J Algebr Comb (2015) 41:493–538
DOI 10.1007/s10801-014-0543-4

# Combinatorial operads from monoids

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Received: 2 October 2013 / Accepted: 26 June 2014 / Published online: 14 August 2014


**Abstract** We introduce a functorial construction which, from a monoid, produces a set-operad. We obtain new (symmetric or not) operads as suboperads or quotients of the operads obtained from usual monoids such as the additive and multiplicative monoids of integers and cyclic monoids. They involve various familiar combinatorial objects: endofunctions, parking functions, packed words, permutations, planar rooted trees, trees with a fixed arity, Schröder trees, Motzkin words, integer compositions, directed animals, and segmented integer compositions. We also recover some already known (symmetric or not) operads: the magmatic operad, the associative commutative operad, the diassociative operad, and the triassociative operad. We provide presentations by generators and relations of all constructed nonsymmetric operads.

**Keywords** Operad · Tree · Monoid · Rewriting

## 1 Introduction

Operads are algebraic structures introduced in the 1970s by Boardman and Vogt [5] and by May [24] in the context of algebraic topology. Informally, an operad is a structure containing operators with $n$ inputs and 1 output, for all positive integers $n$. Two operators $x$ and $y$ can be composed at the $i$th position by grafting the output of $y$ on the $i$th input of $x$. The new operator thus obtained is denoted by $x \circ_i y$. In an operad, one can also switch the inputs of an operator $x$ by letting a permutation $\sigma$ act to obtain a new operator denoted by $x \cdot \sigma$. One of the main relishes of operads comes

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from the fact that they offer a general theory to study in an unifying way different types of algebras, such as associative algebras and Lie algebras.

In recent years, the importance of operads in combinatorics has increased and several new (nonsymmetric) operads were defined on combinatorial objects (see e.g., [7,8,19,20]). The structure thereby added on combinatorial families enables us to see these in a new light and offers original ways to solve some combinatorial problems. For example, the dendriform operad [20] is a nonsymmetric operad on binary trees that plays an interesting role in describing the Hopf algebra of Loday–Ronco of binary trees [16,21]. Besides, this nonsymmetric operad is a key ingredient for the enumeration of intervals of the Tamari lattice [6,7]. There is also a very rich link connecting combinatorial Hopf algebra theory and operad theory: various constructions produce combinatorial Hopf algebras from operads [9,11,23,27,31].

In this paper, we propose a new generic method to build combinatorial operads. The starting point is to pick a monoid $M$. We then consider the set of words whose letters are elements of $M$. The arity of such words are their length, the composition of two words is expressed from the product of $M$, and permutations act on words by permuting letters. In this way, we associate an operad denoted by $\mathsf{T}M$ with any monoid $M$. This construction is rich from a combinatorial point of view since it allows us, by considering suboperads and quotients of $\mathsf{T}M$, to get new (symmetric or not) operads on various combinatorial objects. Our construction is related to two previous ones.

The first one is a construction of Méndez and Nava [26] emerging from the context of the species theory [17]. Roughly speaking, a species is a combinatorial construction $U$ which takes an underlying set $E$ as input and produces a set $U[E]$ of objects by adding some structure on the elements of $E$ (see [4]). This theory has many links with the theory of operads since an operad is a monoid with respect to the operation of substitution of species. In [26], the authors defined the plethystic species, that are species taking as input sets where any element has a colour picked from a fixed monoid $M$. This monoid has to satisfy some precise conditions (as to be left cancellable and without proper divisor of the unity, and such that any element has finitely many factorizations). It appears that the elements of the so-called uniform plethystic species can be seen as words of colours and hence, as elements of $\mathsf{T}M$. Moreover, the composition of this operad is the one of $\mathsf{T}M$. The main difference between the construction of Méndez and Nava and ours lies in the fact that the construction $\mathsf{T}$ can be applied on any monoid.

The second one, introduced by Berger and Moerdijk [3], is a construction which allows to obtain, from a commutative bialgebra $\mathcal{B}$, a cooperad $T\mathcal{B}$. Our construction $\mathsf{T}$ and the construction $T$ of these two authors are different but coincide in many cases. For instance, when $(M, \bullet)$ is a monoid such that for any $x \in M$, the set of pairs $(y, z) \in M^2$ satisfying $y \bullet z = x$ is finite, the operad $\mathsf{T}M$ is the dual of the cooperad $T\mathcal{B}$ where $\mathcal{B}$ is the dual bialgebra of $\mathbb{K}[M]$ endowed with the diagonal coproduct ($\mathbb{K}$ is a field). On the other hand, there are operads that we can build by the construction $\mathsf{T}$ but not by the construction $T$, and conversely. For example, the operad $\mathsf{T}\mathbb{Z}$, where $\mathbb{Z}$ is the additive monoid of integers, cannot be obtained as the dual of a cooperad built by the construction $T$ of Berger and Moerdijk.

Furthermore, our construction is defined in the category of sets and computations are explicit. It is therefore possible given a monoid $M$, to make experiments on the operad $\mathsf{T}M$, using if necessary a computer. In this paper, we study many applications of the construction $\mathsf{T}$ focusing on its combinatorial aspect. More precisely, we define, by starting from very simple monoids like the additive or multiplicative monoids of integers, or cyclic monoids, various nonsymmetric operads involving well-known combinatorial objects.

This paper is organized as follows. In Sect. 2, we set some notations about syntax trees and rewriting systems on syntax trees. We then briefly recall the basics about operads. We also prove in this section two important lemmas used in the sequel of the paper: the first one deals with the form of elements of nonsymmetric operads generated by a set of generators and the second one is a tool to prove presentations by generators and relations of nonsymmetric operads using rewrite rules on syntax trees. Sect. 3 defines the construction $\mathsf{T}$, associating an operad with a monoid and establishes its first properties. We show that this construction is a functor from the category of monoids to the category of operads which respects injections and surjections. Finally we apply this construction in Sect. 4 on various monoids and obtain several new (symmetric or not) operads. We construct in this way some operads on combinatorial objects which were not provided with such a structure: planar rooted trees with a fixed arity, Motzkin words, integer compositions, directed animals, and segmented integer compositions. We also obtain new operads on objects which are already provided with such a structure: endofunctions, parking functions, packed words, permutations, planar rooted trees, and Schröder trees. By using the construction $\mathsf{T}$, we also give an alternative construction for the diassociative operad [20] and for the triassociative operad [22].

This paper is an extended version of [13] and [14]. It contains all proofs and new results like the presentations by generators and relations of the considered nonsymmetric operads.

## 2 Syntax trees and operads

In this section, we set some notation about syntax trees and operads. All our operads are reduced, i.e., all their elements have at least one input. In the same way, all our syntax trees are reduced, i.e., all their internal nodes have at least one child. We also present in this section some notions about rewriting systems on syntax trees that we shall use to prove presentations of operads throughout the rest of the paper.

### 2.1 Rewriting systems on syntax trees

In the same way as the elements of free monoids can be seen as words and also are good objects to study monoids, the elements of free operads can be seen as syntax trees and are useful objects to manipulate these algebraic structures.

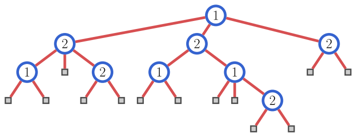
#### *2.1.1 Syntax trees*

We use in the sequel the standard terminology (i.e., *node*, *edge*, *root*, *subtree*, *parent*, *child*, *path*, *etc.*) about trees (see for instance [2]). In our graphical representations, internal nodes are represented by circles, leaves by squares, and edges by segments. Each tree is depicted so that its root is the uppermost node.

Let $S$ be a nonempty set. A *syntax tree on* $S$, or simply a *syntax tree* if $S$ is fixed, is a planar rooted tree such that internal nodes are labeled on $S$. We shall denote by $\mathcal{T}_n^S$ the set of syntax trees on $S$ with $n$ leaves and by $\mathcal{T}^S$ the set $\sqcup_{n \geqslant 1} \mathcal{T}_n^S$.

Let $T$ be a syntax tree. We denote by $\mathrm{n}(T)$ (resp. $\ell(T)$) the number of internal nodes (resp. leaves) of $T$. The *arity* of an internal node is the number of its children. The *depth-first traversal* of $T$ consists in visiting the root of $T$ and then, recursively visiting by a depth-first traversal the subtrees of $T$, from left to right. The *$i$th internal node* (resp. *$i$th leaf*) of $T$ is the $i$th internal node (resp. $i$th leaf) of $T$ visited by a depth-first traversal. The *depth* of a node $x$ of $T$ is the length of the unique path connecting $x$ with the root of $T$. Note that the depth of the root of $T$ is 0. The *weight* $\mathrm{w}(T)$ of $T$ is the sum, for all internal nodes $x$ of $T$, of the number of internal nodes of the rightmost subtree of $x$.

For example,

$T :=$  (2.1)

is a syntax tree on the set $\{1, 2\}$. It has 9 internal nodes, 13 leaves, its weight is 5, and the sequence of the labels of its internal nodes visited by the depth-first traversal is 121221122.

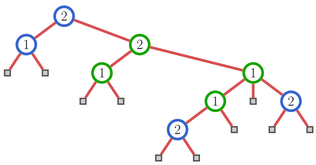
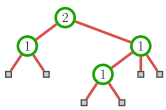
### 2.1.2 Syntax tree patterns

Let $T$ and $\alpha$ be two syntax trees. We say that $T$ *admits an occurrence* of $\alpha$ *at the root* if one of the following two conditions is satisfied:

(i) the tree $\alpha$ consists in exactly one leaf and no internal node;
(ii) the roots of $T$ and $\alpha$ have same arities and same labels, and, by denoting by $(T_1, \ldots, T_k)$ and $(\alpha_1, \ldots, \alpha_k)$ the sequence of the subtrees of $T$ and $\alpha$ from left to right, $T_i$ admits an occurrence of $\alpha_i$ at the root, for any $i \in [k]$.

We say that $T$ *admits an occurrence* of $\alpha$ if there is a node $x$ of $T$ such that the syntax tree rooted on $x$ admits an occurrence of $\alpha$ at the root.

For example, set

$T :=$  and $\alpha :=$ . (2.2)

Then $T$ admits one occurrence of $\alpha$.

*2.1.3 Rewrite rules on syntax trees*

Let $S$ be a nonempty set and $\mathcal{T}$ be a subset of $\mathcal{T}^S$. A *rewrite rule* on $\mathcal{T}$ is a binary relation $\mapsto$ on $\mathcal{T}$ such that

$$\alpha \mapsto \beta \quad \text{implies} \quad \ell(\alpha) = \ell(\beta). \tag{2.3}$$

A syntax tree $T_0$ is *rewritable* into a syntax tree $T_1$ by $\mapsto$ if:

(i) there are two syntax trees $\alpha$ and $\beta$ such that $\alpha \mapsto \beta$;
(ii) there is in $T_0$ an occurrence of $\alpha$;
(iii) we obtain $T_1$ by replacing an occurrence of $\alpha$ in $T_0$ by $\beta$.

We denote this property by $T_0 \to T_1$ and we call the pair $(T_0, T_1)$ a *rewriting*. Moreover, if there is a sequence $S_1, \ldots, S_k$ of syntax trees such that

$$T_0 \to S_1 \to \ldots \to S_k \to T_1, \tag{2.4}$$

we say that $T_0$ is *rewritable* into $T_1$ by $\to$ and we denote this property by $T_0 \overset{*}{\to} T_1$.

If there is no infinite chain

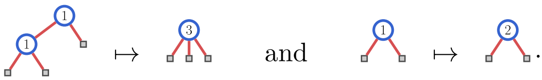
$$T_0 \to T_1 \to \cdots, \tag{2.5}$$

we say that $\mapsto$ is *terminating*. In this case, a syntax tree $T$ that cannot be rewritten is a *normal form* for $\mapsto$. When $\mapsto$ is terminating and there are for all $n \geqslant 1$ finitely many normal forms for $\mapsto$ with $n$ leaves, we denote by $\#(\mapsto_n)$ the number of normal forms for $\mapsto$ with $n$ leaves. In this case, the *generating series* of $\mapsto$ is

$$F_{\mapsto}(t) := \sum_{n \geqslant 1} \#(\mapsto_n) t^n. \tag{2.6}$$

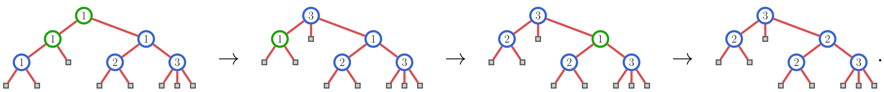
In the sequel, we shall make use of *regular specifications* to describe normal forms of rewrite rules and obtain their generating series. Regular specifications are formal grammars explaining how to build combinatorial objects (see [10] for an introduction on regular specifications).

To give an example, set $S := \{1, 2, 3\}$ and consider the rewrite rule $\mapsto$ on $\mathcal{T}^S$ defined by



(2.7)

Here is a sequence of rewritings:



(2.8)

Since for any rewriting $T_0 \to T_1$, $T_1$ has less nodes labeled by 1 than $T_0$, $\mapsto$ is terminating. The normal forms of $\mapsto$ are the syntax trees on $S$ with no internal node labeled by 1 that has two children.

## 2.2 Operads

Let us now set, in our context, some definitions and notations about operads. We shall use in the next sections the previous notions about syntax trees to handle elements of free nonsymmetric operads and establish presentations by generators and relations of nonsymmetric operads.

### *2.2.1 Nonsymmetric operads*

A *nonsymmetric operad*, or a *ns operad* for short, is a collection

$$\mathcal{P} := \bigsqcup_{n \geqslant 1} \mathcal{P}(n), \tag{2.9}$$

together with *partial composition maps*

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1), \qquad n, m \geqslant 1, i \in [n], \tag{2.10}$$

and a distinguished element $\mathbf{1} \in \mathcal{P}(1)$, the *unit* of $\mathcal{P}$. The above data has to satisfy the following relations:

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \quad x \in \mathcal{P}(n), y \in \mathcal{P}(m), z \in \mathcal{P}(k), i \in [n], j \in [m], \tag{2.11}$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \quad x \in \mathcal{P}(n), y \in \mathcal{P}(m), z \in \mathcal{P}(k), 1 \leqslant i < j \leqslant n, \tag{2.12}$$

$$\mathbf{1} \circ_1 x = x = x \circ_i \mathbf{1}, \quad x \in \mathcal{P}(n), i \in [n]. \tag{2.13}$$

One of the simplest ns operads is the *associative commutative operad* $\mathsf{Com}$. It is defined for all $n \geqslant 1$ by

$$\mathsf{Com}(n) := \{\alpha_n\}, \tag{2.14}$$

and the partial composition maps are defined by

$$\alpha_n \circ_i \alpha_m := \alpha_{n+m-1}, \tag{2.15}$$

for all $n, m \geqslant 1$ and $i \in [n]$.

Let us now fix some terminology about ns operads and recall basic definitions. The *arity* of an element $x$ of $\mathcal{P}(n)$ is $n$ and is denoted by $|x|$. If $\mathcal{Q}$ is a ns operad, a map $\phi : \mathcal{P} \to \mathcal{Q}$ is a *ns operad morphism* if it maps elements of arity $n$ of $\mathcal{P}$

to elements of arity $n$ of $\mathcal{Q}$ and commutes with partial composition maps. The ns operad $\mathcal{Q}$ is a *ns suboperad* of $\mathcal{P}$ if $\mathcal{Q} \subseteq \mathcal{P}$, $\mathbf{1} \in \mathcal{Q}$, and $\mathcal{Q}$ is closed for the partial composition maps.

For any set $G \subseteq \mathcal{P}$, the *ns operad generated by* $G$ is the smallest ns suboperad of $\mathcal{P}$ which contains every element of $G$. When the ns operad generated by $G$ is $\mathcal{P}$ itself and $G$ is minimal with respect to inclusion among the subsets of $\mathcal{P}$ satisfying this property, $G$ is a *generating set* of $\mathcal{P}$ and its elements are *generators* of $\mathcal{P}$. We say that $\mathcal{P}$ is *finitely generated* if it admits a finite generating set.

The *Hilbert series* of a ns operad $\mathcal{P}$ containing, for all $n \geqslant 1$, finitely many elements of arity $n$ is the series

$$F_{\mathcal{P}}(t) := \sum_{n \geqslant 1} \#\mathcal{P}(n)\, t^n. \tag{2.16}$$

A *combinatorial ns operad* is a ns operad which admits a Hilbert series and such that its only element of arity 1 is its unit.

The *composition map* of $\mathcal{P}$ is the mapping

$$\circ : \mathcal{P}(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n), \tag{2.17}$$

defined using partial composition maps $\circ_i$ by

$$x \circ [y_1, \ldots, y_n] := (\ldots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1. \tag{2.18}$$

The ns operad $\mathcal{P}$ is *basic* if for all $y_1, \ldots, y_n \in \mathcal{P}$, the maps

$$\gamma_{y_1,\ldots,y_n} : \mathcal{P}(n) \to \mathcal{P}(|y_1| + \cdots + |y_n|), \tag{2.19}$$

defined by

$$\gamma_{y_1,\ldots,y_n}(x) := x \circ [y_1, \ldots, y_n] \tag{2.20}$$

are injective.

### *2.2.2 Symmetric operads*

Let $\mathfrak{S}_n$ be the group of permutations of $[n]$. Any permutation $\sigma$ is denoted as a word $\sigma_1 \ldots \sigma_n$ in such a way that the $i$th letter $\sigma_i$ is the image of $i$. For instance, the word 312 represents the bijection $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$.

To define what is an operad, we need the following definition. Let $\mathsf{As}$ be the ns operad satisfying for all $n \geqslant 1$,

$$\mathsf{As}(n) := \mathfrak{S}_n, \tag{2.21}$$

and, for all $\sigma \in \mathsf{As}(n)$, $\nu \in \mathsf{As}(m)$, and $i \in [n]$,

$$\sigma \circ_i \nu := \sigma'_1 \ldots \sigma'_{i-1} \nu'_1 \ldots \nu'_m \sigma'_{i+1} \ldots \sigma'_n, \tag{2.22}$$

where

$$\nu'_j := \nu_j + \sigma_i - 1, \qquad \text{for all } j \in [m], \tag{2.23}$$

and

$$\sigma'_j := \begin{cases} \sigma_j & \text{if } \sigma_j < \sigma_i, \\ \sigma_j + m - 1 & \text{otherwise}, \end{cases} \qquad \text{for all } j \in [n]. \tag{2.24}$$

This ns operad is known as the *associative noncommutative operad*.

For instance, here are two examples of compositions in $\mathsf{As}$

$$1\mathbf{2}3 \circ_2 12 = 1234, \tag{2.25}$$

$$741\mathbf{5}623 \circ_4 231 = 941675823. \tag{2.26}$$

A *symmetric operad*, or an *operad* for short, is a ns operad $\mathcal{P}$ together with a map

$$\cdot : \mathcal{P}(n) \times \mathsf{As}(n) \to \mathcal{P}(n), \qquad n \geqslant 1, \tag{2.27}$$

which satisfies the following relation:

$$(x \cdot \sigma) \circ_i (y \cdot \nu) = \left(x \circ_{\sigma_i} y\right) \cdot (\sigma \circ_i \nu), \qquad x \in \mathcal{P}(n), y \in \mathcal{P}(m), \\ \sigma \in \mathsf{As}(n), \nu \in \mathsf{As}(m), i \in [n], \tag{2.28}$$

in such a way that $\cdot$ is also a symmetric group action. Note that any operad $\mathcal{P}$ is also (and thus can be seen as) a ns operad by forgetting its action of $\mathsf{As}$.

If $\mathcal{Q}$ is an operad, a map $\phi : \mathcal{P} \to \mathcal{Q}$ is an *operad morphism* if it is a ns operad morphism that commutes with $\cdot$. The operad $\mathcal{Q}$ is a *suboperad* of $\mathcal{P}$ if $\mathcal{Q}$ is a ns suboperad of $\mathcal{P}$ and $\mathcal{Q}$ is closed for $\cdot$.

For any set $G \subseteq \mathcal{P}$, the *operad generated by* $G$ is the smallest suboperad of $\mathcal{P}$ which contains every element of $G$. When the operad generated by $G$ is $\mathcal{P}$ itself and $G$ is minimal with respect to inclusion among the subsets of $\mathcal{P}$ satisfying this property, $G$ is a *generating set* of $\mathcal{P}$ and its elements are *generators* of $\mathcal{P}$. We say that $\mathcal{P}$ is *finitely generated* if it admits a finite generating set.

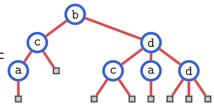
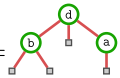
## 2.3 Presentation of nonsymmetric operads

We now focus on ns operads and present the tools we will need to establish presentations by generators and relations of ns operads.

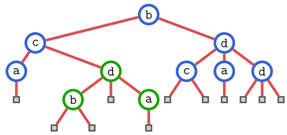
### 2.3.1 *Free ns operads*

Let $S := \sqcup_{n \geqslant 1} S(n)$ be a set. The *free ns operad* over $S$ is the ns operad $\mathcal{F}(S)$ defined as follows. For any $n \geqslant 1$, the set $\mathcal{F}(S)(n)$ is the set of syntax trees on $S$ with $n$ leaves and where internal nodes which have $k$ children are labeled on $S(k)$. The composition $x \circ_i y$ of two elements of $\mathcal{F}(S)$ consists in grafting the root of $y$ on the $i$th leaf of $x$. The unit $\mathbf{1}$ of $\mathcal{F}(S)$ is the tree with no internal node and hence exactly one leaf. The *degree* $\mathrm{d}(x)$ of an element $x$ of $\mathcal{F}(S)$ is its number $\mathrm{n}(x)$ of internal nodes and its *arity* $|x|$ is its number $\ell(x)$ of leaves. If $\mathrm{d}(x) = 1$, the *label* $\mathrm{lbl}(x)$ of $x$ is the element of $S$ which labels the only internal node of $x$.

Let $S := S(1) \sqcup S(2) \sqcup S(3)$, where $S(1) := \{\mathtt{a}\}$, $S(2) := \{\mathtt{b}, \mathtt{c}\}$, and $S(3) := \{\mathtt{d}\}$. Then,

$x :=$  and $y :=$  (2.29)

are two elements of $\mathcal{F}(S)$. The arity of $x$ is 8 and its degree is 7. The arity of $y$ is 4 and its degree is 3. Moreover, one has in $\mathcal{F}(S)$ the following composition

$x \circ_2 y =$  . (2.30)

### 2.3.2 *Evaluations*

Let $\left(\mathcal{P}, \circ_i^{\mathcal{P}}, \mathbf{1}_{\mathcal{P}}\right)$ be a ns operad and $X := \sqcup_{n \geqslant 1} X(n)$ be a set of elements of $\mathcal{P}$, where $X(n) \subseteq \mathcal{P}(n)$ for all $n \geqslant 1$. The *evaluation* ev is the mapping

$$\mathrm{ev} : \mathcal{F}(X) \to \mathcal{P}, \tag{2.31}$$

recursively defined by $\mathrm{ev}(\mathbf{1}) := \mathbf{1}_{\mathcal{P}}$, $\mathrm{ev}(x) := \mathrm{lbl}(x)$ if $\mathrm{d}(x) = 1$, and

$$\mathrm{ev}(x) := \mathrm{ev}(y) \circ_i^{\mathcal{P}} \mathrm{ev}(z) \tag{2.32}$$

if $\mathrm{d}(x) \geqslant 2$, where $y, z \in \mathcal{F}(X) \setminus \{\mathbf{1}\}$, $i \in [|y|]$, and $x = y \circ_i z$.

In other words, we can see an element $x$ of $\mathcal{F}(X)$ as a tree-like expression for an element $\mathrm{ev}(x)$ of $\mathcal{P}$. Moreover, it is easy, by induction on the degree, to prove that ev is a well-defined surjective mapping and hence, is a ns operad morphism.

### 2.3.3 *Ns operadic congruences*

An *ns operadic congruence* over a ns operad $\mathcal{P}$ is an equivalence relation $\equiv$ on its elements such that $x \equiv x'$ implies $|x| = |x'|$, and for all $x, x', y, y' \in \mathcal{P}$ and $i \in [|x|]$,

$$x \equiv x' \text{ and } y \equiv y' \quad \text{implies} \quad x \circ_i y \equiv x' \circ_i y'. \tag{2.33}$$

Given a set $S := \sqcup_{n \geqslant 1} S(n)$ and an operadic congruence $\equiv$ over the free ns operad $\mathcal{F}(S)$, one can construct a *ns quotient operad* $\mathcal{F}(S)/_{\equiv}$ of $\mathcal{F}(S)$ defined as follows. We set

$$\mathcal{F}(S)/_{\equiv}(n) := \{[x]_{\equiv} : x \in \mathcal{F}(S)(n)\}, \quad n \geqslant 1, \tag{2.34}$$

where $[x]_{\equiv}$ is the $\equiv$-equivalence class of $x$, and

$$[x]_{\equiv} \circ_i [y]_{\equiv} := [x \circ_i y]_{\equiv}, \tag{2.35}$$

where $x$ and $y$ are any elements of $\mathcal{F}(S)$ such that $x \in [x]_{\equiv}$ and $y \in [y]_{\equiv}$.

*2.3.4 Presentation by generators and relations*

In the sequel, we shall define ns operadic congruences $\equiv$ over free ns operads $\mathcal{F}(S)$ through equivalence relations $\leftrightarrow$ on the set $\mathcal{F}(S)$. The *congruence generated by* $\leftrightarrow$ is the most refined ns operadic congruence $\equiv$ containing $\leftrightarrow$.

Besides, we say that a relation $\mapsto$ on $\mathcal{F}(S)$ is an *orientation* of $\leftrightarrow$ if $\mapsto$ is the finest relation such that its reflexive, symmetric, and transitive closure is $\leftrightarrow$. The link between rewrite rules on syntax trees and ns operads relies on the fact that orientations can be regarded as rewrite rules.

A *presentation* of a ns operad $\mathcal{P}$ consists of a set $S := \sqcup_{n \geqslant 1} S(n)$ and a ns operadic congruence $\equiv$ over $\mathcal{F}(S)$ such that $\mathcal{P} = \mathcal{F}(S)/_{\equiv}$. When $S(2) \neq \emptyset$ and $S(n) = \emptyset$ for all $n \neq 2$, $\mathcal{P}$ is called *binary*. When $\equiv$ can be generated as a ns operadic congruence by an equivalence relation $\leftrightarrow$ on $\mathcal{F}(S)$ only involving elements of degree 2, $\mathcal{P}$ is called *quadratic*.

The following lemma presents a description of the elements of ns operads generated by a set of generators.

**Lemma 2.1** *Let $\mathcal{P}$ be a ns operad generated by a set $G$ of generators. Then any element $x$ of $\mathcal{P}$ different from the unit of $\mathcal{P}$ can be written as*

$$x = y \circ_i g, \tag{2.36}$$

*where $y \in \mathcal{P}(n)$, $n \geqslant 1$, $g \in G$, and $i \in [n]$.*

*Proof* Since the map $\mathrm{ev} : \mathcal{F}(G) \to \mathcal{P}$ is surjective, $x$ admits a tree-like expression $x' \in \mathcal{F}(G)$ satisfying $\mathrm{ev}(x') = x$. Since $x$ is different from the unit of $\mathcal{P}$, one has $x' = y' \circ_i g'$ for two syntax trees $y'$ and $g'$ of $\mathcal{F}(G)$ such that $g'$ has exactly one internal node. Then, by setting $y := \mathrm{ev}(y')$ and $g := \mathrm{ev}(g')$, we have, since ev is a ns operad morphism, $x = y \circ_i g$. □

We shall use in the sequel Lemma 2.1 to study ns operads $\mathcal{Q}$ generated by a subset of elements of a bigger ns operad $\mathcal{P}$. It allows us to describe $\mathcal{Q}$ arity by arity because any element of $\mathcal{Q}$ can be obtained by composing an element of a smaller arity with a generator.

The following lemma presents a tool for showing that a given combinatorial ns operad admits a specified presentation.

**Lemma 2.2** *Let $\mathcal{P}$ be a combinatorial ns operad generated by a set $G$ of generators and $\equiv$ be a ns operadic congruence over $\mathcal{F}(G)$ generated by an equivalence relation $\leftrightarrow$ on $\mathcal{F}(G)$. If the following two conditions are satisfied together:*

*(i) for all $x, x' \in \mathcal{F}(G)$, $x \leftrightarrow x'$ implies* $\mathrm{ev}(x) = \mathrm{ev}(x')$*;*
*(ii) there exists an orientation $\mapsto$ of $\leftrightarrow$ such that $\mapsto$ is terminating and has as many normal forms of arity $n$ as elements of $\mathcal{P}$ of arity $n$;*

*then $\mathcal{P}$ admits the presentation $\mathcal{P} = \mathcal{F}(G)/_{\equiv}$.*

*Proof* The definition of the evaluation map ev and (i) imply that the map

$$\phi : \mathcal{F}(G)/_{\equiv} \to \mathcal{P} \tag{2.37}$$

defined for any $x \in \mathcal{F}(G)$ by $\phi\,([x]_{\equiv}) := \mathrm{ev}(x)$ is a surjective ns operad morphism.

Since $\mapsto$ is an orientation of $\leftrightarrow$, for any $x \in \mathcal{F}(G)$, there is at least one normal form for $\mapsto$ in $[x]_{\equiv}$ and by (ii), for all $n \geqslant 1$, we have

$$\#\mathcal{P}(n) = \#(\mapsto_n) \;\geqslant \#\mathcal{F}(G)/_{\equiv}(n). \tag{2.38}$$

This, together with the fact that $\phi$ is surjective, implies that $\phi$ is also an isomorphism. Hence, $\mathcal{P}$ admits the claimed presentation. □

## 3 A combinatorial functor from monoids to operads

We describe in this section the main ingredient of this paper, namely the *construction* $\mathsf{T}$. This functorial construction associates an operad $\mathsf{T}M$ with any monoid $M$ and an operad morphism $\mathsf{T}\theta : \mathsf{T}M \to \mathsf{T}N$ with any monoid morphism $\theta : M \to N$.

### 3.1 The construction

#### *3.1.1 From monoids to operads*

Let $(M, \bullet, 1)$ be a monoid. Let us denote by $\mathsf{T}M$ the collection

$$\mathsf{T}M := \bigsqcup_{n \geqslant 1} \mathsf{T}M(n), \tag{3.1}$$

where for all $n \geqslant 1$,

$$\mathsf{T}M(n) := \{(x_1, \ldots, x_n) : x_i \in M \quad \text{for all}\, i \in [n]\}\,. \tag{3.2}$$

We endow the set $\mathsf{T}M$ with maps

$$\circ_i : \mathsf{T}M(n) \times \mathsf{T}M(m) \to \mathsf{T}M(n+m-1), \qquad n, m \geqslant 1, i \in [n], \tag{3.3}$$

defined as follows: for all $x \in \mathsf{T}M(n)$, $y \in \mathsf{T}M(m)$, and $i \in [n]$,

$$x \circ_i y := (x_1, \ldots, x_{i-1}, x_i \bullet y_1, \ldots, x_i \bullet y_m, x_{i+1}, \ldots, x_n). \tag{3.4}$$

Let us also set $\mathbf{1} := (1)$ as a distinguished element of $\mathsf{T}M(1)$. We endow finally each set $\mathsf{T}M(n)$ with a map

$$\cdot : \mathsf{T}M(n) \times \mathsf{As}(n) \to \mathsf{T}M(n), \qquad n \geqslant 1, \tag{3.5}$$

defined as follows: for all $x \in \mathsf{T}M(n)$ and $\sigma \in \mathsf{As}(n)$,

$$x \cdot \sigma := \left(x_{\sigma_1}, \ldots, x_{\sigma_n}\right). \tag{3.6}$$

The elements of $\mathsf{T}M$ are words over $M$ regarded as an alphabet. The arity $|x|$ of an element $x$ of $\mathsf{T}M(n)$ is $n$. For the sake of readability, we shall denote in some cases an element $(x_1, \ldots, x_n)$ of $\mathsf{T}M(n)$ by its *word notation* $x_1 \ldots x_n$.

**Proposition 3.1** *If $M$ is a monoid, then $\mathsf{T}M$ is an operad.*

*Proof* This is a straightforward checking of the relations of operads: (2.11) comes from the fact that the product of $M$ is associative, (2.12) comes from the fact that the elements of $\mathsf{T}M$ are words, (2.13) comes from the fact that $M$ has a unit, and (2.28) comes from the fact that $\cdot$ acts by permuting the letters of the words. □

*3.1.2 From monoids morphisms to operads morphisms*

Let $M$ and $N$ be two monoids and $\theta : M \to N$ be a monoid morphism. Let us denote by $\mathsf{T}\theta$ the map

$$\mathsf{T}\theta : \mathsf{T}M \to \mathsf{T}N, \tag{3.7}$$

defined for all $(x_1, \ldots, x_n) \in \mathsf{T}M(n)$ by

$$\mathsf{T}\theta\,(x_1, \ldots, x_n) := (\theta(x_1), \ldots, \theta(x_n))\,. \tag{3.8}$$

**Proposition 3.2** *If $M$ and $N$ are two monoids and $\theta : M \to N$ is a monoid morphism, then the map $\mathsf{T}\theta : \mathsf{T}M \to \mathsf{T}N$ is an operad morphism.*

*Proof* This is a straightforward checking: the fact that $\theta$ is a monoid morphism implies the statement of the proposition. □

## 3.2 Main properties of the construction

### 3.2.1 Functoriality of $\mathsf{T}$

**Proposition 3.3** *Let $M$ and $N$ be two monoids and $\theta : M \to N$ be a monoid morphism. If $\theta$ is injective (resp. surjective), then $\mathsf{T}\theta$ is injective (resp. surjective).*

*Proof* This is a straightforward checking: the fact that $\mathsf{T}\theta$ acts letter by letter implies the statement of the proposition. □

**Theorem 3.4** *The construction $\mathsf{T}$ is a functor from the category of monoids with monoid morphisms to the category of operads with operad morphisms. Moreover, $\mathsf{T}$ respects injections and surjections.*

*Proof* By Proposition 3.1, $\mathsf{T}$ constructs an operad from a monoid, and by Proposition 3.2, an operad morphism from a monoid morphism. Now, since $\mathsf{T}$ sends identity monoid morphisms to identity operad morphisms and $\mathsf{T}$ commutes with map composition, $\mathsf{T}$ is a functor. Finally, by Proposition 3.3, $\mathsf{T}$ also respects injections and surjections, whence the statement of the theorem. □

### 3.2.2 Miscellaneous properties

Recall that a monoid $(M, \bullet)$ is *right cancellable* if for any $x, y, z \in M$, $y \bullet x = z \bullet x$ implies $y = z$.

**Proposition 3.5** *Let $M$ be a monoid. The operad $\mathsf{T}M$ is basic if and only if $M$ is a right cancellable monoid.*

*Proof* Let us denote by $\bullet$ the product of $M$.

Assume first that $M$ is a right cancellable monoid. Let $y^{(1)}, \dots, y^{(n)} \in \mathsf{T}M$, $x, x' \in \mathsf{T}M(n)$, and assume that

$$\gamma_{y^{(1)},\dots,y^{(n)}}(x) = \gamma_{y^{(1)},\dots,y^{(n)}}(x'). \tag{3.9}$$

Then, for any $i \in [n]$ and $j \in \left[\left|y^{(i)}\right|\right]$,

$$x_i \bullet y^{(i)}_j = x'_i \bullet y^{(i)}_j. \tag{3.10}$$

Since $M$ is right cancellable, $x_i = x'_i$ and then, $x = x'$. This implies that $\gamma_{y^{(1)},\dots,y^{(n)}}$ is injective and that $\mathsf{T}M$ is basic.

Conversely, assume now that $\mathsf{T}M$ is basic. In particular, for any $y \in \mathsf{T}M(1)$, the map $\gamma_y$ is injective. Hence, for any $x, x' \in \mathsf{T}M(1)$, the equality $\gamma_y(x) = \gamma_y(x')$ implies $x = x'$. This is equivalent to say that $x \bullet y = x' \bullet y$ implies $x = x'$. This amounts exactly to say that $M$ is a right cancellable monoid. □

**Proposition 3.6** *Let $M$ be a monoid generated by a set $G$. The ns operad $\mathsf{T}M$ is generated by the set*

$$\{(g) : g \in G\} \cup \{(1, 1)\}, \tag{3.11}$$

*where* $(1, 1) \in \mathsf{T}M(2)$ *and* $1$ *is the unit of* $M$.

*Proof* Any element $x := (x_1, \ldots, x_n)$ of $\mathsf{T}M$ can be generated by the elements of (3.11) in the following way. First, generate the element $y := (1, \ldots, 1)$ of arity $n$ by composing $(1, 1)$ with itself $n - 1$ times. Next, change each letter $y_i$ of $y$ by composing $y$ with a sequence of generators of $G$ to reach $x_i$. This is possible since $M$ is generated by $G$. □

**Theorem 3.7** *Let* $(M, \bullet)$ *be a monoid generated by a set* $G := \{g_1, g_2, \ldots\}$ *of generators satisfying a set* $R$ *of nontrivial relations. Then, any algebra* $S$ *over the ns operad* $\mathsf{T}M$ *is a set equipped with maps*

$$\star : S \times S \to S \tag{3.12}$$

*and*

$$\uparrow_g : S \to S, \qquad g \in G \tag{3.13}$$

*satisfying for all* $a, b, c \in S$, $g \in G$, *and all relations* $g_{i_1} \bullet \ldots \bullet g_{i_n} = g_{j_1} \bullet \ldots \bullet g_{j_m}$ *of* $R$, *the equalities*

$$(a \star b) \star c = a \star (b \star c), \tag{3.14}$$

$$(a \star b) \uparrow_g = a \uparrow_g \star\, b \uparrow_g, \tag{3.15}$$

$$a \uparrow_{g_{i_1}} \ldots \uparrow_{g_{i_n}} = a \uparrow_{g_{j_1}} \ldots \uparrow_{g_{j_m}}. \tag{3.16}$$

*Proof* Proving the statement of the theorem is equivalent to prove that the ns operad $\mathsf{T}M$ admits the presentation by generators and relations obtained by traducing (3.12), (3.13), (3.14), (3.15), and (3.16) in operadic terms. Thereby, this ns operad $\mathcal{P}$ is the quotient of the free operad generated by a binary generator $\star$ and unary generators $\uparrow_g$, $g \in G$, submitted to the relations

$$\star \circ_1 \star = \star \circ_2 \star, \tag{3.17}$$

$$\uparrow_g \circ_1 \star = \star \circ [\uparrow_g, \uparrow_g], \qquad g \in G, \tag{3.18}$$

$$\uparrow_{g_{i_1}} \circ_1 \ldots \circ_1 \uparrow_{g_{i_n}} = \uparrow_{g_{j_1}} \circ_1 \ldots \circ_1 \uparrow_{g_{j_m}} \tag{3.19}$$

for all relations $g_{i_1} \bullet \ldots \bullet g_{i_n} = g_{j_1} \bullet \ldots \bullet g_{j_m}$ of $R$.

Let $\phi : \mathcal{P} \to \mathsf{T}M$ be ns operad morphism defined by $\phi(\star) := (1, 1)$ and $\phi(\uparrow_g) := (g)$ for any $g \in G$, where $1$ denotes the unit of $M$. This morphism is well-defined since the elements $(1, 1)$ and $(g)$ of $\mathsf{T}M$ satisfy the above relations by replacing $\star$ by $(1, 1)$ and $\uparrow_g$ by $(g)$. Proposition 3.6 implies that $\phi$ is surjective since, as a ns operad, $\mathsf{T}M$ is generated by $(1, 1)$ and $(g)$, $g \in G$.

Now, since the equivalence classes of $\mathcal{P}$ are clearly in bijection with the elements of $\mathsf{T}M$, this shows that $\phi$ is an isomorphism. □

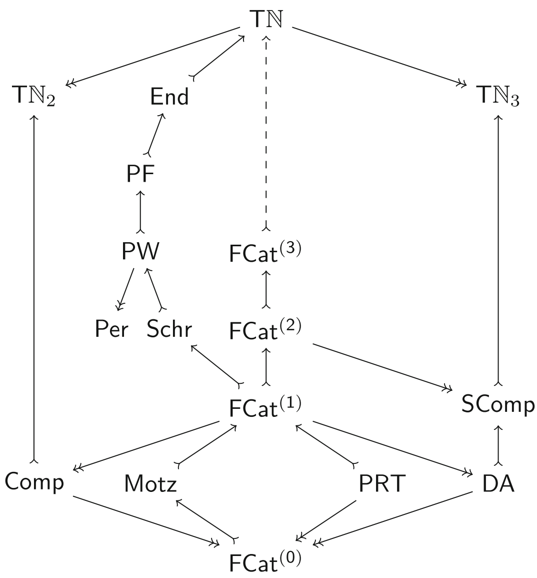



**Fig. 1** The diagram of ns suboperads and quotients of $\mathsf{T}\mathbb{N}$. *Arrows* $\rightarrowtail$ (resp. $\twoheadrightarrow$) are injective (resp. surjective) ns operad morphisms

## 4 Constructing operads

Through this section, we consider examples of applications of the functor $\mathsf{T}$. We shall mainly consider, given a monoid $M$, some suboperads of $\mathsf{T}M$, symmetric or not, which have for all $n \geqslant 1$ finitely many elements of arity $n$.

For the most part of the constructed operads $\mathcal{P}$, we shall establish for all arities $n \geqslant 1$, bijections $\phi : \mathcal{P}(n) \to \mathcal{C}_n$ between the elements of $\mathcal{P}$ of arity $n$ and elements of size $n$ of a set $\mathcal{C} := \sqcup_{n \geqslant 1} \mathcal{C}_n$ of combinatorial objects. These bijections, in addition to showing that $\mathcal{P}$ are operads involving the objects of $\mathcal{C}$, allow us to define composition operations on $\mathcal{C}$ by interpreting the partial composition maps of $\mathcal{P}$ on the elements of $\mathcal{C}$.

Moreover, we shall also establish presentations by generators and relations of the constructed ns operads by using the tools provided by Sect. 2.

### 4.1 Operads from the additive monoid

We shall denote by $\mathbb{N}$ the additive monoid of integers, and for all $\ell \geqslant 1$, by $\mathbb{N}_\ell$ the quotient of $\mathbb{N}$ consisting of the set $\{0, 1, \ldots, \ell - 1\}$ with the addition modulo $\ell$ as the operation of $\mathbb{N}_\ell$.

Note that since, by Theorem 3.4, $\mathsf{T}$ is a functor which respects surjective maps, $\mathsf{T}\mathbb{N}_\ell$ is a quotient operad of $\mathsf{T}\mathbb{N}$. Besides, since the monoids $\mathbb{N}$ and $\mathbb{N}_\ell$ are right cancellable, by Proposition 3.5, the operads $\mathsf{T}\mathbb{N}$ and $\mathsf{T}\mathbb{N}_\ell$ are basic, and since any suboperad of a basic operad is basic, all operads constructed in this section are basic.

The ns operads constructed in this section fit into the diagram of ns operads represented by Fig. 1. Table 1 summarizes some information about these ns operads.

**Table 1** Ground monoids, generators, first dimensions, and combinatorial objects involved in the ns suboperads and quotients of $\mathsf{T}\mathbb{N}$

| Monoid | Ns operad | Generators | First dimensions | Combinatorial objects |
|---|---|---|---|---|
| $\mathbb{N}$ | End | — | 1, 4, 27, 256, 3125 | Endofunctions |
| | PF | — | 1, 3, 16, 125, 1296 | Parking functions |
| | PW | — | 1, 3, 13, 75, 541 | Packed words |
| | Per | — | 1, 2, 6, 24, 120 | Permutations |
| | PRT | 01 | 1, 1, 2, 5, 14, 42 | Planar rooted trees |
| | $\mathsf{FCat}^{(k)}$ | 00, 01, …, 0$k$ | Fuß-Catalan numbers | $k$-leafy trees |
| | Schr | 00, 01, 10 | 1, 3, 11, 45, 197 | Schröder trees |
| | Motz | 00, 010 | 1, 1, 2, 4, 9, 21, 51 | Motzkin words |
| $\mathbb{N}_2$ | Comp | 00, 01 | 1, 2, 4, 8, 16, 32 | Int. compo. |
| $\mathbb{N}_3$ | DA | 00, 01 | 1, 2, 5, 13, 35, 96 | Directed animals |
| | SComp | 00, 01, 02 | 1, 3, 27, 81, 243 | Seg. int. compo. |

*4.1.1 Operads on endofunctions, parking functions, packed words, and permutations*

Recall that an *endofunction* of size $n$ is a word $x$ of length $n$ on the alphabet $\{1, \dots, n\}$. A *parking function* of size $n$ is an endofunction $x$ of size $n$ such that the nondecreasing rearrangement $y$ of $x$ satisfies $y_i \leqslant i$ for all $i \in [n]$. A *packed word* of size $n$ is an endofunction $x$ of size $n$ such that for any letter $x_i \geqslant 2$ of $x$, there is in $x$ a letter $x_j = x_i - 1$.

Note that neither the set of endofunctions nor the set of parking functions, packed words, and permutations are suboperads of $\mathsf{T}\mathbb{N}$. Indeed, one has the following counterexample:

$$1\mathbf{2} \circ_2 12 = 134, \tag{4.1}$$

and, even if 12 is a permutation, 134 is not an endofunction.

Therefore, let us call a word $x$ a *twisted* endofunction (resp. parking function, packed word, permutation) if the word $(x_1 + 1, x_2 + 1, \dots, x_n + 1)$ is an endofunction (resp. parking function, packed word, permutation). For example, the word 2300 is a twisted endofunction since 3411 is an endofunction. Let us denote by End (resp. PF, PW, Per) the set of twisted endofunctions (resp. parking functions, packed words, permutations). Under this reformulation, one has the following result:

**Proposition 4.1** *The sets* End*,* PF*, and* PW *form suboperads of* $\mathsf{T}\mathbb{N}$.

*Proof* First, by definition of the partial composition map of $\mathsf{T}\mathbb{N}$, the set of twisted endofunctions forms a suboperad of $\mathsf{T}\mathbb{N}$.

Let $x$ and $y$ be two twisted parking functions (resp. packed words) and $i \in [|x|]$. Since $x$ and $y$ have by definition at least one occurrence of 0, we have in $\mathsf{T}\mathbb{N}$,

$$\mathrm{Alph}(x \circ_i y) = \mathrm{Alph}(x) \cup \{x_i + a : a \in \mathrm{Alph}(y)\}, \tag{4.2}$$

where $\mathrm{Alph}(u)$ is the set $\{u_j : j \in [|u|]\}$. This, in addition to the fact that any permutation of a twisted parking function (resp. packed word) is still a twisted parking function (resp. packed word), shows that the partial composition maps of $\mathsf{T}\mathbb{N}$ and the map $\cdot$ are still well-defined in $\mathsf{PF}$ (resp. $\mathsf{PW}$). $\square$

For example, we have in $\mathsf{End}$ the following composition

$$2\mathbf{1}23 \circ_2 30313 = 24142423, \tag{4.3}$$

and the following application of the map $\cdot$

$$11210 \cdot 23514 = 12011. \tag{4.4}$$

Note that $\mathsf{End}$ is not a finitely generated operad. Indeed, the twisted endofunctions $x$ of size $n$ satisfying $x_i := n-1$ for all $i \in [n]$ cannot be obtained by compositions involving elements of $\mathsf{End}$ of arity smaller than $n$. Similarly, $\mathsf{PF}$ is not a finitely generated operad since the twisted parking functions $x$ of size $n$ satisfying $x_i := 0$ for all $i \in [n-1]$ and $x_n := n-1$ cannot be obtained by compositions involving elements of $\mathsf{PF}$ of arity smaller than $n$.

However, the operad $\mathsf{PW}$ is a finitely generated operad:

**Proposition 4.2** *The operad* $\mathsf{PW}$ *is the suboperad of* $\mathsf{T}\mathbb{N}$ *generated by the elements* 00 *and* 01.

*Proof* Let $\mathcal{P}$ be the suboperad of $\mathsf{T}\mathbb{N}$ generated by the elements 00 and 01, and let us show that $\mathcal{P} = \mathsf{PW}$.

First, by Proposition 4.1, since 00 and 01 are twisted packed words, the elements of $\mathcal{P}$ also are twisted packed words.

Now let $x$ be a nondecreasing twisted packed word and let us show by induction on the size of $x$ that $x \in \mathcal{P}$. If $|x| = 1$, since $x$ is a twisted packed word, one has $x = 0$ and since 0 is the unit of $\mathsf{T}\mathbb{N}$, $x \in \mathcal{P}$. Otherwise, let $y$ be the prefix of size $n-1$ of $x$. Since $x$ is a nondecreasing word, there are two possibilities to express the last letter $x_n$ of $x$ from the letter $x_{n-1}$. If $x_n = x_{n-1}$, we have $x = y \circ_{n-1} 00$, and if $x_n = x_{n-1} + 1$, we have $x = y \circ_{n-1} 01$. Hence, since by the induction hypothesis $\mathcal{P}$ contains $y$, $\mathcal{P}$ also contains $x$. Finally, since any twisted packed word $z$ can be obtained from a nondecreasing packed word $x$ by permuting its letters, we have $z = x \cdot \sigma$ for a certain permutation $\sigma$ of $\mathsf{As}(n)$, and hence, $\mathcal{P} = \mathsf{PW}$. $\square$

Let $\mathbb{K}$ be a field and let us from now consider that $\mathsf{PW}$ is an operad in the category of $\mathbb{K}$-vector spaces, i.e., $\mathsf{PW}$ is the free $\mathbb{K}$-vector space over the set of twisted packed words with partial composition maps and the map $\cdot$ extended by linearity. For more details on operads in the category of vector spaces, we redirect the reader to [23].

Let $I$ be the free $\mathbb{K}$-vector space over the set of twisted packed words having multiple occurrences of a same letter.

**Proposition 4.3** *The vector space $I$ is an operadic ideal of* $\mathsf{PW}$. *Moreover, the operadic quotient* $\mathsf{PW}/_I$ *is the free vector space over the set of twisted permutations* $\mathsf{Per}$ *and, for all twisted permutations $x$ and $y$, the partial composition map in* $\mathsf{Per}$ *is expressed as*

$$x \circ_i y = \begin{cases} x \circ_i y & \text{if } x_i = |x|, \\ 0_{\mathbb{K}} & \text{otherwise,} \end{cases} \tag{4.5}$$

*where $0_{\mathbb{K}}$ is the null vector of* $\mathsf{Per}$ *and the partial composition map $\circ_i$ in the right-hand side of* (4.5) *is the partial composition map of* $\mathsf{PW}$.

*Proof* Let $x$ be a twisted packed word and $y$ be a twisted packed word having multiple occurrences of a same letter. Since $x$ and $y$ have at least one occurrence of 0, any composition involving $x$ and $y$ also has multiple occurrences of a same letter. Moreover, for any permutation $\sigma$ of $\mathsf{As}$ of size $|y|$, $y \cdot \sigma$ also has multiple occurrences of a same letter. Hence, $I$ is an operadic ideal of $\mathsf{PW}$ and one can consider the operadic quotient $\mathsf{PW}/_I$.

Since twisted packed words with no multiple occurrence of a same letter are twisted permutations, $\mathsf{PW}/_I$ can be identified with the $\mathbb{K}$-vector space over the set of twisted permutations $\mathsf{Per}$ and (4.5) follows from the fact that the composition $x \circ_i y$ of two twisted permutations $x$ and $y$ is still a twisted permutation if and only if $x_i$ is the greatest letter of $x$. □

Here are two examples of compositions in $\mathsf{Per}$

$$\mathbf{2}0431 \circ_1 102 = 0_{\mathbb{K}}, \tag{4.6}$$

$$20\mathbf{4}31 \circ_3 102 = 2054631. \tag{4.7}$$

*4.1.2 A ns operad on planar rooted trees*

Let $\mathsf{PRT}$ be the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 01. The following table shows the first elements of $\mathsf{PRT}$.

| Arity | Elements of $\mathsf{PRT}$ |
|---|---|
| 1 | 0 |
| 2 | 01 |
| 3 | 011, 012 |
| 4 | 0111, 0112, 0121, 0122, 0123 |
| 5 | 01111, 01112, 01121, 01122, 01123, 01211, 01212, 01221, 01222, 01223, 01231, 01232, 01233, 01234 |

One has the following characterization of the elements of $\mathsf{PRT}$:

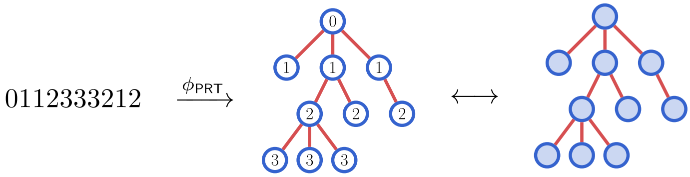



**Fig. 2** Interpretation of an element of the ns operad $\mathsf{PRT}$ in terms of planar rooted trees via the bijection $\phi_{\mathsf{PRT}}$. The *nodes of the planar rooted tree* in the *middle* are labeled by their depth

**Proposition 4.4** *The elements of* $\mathsf{PRT}$ *are exactly the words* $x$ *on the alphabet* $\mathbb{N}$ *satisfying* $x_1 = 0$ *and* $1 \leqslant x_{i+1} \leqslant x_i + 1$ *for all* $i \in [|x| - 1]$.

*Proof* Let us first show by induction on the length of the words that any word $x$ of $\mathsf{PRT}$ satisfies the statement. This is true when $|x| = 1$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of $\mathsf{PRT}$ of length $n := |x| - 1$ and an integer $i \in [n]$ such that $x = y \circ_i 01$. We have

$$x = (y_1, \ldots, y_{i-1}, y_i, y_i + 1, y_{i+1}, \ldots, y_n). \tag{4.8}$$

Since $x_{i+1} = x_i + 1$ and since, by the induction hypothesis, $y$ satisfies the statement, $x$ also satisfies it.

Let us now show by induction on the length of the words that $\mathsf{PRT}$ contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, since $x_1 = 0$ and $x_2 = 1$, $x$ has a factor $x_i x_{i+1}$ where $i$ is the greatest integer such that $x_{i+1} = x_i + 1$. Now, by setting

$$y := (x_1, \ldots, x_i, x_{i+2}, \ldots, x_n), \tag{4.9}$$

we have $x = y \circ_i 01$, and, since $i$ is maximal, if $i + 2 \leqslant n$ we have $x_{i+2} \leqslant x_{i+1}$. This implies that $y$ satisfies the statement. By the induction hypothesis, $\mathsf{PRT}$ contains $y$ and, since $x = y \circ_i 01$, $\mathsf{PRT}$ also contains $x$. □

Recall that there are $\frac{1}{n}\binom{2n-2}{n-1}$ planar rooted trees with $n$ nodes. There is a bijection $\phi_{\mathsf{PRT}}$ between the words of $\mathsf{PRT}$ of arity $n$ and planar rooted trees with $n$ nodes.

To compute $\phi_{\mathsf{PRT}}(x)$ where $x$ is an element of $\mathsf{PRT}$, iteratively insert the letters of $x$ from left to right according to the following procedure. If $|x| = 1$, then $x = 0$ and $\phi_{\mathsf{PRT}}(0)$ is the only planar rooted tree with one node. Otherwise, the insertion of a letter $\mathsf{a} \geqslant 1$ into a planar rooted tree $T$ consists in grafting in $T$ a new node as the rightmost child of the last node of depth $\mathsf{a} - 1$ for the depth-first traversal of $T$.

The inverse bijection is computed as follows. Given a planar rooted tree $T$ of size $n$, one computes an element of $\mathsf{PRT}$ of arity $n$ by labelling each node of $T$ by its depth and then, by reading its labels following a depth-first traversal of $T$.

Since the elements of $\mathsf{PRT}$ satisfy Proposition 4.4, $\phi_{\mathsf{PRT}}$ is well-defined. Hence, we can regard the elements of arity $n$ of $\mathsf{PRT}$ as planar rooted trees with $n$ nodes. Figure 2 shows an example of this bijection.

The bijection $\phi_{\mathsf{PRT}}$ between elements of $\mathsf{PRT}$ and planar rooted trees offers an alternative way to compute the composition of elements of $\mathsf{PRT}$:

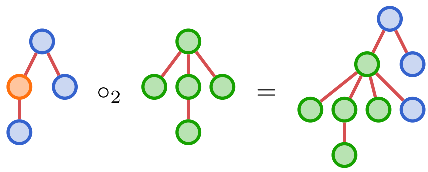


**Fig. 3** Interpretation of the partial composition map of the ns operad PRT in terms of planar rooted trees

**Proposition 4.5** *Let $S$ and $T$ be two planar rooted trees and $s$ be the $i$th node for the depth-first traversal of $S$. The composition $S \circ_i T$ in* PRT *amounts to replace $s$ by the root of $T$ and graft the children of $s$ as rightmost sons of the root of $T$.*

*Proof* Let $x \in \mathsf{PRT}(n)$ and $y \in \mathsf{PRT}(m)$ such that $S := \phi_{\mathsf{PRT}}(x)$ and $T := \phi_{\mathsf{PRT}}(y)$. Let $U := \phi_{\mathsf{PRT}}(x \circ_i y)$. By definition of $\phi_{\mathsf{PRT}}$ and the partial composition maps of PRT, $U$ is obtained by inserting the prefix of length $i - 1$ of $x$, then the letters of $y$ incremented by $x_i$, and finally, the suffix of length $n - i$ of $x$. Since by Proposition 4.4, $y$ starts by 0, the nodes created by inserting the letters of $y$ incremented by $x_i$ are descendants of the node created by inserting $x_i = y_1$. Moreover, the nodes corresponding to the letters of the suffix of length $n-i$ of $x$ have same parents as they have in $T$. This implies the statement. □

Figure 3 shows an example of composition in PRT.

**Proposition 4.6** *The ns operad* PRT *is isomorphic to the free ns operad generated by one element of arity* 2.

*Proof* By the characterization of its elements given by Proposition 4.4 and the bijection $\phi_{\mathsf{PRT}}$, there are as many elements in PRT of arity $n$ as there are of arity $n$ of the free ns operad generated by one element of arity 2. These two ns operads are hence isomorphic. □

Proposition 4.6 also says that PRT is isomorphic to the magmatic operad and hence, that PRT is a realization of the magmatic operad. This result is already known since in [25], Méndez and Yang point out that the species of parenthesizations (binary trees) and the species of planar rooted trees are isomorphic. This isomorphism implies that these species are also isomorphic as ns operads. Moreover, PRT can be seen as a planar version of the *non-associative permutative operad* NAP [25] (see also [19]) seen as a ns operad, which is an operad involving labeled non-planar rooted trees.

#### *4.1.3 A ns operad on leafy trees with a fixed arity*

Let $k \geqslant 0$ be an integer and $\mathsf{FCat}^{(k)}$ be the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by $00, 01, \ldots, 0k$. The following tables, respectively, show the first elements of $\mathsf{FCat}^{(1)}$ and $\mathsf{FCat}^{(2)}$.

| Arity | Elements of $\mathsf{FCat}^{(1)}$ |
|---|---|
| 1 | 0 |
| 2 | 00, 01 |
| 3 | 000, 001, 010, 011, 012 |
| 4 | 0000, 0001, 0010, 0011, 0012, 0100, 0101, 0110, 0111, 0112, 0120, 0121, 0122, 0123 |

| Arity | Elements of $\mathsf{FCat}^{(2)}$ |
|---|---|
| 1 | 0 |
| 2 | 00, 01, 02 |
| 3 | 000, 001, 002, 010, 011, 012, 013, 020, 021, 022, 023, 024 |
| 4 | 0000, 0001, 0002, 0010, 0011, 0012, 0013, 0020, 0021, 0022, 0023, 0024, 0100, 0101, 0102, 0110, 0111, 0112, 0113, 0120, 0121, 0122, 0123, 0124, 0130, 0131, 0132, 0133, 0134, 0135, 0200, 0201, 0202, 0210, 0211, 0212, 0213, 0220, 0221, 0222, 0223, 0224, 0230, 0231, 0232, 0233, 0234, 0235, 0240, 0241, 0242, 0243, 0244, 0245, 0246 |

It is immediate from the definition of $\mathsf{FCat}^{(k)}$ that for any $k \geqslant 0$, $\mathsf{FCat}^{(k)}$ is a ns suboperad of $\mathsf{FCat}^{(k+1)}$. Hence, the ns operads $\mathsf{FCat}^{(k)}$ form an increasing sequence (for inclusion) of ns operads. Note that $\mathsf{FCat}^{(0)}$ is isomorphic to the associative commutative operad $\mathsf{Com}$. Note also that since $\mathsf{FCat}^{(1)}$ is generated by 00 and 01 and since $\mathsf{PRT}$ is generated by 01, $\mathsf{PRT}$ is a ns suboperad of $\mathsf{FCat}^{(1)}$. Moreover, $\mathsf{FCat}^{(0)}$ is a quotient of $\mathsf{PRT}$ by the ns operadic congruence $\equiv$ defined for all $x, y \in \mathsf{PRT}(n)$ by $x \equiv y$.

One has the following characterization of the elements of $\mathsf{FCat}^{(k)}$:

**Proposition 4.7** *The elements of* $\mathsf{FCat}^{(k)}$ *are exactly the words $x$ on the alphabet $\mathbb{N}$ satisfying $x_1 = 0$ and $0 \leqslant x_{i+1} \leqslant x_i + k$ for all $i \in [|x| - 1]$.*

*Proof* Let us first show by induction on the length of the words that any word $x$ of $\mathsf{FCat}^{(k)}$ satisfies the statement. This is true when $|x| = 1$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of $\mathsf{FCat}^{(k)}$ of length $n := |x| - 1$, an integer $i \in [n]$, and $0 \leqslant h \leqslant k$ such that $x = y \circ_i 0h$. We have

$$x = (y_1, \ldots, y_{i-1}, y_i, y_i + h, y_{i+1}, \ldots, y_n). \tag{4.10}$$

Since $x_{i+1} = x_i + h$ and $0 \leqslant h \leqslant k$, and since, by the induction hypothesis, $y$ satisfies the statement, $x$ also satisfies it.

Let us now show by induction on the length of the words that $\mathsf{FCat}^{(k)}$ contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, since $x_1 = 0$ and $0 \leqslant x_2 \leqslant k$, $x$ has a factor $x_i x_{i+1}$ where $i$ is the greatest integer such that $x_i \leqslant x_{i+1}$. Now, by setting $h := x_{i+1} - x_i$ and

$$y := (x_1, \ldots, x_i, x_{i+2}, \ldots, x_n), \tag{4.11}$$

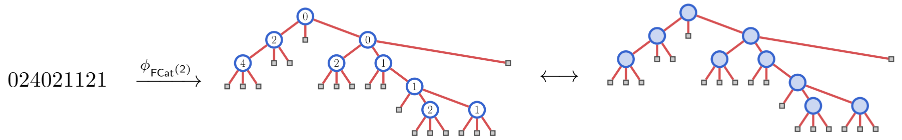



**Fig. 4** Interpretation of an element of the ns operad $\mathsf{FCat}^{(2)}$ in terms of 2-leafy trees via the bijection $\phi_{\mathsf{FCat}^{(2)}}$. The 2-*leafy tree in the middle* is well-labeled

we have $x = y \circ_i 0h$. Since $i$ is maximal, if $i + 2 \leqslant n$ we have $x_{i+2} < x_{i+1}$. This implies that $y$ satisfies the statement. By the induction hypothesis, $\mathsf{FCat}^{(k)}$ contains $y$ and, since $x = y \circ_i 0h$, $\mathsf{FCat}^{(k)}$ also contains $x$. □

A *k-leafy tree* is a planar rooted tree such that each internal node has exactly $k + 1$ children. The size $|T|$ of a $k$-leafy tree $T$ is the number of its internal nodes. It is well-known that there are $\frac{1}{kn+1}\binom{kn+n}{n}$ $k$-leafy trees of size $n$. We say that an internal node $x$ is *smaller* than an internal node $y$ of $T$ if, in the depth-first traversal of $T$, $x$ appears before $y$. We also say that a $k$-leafy tree $T$ is *well-labeled* if its root is labeled by 0, and, for each internal node $x$ of $T$ labeled by $\mathsf{a}$, the children of $x$ are labeled, from left to right, by $\mathsf{a} + k, \ldots, \mathsf{a} + 1, \mathsf{a}$. There is a unique way to label a $k$-leafy tree so that it is well-labeled. There is a bijection $\phi_{\mathsf{FCat}^{(k)}}$ between the words of $\mathsf{FCat}^{(k)}$ of arity $n$ and well-labeled $k$-leafy trees of size $n$.

To compute $\phi_{\mathsf{FCat}^{(k)}}(x)$ where $x$ is an element of $\mathsf{FCat}^{(k)}$, iteratively insert the letters of $x$ from left to right according to the following procedure. If $|x| = 1$, then $x = 0$ and $\phi_{\mathsf{FCat}^{(k)}}(x)$ is the only well-labeled $k$-leafy tree of size 1. Otherwise, the insertion of a letter $\mathsf{a} \geqslant 0$ into a well-labeled $k$-leafy tree $T$ consists in replacing a leaf of $T$ by the $k$-leafy tree $S$ of size 1 labeled by $\mathsf{a}$ so that $S$ is the child of the greatest internal node such that the obtained $k$-tree is still well-labeled.

The inverse bijection is computed as follows. Given a well-labeled $k$-leafy tree $T$, one computes an element of $\mathsf{FCat}^{(k)}$ of arity $n$ by reading its labels following a depth-first traversal of $T$.

Since the elements of $\mathsf{FCat}^{(k)}$ satisfy Proposition 4.7, $\phi_{\mathsf{FCat}^{(k)}}$ is well-defined. Hence, we can regard the elements of arity $n$ of $\mathsf{FCat}^{(k)}$ as $k$-leafy trees of size $n$. Figure 4 shows an example of this bijection.

The bijection $\phi_{\mathsf{FCat}^{(k)}}$ between elements of $\mathsf{FCat}^{(k)}$ and $k$-leafy trees offers an alternative way to compute the composition of elements of $\mathsf{FCat}^{(k)}$:

**Proposition 4.8** *Let $S$ and $T$ be two $k$-leafy trees and $s$ be the $i$th internal node for the depth-first traversal of $S$. The composition $S \circ_i T$ in $\mathsf{FCat}^{(k)}$ amounts to replace $s$ by the root of $T$ and graft the children of $s$ from right to left on the rightmost leaves of $T$.*

*Proof* Let $x \in \mathsf{FCat}^{(k)}(n)$ and $y \in \mathsf{FCat}^{(k)}(m)$ such that $S := \phi_{\mathsf{FCat}^{(k)}}(x)$ and $T := \phi_{\mathsf{FCat}^{(k)}}(y)$. Let $U := \phi_{\mathsf{FCat}^{(k)}}(x \circ_i y)$. By definition of $\phi_{\mathsf{FCat}^{(k)}}$ and the partial composition maps of $\mathsf{FCat}^{(k)}$, $U$ is obtained by inserting the prefix of length $i - 1$ of $x$, then the letters of $y$ incremented by $x_i$, and finally, the suffix of length $n - i$ of $x$. Since by Proposition 4.7, $y$ starts by 0, the internal nodes created by inserting the letters of $y$ incremented by $x_i$ are descendants of the internal node created by inserting $x_i = y_1$. Since the last $n - i$ letters of $x \circ_i y$ are the same as the last $n - i$ letters

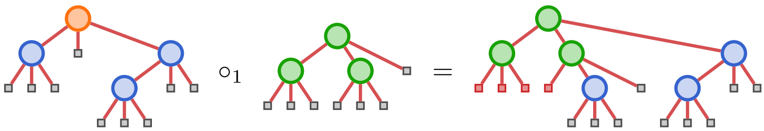



**Fig. 5** Interpretation of the partial composition map of the ns operad $\mathsf{FCat}^{(2)}$ in terms of 2-leafy trees

of $x$, by definition of $\phi_{\mathsf{FCat}^{(k)}}$, the children of the $i$th internal node of $S$ are grafted in $U$ from right to left on the rightmost leaves of $T$. This implies the statement. □

Figure 5 shows an example of composition in $\mathsf{FCat}^{(2)}$.

The next theorem elucidates the structure of $\mathsf{FCat}^{(k)}$:

**Theorem 4.9** *The ns operad* $\mathsf{FCat}^{(k)}$ *admits the presentation*

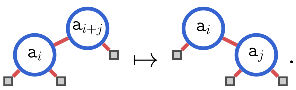
$$\mathsf{FCat}^{(k)} = \mathcal{F}\left(\{\mathsf{a}_0, \dots, \mathsf{a}_k\}\right)/_{\equiv}, \tag{4.12}$$

*where the* $\mathsf{a}_i$ *are of arity* 2 *and* $\equiv$ *is the ns operadic congruence generated by*

$$\mathsf{a}_{i+j} \circ_1 \mathsf{a}_i \;\leftrightarrow\; \mathsf{a}_i \circ_2 \mathsf{a}_j, \qquad i, j \geqslant 0, i + j \leqslant k. \tag{4.13}$$

*Proof* First, note that by replacing $\mathsf{a}_i$ by $0i \in \mathsf{FCat}^{(k)}(2)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the relation $x \leftrightarrow y$ of the statement of the theorem. Indeed, this equivalence class is the one of the element $(0, i, i + j)$ of $\mathsf{FCat}^{(k)}$.

Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by



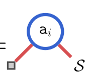
. (4.14)

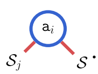
This rewrite rule is terminating. Indeed, it is plain that for any rewriting $T_0 \to T_1$, we have $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

Moreover, the normal forms of $\mapsto$ are all elements of $\mathcal{F}\left(\{\mathsf{a}_0, \dots, \mathsf{a}_k\}\right)$ such that for each node $x$ labeled by $\mathsf{a}_i$ which has a left child $y$ labeled by $\mathsf{a}_j$, one has $i < j$. This set $\mathcal{S}$ of syntax trees admits the following regular specification

$\mathcal{S} =$  $+ \bigsqcup_{0 \leqslant i \leqslant k} \mathcal{S}_i,$ (4.15)

where $\mathcal{S}_i$ is the set of such syntax trees with roots labeled by $\mathsf{a}_i$. These sets satisfy the following regular specification

$\mathcal{S}_i =$ 

$+ \bigsqcup_{i+1 \leqslant j \leqslant k}$ 

. (4.16)

Hence, the generating series $F(t)$ of $\mathcal{S}$ and $F_i(t)$ of $\mathcal{S}_i$ satisfy

$$F(t) = t + \sum_{0 \leqslant i \leqslant k} F_i(t), \tag{4.17}$$

and

$$F_i(t) = tF(t) + F(t) \sum_{i+1 \leqslant j \leqslant k} F_j(t). \tag{4.18}$$

By basic manipulations involving binomial coefficients, we obtain

$$F_i(t) = tF(t) \sum_{0 \leqslant j \leqslant k-i} \binom{k-i}{j} F(t)^j, \tag{4.19}$$

and then,

$$F(t) = t \sum_{0 \leqslant j \leqslant k+1} \binom{k+1}{j} F(t)^j. \tag{4.20}$$

The functional equation (4.20) is an alternative functional equation for the generating series of $k$-leafy trees. By Proposition 4.7, $F(t)$ is also the Hilbert series of $\mathsf{FCat}^{(k)}$.

Hence, by Lemma 2.2, $\mathsf{FCat}^{(k)}$ admits the claimed presentation. □

#### *4.1.4 A ns operad on Schröder trees*

Let $\mathsf{Schr}$ be the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 00, 01, and 10. The following table shows the first elements of $\mathsf{Schr}$.

| Arity | Elements of Schr |
|---|---|
| 1 | 0 |
| 2 | 00, 01, 10 |
| 3 | 000, 001, 010, 011, 012, 021, 100, 101, 110, 120, 210 |
| 4 | 0000, 0001, 0010, 0011, 0012, 0021, 0100, 0101, 0110, 0111, 0112, 0120, 0121, 0122, 0123, 0132, 0210, 0211, 0212, 0221, 0231, 0321, 1000, 1001, 1010, 1011, 1012, 1021, 1100, 1101, 1110, 1120, 1200, 1201, 1210, 1220, 1230, 1320, 2100, 2101, 2110, 2120, 2210, 2310, 3210 |

Since $\mathsf{FCat}^{(1)}$ is generated by 00 and 01, $\mathsf{FCat}^{(1)}$ is a ns suboperad of $\mathsf{Schr}$. Moreover, since $\mathsf{PW}$ is, by Proposition 4.2, generated as an operad by 00 and 01, $\mathsf{Schr}$ is a ns suboperad of $\mathsf{PW}$.

One has the following characterization of the elements of $\mathsf{Schr}$:

**Proposition 4.10** *The elements of* $\mathsf{Schr}$ *are exactly the words* $x$ *on the alphabet* $\mathbb{N}$ *having at least one occurrence of* 0 *for which for every letter* $\mathtt{b} \geqslant 1$ *of* $x$*, there exists* *a letter* $\mathtt{a} := \mathtt{b} - 1$ *such that* $x$ *has a factor* $\mathtt{a}u\mathtt{b}$ *or* $\mathtt{b}u\mathtt{a}$ *where* $u$ *is a word consisting in letters* $\mathtt{c}$ *satisfying* $\mathtt{c} \geqslant \mathtt{b}$.

*Proof* Let us first show by induction on the length of the words that any word $x$ of $\mathsf{Schr}$ satisfies the statement. This is true when $|x| = 1$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of $\mathsf{Schr}$ of length $n := |x| - 1$, an integer $i \in [n]$, and $g \in \{00, 01, 10\}$ such that $x = y \circ_i g$. Then,

$$x = (y_1, \ldots, y_{i-1}, y_i + g_1, y_i + g_2, y_{i+1}, \ldots, y_{n-1}). \tag{4.21}$$

Since by the induction hypothesis $y$ satisfies the statement and $x_i = y_i$ or $x_{i+1} = y_i$, $x$ also satisfies the statement.

Let us now show by induction on the length of the words that $\mathsf{Schr}$ contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, let us observe that if $x$ only consists in letters 0, $\mathsf{Schr}$ contains $x$ because $x$ can be obtained by composing the generator 00 with itself. Hence, let us assume that $x$ contains at least one letter different from 0. Set $\mathtt{b}$ as the greatest letter of $x$ and $\mathtt{a} := \mathtt{b} - 1$. Since $\mathtt{b}$ is the greatest letter of $x$, there is a factor $x_i x_{i+1}$ of $x$ such that $x_i x_{i+1} \in \{\mathtt{bb}, \mathtt{ba}, \mathtt{ab}\}$. Set

$$y := (x_1, \ldots, x_{i-1}, \min\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n), \tag{4.22}$$

and $g$ as the generator 00 if $x_i = x_{i+1}$, as 01 if $x_i = x_{i+1} - 1$, or as 10 when $x_i = x_{i+1} + 1$. Then, we have $x = y \circ_i g$, and, since $y$ is obtained from $x$ by removing one of its greatest letter, $y$ satisfies the statement. By the induction hypothesis, $\mathsf{Schr}$ contains $y$, and since $x = y \circ_i g$, $\mathsf{Schr}$ also contains $x$. □

A *Schröder tree* is a planar rooted tree such that no node has exactly one child. The size $|T|$ of a Schröder tree $T$ is its number of leaves. There is a bijection $\phi_{\mathsf{Schr}}$ between the words of $\mathsf{Schr}$ of arity $n$ and Schröder trees of size $n$.

To compute $\phi_{\mathsf{Schr}}(x)$ where $x$ is an element of $\mathsf{Schr}$, factorize $x$ as $x = x^{(1)}\mathtt{a} \ldots \mathtt{a}x^{(\ell)}$ where $\mathtt{a}$ is the smallest letter occurring in $x$ and the $x^{(i)}$ are factors of $x$ without $\mathtt{a}$. Then, set

$$\phi_{\mathsf{Schr}}(x) := \begin{cases} \mid & \text{if } x = \epsilon, \\ \bigwedge\left(\phi_{\mathsf{Schr}}\left(x^{(1)}\right), \ldots, \phi_{\mathsf{Schr}}\left(x^{(\ell)}\right)\right) & \text{otherwise,} \end{cases} \tag{4.23}$$

where $\epsilon$ denotes the empty word and $\bigwedge(T_1, \ldots, T_\ell)$ is the Schröder tree consisting in a root that has $T_1, \ldots, T_\ell$ as subtrees from left to right.

The inverse bijection is computed as follows. Given a Schröder tree $T$, one computes an element of $\mathsf{Schr}$ by considering each internal node $s$ and two adjacent consecutive edges of $s$ and by assigning to these the depth of $s$. The element of $\mathsf{Schr}$ is obtained by reading the labels from left to right.

Since the elements of $\mathsf{Schr}$ satisfy Proposition 4.10, $\phi_{\mathsf{Schr}}$ is well-defined. Figure 6 shows an example of this bijection.

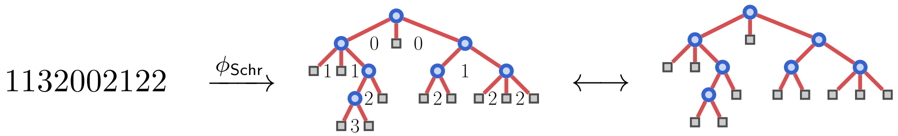
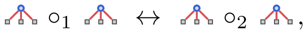


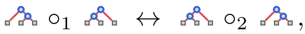

**Fig. 6** Interpretation of an element of the ns operad $\mathsf{Schr}$ in terms of Schröder trees via the bijection $\phi_{\mathsf{Schr}}$

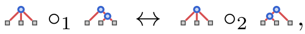
**Theorem 4.11** *The ns operad* $\mathsf{Schr}$ *admits the presentation*

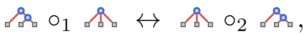
 (4.24)

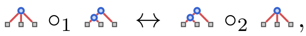
*where* , , *and*  *are of arity* 2, *and* $\equiv$ *is the ns operadic congruence generated by*

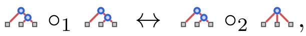
 (4.25)

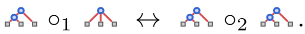
 (4.26)

 (4.27)

 (4.28)

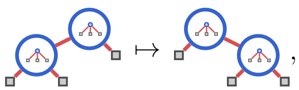
 (4.29)

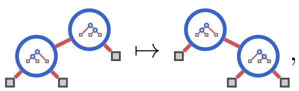
 (4.30)

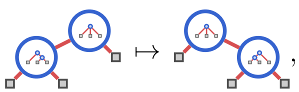
 (4.31)

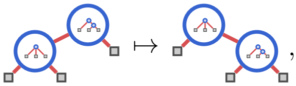
*Proof* First, note that by replacing  by $00 \in \mathsf{Schr}(2)$,  by $01 \in \mathsf{Schr}(2)$, and  by $10 \in \mathsf{Schr}(2)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the seven relations $x \leftrightarrow y$ of the statement of the theorem. Indeed, then seven equivalence classes are, respectively, the ones of the elements 000, 101, 010, 001, 100, 011, and 110 of $\mathsf{Schr}$.

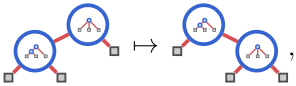
Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by

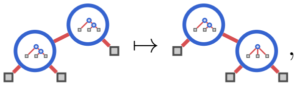
 (4.32)

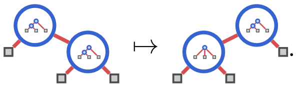
 (4.33)

 (4.34)

 (4.35)

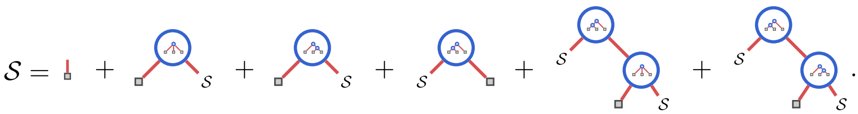
 (4.36)

 (4.37)

 (4.38)

This rewrite rule is terminating. Indeed, let $T$ be an element of $\mathcal{F}(\{\ ,\ ,\ \})$.. By associating with $T$ the pair $(k_T, \mathrm{w}(T))$ where $k_T$ is the number of nodes labeled by in $T$, it is plain that for any rewriting $T_0 \to T_1$, one has $k_{T_0} < k_{T_1}$ or $k_{T_0} = k_{T_1}$ and $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

Moreover, the normal forms of $\mapsto$ are all elements of $\mathcal{F}(\{\ ,\ ,\ \})$ such that nodes labeled by or have no left child and nodes labeled by have no right child labeled by .. This set $\mathcal{S}$ of syntax trees admits the following regular specification



(4.39)

Hence, the generating series $F(t)$ of $\mathcal{S}$ satisfies

$$F(t) = t + 3tF(t) + 2tF(t)^2, \tag{4.40}$$

that is the generating series of Schröder trees. By Proposition 4.10, $F(t)$ is also the Hilbert series of Schr.

Hence, by Lemma 2.2, Schr admits the claimed presentation. □

*4.1.5 A ns operad on Motzkin words*

Let Motz be the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 00 and 010. The following table shows the first elements of Motz.

| Arity | Elements of Motz |
|---|---|
| 1 | 0 |
| 2 | 00 |
| 3 | 000, 010 |
| 4 | 0000, 0010, 0100, 0110 |
| 5 | 00000, 00010, 00100, 00110, 01000, 01010, 01100, 01110, 01210 |
| 6 | 000000, 000010, 000100, 000110, 001000, 001010, 001100, 001110, 001210, 010000, 010010, 010100, 010110, 011000, 011010, 011100, 011110, 011210, 012100, 012110, 012210 |

Since 00 and 01 generate $\mathsf{FCat}^{(1)}$ and since $010 = 00 \circ_1 01$, Motz is a ns suboperad of $\mathsf{FCat}^{(1)}$. Moreover, since $\mathsf{FCat}^{(0)}$ is generated by 00, $\mathsf{FCat}^{(0)}$ is a ns suboperad of Motz.

One has the following characterization of the elements of Motz:

**Proposition 4.12** *The elements of* Motz *are exactly the words $x$ on the alphabet $\mathbb{N}$ which begin and end with* 0 *and satisfy $|x_i - x_{i+1}| \leqslant 1$ for all $i \in [|x| - 1]$.*

*Proof* Let us first show by induction on the length of the words that every word $x$ of Motz satisfies the statement. This is true when $|x| = 1$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of Motz of length $n < |x|$, an integer $i \in [n]$, and $g \in \{00, 010\}$ such that $x = y \circ_i g$. If $g = 00$, then one has

$$x = (y_1, \dots, y_{i-1}, y_i, y_i, y_{i+1}, \dots, y_n). \tag{4.41}$$

Otherwise, we have $g = 010$ and

$$x = (y_1, \dots, y_{i-1}, y_i, y_i + 1, y_i, y_{i+1}, \dots, y_n). \tag{4.42}$$

Since by the induction hypothesis $y$ satisfies the statement, it is immediate that in both cases, $x$ also satisfies the statement.

Let us now show by induction on the length of the words that Motz contains every word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, one has two cases to consider. If $x$ contains a factor $x_i x_{i+1}$ such that $x_i = x_{i+1}$, by setting

$$y := (x_1, \dots, x_i, x_{i+2}, \dots, x_n), \tag{4.43}$$

we have $x = y \circ_i 00$. Otherwise, let $\mathtt{b}$ be the greatest letter of $x$. Since $x$ satisfies the statement, there is in $x$ a factor $x_i x_{i+1} x_{i+2}$ where $x_{i+1} = \mathtt{b}$ and $x_i = x_{i+2} = \mathtt{b} - 1$. By setting

$$y := (x_1, \dots, x_i, x_{i+3}, \dots, x_n), \tag{4.44}$$

we have $x = y \circ_i 010$. Now, for both cases, since by the induction hypothesis, Motz contains $y$, Motz also contains $x$. □

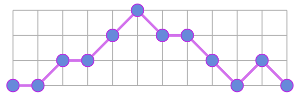
$$001123221010 \xrightarrow{\phi_{\mathsf{Motz}}} 01011\bar{1}0\bar{1}\bar{1}1\bar{1} \longleftrightarrow$$ 

**Fig. 7** Interpretation of an element of the ns operad Motz in terms of Motzkin words and Motzkin paths via the bijection $\phi_{\mathsf{Motz}}$

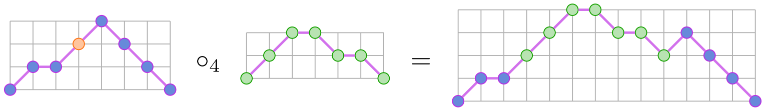


**Fig. 8** Interpretation of the partial composition map of the ns operad Motz in terms of Motzkin paths

A *Motzkin word* is a word $u$ on the alphabet $\{-1, 0, 1\}$ such that the sum of all letters of $u$ is 0 and, for any prefix $u'$ of $u$, the sum of all letters of $u'$ is a nonnegative integer. The size $|u|$ of a Motzkin word $u$ is its length plus one. In the sequel, we shall denote by $\bar{1}$ the letter $-1$. We can represent a Motzkin word $u$ graphically by a *Motzkin path* that is the path in $\mathbb{N}^2$ connecting the points $(0, 0)$ and $(n, 0)$ obtained by drawing a step $(1, -1)$ (resp. $(1, 0)$, $(1, 1)$) for each letter $\bar{1}$ (resp. 0, 1) of $u$. There is a bijection $\phi_{\mathsf{Motz}}$ between the words of Motz of arity $n$ and Motzkin words of size $n$.

To compute $\phi_{\mathsf{Motz}}(x)$ where $x$ is an element of $\mathsf{Motz}(n)$, build the word $u$ of length $n-1$ satisfying $u_i := x_{i+1} - x_i$ for all $i \in [n-1]$.

The inverse bijection is computed as follows. The element of Motz which corresponds to a Motzkin word $u$ is the word $x$ such that $x_i$ is the sum of the letters of the prefix $u_1 \dots u_{i-1}$ of $u$, for all $i \in [n]$.

Since the elements of Motz satisfy Proposition 4.12, $\phi_{\mathsf{Motz}}$ is well-defined. Figure 7 shows an example of this bijection.

The bijection $\phi_{\mathsf{Motz}}$ between elements of Motz and Motzkin words offers an alternative way to compute the composition of elements of Motz:

**Proposition 4.13** *Let $u$ and $v$ be two Motzkin words where $u$ is of size $n$, and $i \in [n]$ be an integer. Then the composition $u \circ_i v$ in* Motz *amounts to insert $v$ at the $i$th position into $u$.*

*Proof* Let $y$ be the element of Motz in bijection by $\phi_{\mathsf{Motz}}$ with $v$. The statement is a direct consequence of the fact that, by Proposition 4.12, $y$ starts and ends by 0. □

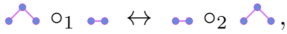
Figure 8 shows an example of composition in Motz.

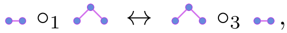
**Theorem 4.14** *The ns operad* Motz *admits the presentation*

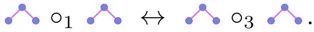
$$\mathsf{Motz} = \mathcal{F}\left(\{ \bullet\!\!-\!\!\bullet, \wedge \}\right) /_{\equiv}, \tag{4.45}$$

*where* $\bullet\!\!-\!\!\bullet$ *is of arity* 2, $\wedge$ *of arity* 3, *and* $\equiv$ *is the ns operadic congruence generated by*

$$\bullet\!\!-\!\!\bullet \circ_1 \bullet\!\!-\!\!\bullet \leftrightarrow \bullet\!\!-\!\!\bullet \circ_2 \bullet\!\!-\!\!\bullet, \tag{4.46}$$

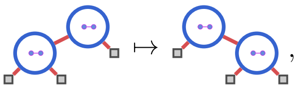
 (4.47)

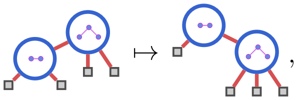
 (4.48)

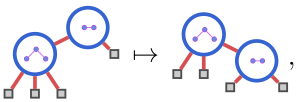
 (4.49)

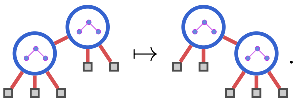
*Proof* First, note that by replacing  by $00 \in \mathsf{Motz}(2)$ and  by $010 \in \mathsf{Motz}(3)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the four relations $x \leftrightarrow y$ of the statement of the theorem. Indeed, the four equivalence classes are, respectively, the ones of the elements 000, 0010, 0100, and 01010 of Motz.

Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by

 (4.50)

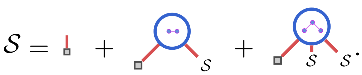
 (4.51)

 (4.52)

 (4.53)

This rewrite rule is terminating. Indeed, it is plain that for any rewriting $T_0 \to T_1$, we have $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

Moreover, the normal forms of $\mapsto$ are all syntax trees of $\mathcal{F}$ ({ ,  }) which have no internal node with an internal node as leftmost child. This set $\mathcal{S}$ of trees admits the following regular specification

 (4.54)

Hence, the generating series $F(t)$ of $\mathcal{S}$ satisfies

$$F(t) = t + tF(t) + tF(t)^2, \tag{4.55}$$

that is the generating series of Motzkin words. By Proposition 4.12, $F(t)$ is also the Hilbert series of Motz.

Hence, by Lemma 2.2, Motz admits the claimed presentation. □

*4.1.6 A ns operad on integer compositions*

Let Comp be the ns suboperad of $\mathsf{T}\mathbb{N}_2$ generated by 00 and 01. The following table shows the first elements of Comp.

| Arity | Elements of Comp |
|---|---|
| 1 | 0 |
| 2 | 00, 01 |
| 3 | 000, 001, 010, 011 |
| 4 | 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111 |
| 5 | 00000, 00001, 00010, 00011, 00100, 00101, 00110, 00111, 01000, 01001, 01010, 01011, 01100, 01101, 01110, 01111 |

Since $\mathsf{FCat}^{(1)}$ is the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 00 and 01, and since $\mathsf{T}\mathbb{N}_2$ is a quotient of $\mathsf{T}\mathbb{N}$, Comp is a quotient of $\mathsf{FCat}^{(1)}$. Moreover, $\mathsf{FCat}^{(0)}$ is a quotient of Comp by the ns operadic congruence $\equiv$ defined for all $x, y \in$ Comp by $x \equiv y$.

One has the following characterization of the elements of Comp:

**Proposition 4.15** *The elements of* Comp *are exactly the words on the alphabet* $\{0, 1\}$ *beginning by* 0.

*Proof* It is immediate, from the definition of Comp and Lemma 2.1, that any element of this ns operad starts by 0 since its generators 00 and 01 all start by 0.

Let us now show by induction on the length of the words that Comp contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, let us observe that if $x$ only consists of letters 0, Comp contains $x$ because $x$ can be obtained by composing the generator 00 with itself. Otherwise, $x$ has at least one occurrence of 1. Since its first letter is 0, there is in $x$ a factor $x_i x_{i+1} = 01$. By setting

$$y := (x_1, \ldots, x_i, x_{i+2}, \ldots, x_n), \tag{4.56}$$

we have $x = y \circ_i 01$. Since $y$ satisfies the statement, by the induction hypothesis Comp contains $y$. Hence, Comp also contains $x$. □

An *integer composition* is a sequence $u_1 \ldots u_k$ of positive integers. The size $|u|$ of an integer composition $u$ is the sum of its letters. It is well-known that there are $2^{n-1}$ integer compositions of size $n$. We shall represent an integer composition $u := u_1 \ldots u_k$ by a *ribbon diagram*, that is the diagram in which each letter $u_i$ of $u$ is encoded by a column consisting in $u_i$ boxes, and the column encoding the letter $u_{i+1}$ is attached on the right edge of the bottommost box of the column encoding $u_i$, for any $i \in [k-1]$. The *$i$th box* of a ribbon diagram $D$ is the $i$th encountered box by traversing $D$ column by column from left to right and from top to bottom. The *transpose* of $D$ is the ribbon diagram obtained by applying on $D$ the reflection through the line passing by its first and its last boxes. There is a bijection $\phi_{\mathsf{Comp}}$ between the words of Comp of arity $n$ and ribbon diagrams of integer compositions of size $n$.

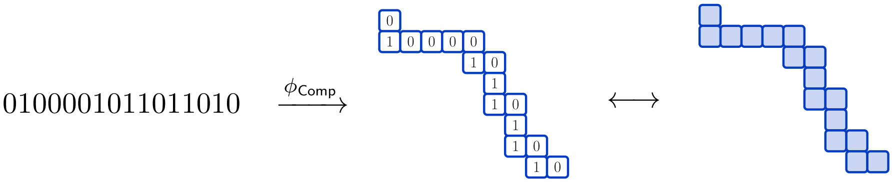



**Fig. 9** Interpretation of an element of the ns operad $\mathsf{Comp}$ in terms of integer compositions via the bijection $\phi_{\mathsf{Comp}}$. *Boxes* of the ribbon diagram in the *middle* are labeled

To compute $\phi_{\mathsf{Comp}}(x)$ where $x$ is an element of $\mathsf{Comp}$, iteratively insert the letters of $x$ from left to right according to the following procedure. If $|x| = 1$, then $x = 0$ and $\phi_{\mathsf{Comp}}(0)$ is the only ribbon diagram consisting in one box. Otherwise, the insertion of a letter $\mathtt{a}$ into $D$ consists in adding a new box below (resp. to the right of) the right bottommost box of $D$ if $\mathtt{a} = 1$ (resp. $\mathtt{a} = 0$).

The inverse bijection is computed as follows. Given a ribbon diagram $D$ of an integer composition of size $n$, one computes an element of $\mathsf{Comp}$ of arity $n$ by labelling the first box of $D$ by 0 and the $i$th box $b$ by 0 if the $(i-1)$st box is on the left of $b$ or by 1 otherwise, for any $1 \leqslant i \leqslant n$. The corresponding element of $\mathsf{Comp}$ is obtained by reading the labels of $D$ from top to bottom and left to right.

Since the elements of $\mathsf{Comp}$ satisfy Proposition 4.15, $\phi_{\mathsf{Comp}}$ is well-defined. Hence, we can regard the elements of arity $n$ of $\mathsf{Comp}$ as ribbon diagrams with $n$ boxes. Figure 9 shows an example of this bijection.

Encoding integer compositions by ribbon diagrams offers an alternative way to compute the composition of elements of $\mathsf{Comp}$:

**Proposition 4.16** *Let $C$ and $D$ be two ribbon diagrams, $i$ be an integer, and $c$ be the $i$th box of $C$. Then, the composition $C \circ_i D$ in* $\mathsf{Comp}$ *amounts to replace $c$ by $D$ if $c$ is the upper box of its column, or to replace $c$ by the transpose ribbon diagram of $D$ otherwise.*

*Proof* Let $x \in \mathsf{Comp}(n)$ and $y \in \mathsf{Comp}(m)$ such that $C := \phi_{\mathsf{Comp}}(x)$ and $D := \phi_{\mathsf{Comp}}(y)$. Let $E := \phi_{\mathsf{Comp}}(x \circ_i y)$. By definition of $\phi_{\mathsf{Comp}}$ and the partial composition maps of $\mathsf{Comp}$, $E$ is obtained by inserting the prefix of length $i-1$ of $x$, then the letters of $y$ incremented by $x_i$, and finally, the suffix of length $n-i$ of $x$. If $c$ is the upper box of its column, then $x_i = 0$, and by definition of $\phi_{\mathsf{Comp}}$, $E$ is obtained by replacing $c$ by $D$ in $C$. Otherwise, $c$ is not the upper box of its column, and then $x_i = 1$. Immediately from the definition of $\phi_{\mathsf{Comp}}$, for any word $z$ on the alphabet $\{0, 1\}$, the ribbon diagram $\phi_{\mathsf{Comp}}(0z)$ is the transpose of $\phi_{\mathsf{Comp}}(0\bar{z})$ where $\bar{z}$ is the complementary of $z$. This implies the statement. □

Figure 10 shows two examples of compositions in $\mathsf{Comp}$.

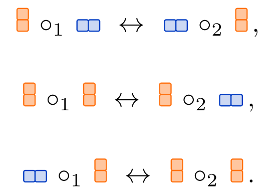
**Theorem 4.17** *The ns operad* $\mathsf{Comp}$ *admits the presentation*

$$\mathsf{Comp} = \mathcal{F}\left(\left\{ \square\square, \boxminus \right\}\right) /_{\equiv}, \tag{4.57}$$

**(a)** Composition in an upper box of a column. **(b)** Composition in a non-upper box of a column.

**Fig. 10** Interpretation of the partial composition map of the ns operad Comp in terms of ribbon diagrams

*where* and *are of arity* 2, *and* $\equiv$ *is the ns operadic congruence generated by*



(4.58)

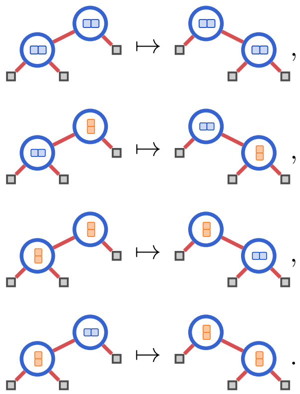
(4.59)

(4.60)

(4.61)

*Proof* First, note that by replacing by $00 \in \mathsf{Comp}(2)$ and by $01 \in \mathsf{Comp}(2)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the four relations $x \leftrightarrow y$ of the statement of the theorem. Indeed, the four equivalence classes are, respectively, the ones of the elements 000, 001, 011, and 100 of Comp.

Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by



(4.62)

(4.63)

(4.64)

(4.65)

This rewrite rule is terminating. Indeed, it is plain that for any rewriting $T_0 \to T_1$, we have $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

Moreover, the normal forms of $\mapsto$ are all syntax trees of $\mathcal{F}(\{ \; , \; \})$ which have no internal node with an internal node as leftmost child. Hence, the generating series $F(t)$ of the normal forms of $\mapsto$ is

$$F(t) = \sum_{n \geqslant 1} 2^{n-1} t^n. \tag{4.66}$$

By Proposition 4.15, $F(t)$ is also the Hilbert series of Comp.

Hence, by Lemma 2.2, Comp admits the claimed presentation. □

#### *4.1.7 A ns operad on directed animals*

Let DA be the ns suboperad of $\mathsf{T}\mathbb{N}_3$ generated by 00 and 01. We shall here denote by $\bar{1}$ the representative of the equivalence class of 2 in $\mathbb{N}_3$. The following table shows the first elements of DA.

| Arity | Elements of DA |
|---|---|
| 1 | $0$ |
| 2 | $00, 01$ |
| 3 | $000, 001, 010, 011, 01\bar{1}$ |
| 4 | $0000, 0001, 0010, 0011, 001\bar{1}, 0100, 0101, 0110, 0111, 011\bar{1}, 01\bar{1}0, 01\bar{1}1, 01\bar{1}\bar{1}$ |
| 5 | $00000, 00001, 00010, 00011, 0001\bar{1}, 00100, 00101, 00110, 00111, 0011\bar{1}, 001\bar{1}0, 001\bar{1}1, 001\bar{1}\bar{1}, 01000, 01001, 01010, 01011, 0101\bar{1}, 01100, 01101, 01110, 01111, 0111\bar{1}, 011\bar{1}0, 011\bar{1}1, 011\bar{1}\bar{1}, 01\bar{1}00, 01\bar{1}01, 01\bar{1}0\bar{1}, 01\bar{1}10, 01\bar{1}11, 01\bar{1}1\bar{1}, 01\bar{1}\bar{1}0, 01\bar{1}\bar{1}1, 01\bar{1}\bar{1}\bar{1}$ |

Since $\mathsf{FCat}^{(1)}$ is the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 00 and 01, and since $\mathsf{T}\mathbb{N}_3$ is a quotient of $\mathsf{T}\mathbb{N}$, DA is a quotient of $\mathsf{FCat}^{(1)}$. Moreover, DA is a quotient of $\mathsf{FCat}^{(0)}$ by the ns operadic congruence $\equiv$ defined for all $x, y \in \mathsf{DA}$ by $x \equiv y$.

**Proposition 4.18** *Let* $\phi_{\mathsf{DA}} : \mathsf{DA}(n) \to \{\bar{1}, 0, 1\}^{n-1}$ *be the mapping defined for any element $x$ of arity $n$ of* DA *by*

$$\phi_{\mathsf{DA}}(x) := (x_1 * x_2, x_2 * x_3, \dots, x_{n-1} * x_n), \tag{4.67}$$

*where* $x_i * x_{i+1} := x_{i+1} - x_i \mod 3$. *Then,* $\phi_{\mathsf{DA}}$ *is a bijection between the elements of arity $n$ of* DA *and prefixes of Motzkin words of length* $n - 1$.

*Proof* Let us first show by induction on the length of the words that for any $x \in \mathsf{DA}$, $\phi_{\mathsf{DA}}(x)$ is a prefix of a Motzkin word of length $|x| - 1$. This is true when $|x| = 1$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of DA of length $n := |x| - 1$, an integer $i \in [n]$, and $g \in \{00, 01\}$ such that $x = y \circ_i g$. We now have two cases depending on $g$.

Case 1: If $g = 00$, then

$$x = (y_1, \dots, y_i, y_i, y_{i+1}, \dots, y_n). \tag{4.68}$$

By the induction hypothesis, $\phi_{\mathsf{DA}}(y)$ is a prefix of a Motzkin word of length $n - 1$. Since $x$ is obtained from $y$ by duplicating its $i$th letter, $\phi_{\mathsf{DA}}(x)$ is obtained from $\phi_{\mathsf{DA}}(y)$ by inserting a 0 at an appropriate place. Hence, $\phi_{\mathsf{DA}}(x)$ is a prefix of a Motzkin word of length $n$.

Case 2: Otherwise, we have $g = 01$ and then,

$$x = (y_1, \dots, y_i, y_i + 1, y_{i+1}, \dots, y_n), \tag{4.69}$$

where $+$ denotes the addition in $\mathbb{N}_3$. We have now two sub-cases whether $y_i$ is the last letter of $y$.

Case 2.1: If it is the case, then $x_{i+1} = y_i + 1$ is the last letter of $x$ and $\phi_{\mathsf{DA}}(x)$ is obtained from $\phi_{\mathsf{DA}}(y)$ by concatenating a 1 on the right. Hence, since by the induction hypothesis, $\phi_{\mathsf{DA}}(y)$ is a prefix of a Motzkin word of length $n - 1$, $\phi_{\mathsf{DA}}(x)$ is a prefix of a Motzkin word of length $n$.

Case 2.2: Otherwise, we have $i < n$. We observe that $\phi_{\mathsf{DA}}(x)$ is obtained from $\phi_{\mathsf{DA}}(y)$ by replacing a letter 0 (resp. 1, $\bar{1}$) by a factor $1\bar{1}$ (resp. 10, 11) at an appropriate place. Hence, since by the induction hypothesis, $\phi_{\mathsf{DA}}(y)$ is a prefix of a Motzkin word of length $n - 1$, $\phi_{\mathsf{DA}}(x)$ is a prefix of a Motzkin word of length $n$.

Let us now show that $\phi_{\mathsf{DA}}$ is a bijection between the elements of arity $n$ of $\mathsf{DA}$ and prefixes of Motzkin words of length $n - 1$.

The injectivity of $\phi_{\mathsf{DA}}$ is a direct consequence of the fact that, given an element $x$ of $\mathsf{DA}$ and a letter $\mathtt{a} \in \{\bar{1}, 0, 1\}$, there is at most one letter $\mathtt{b} \in \{\bar{1}, 0, 1\}$ such that $\phi_{\mathsf{DA}}(x\mathtt{b}) = \phi_{\mathsf{DA}}(x)\mathtt{a}$.

Let us finally show that $\phi_{\mathsf{DA}}$ is a surjection. We proceed by induction on the length of the words to construct for any prefix of a Motzkin word $u$ an element $x$ of $\mathsf{DA}$ such that $\phi_{\mathsf{DA}}(x) = u$. When $u$ is the empty word, $x := 0$ is an element of $\mathsf{DA}(1)$ and since $\phi_{\mathsf{DA}}(x)$ is the empty word, the property is satisfied. When $n := |u| \geqslant 1$, one has two cases to consider depending on the last letter $u_n$ of $u$.

Case 1′: If $u_n \in \{0, 1\}$, by the induction hypothesis, there is an element $y$ of $\mathsf{DA}(n)$ such that $\phi_{\mathsf{DA}}(y) = u_1 \dots u_{n-1}$. Hence, by setting $x := y \circ_n 0u_n$, it follows, by definition of $\phi_{\mathsf{DA}}$, that $x$ is a preimage of $u$ for $\phi_{\mathsf{DA}}$.

Case 2′: Otherwise, we have $u_n = \bar{1}$ and there is at least one occurrence of a 1 in $u$. Hence, let $i \in [n - 1]$ be the greatest integer such that $u_i = 1$. We now have two sub-cases depending on the value of $u_{i+1}$.

Case 2′1.: If $u_{i+1} = 0$, the word

$$u' := u_1 \dots u_i u_{i+2} \dots u_n \tag{4.70}$$

is still a prefix of a Motzkin word. Then, by the induction hypothesis, there is an element $y$ of $\mathsf{DA}(n)$ such that $\phi_{\mathsf{DA}}(y) = u'$. Hence, by setting $x := y \circ_i 00$, it follows, by definition of $\phi_{\mathsf{DA}}$, that $x$ is a preimage of $u$ for $\phi_{\mathsf{DA}}$.

Case 2′.2: Otherwise, we have $u_{i+1} = \bar{1}$. Then, the word

$$u' := u_1 \dots u_{i-1} 0 u_{i+2} \dots u_n \tag{4.71}$$

is still a prefix of a Motzkin word. Then, by the induction hypothesis, there is an element $y$ of $\mathsf{DA}(n)$ such that $\phi_{\mathsf{DA}}(y) = u'$. Hence by setting $x := y \circ_i 01$, it follows, by definition of $\phi_{\mathsf{DA}}$, that $x$ is a preimage of $u$ for $\phi_{\mathsf{DA}}$.

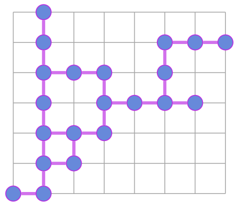


**Fig. 11** A directed animal of size 21. The point (0, 0) is the lowest and leftmost point

We then have proved that $\phi_{\mathsf{DA}}$ is well-defined, injective, and surjective. Hence, it is a bijection between elements of arity $n$ of $\mathsf{DA}$ and prefixes of Motzkin words of length $n - 1$. □

Here are two examples of images by $\phi_{\mathsf{DA}}$ of elements of $\mathsf{DA}$.

$$\phi_{\mathsf{DA}}(011\bar{1}\bar{1}0\bar{1}01) = 10101\bar{1}11, \tag{4.72}$$

$$\phi_{\mathsf{DA}}(010010101\bar{1}1) = 1\bar{1}01\bar{1}1\bar{1}11\bar{1}. \tag{4.73}$$

Recall that a *directed animal* is a subset $A$ of $\mathbb{N}^2$ such that $(0, 0) \in A$ and $(i, j) \in A$ with $i \geqslant 1$ or $j \geqslant 1$ implies $(i - 1, j) \in A$ or $(i, j - 1) \in A$. The size of a directed animal $A$ is its cardinality. Figure 11 shows a directed animal.

According to [15], there is a bijection $\alpha$ between the set of prefixes of Motzkin words of length $n - 1$ and the set of directed animals of size $n$. Hence, by Proposition 4.18, the map $\alpha \circ \phi_{\mathsf{DA}}$ is a bijection between the elements of $\mathsf{DA}$ of arity $n$ and directed animals of size $n$ and moreover, $\mathsf{DA}$ can be seen as a ns operad on directed animals.

**Theorem 4.19** *The ns operad* $\mathsf{DA}$ *admits the presentation*

$$\mathsf{DA} = \mathcal{F}\left(\{\bullet\!\!-\!\!\bullet, \overset{\bullet}{\underset{\bullet}{|}}\}\right)/_{\equiv},$$

*where* $\bullet\!\!-\!\!\bullet$ *and* $\overset{\bullet}{\underset{\bullet}{|}}$ *are of arity* 2, *and* $\equiv$ *is the ns operadic congruence generated by*

$$\bullet\!\!-\!\!\bullet \circ_1 \bullet\!\!-\!\!\bullet \leftrightarrow \bullet\!\!-\!\!\bullet \circ_2 \bullet\!\!-\!\!\bullet, \tag{4.74}$$

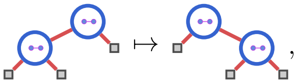
$$\overset{\bullet}{\underset{\bullet}{|}} \circ_1 \bullet\!\!-\!\!\bullet \leftrightarrow \bullet\!\!-\!\!\bullet \circ_2 \overset{\bullet}{\underset{\bullet}{|}}, \tag{4.75}$$

$$\overset{\bullet}{\underset{\bullet}{|}} \circ_1 \overset{\bullet}{\underset{\bullet}{|}} \leftrightarrow \overset{\bullet}{\underset{\bullet}{|}} \circ_2 \bullet\!\!-\!\!\bullet, \tag{4.76}$$

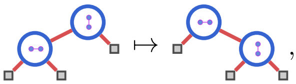
$$\left(\bullet\!\!-\!\!\bullet \circ_1 \overset{\bullet}{\underset{\bullet}{|}}\right) \circ_2 \overset{\bullet}{\underset{\bullet}{|}} \leftrightarrow \left(\overset{\bullet}{\underset{\bullet}{|}} \circ_2 \overset{\bullet}{\underset{\bullet}{|}}\right) \circ_3 \overset{\bullet}{\underset{\bullet}{|}}. \tag{4.77}$$

*Proof* First, note that by replacing $\bullet\!\!-\!\!\bullet$ by $00 \in \mathsf{DA}(2)$ and $\overset{\bullet}{\underset{\bullet}{|}}$ by $01 \in \mathsf{DA}(2)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the four relations $x \leftrightarrow y$ of the statement of the theorem. Indeed, the four equivalence classes are, respectively, the ones of the elements 000, 001, 011, and $01\bar{1}0$ of $\mathsf{DA}$.

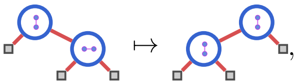
Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by



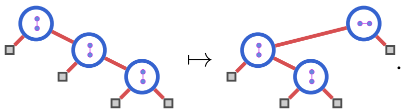
(4.78)



(4.79)



(4.80)

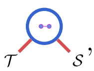


(4.81)

This rewrite rule is terminating. Indeed, let $T$ be a syntax tree of $\mathcal{F}(\{\ ,\ \})$. By associating the pair $(-k_T, \mathrm{w}(T))$ with $T$, where $k_T$ is the sum, for all internal nodes $x$ of $T$ labeled by , of the number of internal nodes constituting the right subtree of $x$, it is plain that for any rewriting $T_0 \to T_1$, we have $k_{T_0} < k_{T_1}$, or $k_{T_0} = k_{T_1}$ and $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

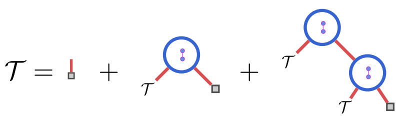
Moreover, the normal forms of $\mapsto$ are all syntax trees of $\mathcal{F}(\{\ ,\ \})$ such that no internal node labeled by has a left child labeled by , no internal node labeled by has a child labeled by , and no internal node labeled by has a right child labeled by which has a right child labeled by . This set $\mathcal{S}$ of syntax trees admits the following regular specification

$\mathcal{S} = \mathcal{T}$ +



(4.82)

where $\mathcal{T}$ is the set of syntax trees admitting the following regular specification



(4.83)

Hence, the generating series $F(t)$ of $\mathcal{S}$ satisfies

$$F(t) = \frac{1 - 3t - (1 - 2t - 3t^2)^{1/2}}{6t - 2}, \tag{4.84}$$

which is the generating function of directed animals. By Proposition 4.18, $F(t)$ is also the Hilbert series of DA.

Hence, by Lemma 2.2, DA admits the claimed presentation. □

Since the nontrivial relation (4.77) has degree 3, the presentation of DA exhibited by Theorem 4.19 is not quadratic. Moreover, DA is not a quadratic ns operad since, as an exhaustive inspection can show, there is no quadratic ns operad generated by two generators of arity 2 which has the same dimensions as DA.

### 4.1.8 A ns operad on segmented integer compositions

Let SComp be the ns suboperad of $\mathsf{T}\mathbb{N}_3$ generated by 00, 01, and 02. The following table shows the first elements of SComp.

| Arity | Elements of SComp |
|---|---|
| 1 | 0 |
| 2 | 00, 01, 02 |
| 3 | 000, 001, 002, 010, 011, 012, 020, 021, 022 |
| 4 | 0000, 0001, 0002, 0010, 0011, 0012, 0020, 0021, 0022, 0100, 0101, 0102, 0110, 0111, 0112, 0120, 0121, 0122, 0200, 0201, 0202, 0210, 0211, 0212, 0220, 0221, 0222 |

Since $\mathsf{FCat}^{(2)}$ is the ns suboperad of $\mathsf{T}\mathbb{N}$ generated by 00, 01, and 02, and since $\mathsf{T}\mathbb{N}_3$ is a quotient of $\mathsf{T}\mathbb{N}$, SComp is a quotient of $\mathsf{FCat}^{(2)}$. Moreover, since DA is generated by 00 and 01, DA is a ns suboperad of SComp.

One has the following characterization of the elements of SComp:

**Proposition 4.20** *The elements of* SComp *are exactly the words on the alphabet* $\{0, 1, 2\}$ *beginning with* 0.

*Proof* It is immediate, from the definition of SComp and Lemma 2.1, that any element of this ns operad starts by 0 since its generators 00, 01, and 02 all start by 0.

Let us now show by induction on the length of the words that SComp contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, let us observe that if $x$ only consists of letters 0, SComp contains $x$ because $x$ can be obtained by composing the generator 00 with itself. Otherwise, $x$ has at least one occurrence of a 1 or a 2. Since its first letter is 0, there is in $x$ a factor $x_i x_{i+1} =: g$ such that $g \in \{01, 02\}$. By setting

$$y := (x_1, \ldots, x_i, x_{i+2}, \ldots, x_n), \tag{4.85}$$

we have $x = y \circ_i g$. Since $y$ satisfies the statement, by the induction hypothesis SComp contains $y$. Hence, SComp also contains $x$. □

A *segmented integer composition* is a sequence $(S_1, \ldots, S_\ell)$ of integers compositions. The size $|S|$ of a segmented integer composition is the sum of the sizes of the integer compositions which constitute $S$. It is well-known that there are $3^{n-1}$ segmented integer compositions of size $n$. We shall represent a segmented integer composition $S$ by a *ribbon diagram*, that is the diagram consisting of the sequence of the ribbon diagrams of the integer compositions that constitute $S$. There is a bijection

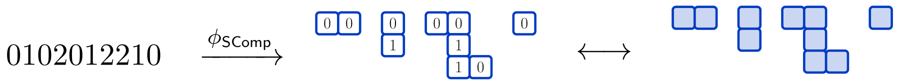


**Fig. 12** Interpretation of an element of the operad SComp in terms of a segmented composition via the bijection $\phi_{\mathsf{SComp}}$. *Boxes* of the ribbon diagram in the *middle* are labeled

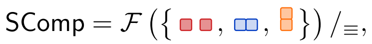
between the words of SComp of arity $n$ and ribbon diagrams of segmented compositions of size $n$.

To compute $\phi_{\mathsf{SComp}}(x)$ where $x$ is an element of SComp, factorize $x$ as $x = 0x^{(1)} \ldots 0x^{(\ell)}$ such that for any $i \in [\ell]$, the factor $x^{(i)}$ has no occurrence of 0, and compute the sequence $\left(\phi_{\mathsf{Comp}}\left(0\bar{x}^{(1)}\right), \ldots, \phi_{\mathsf{Comp}}\left(0\bar{x}^{(\ell)}\right)\right)$, where for any $i \in [\ell]$, $\bar{x}^{(i)}$ is the word obtained from $x^{(i)}$ by decreasing all letters.

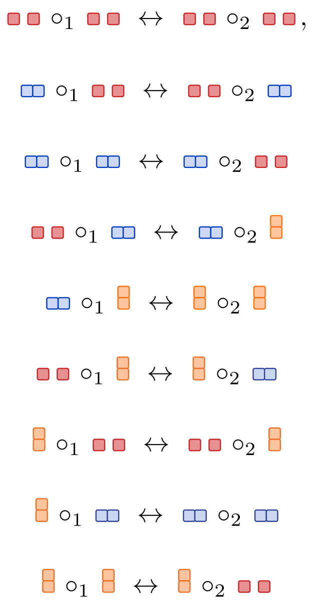
The inverse bijection is computed as follows. Given a ribbon diagram $S := (S_1, \ldots, S_\ell)$ of a segmented integer composition of size $n$, one computes an element of SComp of arity $n$ by computing the sequence $\left(u^{(1)}, \ldots, u^{(\ell)}\right)$ where for any $i \in [|\ell|]$, $u^{(i)}$ is the word of Comp obtained by applying the inverse bijection of $\phi_{\mathsf{Comp}}$ on $u^{(i)}$, then by incrementing in each $u^{(i)}$ all letters, excepted the first one, and finally by concatenating all words of the sequence.

Since the elements of SComp satisfy Proposition 4.20, $\phi_{\mathsf{SComp}}$ is well-defined. Figure 12 shows an example of this bijection.

**Theorem 4.21** *The ns operad* SComp *admits the presentation*

 (4.86)

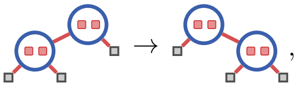
*where* , , *and*  *are of arity* 2, *and* $\equiv$ *is the ns operadic congruence generated by*

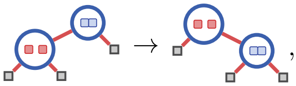


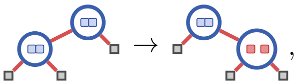
(4.87)
(4.88)
(4.89)
(4.90)
(4.91)
(4.92)
(4.93)
(4.94)
(4.95)

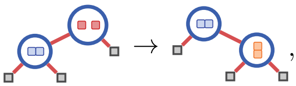
*Proof* First, note that by replacing by $00 \in \mathsf{SComp}(2)$, by $01 \in \mathsf{SComp}(2)$, and by $02 \in \mathsf{SComp}(2)$, we have $\mathrm{ev}(x) = \mathrm{ev}(y)$ for the nine relations $x \leftrightarrow y$ of the statement of the theorem. Indeed, the nine equivalence classes are, respectively, the ones of the elements 000, 001, 011, 010, 021, 020, 002, 012, and 022.

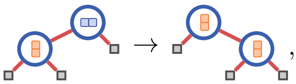
Consider now the orientation of $\leftrightarrow$ into the rewrite rule $\mapsto$ defined by

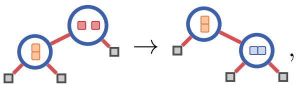

(4.96)

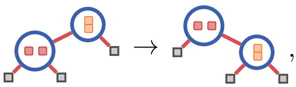

(4.97)

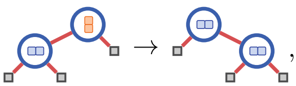

(4.98)

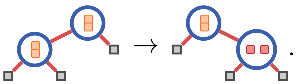

(4.99)


(4.100)


(4.101)


(4.102)


(4.103)


(4.104)

This rewrite rule is terminating. Indeed, it is plain that for any rewriting $T_0 \to T_1$, we have $\mathrm{w}(T_0) < \mathrm{w}(T_1)$.

Moreover, the normal forms of $\mapsto$ are all syntax trees of $\mathcal{F}(\{\ ,\ ,\ \})$ such that each internal node has no internal node as left son. Hence, the generating series $F(t)$ of the normal forms of $\mapsto$ is

$$F(t) = \sum_{n \geqslant 1} 3^{n-1} t^n. \tag{4.105}$$

By Proposition 4.20, $F(t)$ is also the Hilbert series of $\mathsf{SComp}$.

Hence, by Lemma 2.2, $\mathsf{SComp}$ admits the claimed presentation. □

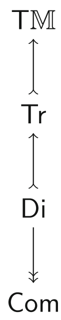



**Fig. 13** The diagram of ns suboperads and quotients of $\mathsf{T}\mathbb{M}$. *Arrows* $\rightarrowtail$ (resp. $\twoheadrightarrow$) are injective (resp. surjective) ns operad morphisms

**Table 2** Ground monoids, generators, first dimensions, and combinatorial objects involved in the ns suboperads and quotients of $\mathsf{T}\mathbb{M}$

| Monoid | Ns operad | Generators | First dimensions | Combinatorial objects |
|---|---|---|---|---|
| $\mathbb{M}$ | Tr | 01, 10, 11 | 1, 3, 7, 15, 31 | Binary words with at least one 1 |
| | Di | 01, 10 | 1, 2, 3, 4, 5 | Binary words with exactly one 1 |

## 4.2 Operads from the multiplicative monoid

We shall denote by $\mathbb{M}$ the multiplicative monoid of integers.

Note that the ns suboperad of $\mathsf{T}\mathbb{M}$ generated by 00 and the ns suboperad of $\mathsf{T}\mathbb{M}$ generated by 11 are both isomorphic to the associative commutative operad Com.

The operads constructed in this section fit into the diagram of ns operads represented by Fig. 13. Table 2 summarizes some information about these ns operads.

### *4.2.1 The diassociative operad*

Let Di be the ns suboperad of $\mathsf{T}\mathbb{M}$ generated by 01 and 10. The following table shows the first elements of Di.

| Arity | Elements of Di |
|---|---|
| 1 | 1 |
| 2 | 01, 10 |
| 3 | 001, 010, 100 |
| 4 | 0001, 0010, 0100, 1000 |
| 5 | 00001, 00010, 00100, 01000, 10000 |
| 6 | 000001, 000010, 000100, 001000, 010000, 100000 |

Note that Com is a quotient of Di by the ns operadic congruence $\equiv$ defined for all $x, y \in \mathsf{Di}(n)$ by $x \equiv y$.

One has the following characterization of the elements of Di:

**Proposition 4.22** *The elements of* Di *are exactly the words on the alphabet* $\{0, 1\}$ *containing exactly one* 1.

*Proof* Let us first show by induction on the length of the words that every word $x$ of Di satisfies the statement. This is true when $|x| = 1$ since 1 is the unit of $\mathbb{M}$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of Di of length $n := |x| - 1$, an integer $i \in [n]$, and $g \in \{01, 10\}$ such that $x = y \circ_i g$. In all cases, $x$ is obtained from $y$ by inserting a 0 at an appropriate position. Since, by the induction hypothesis, $y$ satisfies the statement, $x$ also satisfies the statement.

Let us now show by induction on the length of the words that Di contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, there is in $x$ a factor $x_i x_{i+1} =: g$ such that $g \in \{01, 10\}$. Assume without lost of generality that $g = 01$. Then, by setting

$$y := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{4.106}$$

we have $x = y \circ_i g$. Since $y$ satisfies the statement, by the induction hypothesis Di contains $y$. Hence, Di also contains $x$. □

Recall that the *diassociative operad* [20] Dias is the ns operad admitting the presentation

$$\mathsf{Dias} := \mathcal{F}(\{\dashv, \vdash\})/_{\equiv}, \tag{4.107}$$

where $\dashv$ and $\vdash$ are of arity 2, and $\equiv$ is the ns operadic congruence generated by

$$\dashv \circ_1 \dashv \;\leftrightarrow\; \dashv \circ_2 \dashv \;\leftrightarrow\; \dashv \circ_2 \vdash, \tag{4.108}$$

$$\vdash \circ_2 \vdash \;\leftrightarrow\; \vdash \circ_1 \vdash \;\leftrightarrow\; \vdash \circ_1 \dashv, \tag{4.109}$$

$$\dashv \circ_1 \vdash \;\leftrightarrow\; \vdash \circ_2 \dashv\,. \tag{4.110}$$

**Proposition 4.23** *The ns operads* Di *and* Dias *are isomorphic and the map*

$$\phi : \mathsf{Dias} \to \mathsf{Di} \tag{4.111}$$

*satisfying* $\phi(\dashv) = 10$ *and* $\phi(\vdash) = 01$ *is an isomorphism.*

*Proof* By replacing each generator $g$ of Dias by $\phi(g)$, the generators 01 and 10 of Di satisfy the same relations as the generators $\dashv$ and $\vdash$ of Dias. Moreover, by Proposition 4.22, the Hilbert series of Di is

$$F(t) = \sum_{n \geqslant 1} n t^n, \tag{4.112}$$

which is also the Hilbert series of Dias. Then, there is no nontrivial relation of degree greater than two involving generators of Di. □

Proposition 4.23 also shows that Di is a realization of the diassociative operad.

*4.2.2 The triassociative operad*

Let $\mathsf{Tr}$ be the ns suboperad of $\mathsf{T}\mathbb{M}$ generated by 01, 10, and 11. The following table shows the first elements of $\mathsf{Tr}$.

| Arity | Elements of $\mathsf{Tr}$ |
|---|---|
| 1 | 1 |
| 2 | 01, 10, 11 |
| 3 | 001, 010, 011, 100, 101, 110, 111 |
| 4 | 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111 |

Since $\mathsf{Di}$ is generated by 01 and 10, $\mathsf{Di}$ is a ns suboperad of $\mathsf{Tr}$.

One has the following characterization of the elements of $\mathsf{Tr}$:

**Proposition 4.24** *The elements of* $\mathsf{Tr}$ *are exactly the words on the alphabet* $\{0, 1\}$ *containing at least one* 1.

*Proof* Let us first show by induction on the length of the words that every word $x$ of $\mathsf{Tr}$ satisfies the statement. This is true when $|x| = 1$ since 1 is the unit of $\mathbb{M}$. When $|x| \geqslant 2$, by Lemma 2.1, there is an element $y$ of $\mathsf{Tr}$ of length $n := |x| - 1$, an integer $i \in [n]$, and $g \in \{01, 10, 11\}$ such that $x = y \circ_i g$. By the induction hypothesis, $y$ contains at least one 1. Since all generators of $\mathsf{Tr}$ contain at least one 1, $x$ also contains at least one 1.

Let us now show by induction on the length of the words that $\mathsf{Tr}$ contains any word $x$ satisfying the statement. This is true when $|x| = 1$. When $n := |x| \geqslant 2$, there is in $x$ a factor $x_i x_{i+1} =: g$ such that $g \in \{01, 10, 11\}$. Then, by setting

$$y := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \tag{4.113}$$

if $g = 01$, or

$$y := (x_1, \ldots, x_i, x_{i+2}, \ldots, x_n) \tag{4.114}$$

if $g \in \{10, 11\}$, we have $x = y \circ_i g$. Since $y$ satisfies the statement, by the induction hypothesis $\mathsf{Tr}$ contains $y$. Hence, $\mathsf{Tr}$ also contains $x$. □

Recall that the *triassociative operad* [22] $\mathsf{Trias}$ is the ns operad admitting the presentation

$$\mathsf{Trias} := \mathcal{F}(\{\dashv, \perp, \vdash\})/_{\equiv}, \tag{4.115}$$

where $\dashv$, $\perp$, and $\vdash$ are of arity 2, and $\equiv$ is the ns operadic congruence generated by

$$\dashv \circ_1 \vdash \;\leftrightarrow\; \vdash \circ_2 \dashv, \tag{4.116}$$
$$\perp \circ_1 \perp \;\leftrightarrow\; \perp \circ_2 \perp, \tag{4.117}$$
$$\dashv \circ_1 \perp \;\leftrightarrow\; \perp \circ_2 \dashv, \tag{4.118}$$
$$\perp \circ_1 \dashv \;\leftrightarrow\; \perp \circ_2 \vdash, \tag{4.119}$$
$$\perp \circ_1 \vdash \;\leftrightarrow\; \vdash \circ_2 \perp, \tag{4.120}$$
$$\dashv \circ_1 \dashv \;\leftrightarrow\; \dashv \circ_2 \dashv \;\leftrightarrow\; \dashv \circ_2 \vdash \;\leftrightarrow\; \dashv \circ_2 \perp, \tag{4.121}$$
$$\vdash \circ_2 \vdash \;\leftrightarrow\; \vdash \circ_1 \vdash \;\leftrightarrow\; \vdash \circ_1 \dashv \;\leftrightarrow\; \vdash \circ_1 \perp . \tag{4.122}$$

**Proposition 4.25** *The ns operads* Tr *and* Trias *are isomorphic and the map*

$$\phi : \mathsf{Trias} \to \mathsf{Tr} \tag{4.123}$$

*satisfying* $\phi(\dashv) = 10$, $\phi(\vdash) = 01$, *and* $\phi(\perp) = 11$ *is an isomorphism.*

*Proof* By replacing each generator $g$ of Trias by $\phi(g)$, the generators 01, 10, and 11 of Tr satisfy the same relations as the generators $\dashv$, $\vdash$, and $\perp$ of Trias. Moreover, by Proposition 4.24, the Hilbert series of Tr is

$$F(t) = \sum_{n \geqslant 1} (2^n - 1)t^n, \tag{4.124}$$

which is also the Hilbert series of Trias. Then, there is no nontrivial relation of degree greater than two involving generators of Tr. □

Proposition 4.25 also shows that Tr is a realization of the triassociative operad.

## 5 Concluding remarks

We have presented the functorial construction T producing an operad given a monoid. As we have seen, this construction is very rich from a combinatorial point of view since most of the obtained operads coming from usual monoids involve a wide range of combinatorial objects. There are various way to continue this work. Let us address here the main directions.

In the first place, it appears that we have somewhat neglected the fact that T is a functor to operads and not only to ns ones. Indeed, except for the operads End, PF, PW, and Per, we only have regarded the obtained operads as ns ones. Computer experiments let us think that the dimensions of the operads PRT, $\mathsf{FCat}^{(2)}$, Motz, DA and SComp seen as symmetric ones are, respectively, Sequences **A052882**, **A050351**, **A032181**, **A101052**, and **A001047** of [28]. Bijections between elements of these operads and combinatorial objects enumerated by these sequences, together with presentations by generators and relations in this symmetric context, would be worthwhile.

Furthermore, we have considered T only in the category of sets, i.e., it takes a monoid as input and constructs a set-operad as output. We can obviously extend the definition of T over the category of vector spaces. In that event, T would be a functor from the category of unital associative algebras to the category of operads in the category of vector spaces. It is thus natural to ask what operads T produces in this category.

Another line of research is the following. It is well-known that the Koszul dual (see [12] for Koszul duality of operads) of the operads Dias and Trias are, respectively, the dendriform Dendr [20] and the tridendriform TDendr [22] operads. The tridendriform operad is a generalization of the dendriform operad and further generalizations were proposed, like the operads Quad [1] and Ennea [18]. Since the operads Di and Tr, obtained from the T construction, are respectively isomorphic to the operads Dias and Trias, we can ask if there are generalizations of Di and Tr so that their Koszul duals provide generalizations of the operads Dendr and TDendr.

**Acknowledgments** The author would like to thank N. Bergeron, F. Chapoton, F. Hivert, Y. Le Borgne, M. Méndez, J.-C. Novelli, F. Saliola, and Y. Vargas for their suggestions and interesting discussions. The computations of this work have been done with the open-source mathematical software Sage [29] and one of its extensions, Sage-Combinat [30]. The author also wishes to thank the anonymous referees for their highly valuable comments and suggestions.

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