $1 \leq p \leq n-1$, the relevant entries are $\mathcal{P} := \{p, \ p+1, \ \overline{p+1}, \ \overline{p}\}$. If $a_i = 1$ (resp. $a_i = -1$), then C_i contains both p and $\overline{p+1}$ (resp. $p+1$ and $\overline{p}$). If $a_i = \frac{1}{2}$ (resp. $a_i = -\frac{1}{2}$), then C_i contains only one of p, $\overline{p+1}$ (resp. only one of $p+1$, $\overline{p}$), while if $a_i = 0$ then C_i contains both p and $p+1$, or both $\overline{p+1}$ and $\overline{p}$, or none of these elements. For $p = n$, the relevant entries are n and $\overline{n}$. If $a_i = \frac{1}{2}$ (resp. $a_i = -\frac{1}{2}$), then C_i contains n (resp. $\overline{n}$), while if $a_i = 0$ then C_i contains both n and $\overline{n}$, or none of these elements. The case $p = 0$ is similar to $p = n$: just replace n and $\overline{n}$ with $\overline{1}$ and 1, respectively.

(2) The sequence a_i corresponds to the p-signature of the filling σ. To be more precise, associate with this sequence a $(+, -)$-word by replacing $\pm \frac{1}{2}$ with $\pm$ and ± 1 with $\pm\pm$ (the 0's are ignored). If $1 \leq p \leq n-1$, this is the same as the $(+, -)$-word associated with σ (see Sect. 2.1) after canceling $-+$ pairs corresponding to the entries p and $p+1$ (or $\overline{p+1}$ and $\overline{p}$) in a column; similarly for $p = n$ and $p = 0$.

Let $h_j := \langle \mathrm{ct}(\sigma[j]), \alpha_p^\vee \rangle = \sum_{i=0}^{j} a_i$, with $a_0 = h_0 := 0$. Like in type A, let $M' \geq 0$ be the maximum of h_j, and let m' be minimal with the property $h_{m'} = M'$. If $M' > 0$ then m' is the number of the column containing the entry changed by f_p. Recall that we need to apply f_p twice; the way in which this can happen is described below.

Proposition 4.28 *If $1 \leq p \leq n-1$, then we always have one of the following cases related to the action of f_p^2 on the filling σ.*

(i) *$m' = 2i - 1$ and $a_{m'} = 1$: p and $\overline{p+1}$ in column m' are changed to $p+1$ and $\overline{p}$.*
(ii) *same as (i) with $m' = 2i$.*
(iii) *$m' = 2i$ and $a_{m'} = a_{m'-1} = \frac{1}{2}$: columns m' and $m'-1$ both contain an entry p (or both contain $\overline{p+1}$), and these entries are changed to $p+1$ (resp. $\overline{p}$).*

If $p = n$ or $p = 0$, then the analog of case (iii) always holds, with n changed to $\overline{n}$, resp. $\overline{1}$ changed to 1.

Proof We implicitly use the following observation, which is immediate from the construction of the splitting (lC, rC) of a column C in Definition 2.7: given $x \in [n]$, the column lC contains x or $\overline{x}$ if and only if rC does.

We consider only $1 \le p \le n - 1$, as the proof is simpler for $p = n$ and $p = 0$. We first prove the following claim: if $a_{2i-1} = \pm\frac{1}{2}$ (or $a_{2i} = \pm\frac{1}{2}$), then C_{2i-1} and C_{2i} contain a single element in $\mathcal{P}$, the two elements have the same absolute value, and the pair (a_{2i-1}, a_{2i}) can take only the following values: $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, or $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

We consider only the case $a_{2i-1} = \frac{1}{2}$, as the others are completely similar. The assumption implies that column C_{2i-1} contains a single element in $\mathcal{P}$, namely p or $\overline{p+1}$. In the first case, it is clear that C_{2i} does not contain $p + 1$ or $\overline{p+1}$, but it contains either p or $\overline{p}$. It suffices to rule out the occurence of $\overline{p}$. Assuming it, we deduce that the column C whose splitting is (C_{2i-1}, C_{2i}) contains $z_j > p$ and $\overline{z_j}$, and the corresponding $t_j < z_j$ is p (cf. Definition 2.7). But C cannot contain $p + 1$ or $\overline{p+1}$, so $z_j > p + 1$, and the maximality of t_j is contradicted. In the second case, we need to rule out the occurence of $p + 1$ in C_{2i}. Assuming it, we deduce that C contains $z_j = p + 1$ and $\overline{z_j}$, and we have $t_j < p$, so again the maximality of t_j is contradicted.

Now consider the $(+, -)$-word associated with the sequence a_j, see Remark 4.27 (2). Cancel pairs $-+$ corresponding to the case $(a_{2i-1}, a_{2i}) = \left(-\frac{1}{2}, \frac{1}{2}\right)$ mentioned above. The above claim implies that the resulting word is a concatenation of pairs $++$ and $--$ which come from $a_j = \pm 1$ and $(a_{2i-1}, a_{2i}) = \left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$; recall that for the latter pairs, the claim also gives the corresponding entries in $\mathcal{P}$. The statement of the proposition now follows. $\qquad\square$

The following is the analog of Lemma 4.14.

Lemma 4.29 *If $\alpha_p = \gamma_k \in \Delta^{2i-1}$ with $k \notin J$ then we have either $a_{2i-1} = \frac{1}{2}$ and $a_{2i} = \frac{1}{2}$, or $a_{2i-1} = 1$. If $\alpha_p = \gamma_k \in \Delta^{2i}$ with $k \notin J$ then $a_{2i} = 1$.*

Proof We only consider the case corresponding to $\alpha_p \in \Delta^{2i-1}$ and $1 \le p \le n - 1$, as the others are simpler. We use freely the structure of the chain of roots $\widetilde{\Gamma}^{2i-1}$, see (27). Recall that $\alpha_p = \gamma_k = w(\beta_k)$ for the corresponding w defined in (9). We have the following cases:

(1) $\beta_k = (b, c)$ with $b \le \lambda_i'$, $c > \lambda_i'$, and $w(b) = p$, $w(c) = p + 1$ (or $w(b) = \overline{p+1}$, $w(c) = \overline{p}$);
(2) $\beta_k = (b, \overline{c})$ with $b \le \lambda_i'$, $c > \lambda_i'$, and $w(b) = p$, $w(c) = \overline{p+1}$ (or $w(b) = \overline{p+1}$, $w(c) = p$);
(3) $\beta_k = (b, \overline{c})$ with $c < b \le \lambda_i'$, and $w(b) = p$, $w(c) = \overline{p+1}$ (or $w(b) = \overline{p+1}$, $w(c) = p$).

We will only consider the first case with $w(b) = p$, $w(c) = p + 1$, as the others are completely similar. We first claim that $w(b) = \pi_{2i-1}(b)$ (cf. Definition 4.4), i.e., the entry p is not moved by the reflections $s_{\beta_{k'}}$ for $\beta_{k'}$ in $\widetilde{\Gamma}^{2i-1}$ with $k' > k$. These reflections are (b, c') with $c' > c$, $(b, \overline{b})$, $(b, \overline{c'})$ with $c' > \lambda_i'$, and $(b, \overline{c'})$ with $c' < b$. So if the claim failed, the quantum Bruhat graph criterion in Proposition 4.17 would be violated, because the entry $p + 1$ is still in position c when these reflections are applied.

Let us now track the entry $p+1$ as we apply the subsequent reflections not involving position b, by freely using the quantum Bruhat graph criterion. If this entry is not moved by any of these reflections, then $\pi_{2i-1}(c) = p + 1$, so $a_{2i-1} = \frac{1}{2}$. The first reflection which can move $p + 1$ is of the form $(b', \overline{c})$ with $b' < b$, which means that in position b' we will now have the entry $\overline{p+1}$. If this entry is not moved by any of the subsequent reflections $(b', \overline{c'})$ with $\lambda'_i < c' < c$, then it is not moved by any of the remaining reflections either, so $\pi_{2i-1}(b') = \overline{p+1}$, and $a_{2i-1} = 1$. Otherwise, we will have the entry $p + 1$ in a position c' with $\lambda'_i < c' < c$, and the above reasoning can be applied again (a finite number of times).

Finally, the fact that if $a_{2i-1} = \frac{1}{2}$ then $a_{2i} = \frac{1}{2}$ was deduced in the proof of Proposition 4.28.   □

Recall Proposition 3.22 and the notation therein. M is the maximum of g_{α_p}, and suppose $M > \delta_{p,0}$; then $\gamma_k = \alpha_p$ with $k \notin J$, $\mathrm{sgn}(\alpha_p)l_k^J = M - 1$, and if $m \neq \infty$ then $\gamma_m = \alpha_p$ with $m \in J$. The following result is the analog of Proposition 4.15, and its proof is identical. Indeed, the following key fact is still true: if $a_i \neq 0$ then $\mathrm{sgn}(a_i) = \mathrm{sgn}(\pi_i^{-1}(\alpha_p))$, where π_i is given in Definition 4.19 (simply note that $a_i = \langle \pi_i(\omega_k), \alpha_p^\vee \rangle = \langle \omega_k, \pi_i^{-1}(\alpha_p^\vee) \rangle$, where $k = \lambda'_{\lceil \frac{i}{2} \rceil}$). This will be needed in the proof of Proposition 4.31 as well.

Proposition 4.30 *We have $M \geq M'$. If $M \geq \delta_{p,0}$, then $M = M'$.*

The following result is the analog of Proposition 4.16.

Proposition 4.31 *Assume that $M > \delta_{p,0}$, so $M = M'$ (by Proposition 4.30) and $f_p(J) \neq \mathbf{0}$. If $a_{m'} = 1$ then $\gamma_k \in \Delta^{m'}$, otherwise $\gamma_k \in \Delta^{m'-1}$. If $m \neq \infty$, so $\gamma_m \in \Delta^{m''}$, then $a_i = 0$ for $i \in (m', m'')$; if $m = \infty$, then $a_i = 0$ for $i > m'$.*

Proof Assuming $\gamma_k \in \Delta^j$, by Proposition 3.22 (2), Lemma 4.29, and Lemma 4.26, we have $\mathrm{sgn}(\alpha_p)l_k^J = M - 1 = h_{j-1}$, and either $a_j = 1$ or $a_j = a_{j+1} = \frac{1}{2}$. In the first case, the rest of the proof is essentially identical to that of Proposition 4.16; in particular, we show that $j = m'$. In the second case, we have $j = 2i - 1$, so it follows that $h_{2i} = M = M'$. Once again, essentially the same proof as that of Proposition 4.16 applies; in particular, we show that $2i = m'$. Note that in both situations we implicitly used the cases in Proposition 4.28.   □

Proof of Theorem 4.22 The proof is similar to that of Theorem 4.7, so we only point out the extra complexity in type C. This has to do with showing that, if $b = \mathrm{sfill}(J)$ and $f_p(J) \neq \mathbf{0}$, then $f_p^2(b) = \mathrm{sfill}(f_p(J))$. We continue to use the notation from the above setup, and we consider only the case $1 \leq p \leq n - 1$, as the others are simpler.

Since $f_p(b) \neq \mathbf{0}$, we have $M' > 0$, and f_p^2 acts on b in one of the ways indicated in Proposition 4.28, cases (i)–(iii). Now let us turn to $f_p(J)$, and use the same setup as in the proof of Theorem 4.7, to which we refer. By the first part of Proposition 4.31, we conclude that $\mathrm{sfill}(f_p(J))$ is obtained from $\mathrm{sfill}(J)$ by applying s_p to columns i for $i \in [m', m'')$ in cases (i)–(ii), resp. $i \in [m' - 1, m'')$ in case (iii). By the second part of Proposition 4.31, this amounts to applying s_p to column m', resp. to columns $m' - 1$

and m'. By Proposition 4.28, this is the same as the action of f_p^2 on sfill(J), which concludes the proof. Note that in the above reasoning we implicitly used Remark 4.27 (1). □

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