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Ordering families using Lusztig's symbols in type B: the integer case

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Abstract Let $\mathrm{Irr}(W)$ be the set of irreducible representations of a finite Weyl group W. Following an idea from Spaltenstein, Geck has recently introduced a preorder $\preceq_L$ on $\mathrm{Irr}(W)$ in connection with the notion of Lusztig families. In a later paper with Iancu, they have shown that in type B (in the asymptotic case and in the equal parameter case) this preorder coincides with the preorder on Lusztig symbols as defined by Geck and the second author in 2011. In this paper, we show that this characterisation extends to the so-called integer case, that is, when the ratio of the parameters is an integer.

Keywords Lusztig's families · Lusztig's symbols · Representation of Weyl groups

1 Introduction

Let W be a finite Weyl group together with a weight function L and let $\mathrm{Irr}(W)$ be the set of irreducible representations of W over $\mathbb{C}$. On the one hand, Lusztig has defined a function $\boldsymbol{a}$ on the set $\mathrm{Irr}(W)$ (now known as the Lusztig $\boldsymbol{a}$-function) which allowed him to define a partition of $\mathrm{Irr}(W)$ into the so-called families. On the other hand, Kazhdan-Lusztig theory naturally yields a preorder $\leq_{\mathcal{LR}}$ on $\mathrm{Irr}(W)$ which in turn gives rise to a partition of $\mathrm{Irr}(W)$. When the weight function is equal to the length function (the equal parameter case), these two partitions turn out to be the same: the proof relies on some geometric interpretation. It is conjectured that the two partitions should coincide in the general case of unequal parameters, that is for any choice of weight function L. The

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notion of families plays a fundamental role in the work of Lusztig on the characters of reductive groups over finite fields. It also naturally appears in the work of Geck [2] on the cellular structure of Iwahori–Hecke algebras.

Despite a fairly simple definition, the preorder $\leq_{\mathcal{LR}}$ is particularly hard to handle. To understand it and to give a combinatorial description, Geck has introduced a new preorder $\preceq_L$ in [3] on $\operatorname{Irr}(W)$ which is defined using only standard operations on the irreducible representations of W, such as truncated induction from parabolic subgroups or tensoring with the sign representation. He then proved that this preorder coincides with the Kazhdan–Lusztig preorder in the equal parameter case. He conjectured that this should also hold in the general case of unequal parameters. Later on, Geck and Iancu have studied $\preceq_L$ in type B, and they have given a complete combinatorial description of it in the equal parameter case and in the asymptotic case, using the combinatorics of Lusztig symbols. Their result brings in a preorder on symbols, which generalises the dominance order on partitions. This preorder was introduced, as far as we know, in [4, §5.7.5] and since then it has naturally appeared in various contexts not only in the representation theory of Hecke algebras [7] but also in the theory of canonical bases for Fock spaces [4, Ch. 6] and in the theory of Cherednik algebras [1,8,9].

The main purpose of this paper is to show that the results of Geck and Iancu remain valid in the integer case; that is when the weight function L satisfies some integer condition. The proof relies on some combinatorial properties of Lusztig symbols which may be of independent interest. The paper is organised as follows. In Sect. 2, we introduced the basic concepts such as the preorder $\preceq_L$, the notion of symbols and the dominance order on the set of symbols. In Sect. 3, we characterise adjacency of two symbols for the dominance order. This will play a crucial role in the proof of the main result in Sect. 4.

2 Ordering Lusztig families

2.1 Lusztig families

Let W be a finite Coxeter group with generating set S, and let L be a weight function on W, that is a function $L : W \to \mathbb{N}$ such that $L(w_1w_2) = L(w_1) + L(w_2)$ whenever $w_1, w_2 \in W$ satisfy $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Here ℓ denotes the usual length function. It is easily seen that L only depends on the set $\{L(s) \mid s \in S\}$ which is called the set of parameters.

Let $\operatorname{Irr}(W)$ be the set of complex irreducible representations of W. Using the 'generic degrees', Lusztig [11] defined a function

$$\begin{aligned} \boldsymbol{a} : \operatorname{Irr}(W) &\longrightarrow \mathbb{Z} \\ E &\longmapsto \boldsymbol{a}(E) \end{aligned}$$

which plays an important role in the representation theory of Weyl groups; see for example [4, Ch. 1,2,3]. Using this function, Lusztig [10, §4.2] showed that the set $\operatorname{Irr}(W)$ can be naturally partitioned into the so-called 'families'. In Example 2.5 and

Proposition 2.11, we give explicit formulae for the values of the $\boldsymbol{a}$-function on the irreducible representations of the Weyl groups of type A and B.

We now briefly recall the definition of families. We use the following notation: if $J \subset S$, $M \in \mathrm{Irr}(W_J)$ and $E \in \mathrm{Irr}(W)$, we write $M \uparrow E$ if E is a constituent of $\mathrm{Ind}_J^S(M)$, and we write $M \rightsquigarrow_L E$ if $M \uparrow E$ and $\boldsymbol{a}(M) = \boldsymbol{a}(E)$ (where $\boldsymbol{a}(M)$ is the value of the $\boldsymbol{a}$-function within the group W_J).

Definition 2.1 When $W = \{1\}$, there is only one family which consists of the unit representation. When $W \neq \{1\}$, $E \in \mathrm{Irr}(W)$ and $E' \in \mathrm{Irr}(W)$ are in the same family if there exists a sequence

$$E = E_0, E_1, \ldots, E_m = E'$$

in $\mathrm{Irr}(W)$ such that for all $i = 1, 2, \ldots, m$, the following condition is satisfied: there exist a subset $I_i \subset S$ and two simple modules $M_i, M_i' \in \mathrm{Irr}(W_{I_i})$ which belong to the same family in $\mathrm{Irr}(W_{I_i})$ such that

- either $M_i \rightsquigarrow_L E_{i-1}$ and $M_i' \rightsquigarrow_L E_i$,
- or $M_i \rightsquigarrow_L E_{i-1} \otimes \varepsilon$ and $M_i' \rightsquigarrow_L E_i \otimes \varepsilon$,

where ε denotes the sign representation of W.

Example 2.2 Let $W = \langle s, t \rangle$ be the dihedral group of order 12, that is, the element st is of order 6. We refer to [6, §5.3.4,§6.3.5,§6.5.10] for details of the computations. We have $\mathrm{Irr}(W) = \{1_W, \varepsilon, \varepsilon_s, \varepsilon_t, \varphi_1, \varphi_2\}$ where

- 1_W is the trivial representation,
- ε is the sign representation, i.e. $\varepsilon(w) = (-1)^{\ell(w)}$,
- ε_s and ε_t are two linear representations defined by $\varepsilon_x(w) = (-1)^{\ell_x(w)}$ where ℓ_x for $x = s, t$ is the number of x in any reduced expression of w,
- φ_i for $i = 1, 2$ are defined by

$$\varphi_i(st) = \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^i \end{pmatrix} \quad \text{and } \varphi_i(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{where } \xi = e^{-\frac{i2\pi}{6}}.$$

There are only 4 parabolic subgroups, namely $W_\emptyset = \{1\}$, $W_s = \langle s \rangle$, $W_t = \langle t \rangle$ and W itself. One can check that we obtain the following relations, inducing the trivial representation of each of the parabolic subgroups to W:

$$\begin{aligned} 1_W^W &= 1_W, \\ 1_{W_s}^W &= 1_W + \varepsilon_s + \sum_i \varphi_i, \\ 1_{W_t}^W &= 1_W + \varepsilon_t + \sum_i \varphi_i, \\ 1_{W_\emptyset}^W &= 1_W + \varepsilon + \varepsilon_s + \varepsilon_t + 2\sum_i \varphi_i. \end{aligned}$$

Next we obtain the following values for Lusztig $\boldsymbol{a}$-function

$E \in \mathrm{Irr}(W)$	1_W	ε_s	ε_t	φ_1	φ_2	ε
$\boldsymbol{a}(E)$	0	1	1	1	1	6

From there, we see that the families are

$$\{1_W\}, \{\varepsilon_s, \varepsilon_t, \varphi_1, \varphi_2\}, \{\varepsilon\}.$$

□

Kazhdan–Lusztig theory allows one to define another preorder on irreducible representations that we will denote by $\leq_{\mathcal{LR}}$. More precisely, one can define a preorder $\leq_{\mathcal{LR}}$ on W using the Hecke algebra associated to W. This preorder yields a partition of W into the so-called two-sided cells which are then naturally equipped with a partial order that we will still denote by $\leq_{\mathcal{LR}}$. Next, each two-sided cell $\mathbf{c}$ affords a representation of W, not necessarily irreducible, that we will denote by $M_{\mathbf{c}}$. It turns out that any irreducible representation appears in a unique cell representation and therefore, the partial order $\leq_{\mathcal{LR}}$ on cells induces a preorder on $\mathrm{Irr}(W)$ as follows: $E \leq_{\mathcal{LR}} E'$ if and only if there exist two two-sided cells $\mathbf{c}$ and $\mathbf{c}'$ such that $\mathbf{c} \leq_{\mathcal{LR}} \mathbf{c}'$, E is a constituent of $M_{\mathbf{c}}$, and E' is a constituent of $M_{\mathbf{c}'}$. In turn, this preorder induces a partition of $\mathrm{Irr}(W)$.

As shown in [10, Chapter 5] these two partitions turn out to be the same in the equal parameter case. The proof relies on some geometric interpretation which does not exist in the general case of unequal parameters.

Conjecture 2.3 *(Lusztig) Let W be a finite Coxeter group together with a weight function L. The partition of $\mathrm{Irr}(W)$ into families agrees with the partition induced by the Kazhdan–Lusztig preorder $\leq_{\mathcal{LR}}$.*

As far as unequal parameters are concerned, this conjecture has been verified by explicit computations in type $I_2(m)$ and F_4 by Geck. In type B, it holds in the asymptotic case; see next section.

2.2 The preorder $\preceq_L$

To have a better understanding of the preorder $\leq_{\mathcal{LR}}$ in Conjecture 2.3, Geck [3] has introduced a preorder $\preceq_L$ (see the definition below) satisfying the following condition: $E, E' \in \mathrm{Irr}(W)$ lie in the same family if and only if $E \preceq_L E'$ and $E' \preceq_L E$. Then with Iancu, they studied this preorder in type B and gave a complete combinatorial description in the asymptotic case and in the equal parameter case [5]. Finally, they deduced that Conjecture 2.3 holds in the asymptotic case.

We now recall the definition of $\preceq_L$.

Definition 2.4 When $W = \{1\}$, $\mathrm{Irr}(W)$ contains only the unit representation which is related with itself. When $W \neq \{1\}$, $E \in \mathrm{Irr}(W)$ and $E' \in \mathrm{Irr}(W)$ satisfy $E \preceq_L E'$ if and only if there exists a sequence

$$E = E_0, E_1, \ldots, E_m = E'$$

in $\mathrm{Irr}(W)$ such that for all $i = 1, 2, \ldots, m$, the following condition is satisfied: there exist a subset $I_i \subset S$ and two simple modules $M_i, M_i' \in \mathrm{Irr}(W_{I_i})$ satisfying $M_i \preceq_L M_i'$ in $\mathrm{Irr}(W_{I_i})$ such that

- either $E_{i-1} \uparrow \mathrm{Ind}_{I_i}^S(M_i)$ and $M_i' \rightsquigarrow_L E_i$,
- or $E_i \otimes \varepsilon \uparrow \mathrm{Ind}_{I_i}^S(M_i)$ and $M_i' \rightsquigarrow_L E_{i-1} \otimes \varepsilon$,

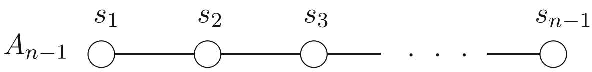
where ε denotes the sign representation. We write $E \simeq_L E'$ if we have $E \preceq_L E'$ and $E' \preceq_L E$.

Directly from the definition of $\preceq_L$, one can easily check that

1. if E and E' belong to the same family, then $E \simeq_L E'$;
2. $E \preceq_L E' \Rightarrow \boldsymbol{a}(E) \geq \boldsymbol{a}(E')$.

The fact that this preorder gives rise to families (in other words that the converse of 1. holds) is not straightforward by any means: it is one of the main results of [3] and [5].

Example 2.5 Let $n \geq 2$ and W be a Weyl group of type A_{n-1} with diagram as follows:



The description of $\preceq_L$ is given in [5, Ex. 2.11]. We explain it for the convenience of the reader as our proof in type B_n will roughly follow the same pattern. The group W can be identified with the symmetric group $\mathfrak{S}_n$, and it is a well-known fact that the irreducible representations of W are parametrised by partitions of n. We use the labelling as in [6] where, for instance, the unit representation is parametrised by the partition with one part equal to n and the sign representation by the partition with n parts equal to 1. For a partition λ of n, we denote by E^λ the corresponding element of $\mathrm{Irr}(W)$. We will denote by $\mathfrak{S}_k$ the parabolic subgroup of W generated by $\{s_1, \ldots, s_{k-1}\}$ and by H_ℓ the parabolic subgroup of W generated by $\{s_{n-\ell+1}, \ldots, s_{n-1}\}$ where, by convention, we set $\mathfrak{S}_1 = \{1\}$ and $H_1 = \{1\}$. With these notations, if $k, \ell \geq 1$ are such that $n = k + \ell$, then the subgroup $\mathfrak{S}_k \times H_\ell$ is a maximal parabolic subgroup of W.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Then we have

$$\boldsymbol{a}(E^\lambda) = \sum_{i=1}^{m} (i-1)\lambda_i.$$

We claim that the order $\preceq_L$ is the usual dominance order $\trianglelefteq$ on partitions. Recall that, for two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ of n, we have $\lambda \trianglelefteq \mu$ if and only if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \text{ for all } k \geq 1$$

where, by convention, we set $\lambda_i = 0$ if $i > r$ and $\mu_j = 0$ if $j > s$.

We will need the following results, and we refer to [6] for proofs and details. In the following, ε_{ℓ_i} denotes the sign representation of the parabolic subgroup H_{ℓ_i}, while ε denotes the sign representation of W.

(1) We have $E \preceq_L E'$ if and only if there exists a sequence $E = E_0, \dots, E_m = E'$ such that for each $1 \leq i \leq m$ there is a decomposition $n = k_i + \ell_i$ and two irreducible representations $M_i, M_i' \in \mathrm{Irr}(\mathfrak{S}_{k_i})$ such that $M_i \preceq_L M_i'$ and either
- $M_i \boxtimes \varepsilon_{\ell_i} \uparrow E$ and $M_i' \boxtimes \varepsilon_{\ell_i} \rightsquigarrow_L E'$
- $M_i \boxtimes \varepsilon_{\ell_i} \uparrow E' \otimes \varepsilon$ and $M_i' \boxtimes \varepsilon_{\ell_i} \rightsquigarrow_L E \otimes \varepsilon$

(2) (Pieri's Rule) Let $n = k + \ell$, and let λ be a partition of k and μ a partition of n. Then

$$E^\lambda \boxtimes \varepsilon_\ell \uparrow E^\mu$$

if and only if μ can be obtained from λ by adding one box to ℓ different parts of λ.

(3) (truncated Pieri's Rule) Let $n = k + \ell$, and let λ be a partition of k and μ a partition of n. Then

$$E^\lambda \boxtimes \varepsilon_\ell \rightsquigarrow_L E^\mu$$

if and only if μ can be obtained from λ by adding one box to the ℓ greatest parts of λ.

We are now ready to prove the claim. Let λ, μ be such that $E^\lambda \preceq_L E^\mu$. It is enough to consider an elementary step in the definition, so by (1), we may assume that there exist $k \leq n$ and $E^{\lambda'}, E^{\mu'} \in \mathrm{Irr}(\mathfrak{S}_k)$ such that the following is satisfied: $E^{\lambda'} \preceq_L E^{\mu'}$ within $\mathfrak{S}_k$, and we have either

- $E^{\lambda'} \boxtimes \varepsilon_\ell \uparrow E^\lambda$ and $E^{\mu'} \boxtimes \varepsilon_\ell \rightsquigarrow_L E^\mu$;
- $E^{\lambda'} \boxtimes \varepsilon_\ell \uparrow E^\mu \otimes \varepsilon$ and $E^{\mu'} \boxtimes \varepsilon_\ell \rightsquigarrow_L E^\lambda \otimes \varepsilon$.

Arguing by induction we may assume that $\lambda' \trianglelefteq \mu'$. Suppose that we are in the first case. Then, by (2), λ can be obtained from λ' by adding one box to ℓ different parts of λ', while, by (3), μ can be obtain from μ' by adding one box to the ℓ greatest parts of μ. It is clear that in this case we still have $\lambda \trianglelefteq \mu$. The argument is similar in the second case, since $E^\lambda \otimes \varepsilon = E^{\overline{\lambda}}$ where $\overline{\lambda}$ is the transposed partition of λ.

Conversely, assume that $\lambda \trianglelefteq \mu$. We may assume that λ and μ are adjacent in the dominance order, that is, if ν satisfies $\lambda \preceq \nu \preceq \mu$ then either $\nu = \lambda$ or $\nu = \mu$. By [12, §1.16], we know that μ differs from λ by only one box. That is there exists $i_1 < j_1 \in \mathbb{N}$ such that

$$\mu_{i_1} = \lambda_{i_1} + 1, \quad \mu_{j_1} = \lambda_{j_1} - 1 \quad \text{and } \mu_i = \lambda_i \text{ for all } i \neq i_1, j_1.$$

Let ν be the partition defined by

$$\nu_k = \begin{cases} \mu_k - 1 & \text{if } k < j_1, \\ \mu_k & \text{if } k \geq j_1. \end{cases}$$

Then, one sees that μ can be obtained from ν by adding one box to the $j_1 - 1$ greatest parts of μ, while λ can be obtained by adding one box to $j_1 - 1$ different parts of ν (namely the part indexed by $\{1, \dots, j_1\} - \{i_1\}$). Therefore we have $E^\nu \boxtimes \varepsilon_{j_1-1} \uparrow E^\lambda$ and $E^\nu \boxtimes \varepsilon_{j_1-1} \rightsquigarrow_L E^\mu$ and $\lambda \preceq_L \mu$ as required using (1). □

Finally, Geck conjectures the following relation.

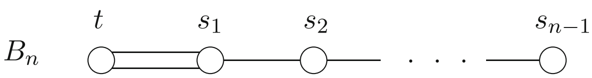
Conjecture 2.6 (Geck [3]) *Let $E, E' \in Irr(W)$. We have $E \preceq_L E'$ if and only if $E \preceq_{\mathcal{LR}} E'$.*

It has been shown in [3, Prop. 3.4] that the implication $E \preceq_L E' \Rightarrow E \preceq_{\mathcal{LR}} E'$ always holds. The conjecture has been proved in the following cases:

- in the equal parameter case by Geck [3, Th. 4.11]. The proof relies on a geometric interpretation of the Kazhdan–Lusztig theory;
- in type F_4 and $I_2(m)$ for all choices of L by Geck [3, §3];
- in type B_n, in the so-called asymptotic case by Geck and Iancu [5].

2.3 Type B_n

Let W be a the Weyl group of type B_n with diagram as follows:



together with a weight function $L : W \to \mathbb{Z}$.

Remark 2.7 The case where $L(t) > (n-1)L(s_i) > 0$ for all $i = 1, \dots, n-1$ is called the 'asymptotic case'.

> *In the rest of the paper, we will assume that we are in the 'integer case', that is, $L(t) = b \in \mathbb{N}$ and $L(s_i) = 1$ if $i = 1, \dots, n-1$; we will denote by L_b such a weight function.*

We now show how the irreducible representations, the $\boldsymbol{a}$-function and the families can be described in the integer case, for all choices of weight function L_b.

The set of irreducible representations of W is naturally labelled by the set Π_n^2 of bipartitions of rank n. Recall that for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ we have set $|\lambda| = \sum_{1 \le i \le r} \lambda_i$. A bipartition of n is just a pair of partitions $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ such that $|\lambda^1| + |\lambda^2| = n$. We have

$$\mathrm{Irr}(W) = \{E^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Pi_n^2\}.$$

We refer to [4] for more details and for an explicit description of the irreducible representations $E^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in \Pi_n^2$.

To each bipartition, one can associate an important object: its symbol. This notion depends on the weight function that we have chosen on W. Let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ be a bipartition of $n \in \mathbb{N}$ where $\lambda^i = (\lambda_1^i, \dots, \lambda_{r_i}^i)$ and $\lambda_1^i \ge \lambda_2^i \ge \dots \ge \lambda_{r_i}^i \ge 0$ for $i = 1, 2$. Let $N \in \mathbb{N}$ be such that

$$N \ge \max\{k \in \mathbb{N} \mid \max\{\lambda_k^1, \lambda_k^2\} \neq 0\}$$

where, by convention, we set $\lambda^1_k = 0$ if $k > r_1$ and $\lambda^2_k = 0$ if $k > r_2$. Such an integer will be called admissible for the bipartition $\boldsymbol{\lambda}$. Then, we associate to $\boldsymbol{\lambda}$ and N the (b, N)-*symbol*

$$\mathfrak{B}_{(b,N)}(\boldsymbol{\lambda}) = \begin{pmatrix} \mathfrak{B}^2_N & \dots & \mathfrak{B}^2_1 & & \\ \mathfrak{B}^1_{N+b} & \dots & \mathfrak{B}^1_2 & \dots & \mathfrak{B}^1_1 \end{pmatrix}$$

where

$$\begin{aligned} \mathfrak{B}^1_j &:= \lambda^1_j - j + N + b \text{ for } j = 1, \dots, N + b, \\ \mathfrak{B}^2_j &:= \lambda^2_j - j + N \text{ for } j = 1, \dots, N. \end{aligned}$$

We will denote by $\kappa_{(b,N)}(\boldsymbol{\lambda})$ the sequence of $2N + b$ elements in $\mathfrak{B}_{(b,N)}(\boldsymbol{\lambda})$ written in decreasing order. A straightforward computation shows that it is a partition of the integer

$$f(b, N, n) = n + (N(N-1))/2 + (N+b)(N+b-1)/2.$$

We set $\kappa_{(b,N)}(\boldsymbol{\lambda}) = (\kappa_1, \dots, \kappa_{2N+b})$ and

$$n_{(b,N)}(\boldsymbol{\lambda}) = \sum_{1 \le i \le 2N+b} (i-1)\kappa_i.$$

Example 2.8 Let $b = 2$, $\lambda^1 = (5, 1)$, $\lambda^2 = (2, 2, 1)$ and $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$. Then, $N = 3$ is admissible for $\boldsymbol{\lambda}$, and we have

$$\mathfrak{B}_{(2,3)}(\boldsymbol{\lambda}) = \begin{pmatrix} 1\ 3\ 4 & & \\ 0\ 1\ 2\ 4\ 9 \end{pmatrix}.$$

Then $\kappa_{(2,3)}(\boldsymbol{\lambda}) = (9, 4, 4, 3, 2, 1, 1, 0)$ which is a partition of $f(2, 3, 11) = 24$. □

From now on and until the end of this section, we fix a weight function L_b on W.

Using symbols, one can easily describe the $\boldsymbol{a}$-function and the families in type B_n.

Proposition 2.9 ([11, §22.14, §23.1]) *Let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2), \boldsymbol{\mu} = (\mu^1, \mu^2) \in \Pi^2_n$ and assume that $N \in \mathbb{N}$ is admissible for both $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Then the irreducible representations $E^{\boldsymbol{\lambda}}$ and $E^{\boldsymbol{\mu}}$ belong to the same family in W with respect to the weight function L_b if and only if $\kappa_{(b,N)}(\boldsymbol{\lambda}) = \kappa_{(b,N)}(\boldsymbol{\mu})$.*

Example 2.10 (1) As noted in [5, Ex. 8.1], if we have $b > n - 1$, then all the families are singletons.
(2) Consider the Weyl group of type B_3 together with the weight function L_1. There are 10 bipartitions of 3. In Fig. 1, we list all the bipartitions of 3 together with their (3, 1)-symbols and the partitions $\kappa_{3,1}$.

Bipartition of 3	Symbol	$\kappa_{1,3}$
$((1,1,1), \emptyset)$	$\begin{pmatrix} 0 & 1 & 2 & \\ 0 & 2 & 3 & 4 \end{pmatrix}$	$(4,3,2,2,1,0,0)$
$((2,1), \emptyset)$	$\begin{pmatrix} 0 & 1 & 2 & \\ 0 & 1 & 3 & 5 \end{pmatrix}$	$(5,3,2,1,1,0,0)$
$((3), \emptyset)$	$\begin{pmatrix} 0 & 1 & 2 & \\ 0 & 1 & 2 & 6 \end{pmatrix}$	$(6,2,2,1,1,0,0)$
$(\emptyset, (1,1,1))$	$\begin{pmatrix} 1 & 2 & 3 & \\ 0 & 1 & 2 & 3 \end{pmatrix}$	$(3,3,2,2,1,1,0)$
$(\emptyset, (2,1))$	$\begin{pmatrix} 0 & 2 & 4 & \\ 0 & 1 & 2 & 3 \end{pmatrix}$	$(4,3,2,2,1,0,0)$
$(\emptyset, (3))$	$\begin{pmatrix} 0 & 1 & 5 & \\ 0 & 1 & 2 & 3 \end{pmatrix}$	$(5,3,2,1,1,0,0)$
$((1), (1,1))$	$\begin{pmatrix} 0 & 2 & 3 & \\ 0 & 1 & 2 & 4 \end{pmatrix}$	$(4,3,2,2,1,0,0)$
$((1), (2))$	$\begin{pmatrix} 0 & 1 & 4 & \\ 0 & 1 & 2 & 4 \end{pmatrix}$	$(4,4,2,1,1,0,0)$
$((1,1), (1))$	$\begin{pmatrix} 0 & 1 & 3 & \\ 0 & 1 & 3 & 4 \end{pmatrix}$	$(4,3,3,1,1,0,0)$
$((2), (1))$	$\begin{pmatrix} 0 & 1 & 3 & \\ 0 & 1 & 2 & 5 \end{pmatrix}$	$(5,3,2,1,1,0,0)$

Fig. 1 Bipartitions and symbols for (B_3, L_1)

One can easily check that the families are then given by

$$\left\{((1,1,1), \emptyset), (\emptyset, (2,1)), ((1), (1,1))\right\}, \{((2,1), \emptyset), (\emptyset, (3)), ((2), (1))\}, \{((3), \emptyset)\},$$
$$\left\{(\emptyset, (1,1,1))\right\}, \{((1), (2))\}, \{((1,1), (1))\}. \blacksquare$$

Proposition 2.11 (Lusztig) *Let* $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \in \Pi_n^2$ *and assume that* $N \in \mathbb{N}$ *is admissible for* $\boldsymbol{\lambda}$. *Then*

$$\boldsymbol{a}(E^{\boldsymbol{\lambda}}) = n_{(b,N)}(\boldsymbol{\lambda}) - n_{(b,N)}(\boldsymbol{\emptyset})$$

where $\boldsymbol{\emptyset}$ *denotes the empty bipartition.*

There is a similar description of the $\boldsymbol{a}$-function in the case where W is the complex reflection group $(\mathbb{Z}/\ell\mathbb{Z})^n \rtimes \mathfrak{S}_n$ (see [4, Prop 5.5.11]).

Recall that our main aim is to give an explicit description of $\preceq_L$. We already know that this preorder is a refinement of the preorder induced by the $\boldsymbol{a}$-function. Actually, looking at the above formula for the $\boldsymbol{a}$-function, we see that the preorder induced by $\boldsymbol{a}$ admits another natural refinement which has first been introduced in [4], again in the wider context of complex reflection groups of types $(\mathbb{Z}/\ell\mathbb{Z})^n \rtimes \mathfrak{S}_n$.

Proposition 2.12 (Geck-J.) *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_n^2$ *and assume that* $N \in \mathbb{N}$ *is admissible for both* λ *and* μ. *Then we have*

$$\kappa_{(b,N)}(\boldsymbol{\lambda}) \trianglelefteq \kappa_{(b,N)}(\boldsymbol{\mu}) \Longrightarrow \boldsymbol{a}(E^{\boldsymbol{\lambda}}) \geq \boldsymbol{a}(E^{\boldsymbol{\mu}}).$$

It is then natural to ask if the preorder defined by the dominance order on the $\kappa_{(b,N)}(\boldsymbol{\lambda})$ coincides with $\preceq_L$. This is the main result of this paper.

Theorem 2.13 *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_n^2$ *and assume that* $N \in \mathbb{N}$ *is admissible for both* λ *and* μ. *Then*

$$\kappa_{(b,N)}(\boldsymbol{\lambda}) \trianglelefteq \kappa_{(b,N)}(\boldsymbol{\mu}) \iff \boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}.$$

Note that the implication

$$\kappa_{(b,N)}(\boldsymbol{\lambda}) \trianglelefteq \kappa_{(b,N)}(\boldsymbol{\mu}) \Longleftarrow \boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}$$

has already been proved in [5, Theorem 7.11]. So this article is devoted to the proof of the reverse implication. Note that the result has also been established in the case where:

- $b = 0$ and $b = 1$ by Geck and Iancu [5, Ex. 8.2] using results of Spaltenstein [13],
- $b > n - 1$ by Geck and Iancu [5, §6].

3 On the adjacency of Lusztig symbols with respect to $\trianglelefteq$

As we have seen in the previous section, to any bipartition $\boldsymbol{\lambda}$ of n we can associate the Lusztig's symbol $\mathfrak{B}_{(b,N)}(\boldsymbol{\lambda})$ and a partition $\kappa_{(b,N)}(\boldsymbol{\lambda})$. Therefore, the usual dominance order $\trianglelefteq$ on partitions yields an order on Lusztig's symbols and bipartitions of n. We will still denote this order by $\trianglelefteq$. In this section we study adjacent bipartitions for this order. Let us first clarify what we mean by adjacent: let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions of n, and let $N \in \mathbb{N}$ be admissible for both $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. We say that $\boldsymbol{\lambda} \triangleleft \boldsymbol{\mu}$ are adjacent if $\kappa_{(b,N)}(\boldsymbol{\lambda}) \triangleleft \kappa_{(b,N)}(\boldsymbol{\mu})$ and there is no bipartition $\boldsymbol{\nu}$ of n such that $\kappa_{(b,N)}(\boldsymbol{\lambda}) \triangleleft \kappa_{(b,N)}(\boldsymbol{\nu}) \triangleleft \kappa_{(b,N)}(\boldsymbol{\mu})$. Equivalently, if a bipartition $\boldsymbol{\nu}$ satisfies

$$\kappa_{(b,N)}(\boldsymbol{\lambda}) \trianglelefteq \kappa_{(b,N)}(\boldsymbol{\nu}) \trianglelefteq \kappa_{(b,N)}(\boldsymbol{\mu})$$

then we have either $\boldsymbol{\nu} = \boldsymbol{\lambda}$ or $\boldsymbol{\nu} = \boldsymbol{\mu}$.

It is a well-known fact, see, for example, [12, (1.16)], that if two partitions are adjacent for the dominance order then one can be obtained from the other by moving a single box in their Young diagram. The aim of this section is to show that a similar result holds for the dominance order on bipartitions and symbols: if $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ are adjacent bipartitions then the partition $\kappa_{(b,N)}(\boldsymbol{\lambda})$ can be obtained from $\kappa_{(b,N)}(\boldsymbol{\mu})$ by moving a single box in their Young diagram.

3.1 Sympartitions and symbols

In this section, we study the following problem: given a partition λ, under which condition can we find a bipartition $\boldsymbol{\lambda} \in \Pi_n^2$ such that $\lambda = \kappa_{(b,N)}(\boldsymbol{\lambda})$ for some $b, N \in \mathbb{N}$? We start by introducing two definitions.

Definition 3.1 An *ℓ-overlap* in a partition λ is a repetition of exactly ℓ non-zero elements.

For example, there is one 2-overlap in the partition $\lambda = (4, 4, 2, 2, 2, 1)$ and one 3-overlap.

Definition 3.2 Let $b, N, n \in \mathbb{N}^*$. We say that λ is a (b, N, n)-sympartition if and only if

- λ is a partition of $f(b, N, n)$,
- there is no 3-overlap in λ,
- the number of 2-overlaps is at most N.

When the triplet (b, N, n) is clear from the context, we will simply write sympartition instead of (b, N, n)-sympartition.

Example 3.3 The partition $(7, 4, 4, 3, 2, 1, 1, 0)$ is a $(2, 3, 9)$-sympartition and also a $(4, 2, 6)$-sympartition. However, it cannot be a sympartition associated to a triplet of the form $(b, 1, n)$ as there are two 2-overlaps.

Recall the definition of admissible in Sect. 2.3.

Proposition 3.4 *Let $\boldsymbol{\lambda}$ be a bipartition of n, and let $N \in \mathbb{N}$ be admissible for $\boldsymbol{\lambda}$. Then the partition $\kappa_{(b,N)}(\boldsymbol{\lambda})$ is a (b, N, n)-sympartition. Conversely, let λ be a (b, N, n)-sympartition. Then there exists a bipartition $\boldsymbol{\lambda}$ of n such that $\kappa_{(b,N)}(\boldsymbol{\lambda}) = \lambda$.*

Proof Let $\boldsymbol{\lambda}$ be a bipartition of n. Then by definition of the (b, N)-symbol, the sequences $(\mathfrak{B}_i^1)_{i=1,\dots,N+b}$ and $(\mathfrak{B}_i^2)_{i=1,\dots,N}$ are strictly increasing. It follows easily that there cannot be any 3-overlap in $\kappa_{(b,N)}(\boldsymbol{\lambda})$ and that the number of 2-overlaps is less than or equal to N. We have seen in Sect. 2.3 that $\kappa_{(b,N)}(\boldsymbol{\lambda})$ is a partition of $f(b, N, n)$. Hence $\kappa_{(b,N)}(\boldsymbol{\lambda})$ is a (b, N, n)-sympartition.

Assume now that λ is a (b, N, n)-sympartition. Then there exist two sequences of strictly increasing integers $\mathfrak{B}^1$ and $\mathfrak{B}^2$ such that

- there are N elements in $\mathfrak{B}^1$ and $N + b$ elements in $\mathfrak{B}^2$,
- $\mathfrak{B}^1 \cap \mathfrak{B}^2$ contains all the 2-overlaps of λ,
- $\mathfrak{B}^1 \cup \mathfrak{B}^2 = \lambda$ (as a multiset).

Now there exist two partitions λ^1 and λ^2 such that the increasing sequences $(\lambda_i^1 - i + N + b)_{i=1,\dots,N+b}$ and $(\lambda_i^2 - i + N)_{i=1,\dots,N}$ are, respectively, equal to $\mathfrak{B}^1$ and $\mathfrak{B}^2$. If we set $\boldsymbol{\lambda} := (\lambda^1, \lambda^2)$, then one can check that λ is a bipartition of n and that

$$\mathfrak{B}_{(b,N)}(\boldsymbol{\lambda}) = \begin{pmatrix} \mathfrak{B}^2 \\ \mathfrak{B}^1 \end{pmatrix} \quad \text{and} \quad \kappa_{(b,N)}(\boldsymbol{\lambda}) = \lambda.$$

3.2 Raising operators

Following [12], we introduce the raising operators. To this end, we will work on M-uplets of integers instead of partitions or sympartitions.

Definition 3.5 Let $a = (a_1, \ldots, a_M) \in \mathbb{Z}^M$. For $1 \leq k_1 < k_2 \leq M$, we set

$$\begin{aligned}\mathrm{Up}_{k_1,k_2}(a) &= (a_1, \ldots, a_{k_1} + 1, \ldots, a_{k_2} - 1, \ldots, a_M),\\ \mathrm{Down}_{k_1,k_2}(a) &= (a_1, \ldots, a_{k_1} - 1, \ldots, a_{k_2} + 1, \ldots, a_M).\end{aligned}$$

Let λ be a partition of n and assume that $\mathrm{Up}_{k_1,k_2}(\lambda)$ is also a partition of n. Note that this would be the case whenever $\lambda_{k_2} > \lambda_{k_2+1}$ and $\lambda_{k_1-1} > \lambda_{k_1}$. Then, looking at the associated Young tableau, $\mathrm{Up}_{k_1,k_2}(\lambda)$ is obtained from λ simply by moving the (k_2, λ_{k_2})-box to the k_1^{th}-part of λ.

Lemma 3.6 *Let $\lambda \lhd \lambda'$ be two partitions of n and let*

$$i := \min\{k \in \mathbb{N} \mid \lambda_k < \lambda'_k\} \quad \text{and} \quad j := \min\{m \in \mathbb{N} \mid m > i, \sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \lambda'_k\}.$$

We have $\lambda_j > \lambda'_j \geq \lambda'_{j+1} \geq \lambda_{j+1}$. Further, for all $i \leq k_1 < k_2 \leq j$ such that $\mathrm{Up}_{k_1,k_2}(\lambda)$ is a partition of n we have

$$\lambda \lhd \mathrm{Up}_{k_1,k_2}(\lambda) \unlhd \lambda'.$$

Similarly, for all $i \leq k_1 < k_2 \leq j$ such that $\mathrm{Down}_{k_1,k_2}(\lambda')$ is a partition of n we have

$$\lambda \unlhd \mathrm{Down}_{k_1,k_2}(\lambda') \lhd \lambda'.$$

Proof By minimality of j we know that

$$\sum_{k=1}^{j-1} \lambda_k < \sum_{k=1}^{j-1} \lambda'_k$$

therefore to have $\sum_{k=1}^{j} \lambda_k = \sum_{k=1}^{j} \lambda'_k$ we must have $\lambda_j > \lambda'_j$. The fact that $\lambda'_j \geq \lambda'_{j+1}$ is clear, since λ' is a partition. Finally we need to have $\lambda_{j+1} \leq \lambda'_{j+1}$, and this follows from the fact that $\lambda \lhd \lambda'$.

Let $k_1, k_2 \in \mathbb{N}$ be such that $i \leq k_1 < k_2 \leq j$ and assume that $\mathrm{Up}_{k_1,k_2}(\lambda)$ is a partition of n. It is clear that $\lambda \lhd \mathrm{Up}_{k_1,k_2}(\lambda)$ since we moved a box in the upward direction. Let us show that $\lambda'' := \mathrm{Up}_{k_1,k_2}(\lambda) \unlhd \lambda'$. Let $i \leq m \leq j$. Note that $\lambda''_m = \lambda_m$ for all $m < k_1$ thus

$$\sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \lambda''_k \quad \text{for all } m < k_1.$$

Further, for all $m \geq k_2$ we have

$$\sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \lambda''_k.$$

From those two inequalities and since $\lambda \trianglelefteq \lambda'$, we get

$$\sum_{k=1}^{m} \lambda''_k \leq \sum_{k=1}^{m} \lambda'_k \quad \text{for all } m < k_1 \text{ and all } m \geq k_2.$$

Let $k_1 \leq m < k_2$. By minimality of j and since $m < j$, we have

$$\sum_{k=1}^{m} \lambda_k < \sum_{k=1}^{m} \lambda'_k.$$

By definition of λ'', we know that

$$\sum_{k=1}^{m} \lambda''_k = \sum_{k=1}^{m} \lambda_k + 1,$$

and therefore we get

$$\sum_{k=1}^{m} \lambda''_k \leq \sum_{k=1}^{m} \lambda'_k.$$

The result follows. The proof of the second part of the lemma is similar. □

3.3 A first simplification

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of some integer. By convention, if $i > r$ we set $\lambda_i = 0$ and if $i \leq 0$ we set $\lambda_i = +\infty$. For $i < j \in \mathbb{N}$, we set

$$\lambda_{[i,j]} = (\lambda_i, \ldots, \lambda_j), \lambda_{\leq i} = (\lambda_1, \ldots, \lambda_i) \quad \text{and} \quad \lambda_{\geq i} = (\lambda_i, \ldots, \lambda_r).$$

We define $\lambda_{<i}, \lambda_{>i}$ in a similar and obvious fashion.

Proposition 3.7 *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions of n, and let $N \in \mathbb{N}$ be admissible for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Assume that $\boldsymbol{\lambda} \lhd \boldsymbol{\mu}$ are adjacent. Let $\kappa := \kappa_{(b,N)}(\boldsymbol{\lambda}) = (\kappa_1, \ldots, \kappa_r)$ and $\kappa' := \kappa_{(b,N)}(\boldsymbol{\mu}) = (\kappa'_1, \ldots, \kappa'_r)$ and set*

$$i := \min\{k \in \mathbb{N} \mid \kappa_k < \kappa'_k\} \quad \text{and} \quad j := \min\{m \in \mathbb{N} \mid m > i, \sum_{k=1}^{m} \kappa_k = \sum_{k=1}^{m} \kappa'_k\}.$$

Then we have $\kappa_{<i} = \kappa'_{<i}$ *and* $\kappa_{>j} = \kappa'_{>j}$.

Proof It is clear that $\kappa_{<i} = \kappa'_{<i}$ by definition of i. Let us show that $\kappa_{>j} = \kappa'_{>j}$. First, by minimality of j we have $\kappa_j > \kappa'_j$, and since $\kappa \lhd \kappa'$, we also have $\kappa'_{j+1} \geq \kappa_{j+1}$. In particular, it follows that $\kappa_j > \kappa_{j+1}$. This implies that both $\alpha := (\kappa_{\leq j}, \kappa'_{\geq j+1})$ and $\beta := (\kappa'_{\leq j}, \kappa_{\geq j+1})$ are partitions. In addition, we have

$$\begin{aligned}|\alpha| &= \textstyle\sum_{1\leq k\leq j} \kappa_k + \sum_{j+1\leq k\leq r} \kappa'_k\\ &= \textstyle\sum_{1\leq k\leq j} \kappa'_k + \sum_{j+1\leq k\leq r} \kappa'_k\\ &= |\kappa'|\end{aligned}$$

and

$$\begin{aligned}|\beta| &= \textstyle\sum_{1\leq k\leq j} \kappa'_k + \sum_{j+1\leq k\leq r} \kappa_k\\ &= \textstyle\sum_{1\leq k\leq j} \kappa_k + \sum_{j+1\leq k\leq r} \kappa_k\\ &= |\kappa|.\end{aligned}$$

So α and β are both partitions of $f(b, N, n)$. We claim that either α or β is a (b, N, n)-sympartition. First let us show that there is no 3-overlap in these two partitions.

- Since $\kappa_j > \kappa'_j \geq \kappa'_{j+1}$, it is easy to see that α cannot contain any 3-overlap as κ and κ' does not; see Proposition 3.4.
- Assume that there is a 3-overlap in β. Then we must have $\kappa'_j = \kappa_{j+1}$ and either $\kappa'_{j-1} = \kappa'_j$ or $\kappa_{j+2} = \kappa_{j+1}$. First $\kappa'_j = \kappa_{j+1}$ implies that $\kappa'_j = \kappa_{j+1} = \kappa'_{j+1}$, since $\kappa'_j \geq \kappa'_{j+1} \geq \kappa_{j+1}$. Second, since $\kappa'_{j+1} = \kappa_{j+1}$, we must have $\kappa'_{j+2} \geq \kappa_{j+2}$ as $\kappa \lhd \kappa'$. Finally, we get $\kappa_{j+1} = \kappa'_{j+1} \geq \kappa'_{j+2} \geq \kappa_{j+2}$.
  - Assume that $\kappa'_{j-1} = \kappa'_j$. Then $\kappa'_{j-1} = \kappa'_j = \kappa'_{j+1}$, and we have a 3-overlap in κ' which is a contradiction.
  - Assume that $\kappa_{j+1} = \kappa_{j+2}$. Then the inequality $\kappa_{j+1} = \kappa'_{j+1} \geq \kappa'_{j+2} \geq \kappa_{j+2}$ implies that $\kappa'_{j+1} = \kappa'_{j+2}$, and since we have seen that $\kappa'_j = \kappa'_{j+1}$, we have a 3-overlap in κ' which is a contradiction.

Thus, as claimed, there is no 3-overlap in α and β.

For $\boldsymbol{\nu}$ a partition, we denote by $\mathfrak{O}(\boldsymbol{\nu})$ the number of 2-overlaps in $\boldsymbol{\nu}$. Then we have

$$\begin{aligned}\mathfrak{O}(\kappa) &= \mathfrak{O}(\kappa_{<j+1}) + \mathfrak{O}(\kappa_{>j}) \quad \text{(we have seen that } \kappa_j > \kappa_{j+1})\\ \mathfrak{O}(\kappa') &= \mathfrak{O}(\kappa'_{<j+1}) + \mathfrak{O}(\kappa'_{>j}) + \delta_{\kappa'_j,\kappa'_{j+1}}\end{aligned}$$

where δ stands for the Kronecker symbol. We also have

$$\begin{aligned}\mathfrak{O}(\alpha) &= \mathfrak{O}(\kappa_{<j+1}) + \mathfrak{O}(\kappa'_{>j}) \quad \text{and}\\ \mathfrak{O}(\beta) &= \mathfrak{O}(\kappa'_{<j+1}) + \mathfrak{O}(\kappa_{>j}) + \delta_{\kappa'_j,\kappa_{j+1}}.\end{aligned}$$

Note that if $\delta_{\kappa'_j,\kappa_{j+1}} = 1$, then we also have $\delta_{\kappa'_j,\kappa'_{j+1}} = 1$ thus in any case $\delta_{\kappa'_j,\kappa_{j+1}} \leq \delta_{\kappa'_j,\kappa'_{j+1}}$. We want to show that either $\mathfrak{O}(\alpha) \leq N$ or $\mathfrak{O}(\beta) \leq N$. We argue by contradiction assuming that $\mathfrak{O}(\alpha) > N$ and $\mathfrak{O}(\beta) > N$. Then

$$\mathfrak{O}(\kappa) + \mathfrak{O}(\kappa') \geq \mathfrak{O}(\alpha) + \mathfrak{O}(\beta) \geq N + 2$$

so

$$\mathfrak{O}(\kappa) + \mathfrak{O}(\kappa') \geq 2N + 2$$

which is a contradiction, because both κ and κ' are (b, N, n)-sympartitions and so we have $\mathfrak{O}(\kappa) \leq N$ and $\mathfrak{O}(\kappa') \leq N$. Since neither α nor β contains a 3-overlap, we see that one of these partitions is a (b, N, n)-sympartition.

Finally, the hypotheses imply that $\kappa_{\leq j} \lhd \kappa'_{\leq j}$ (they are both partitions of the same rank) and $\kappa_{>j} \unlhd \kappa'_{>j}$ (they are both partitions of the same rank). Thus we are in one of the following configurations:

- either α is a (b, N, n)-sympartition, and we have

$$\kappa \unlhd \alpha \unlhd \kappa'.$$

  Then $\kappa = \alpha$ or $\kappa' = \alpha$ and $\kappa_{>j} = \kappa'_{>j}$ since $\kappa_{\leq j} \neq \kappa'_{\leq j}$.
- or β is a (b, N)-sympartition and we have

$$\kappa \unlhd \beta \unlhd \kappa'.$$

  Then $\kappa = \beta$ or $\kappa' = \beta$ and $\kappa_{>j} = \kappa'_{>j}$ since $\kappa_{\leq j} \neq \kappa'_{\leq j}$.

This concludes the proof. □

3.4 The double break Lemma

From now on and until the end of this section, we will assume that $\boldsymbol{\lambda} \lhd \boldsymbol{\mu}$ are two adjacent bipartitions of n and that $N \in \mathbb{N}$ is admissible for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. We set

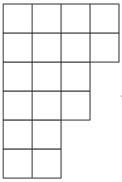
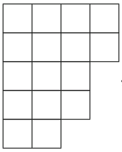
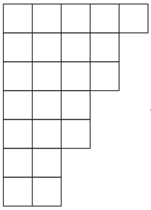
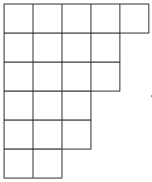
$$\kappa := \kappa_{(b,N)}(\boldsymbol{\lambda}) = (\kappa_1, \ldots, \kappa_r), \quad \kappa' := \kappa_{(b,N)}(\boldsymbol{\mu}) = (\kappa'_1, \ldots, \kappa'_r)$$
$$i = \min\{k \in \mathbb{N} \mid \kappa_k < \kappa'_k\} \quad \text{and} \quad j := \min\left\{m \in \mathbb{N} \mid m > i, \sum_{k=1}^{m} \kappa_k = \sum_{k=1}^{m} \kappa'_k\right\}.$$

For a partition ν we set $\mathfrak{J}_k^{\nu} := \nu_k - \nu_{k+1} \geq 0$. We then say that k is a break point of ν if and only if we have $\mathfrak{J}_{k-1}^{\nu} \geq 1$ and $\mathfrak{J}_k^{\nu} \geq 1$.

Lemma 3.8 *Assume that for all $i \leq m \leq j - 1$ we have $\mathfrak{J}_m^{\kappa'} = 0$ or 1. Then κ' has at least two break points k_1 and k_2 such that $i + 1 \leq k_1, k_2 \leq j - 1$.*

Proof Assume that κ' has no break point. There are 4 cases to consider, whether $\mathfrak{J}_i^{\kappa'}$ and $\mathfrak{J}_{j-1}^{\kappa'}$ are equal to 0 or 1. In the figure below, we give four examples of partitions

κ' corresponding to the four cases we need to consider. The first line corresponds to the ith-part and the last one to the jth-part.

(1)  , (2)  , (3)  , (4)  .

Let $\ell \in \mathbb{N}$ be such that $j - i + 1 = 2\ell + \varepsilon$ where $\varepsilon = 0$ or 1. A straightforward computation in each case yields

$$\begin{aligned}
&(1) \quad \sum_{k=i}^{j} \kappa'_k = \sum_{k=1}^{\ell} 2(\kappa'_i - k + 1) = 2\ell\kappa'_i - \ell(\ell - 1),\\
&(2) \quad \sum_{k=i}^{j} \kappa'_k = \sum_{k=1}^{\ell} 2(\kappa'_i - k + 1) + (\kappa'_i - \ell) = (2\ell + 1)\kappa'_i - \ell^2,\\
&(3) \quad \sum_{k=i}^{j} \kappa'_k = \kappa'_i + \sum_{k=1}^{\ell} 2(\kappa'_i - k) = (2\ell + 1)\kappa'_i - \ell(\ell + 1),\\
&(4) \quad \sum_{k=i}^{j} \kappa'_k = \kappa'_i + \sum_{k=1}^{\ell-1} 2(\kappa'_i - k) + (\kappa'_i - \ell) = 2\ell\kappa'_i - \ell^2.
\end{aligned}$$

From there, we see that the smallest value of $\sum_{k=i}^{j} \kappa'_i$ that can be achieved when there is no break point is

$$\begin{array}{ll}
2\ell\kappa'_i - \ell^2 & \text{if } j - i + 1 \text{is even},\\
(2\ell + 1)\kappa'_i - \ell(\ell + 1) & \text{if } j - i + 1 \text{ is odd}.
\end{array}$$

Now the largest sympartition that one can construct starting with a ith-part equal to $\kappa_i < \kappa'_i$ will satisfy

$$\sum_{k=i}^{j} \kappa_k = \begin{cases} 2\ell\kappa_i - \ell(\ell - 1) & \text{if } j - i + 1 \text{ is even},\\ (2\ell + 1)\kappa_i - \ell^2 & \text{if } j - i + 1 \text{ is odd}. \end{cases}$$

If $j - i + 1$ is even, then we would have

$$\begin{aligned}
\sum_{k=i}^{j} \kappa_k &\le 2\ell\kappa_i - \ell(\ell - 1)\\
&\le 2\ell(\kappa'_i - 1) - \ell(\ell - 1)
\end{aligned}$$

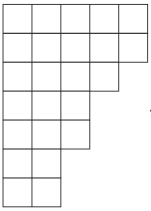
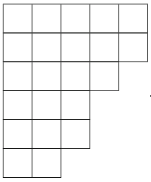
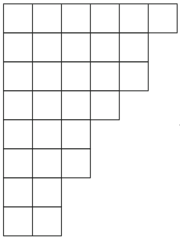
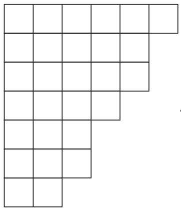
$$
\begin{aligned}
&= 2\ell\kappa'_i - \ell(\ell+1)\\
&< \sum_{k=i}^{j} \kappa'_i
\end{aligned}
$$

contradicting the equality $\sum_{k=i}^{j} \kappa'_i = \sum_{k=i}^{j} \kappa_i$.
If $j-i+1$ is odd, then we would have

$$
\begin{aligned}
\sum_{k=i}^{j} \kappa_i &\leq (2\ell+1)\kappa_i - \ell^2\\
&\leq (2\ell+1)(\kappa'_i - 1) - \ell^2\\
&= (2\ell+1)\kappa'_i - (\ell+1)^2\\
&< \sum_{k=i}^{j} \kappa'_i
\end{aligned}
$$

once again contradicting the equality $\sum_{k=i}^{j} \kappa'_i = \sum_{k=i}^{j} \kappa_i$. So we see that κ' has at least one break point.

Assume that κ' has a unique break point. Let N be the length of the part corresponding to the break point. Note that we have $\kappa'_j < N < \kappa'_i$. There are 4 cases to consider, whether $\mathfrak{I}_i^{\kappa'}$ and $\mathfrak{I}_{j-1}^{\kappa'}$ are equal to 0 or 1. As before, we give four examples corresponding to the four cases we need to consider.

(1)  , (2)  , (3)  , (4)  .

To use the computation done before, we will set ℓ to be such that $j-i+2 = 2\ell+\varepsilon$ where $\varepsilon = 0$ or 1. A straightforward computation in each case yields

$$
\begin{aligned}
&(1) \quad \sum_{k=i}^{j} \kappa'_k = 2\ell\kappa'_i - \ell(\ell-1) - N,\\
&(2) \quad \sum_{k=i}^{j} \kappa'_k = (2\ell+1)\kappa'_i - \ell^2 - N,\\
&(3) \quad \sum_{k=i}^{j} \kappa'_k = (2\ell+1)\kappa'_i - \ell(\ell+1) - N,\\
&(4) \quad \sum_{k=i}^{j} \kappa'_k = 2\ell\kappa'_i - \ell^2 - N.
\end{aligned}
$$

From there, using the fact that $\kappa'_j < N < \kappa'_i$ we see that the smallest value of $\sum_{k=i}^{j} \kappa'_k$ that can be achieved when there is exactly one break point is strictly less than

$$2\ell\kappa'_i - \ell(\ell+1) \text{ if } j-i+1 \text{ is even,}$$
$$(2\ell-1)\kappa'_i - \ell^2 \text{ if } j-i+1 \text{ is odd.}$$

Now the largest sympartition that we can construct starting with a ith-part equal to $\kappa_i < \kappa'_i$ will satisfy

$$\sum_{k=i}^{j} \kappa_k \le \begin{cases} 2\ell\kappa_i - \ell(\ell-1) & \text{if } j-i+1 \text{ is even,} \\ (2\ell-1)\kappa_i - (\ell-1)^2 & \text{if } j-i+1 \text{ is odd.} \end{cases}$$

If $j-i+1$ is even, then we would have

$$\begin{aligned} \sum_{k=i}^{j} \kappa_k &\le 2\ell\kappa_i - \ell(\ell-1) \\ &\le 2\ell(\kappa'_i - 1) - \ell(\ell-1) \\ &= 2\ell\kappa'_i - \ell(\ell-1) - 2\ell \\ &= 2\ell\kappa'_i - \ell(\ell+1) \\ &< \sum_{k=i}^{j} \kappa'_k \end{aligned}$$

contradicting the equality $\sum_{k=i}^{j} \kappa'_k = \sum_{k=i}^{j} \kappa_k$.
If $j-i+1$ is odd, then we would have

$$\begin{aligned} \sum_{k=i}^{j} \kappa_k &\le (2\ell-1)\kappa_i - (\ell-1)^2 \\ &\le (2\ell-1)(\kappa'_i - 1) - (\ell-1)^2 \\ &= (2\ell-1)\kappa'_i - (\ell-1)^2 - (2\ell-1) \\ &= (2\ell-1)\kappa'_i - \ell^2 \\ &< \sum_{k=i}^{j} \kappa'_k \end{aligned}$$

once again contradicting the equality $\sum_{k=i}^{j} \kappa'_k = \sum_{k=i}^{j} \kappa_k$. This concludes the proof. □

3.5 Adjacency of two sympartitions

Theorem 3.9 *Let $b, N, n \in \mathbb{N}$, and let κ, κ' be two adjacent (b, N, n)-sympartitions with respect to $\lhd$. Then there exists $k_1, k_2 \in \mathbb{N}$ such that $\kappa' = \mathrm{Up}_{k_1,k_2}(\kappa)$.*

Proof The proof of this theorem is rather long and tedious and requires a case by case analysis. First, as before, we set

$$i := \min\{k \in \mathbb{N} \mid \kappa_k < \kappa'_k\} \quad \text{and} \quad j := \min\left\{m \in \mathbb{N} \mid m > i, \sum_{k=1}^{m} \kappa_k = \sum_{k=1}^{m} \kappa'_k\right\}.$$

Then by Proposition 3.7 we know that $\kappa_{<i} = \kappa_{<i}$ and $\kappa_{>j} = \kappa'_{>j}$. The idea of the proof is to construct a sympartition κ'' either of the form $\mathrm{Up}_{k_1,k_2}(\kappa)$ or $\mathrm{Down}_{k_1,k_2}(\kappa')$ for some $i \le k_1 \le k_2 \le j$. Assume that we have constructed such a sympartition κ''. Then by Lemma 3.6, we would have

$$\kappa \lhd \kappa'' \unlhd \kappa' \quad \text{or} \quad \kappa \unlhd \kappa'' \lhd \kappa'.$$

Since κ, κ' are adjacent, this would imply that $\kappa'' = \kappa'$ or $\kappa'' = \kappa$ as required.

Case 1. $\mathfrak{I}^{\kappa}_{i-1} \ge 2$ and $\mathfrak{I}^{\kappa}_{j} \ge 2$. In that case, we do not create any overlap by removing the (κ_j, j)-box in κ nor by adding a box to the i^{th} part of κ. Then if we set $\kappa'' = \mathrm{Up}_{i,j}(\kappa)$, we have $\mathfrak{O}(\kappa'') \le \mathfrak{O}(\kappa)$, and the result follows. (Recall that $\mathfrak{O}(\nu)$ denotes the number of 2-overlaps in ν.)

Case 2. Assume that $\mathfrak{I}^{\kappa}_{i-1} \ge 2$ and $\mathfrak{I}^{\kappa}_{j} = 1$. We have $\kappa_j = \kappa_{j+1} + 1$, and by Lemma 3.6 we get $\kappa'_j = \kappa'_{j+1} = \kappa_{j+1}$. The shapes of κ and κ' are described in Figure 2.

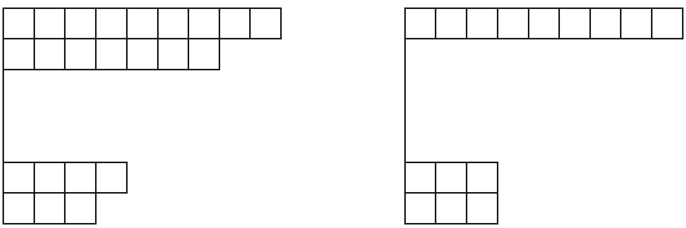
Of course we may have $\kappa_{i-1} - \kappa_i > 2$. Since κ' is a sympartition, we have $\kappa_{j+1} = \kappa'_{j+1} > \kappa'_{j+2} = \kappa_{j+2}$.

Subcase 1: There exists an overlap in $\kappa[i, j]$. We set

$$k := \max\{m \in \mathbb{N} \mid i \le m < j, \mathfrak{I}^{\kappa}_{m} = 0\}.$$

Let us show that $\kappa_{k+2} > \kappa_{k+3}$. Since $\kappa_{j+1} > \kappa_{j+2}$ and $\kappa_j > \kappa_{j+1}$, this is true if $k = j - 1$ or $k = j - 2$. If $k < j - 2$, we cannot have $\kappa_{k+2} = \kappa_{k+3}$, since $k + 2 < j$, and this would contradict the maximality of k.

Consider the partition $\kappa'' = \mathrm{Up}_{k+1,i}(\kappa)$. We claim that κ'' is a sympartition. To see this, note that since $\kappa_{k+2} > \kappa_{k+3}$ and $\kappa_{i-1} - \kappa_i \ge 2$, there is no 3-overlap in κ''. Then by removing the $(k+1, \kappa_{k+1})$-box, we eliminate the 2-overlap at place $(k, k+1)$ while



(a) Sympartition $\kappa[i-1, j+1]$ **(b)** Sympartition $\kappa'[i-1, j+1]$

Fig. 2 Shape of κ and κ'

possibly creating one at place $(k+1, k+2)$. Also, by adding a box at the i^{th}-part of κ, we cannot have created a 2-overlap since $\kappa_{i-1} - 2 \geq \kappa_i$. So the number of 2-overlaps in κ'' is less than or equal to the number of 2-overlaps in κ. This shows that κ'' is a sympartition, and we get the result in this case.

Subcase 2: There is no overlap between i and j. Set $\kappa'' = \mathrm{Up}_{i,j}(\kappa)$. First, since $\kappa_{j+1} > \kappa_{j+2}$, we do not create a 3-overlap removing the (j, κ_j)-box in κ. Then we have

$$\begin{aligned} \mathfrak{D}(\kappa'') &= \mathfrak{D}(\kappa[1, i-1]) + 1 + \mathfrak{D}(\kappa[j+2, \ell(\kappa)]) \\ &= \mathfrak{D}(\kappa'[1, i-1]) + 1 + \mathfrak{D}(\kappa'[j+2, \ell(\kappa')]) \\ &\leq \mathfrak{D}(\kappa') \end{aligned}$$

The last inequality holds because κ' has a 2-overlap at place $(j, j+1)$ and may have more between i and j. Since κ' is a sympartition, so is κ''.

From now on and until the end of the proof, we will assume that $\mathfrak{I}^{\kappa}_{i-1} = 1$. In that case, since $\kappa'_{i-1} = \kappa_{i-1}$ and $\kappa'_i > \kappa_i$, we have $\kappa'_i = \kappa'_{i-1} = \kappa_i + 1$. That is, the sympartitions κ and κ' have shapes as shown in Fig. 3.

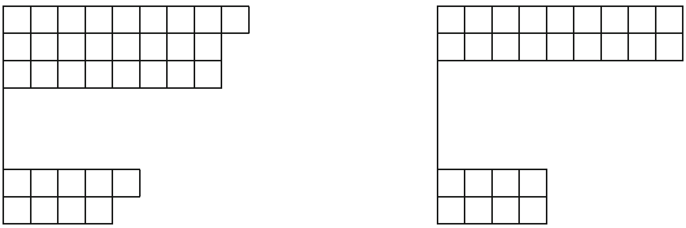
Case 3. $\mathfrak{I}^{\kappa}_i = 0$ and $\mathfrak{I}^{\kappa}_j = 1$. Note that by Lemma 3.6, we have $\kappa'_j = \kappa'_{j+1} = \kappa_{j+1}$. The shape of κ and κ' are described in Fig. 4.

Let $k := \max\{m \in \mathbb{N} \mid i \leq m < j, \mathfrak{I}^{\kappa}_m = 0\}$. Note that k is well defined, since there is a 2-overlap at place $(i, i+1)$. Set $\kappa'' = \mathrm{Up}_{i,k+1}(\kappa)$. Arguing as in Case 2.1, using the maximality of k, one can show that we do not create a 3-overlap by removing the $(k+1, \kappa_{k+1})$-box and that the number of 2-overlaps remains constant. Also, we have $\kappa_{i-2} = \kappa'_{i-2} > \kappa'_{i-1} = \kappa_{i-1}$, so that we can add a box at the ith-part of κ without creating a 3-overlap while keeping the number of 2-overlaps constant. It follows that κ'' is a sympartition as required.

Case 4: $\mathfrak{I}^{\kappa}_i = 0$ and $\mathfrak{I}^{\kappa}_j > 1$. The shapes of κ and κ' are described in Fig. 5. Of course, we may have $\kappa_j - \kappa_{j-1} > 2$. As in the previous case we have $\kappa_{i-2} > \kappa_{i-1}$, so

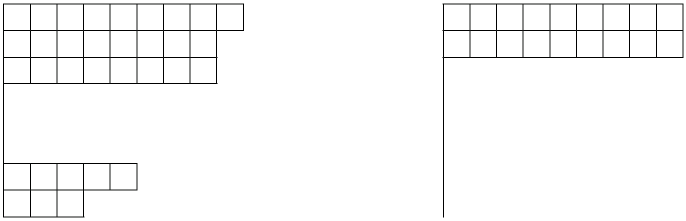
(a) Sympartition $\kappa[i-1, i]$ **(b)** Sympartition $\kappa'[i-1, i]$

Fig. 3 Shape of κ and κ'



(a) Sympartition $\kappa[i-1, j+1]$ **(b)** Sympartition $\kappa'[i-1, j+1]$

Fig. 4 Shape of κ and κ'



(a) Sympartition $\kappa[i-1, j+1]$ **(b)** Sympartition $\kappa'[i-1, i]$

Fig. 5 Shapes of κ and κ'

that we can add a box at the ith-part of κ without creating a 3-overlap while keeping the number of 2-overlaps constant. Then setting $\kappa'' = \mathrm{Up}_{i,j}(\kappa)$ easily yields the result.

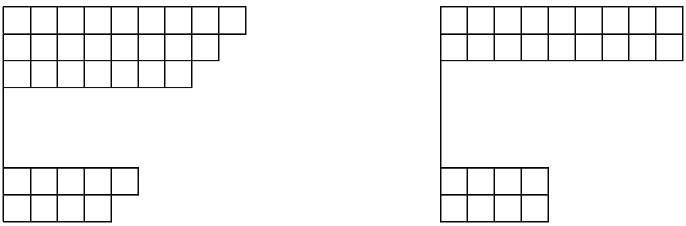
Case 5: $\mathfrak{J}_i^\kappa \geq 1$ and $\mathfrak{J}_j^\kappa = 1$. The shapes of κ and κ' are described in Figure 6.

We may have $\kappa_i - \kappa_{i-1} \geq 1$. Arguing as in Case 3, we can show that $\kappa_{j+1} > \kappa_{j+2}$. First assume that there exists $i \leq m < j$ such that $\kappa'_m - \kappa'_{m+1} \geq 2$. Then set

$$k := \max\{\ell \in \mathbb{N} \mid \ell < m, \mathfrak{J}_\ell^{\kappa'} = 0\}.$$

Note that k is well-defined as there is an overlap in κ' at place $(i-1, i)$ and $k \geq i-1$. Consider $\kappa'' = \mathrm{Down}_{k+1,m+1}(\kappa')$. We show that there is no 3-overlap in κ''. If $k < m-1$, then we cannot have $\kappa'_{k+1} = \kappa'_{k+2}$, because this would contradict the maximality of k. If $k = m-1$, then we do not create a 3-overlap, since $\kappa'_m - \kappa'_{m+1} \geq 2$. We show that $\mathfrak{O}(\kappa'') \leq \mathfrak{O}(\kappa')$. If $k < m-1$, then by removing the $(k+1, \kappa'_{k+1})$-box from κ', we remove an overlap at place $(k, k+1)$ while possibly creating one at place $(k+1, k+2)$. By adding a box at the $m+1$th-part of κ'' we do not create a 2-overlap at place $(m, m+1)$, since $\kappa'_m - \kappa'_{m+1} \geq 2$, hence the result in this case. If $k = m-1$, then we are in the situation described in Fig. 7, and the result follows easily.

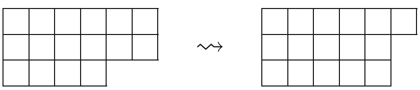
Second assume that for all $i \leq m < j$ we have $\mathfrak{J}_m^{\kappa'} = 0$ or 1. Then by the double break lemma, there exist at least two break points in κ'. Let k_1 (respectively k_2) be

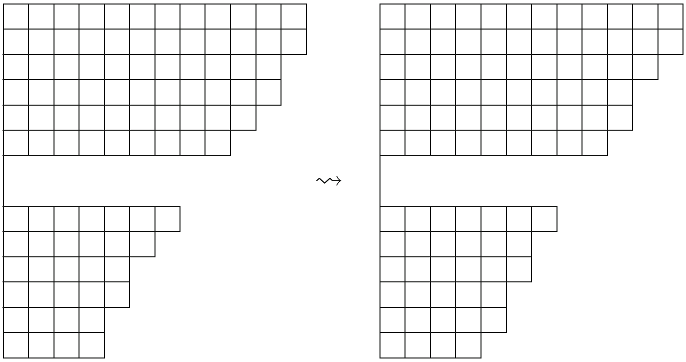


(a) Sympartition $\kappa[i-1, j+1]$ **(b)** Sympartition $\kappa'[i-1, i]$

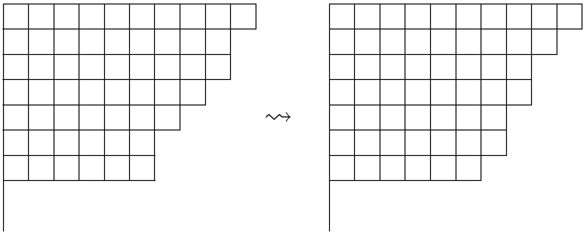
Fig. 6 Shape of κ and κ'

Fig. 7 From $\kappa'[k, k+2]$ to $\kappa''[k, k+2]$





(a) Case $k_1 < k_2 - 1$

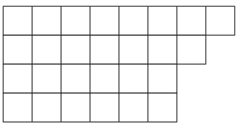


(b) Case $k_1 = k_2 - 1$

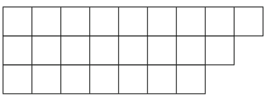
Fig. 8 From κ' to $\kappa'' = \text{Down}_{k_1-1,k_2+1}(\kappa')$

the highest (respectively lowest) one. Note that it is necessarily preceded (respectively followed) by an overlap. Then set $\kappa'' = \text{Down}_{k_1-1,k_2+1}(\kappa')$. It can be seen in Fig. 8 that $\mathfrak{O}(\kappa'') = \mathfrak{O}(\kappa')$, so that κ'' is a sympartition, and the result follows.

Case 6: $\mathfrak{I}_i^\kappa \geq 1$ and $\mathfrak{I}_j^\kappa > 1$. In that case, we can remove the (j, κ_j)-box of κ without creating a 2-overlap nor a 3-overlap. If there exists $i \leq k < j - 1$ such that $\mathfrak{I}_{k-1}^\kappa \geq 2$, then we set $\kappa'' = \text{Up}_{k,j}(\kappa)$, and κ'' is a sympartition. Otherwise, there is no such k and $\mathfrak{I}_{k-1}^\kappa = 0$ or 1 for all $i \leq k < j - 1$. If between i and j there is a sequence of shape,

$(*)$ 

then we can put the (j, κ_j)-box at the 3rd-line starting from the top to obtain the desired sympartition κ''. Since, the top of κ has the following shape:



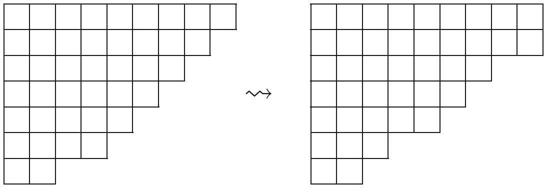


Fig. 9 From κ to κ''

we see that either $\kappa[i, j-1]$ is a 'staircase' or one can find a shape as in $(*)$. If it is a staircase, then we set $\kappa'' = \mathrm{Up}_{i,j}(\kappa)$. This situation is described in Fig. 9 in the case where $\mathfrak{J}^{\kappa}_{j-1} = 1$ though possibly $\mathfrak{J}^{\kappa}_{j-1} \geq 1$.

The number of overlaps in κ'' is then

$$\begin{aligned}\mathfrak{O}(\kappa'') &= \mathfrak{O}(\kappa[1, i-1]) + 1 + \mathfrak{O}(\kappa[j+1, \ell(\kappa)])\\ &= \mathfrak{O}(\kappa'[1, i-1]) + 1 + \mathfrak{O}(\kappa'[j+1, \ell(\kappa')])\\ &\leq \mathfrak{O}(\kappa')\end{aligned}$$

where $\ell(\kappa)$ denotes the length of κ. It follows that κ'' is a sympartition, and this concludes the proof of the theorem. $\square$

4 Proof of the main result

We start by giving an explicit characterisation of the preorder $\preceq_L$ as in [5], which is in spirit very similar to the one given in Example 2.5 in type A. Then, using our characterisation of the adjacency of bipartitions we will prove the main result of this paper, that is, the order $\preceq_L$ on bipartition is the same as the order $\trianglelefteq$ in the integer case.

4.1 On the preorder $\preceq_L$ in type B_n

We refer to [5, Lemma 7.5] for details in this section. Let W be a Weyl group of type B_n with diagram as in sect. 2.3. The maximal parabolic subgroups of W are of the form $W_k \times H_\ell$ where $n = k + \ell$ ($k \geq 0$ and $\ell \geq 1$). Here, W_k is of type B_k generated by $t, s_1, \ldots, s_{k-1}$, and H_ℓ is of type $A_{\ell-1}$ generated by $s_{k+1}, s_{k+2}, \ldots, s_{n-1}$ (with $W_0 = H_1 = 1$). We denote by ε_ℓ the sign representation of H_ℓ.

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions of n. We have $\boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}$ if and only if there exists a sequence

$$\boldsymbol{\lambda} := \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m := \boldsymbol{\mu}$$

such that for each $i \in \{1, \ldots, m\}$, the following condition is satisfied: there exist a decomposition $n = k_i + \ell_i$ and bipartitions $\boldsymbol{\nu}_i, \boldsymbol{\nu}'_i$ of k_i such that $E^{\boldsymbol{\nu}_i} \preceq_L E^{\boldsymbol{\nu}'_i}$ and

- either $E^{\boldsymbol{\nu}_i} \boxtimes \varepsilon_{\ell_i} \uparrow E^{\boldsymbol{\lambda}_{i-1}}$ and $E^{\boldsymbol{\nu}'_i} \boxtimes \varepsilon_{\ell_i} \rightsquigarrow_L E^{\boldsymbol{\lambda}_i}$
- or $E^{\boldsymbol{\nu}_i} \boxtimes \varepsilon_{\ell_i} \uparrow E^{\overline{\boldsymbol{\lambda}_i}}$ and $E^{\boldsymbol{\nu}'_i} \boxtimes \varepsilon_{\ell_i} \rightsquigarrow_L E^{\overline{\boldsymbol{\lambda}_{i-1}}}$ (where $\overline{\boldsymbol{\mu}}$ means the transpose of $\boldsymbol{\mu}$).

Lemma 4.1 *Let $n = k + \ell$ where $k \geq 0$ and $\ell \geq 1$. Let $\boldsymbol{\lambda}$ be a bipartition of k, and let $\boldsymbol{\mu}$ be a bipartition of n. Then $E^{\boldsymbol{\mu}}$ is a constituent of $Ind(E^{\boldsymbol{\lambda}} \boxtimes \varepsilon_\ell)$ if and only if $\kappa_{(b,N)}(\boldsymbol{\mu})$ can be obtained from $\kappa_{(b,N)}(\boldsymbol{\lambda})$ by increasing ℓ parts by 1. In addition, we have $\boldsymbol{a}(E^{\boldsymbol{\lambda}} \boxtimes \varepsilon_\ell) = \boldsymbol{a}(E^{\boldsymbol{\mu}})$ if and only if these parts are the ℓ largest entries of $\kappa_{(b,N)}(\boldsymbol{\lambda})$.*

Proof The first assertion is proved in [5, Lemma 7.6]. The 'if' part of the second assertion is a result of Lusztig [5, Lemma 7.10]. Let us show that if we increase the l largest entries of $\kappa_{(b,N)}(\boldsymbol{\lambda})$ by 1, then $\boldsymbol{a}(E^{\boldsymbol{\lambda}} \boxtimes \varepsilon_\ell) = \boldsymbol{a}(E^{\boldsymbol{\mu}})$. First, by [5, Remark 2.8], we have

$$\boldsymbol{a}(E^{\boldsymbol{\lambda}} \boxtimes \varepsilon_\ell) = \boldsymbol{a}(E^{\boldsymbol{\lambda}}) + \boldsymbol{a}(\varepsilon_\ell).$$

Next by [4, Ex. 1.3.8], we have

$$\boldsymbol{a}(\varepsilon_\ell) = \sum_{1 \leq i \leq \ell} (i-1).$$

If $\kappa_{(b,N)}(\boldsymbol{\mu})$ is obtained from $\kappa_{(b,N)}(\boldsymbol{\lambda})$ by increasing the ℓ largest entries of $\kappa_{(b,N)}(\boldsymbol{\lambda})$ then, by Proposition 2.11, we obtain

$$\boldsymbol{a}(E^{\boldsymbol{\mu}}) = \boldsymbol{a}(E^{\boldsymbol{\lambda}}) + \sum_{1 \leq i \leq \ell} (i-1)$$

as required. □

4.2 Proof of Theorem 2.13

Assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two bipartitions such that $\kappa := \kappa(\boldsymbol{\lambda}) \trianglelefteq \kappa' := \kappa(\boldsymbol{\mu})$. If $\kappa = \kappa'$, then we already know that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are in the same family so that $\boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}$. From now on we assume that $\kappa \lhd \kappa'$.

We want to show that $\boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}$. To do so, it is enough to consider the case where κ and κ' are adjacent. In the previous section, we have seen that there exist i_1 and j_1 such that $\kappa' = \mathrm{Up}_{i_1,j_1}(\kappa)$. In other words

$$\begin{aligned} \kappa &= (\ldots, \kappa_{i_1-1}, \kappa_{i_1}, \kappa_{i_1+1}, \ldots, \kappa_{j_1-1}, \kappa_{j_1}, \kappa_{j_1+1}, \ldots) \quad \text{and} \\ \kappa' &= (\ldots, \kappa_{i_1-1}, \kappa_{i_1}+1, \kappa_{i_1+1}, \ldots, \kappa_{j_1-1}, \kappa_{j_1}-1, \kappa_{j_1+1}, \ldots). \end{aligned}$$

Note that this implies that there is at most one element in κ which is equal to $\kappa'_{j_1} = \kappa_{j_1} - 1$. Indeed, since the (j_1, κ_{j_1})-box can be removed in κ, it implies that $\kappa_{j_1+1} \leq \kappa_{j_1} - 1$. If the inequality is strict, then the result is obvious, since no element of κ is equal to $\kappa_{j_1} - 1$. Next if $\kappa_{j_1+1} = \kappa_{j_1} - 1$, then we see that we must have $\kappa_{j_1+2} < \kappa_{j_1+1}$, otherwise we would create a 3-overlap by removing the (j_1, κ_{j_1})-box.

Case 1: $\kappa_{j_1-1} \neq \kappa_{j_1}$.

We will construct a $(N, b, n-\ell)$-sympartition κ'' for some $\ell \in \mathbb{N}$ satisfying the two following properties:

- κ is obtained from κ'' by increasing ℓ parts,
- κ' is obtained from κ'' by increasing the ℓ largest parts of κ''.

Consider the partition κ'' such that

$$\kappa''_j = \begin{cases} \kappa'_j - 1 & \text{if } j < j_1, \\ \kappa'_j & \text{if } j \geq j_1. \end{cases}$$

Note that this is indeed a partition since we have seen that there is at most one element in κ which is equal to $\kappa'_{j_1} = \kappa_{j_1} - 1$. In other words, we have

$$\kappa'' = (\ldots, \kappa'_{i_1-1} - 1, \kappa'_{i_1} - 1, \kappa'_{i_1+1} - 1, \ldots, \kappa'_{j_1-1} - 1, \kappa'_{j_1}, \kappa'_{j_1+1}, \ldots)$$

and

$$\kappa'' = (\ldots, \kappa_{i_1-1} - 1, \kappa_{i_1}, \kappa_{i_1+1} - 1, \ldots, \kappa_{j_1-1} - 1, \kappa_{j_1} - 1, \kappa_{j_1+1}, \ldots).$$

We show that κ'' is a $(N, b, n-\ell)$-sympartition. It is clear that κ'' is a partition of $f(N, b, n-\ell)$. So we only need to check that (1) there is no 3-overlap, and (2) the number of 2-overlaps is less than or equal to N.

(1) There is no 3-overlap in κ', thus the only possibility to have created one in κ'' is to have $\kappa'_{j_1-1} - 1 = \kappa'_{j_1} = \kappa'_{j_1+1}$, but this would imply that $\kappa_{j_1-1} = \kappa_{j_1}$, contradicting the hypothesis.

(2) The number of 2-overlaps in κ'' satisfies

$$\begin{aligned}\mathfrak{O}(\kappa'') &= \mathfrak{O}(\kappa''_{<j_1}) + \delta_{\kappa''_{j_1-1},\kappa''_{j_1}} + \mathfrak{O}(\kappa''_{\geq j_1}) \\ &= \mathfrak{O}(\kappa'_{<j_1}) + \delta_{\kappa'_{j_1-1}-1,\kappa'_{j_1}} + \mathfrak{O}(\kappa'_{>j_1}).\end{aligned}$$

As we have $\kappa'_{j_1-1} - 1 = \kappa_{j_1-1} - 1 \neq \kappa_{j_1} - 1 = \kappa'_{j_1}$ by hypothesis, we conclude that

$$\mathfrak{O}(\kappa'') \leq \mathfrak{O}(\kappa') \leq N.$$

The $j_1 - 1$ largest parts in κ'' have size $\kappa'_i - 1$ with $i < j_1$. Thus κ' can be obtained from κ'' by adding 1 to the $j_1 - 1$ largest parts. Further, κ can be obtained from κ'' by adding 1 to $j_1 - 1$ part. It follows that $\boldsymbol{\lambda} \preceq_L \boldsymbol{\mu}$ by Lemma 4.1.

Case 2: $\kappa_{j_1-1} = \kappa_{j_1}$.

The partitions κ, κ' have the following shape around j_1 (Fig. 10):

We consider the transposed bipartitions $\overline{\boldsymbol{\lambda}}$ and $\overline{\boldsymbol{\mu}}$. We choose t so that we have the following inclusion of multisets:

$$(t - \kappa_1, \ldots, t - \kappa_r) \subseteq (0, 1, \ldots, t, 0, 1, \ldots, t) \tag{1}$$

and

$$(t - \kappa'_1, \ldots, t - \kappa'_r) \subseteq (0, 1, \ldots, t, 0, 1, \ldots, t). \tag{2}$$

(a) Sympartition $\kappa[j_1 - 2, j_1 + 1]$ **(b)** Sympartition $\kappa'[j_1 - 2, j_1 + 1]$

Fig. 10 Shape of κ and κ'

(a) Sympartition $\kappa[i_2 - 1, i_2 + 1]$ **(b)** Sympartition $\kappa'[i_2 - 1, i_2 + 1]$

Fig. 11 Shape of κ and κ'

Then the sympartition $\overline{\kappa}$ (respectively $\overline{\kappa}'$) associated to $\overline{\boldsymbol{\lambda}}$ (respectively $\overline{\boldsymbol{\mu}}$) is the partition obtained by reordering the complement of the first multiset into the second in (1) (respectively in (2)); see [11, §22]. There exist $i_2 < j_2$ such that

$$\begin{array}{l} \overline{\kappa'} = (\overline{\kappa'}_1, \ldots, \overline{\kappa'}_{i_2-1}, \overline{\kappa'}_{i_2}, \overline{\kappa'}_{i_2+1}, \ldots, \overline{\kappa'}_{j_2-1}, \overline{\kappa'}_{j_2}, \overline{\kappa'}_{j_2+1}, \ldots) \\ \overline{\kappa} = (\overline{\kappa'}_1, \ldots, \overline{\kappa'}_{i_2-1}, \overline{\kappa'}_{i_2} + 1, \overline{\kappa'}_{i_2+1}, \ldots, \overline{\kappa'}_{j_2-1}, \overline{\kappa'}_{j_2} - 1, \overline{\kappa'}_{j_2+1}, \ldots) \end{array}.$$

Looking at the shape of κ, κ' around j_1, we see that the shape of $\overline{\kappa}$ and $\overline{\kappa}'$ around i_2 is the following (Fig. 11):

Consider the partition κ'' defined by

$$\kappa''_j = \begin{cases} \kappa_j - 1 & \text{if } j \leq i_2, \\ \kappa_j & \text{if } j > i_2. \end{cases}$$

Since $\kappa_{i_2} - \kappa_{i_2+1} \geq 2$, we see that κ'' is a sympartition. Then κ is obtained from κ'' by adding 1 to the i_2 largest parts, and κ' is obtained by adding 1 to i_2 parts. This concludes the proof of the theorem.

Remark 4.2 In the non-integer case, the definition of symbol is different (see [5, Definition 3.1]), and it is not clear, at least to us, how to characterise adjacency of symbols in this case.

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