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# Action of the symmetric groups on the homology of the hypertree posets

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**Abstract** The set of hypertrees on $n$ vertices can be endowed with a poset structure. J. McCammond and J. Meier computed the dimension of the unique non-zero homology group of the hypertree poset. We give another proof of their result and use the theory of species to determine the action of the symmetric group on this homology group, which is linked with the anti-cyclic structure of the PreLie operad. We also compute the action on the Whitney homology of the poset.

**Keywords** Hypertree · Poset homology · Whitney homology · Species · Symmetric group action

## 1 Introduction

Hypergraphs were introduced by C. Berge in [1] during the 1980s. They are a generalization of graphs whose edges can contain more than two vertices. Hypertrees are connected and acyclic hypergraphs. Several studies on hypertrees have been led such as the computation of the number of hypertrees on $n$ vertices by L. Kalikow in [8] and by W.D. Smith and D.M. Warme in [14]. For a finite set $I$, we can endow the set of hypertrees on the vertex set $I$ with a structure of poset: given two hypertrees $H$ and $K$, $H \preceq K$ if each edge in $K$ is a subset of some edge in $H$. These hypertree posets have been used for the study of automorphisms of free groups and free products in papers of D. McCullough–A. Miller [9], N. Brady–J. McCammond–J. Meier–A. Miller [3], J. McCammond–J. Meier [10] and C. Jensen–J. McCammond–J. Meier [7] and [6]. In the article [3], the Cohen–Macaulayness of the poset is proven: the poset has only one

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nontrivial homology group. The reduced Euler characteristic of the poset of hypertrees on $n$ vertices has then been computed in [10]: the unique nontrivial homology group has dimension $(n-1)^{n-2}$. Taking the set $\{1,\ldots,n\}$ for $I$, the action of the symmetric group $\mathfrak{S}_n$ on $I$ induces an action on the poset of hypertrees on $I$ compatible with the differential: this provides an action of the symmetric group on the unique nontrivial homology group of the poset. In the article [5], F. Chapoton computed the characteristic polynomial of the poset and gave a conjecture for the representation of the symmetric group $\mathfrak{S}_n$ on the homology and on the Whitney homology of the poset.

This article solves the conjecture of F. Chapoton in Theorems 5.2 and 6.11. The dimension computed by J. McCammond and J. Meier turns out to be also the number of labeled rooted trees on $n-1$ vertices, which is the dimension of the vector space $\mathrm{PreLie}(n-1)$, the component of arity $n-1$ of the PreLie operad. Operad PreLie is an anti-cyclic operad, as proven in [4]. Therefore, the action of the symmetric group $\mathfrak{S}_{n-1}$ on $\mathrm{PreLie}(n-1)$ induces an action of $\mathfrak{S}_n$ on $\mathrm{PreLie}(n-1)$. In Theorem 5.2, we prove that the representation of $\mathfrak{S}_n$ on $\mathrm{PreLie}(n-1)$ and the representation of $\mathfrak{S}_n$ on the poset homology are isomorphic up to tensor product by the sign representation. Theorem 6.11 is a refinement of this theorem in which a type of hypertrees decorated by $\Sigma\mathrm{Lie}$ appears: the action of the symmetric group on the unique nontrivial homology group of the poset is the same as the action of the symmetric group on these decorated hypertrees.

We recommend to read Appendix A and the first two chapters of the book [2] for an introduction to species theory which will be used in this article. In the first part of the article, we recall the construction of the homology group of a poset. In the second part, we determine relations between hypertrees and rooted and/or pointed hypertrees species and then, give a new proof for J. McCammond and J. Meier's result on the dimension of the poset homology group in the third part. In the fourth part, we use the relations between species, established in the first part, to compute the action of the symmetric group on this homology group. In the last part, we compute the action of the symmetric group on Whitney homology.

## 2 Construction of the homology of the hypertree poset

### 2.1 Definition of the poset

The hypertrees and the associated poset are described by F. Chapoton in the article [5]. We briefly recall their definitions.

#### *2.1.1 Hypergraphs and hypertrees*

**Definition 2.1** A *hypergraph* (*on a set* $V$) is an ordered pair $(V,E)$ where $V$ is a finite set and $E$ is a collection of elements of cardinality at least two, belonging to the power set $\mathcal{P}(V)$. The elements of $V$ are called *vertices* and those of $E$ are called *edges*.

An example of a hypergraph is presented in Fig. 1.

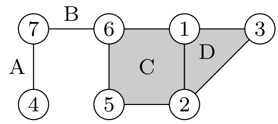



**Fig. 1** An example of a hypergraph on $\{1, 2, 3, 4, 5, 6, 7\}$

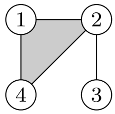



**Fig. 2** An example of a hypertree on $\{1, 2, 3, 4\}$

**Definition 2.2** Let $H = (V, E)$ be a hypergraph.

A *walk from a vertex or an edge d to a vertex or an edge f in H* is an alternating sequence of vertices and edges beginning by $d$ and ending by $f$ $(d, \dots, e_i, v_i, e_{i+1}, \dots, f)$ where for all $i$, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$. The *length* of a walk is the number of edges and vertices in the walk.

*Example 2.3* In the previous example, there are several walks from 4 to 2: $(4, A, 7, B, 6, C, 2)$ and $(4, A, 7, B, 6, C, 1, D, 3, D, 2)$. A walk from $C$ to 3 is $(C, 1, D, 3)$.

**Definition 2.4** A *hypertree* is a non-empty hypergraph $H$ such that, given any distinct vertices $v$ and $w$ in $H$,

– there exists a walk from $v$ to $w$ in $H$ with distinct edges $e_i$, i.e. $H$ is *connected*,
– and this walk is unique, i.e. $H$ has *no cycles*.

The pair $H = (V, E)$ is called *hypertree on* $V$. If $V$ is the set $\{1, \dots, n\}$, then $H$ is called a *hypertree on n vertices*.

Denote the hypertrees species by $\mathcal{H}$. An example of a hypertree is presented in Fig. 2.

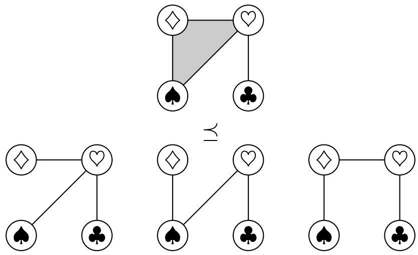
We have the following proposition:

**Proposition 2.5** *Given a hypertree H, a vertex or an edge d of H and a distinct vertex f of H, there is a unique minimal walk from d to f and this walk is the unique walk having distinct edges.*

*Proof* If $d$ is a vertex, there exists a unique walk to $f$ with distinct edges as $H$ is a hypertree. Let us consider another walk $(d = v_0, e_1, v_1, \dots, e_k, v_k = f)$ with $e_i = e_j$ for some $i < j$. Then the walk $(d = v_0, \dots, e_i, v_j, \dots, v_k = f)$ obtained by deleting $(v_i, \dots, e_j)$ in the walk, is a shorter walk. Therefore, a minimal walk has distinct edges and is thus unique.

If $d$ is an edge, we consider a pair of vertices $(v, v')$ in $d$. If $d$ is not on the unique minimal walk $(v = v_0, e_1, v_1, \dots, v_n = f)$ from $v$ to $f$, then $(v', d, v_0, e_1, \dots, v_n = f)$ is a walk from $v'$ to $f$ with distinct edges so it is the unique minimal walk from $v'$ to $f$. Otherwise, let us exchange $v$ and $v'$ so that the edge $d$ is the first edge on the unique minimal walk from $v'$ to $f$. This walk gives a walk $w$ from $d$ to $f$ by deleting

**Fig. 3** An example in the hypertree poset on four vertices on $I = (\diamondsuit, \heartsuit, \clubsuit, \spadesuit)$




the vertex $v'$. Suppose that there is another walk from $d$ to $f$, different from $w$ and shorter. By adding $v'$ at the beginning of the walk, this gives a shorter walk from $v'$ to $f$ than the unique minimal walk from $v'$ to $f$. As this walk is different from the minimal walk, there is a contradiction. Thus, there is a unique minimal walk from $d$ to $f$ and this walk has distinct edges. □

### 2.1.2 *The hypertree poset on n vertices*

Let $I$ be a finite set of cardinality $n$, $S$ and $T$ be two hypertrees on $I$. We say that $S \preceq T$ if each edge of $S$ is the union of edges of $T$, and that $S \prec T$ if $S \preceq T$ but $S \neq T$. An example is presented in Fig. 3.

The set $(\mathcal{H}(I), \preceq)$ is a partially ordered set (or poset), written HT(I). We denote by $\widehat{\mathrm{HT(I)}}$ the poset obtained by adding to HT(I) a formal element $\hat{1}$ above all the other elements of the poset. For every positive integer $n$, we moreover write $\mathrm{HT_n}$ for the poset $\mathrm{HT(\{1, \dots, n\})}$.

**Definition 2.6** Given a relation $\preceq$, the *cover relation* $\lhd$ is defined by $x \lhd y$ ($y$ covers $x$ or $x$ is covered by $y$) if and only if $x \prec y$ and there is no $z$ such that $x \prec z \prec y$.

In HT(I), we define the *rank* $r(h)$ of a hypertree $h$ with $A$ edges by

$$r(h) = A - 1.$$

Each cover relation increases the rank by one, so the poset HT(I) is graded by the number of edges in hypertrees.

## 2.2 Chain complex and homology associated to a poset

We now define the homology associated to a poset $\mathcal{P}$ with a minimum and a maximum. The reader may read Wachs' article [13] for a deeper treatment of this subject and Munkres' book [11] for more details on simplicial homology. We introduce the following terminology:

**Definition 2.7** A *strict m-chain* is a $m$-tuple $(a_1, \dots, a_m)$ where $a_i$ are elements of $\mathcal{P}$, neither maximum nor minimum in $\mathcal{P}$, and $a_i \prec a_{i+1}$, for all $i \geq 1$. We write $\mathcal{C}_m(\mathcal{P})$ for the set of strict $m+1$-chains and $C_m(\mathcal{P})$ for the $\mathbb{C}$-vector space generated by all strict $m+1$-chains.

The set $\bigcup_{m\geq 0}\mathcal{C}_m(\mathcal{P})$ is then a simplicial complex.

Define the linear map $d_m : C_{m+1}(\mathcal{P}) \to C_m(\mathcal{P})$ which maps a $m+1$-simplex to its boundary. These maps satisfy $d_{m-1}\circ d_m = 0$. The pairs $(C_m(\mathcal{P}), d_m)_{m>0}$ obtained form a *chain complex*. Thus, we can define the homology of the poset.

**Definition 2.8** The homology group of dimension $m$ of the poset $\mathcal{P}$ is

$$H_m(\mathcal{P}) = \ker d_m / \operatorname{Im} d_{m+1}.$$

We consider in this article the reduced homology, written $\tilde{H}_i$. Having $C_{-1}(\mathcal{P}) = \mathbb{C}.e$, and $d : C_0 \to C_{-1}$, the trivial linear map which maps every singleton to the element $e$, we obtain

$$\dim\big(\tilde{H}_0(\mathcal{P})\big) = \dim\big(H_0(\mathcal{P})\big) - 1.$$

Dimensions of the homology spaces satisfy the following well-known property:

**Lemma 2.9** *The Euler characteristic of the homology satisfies*

$$\chi = \sum_{m\geq 0}(-1)^m \dim \tilde{H}_m(\mathcal{P}) = \sum_{m\geq -1}(-1)^m \dim C_m(\mathcal{P}). \tag{2.1}$$

## 2.3 Homology of the $\widehat{\mathrm{HT_n}}$ poset

Let us apply the previous subsection to the poset $\widehat{\mathrm{HT_n}}$. The vector spaces $C_m(\mathcal{P})$ and $\tilde{H}_m(\mathcal{P})$ are denoted by $C_m^n$ and $\tilde{H}_m^n$. The reader may consult Sundaram's article [12] for general points on the notion of Cohen–Macaulay poset. The following notion is needed:

**Definition 2.10** Let $\mathcal{P}$ be a poset and $\sigma$ be a closed simplex of the geometric realization $|\mathcal{P}|$ of $\mathcal{P}$. The *link* of $\sigma$ is the subcomplex:

$$Lk(\sigma) = \big\{s \in |\mathcal{P}| : s \cup \sigma \in |\mathcal{P}|, s \cap \sigma = \emptyset\big\}.$$

Let us remark that $s$ denotes simplices in the geometric realization and not only points.

**Definition 2.11** ([10, Definition 2.8]) A poset $\mathcal{P}$ is *Cohen–Macaulay* if its geometric realization $|\mathcal{P}|$ is Cohen–Macaulay. That is, for every closed simplex $\sigma$ in $|\mathcal{P}|$, we have

$$\tilde{H}_i\big(Lk(\sigma)\big) = \begin{cases} 0, & \text{for } i \neq \dim(|\mathcal{P}|) - \dim(\sigma) - 1,\\ \mathbb{C}^{\dim \tilde{H}_i(Lk(\emptyset))}, & \text{for } i = \dim(|\mathcal{P}|) - \dim(\sigma) - 1,\end{cases}$$

where the dimension of the empty simplex is $-1$ by convention.

**Theorem 2.12** ([10, Theorem 2.9]) *For each $n \geq 1$, the poset $\widehat{\mathrm{HT_n}}$ is Cohen–Macaulay.*

**Corollary 2.13** *The homology of* $\widehat{\mathrm{HT}_\mathrm{n}}$ *is concentrated in maximal degree*:

$$\tilde{H}_i\big(Lk(\emptyset)\big) = \begin{cases} 0, & \textit{for } i \neq \dim(|\widehat{\mathrm{HT}_\mathrm{n}}|), \\ \mathbb{C}^{\dim \tilde{H}_i(Lk(\emptyset))}, & \textit{for } i = \dim(|\widehat{\mathrm{HT}_\mathrm{n}}|). \end{cases}$$

As $\dim \widehat{\mathrm{HT}_\mathrm{n}} = n-3$, (2.1) can thus be rewritten as:

$$\dim \tilde{H}^n_{n-3} = \sum_{m\geq -1} (-1)^m \dim C^n_m. \tag{2.2}$$

Moreover, as the differential is compatible with the symmetric group action, the action of the symmetric group on $(C^n_m)_{m\geq -1}$ induces an action on $\tilde{H}^n_{n-3}$. Hence we obtain the following relation, with $\chi^s_{i+1}$ the character of the action of the symmetric group on the vector space $C^n_i$ and $\chi_{\tilde{H}^n_{n-3}}$ the character of the action of the symmetric group on the vector space $\tilde{H}^n_{n-3}$:

$$\chi_{\tilde{H}^n_{n-3}} = \sum_{m\geq -1} (-1)^m \chi^s_{m+1}. \tag{2.3}$$

### 2.4 From large to strict chains

According to (2.3), it is sufficient to compute the alternating sum of characters on $C^n_m$ to determine the character on the only nontrivial homology group.

Let $k$ be a natural number and $I$ be a finite set. The set of *large* $k$*-chains* of hypertrees on $I$ is the set $\mathrm{HL}^I_k$ of $k$-tuples $(a_1,\dots,a_k)$ where $a_i$ are elements of HT(I) and $a_i \preceq a_{i+1}$. The set of *strict* $k$*-chains* of hypertrees on $I$ is the set $\mathrm{HS}^I_k$ of $k$-tuples $(a_1,\dots,a_k)$ where $a_i$ are non-minimum elements of HT(I) and $a_i \prec a_{i+1}$. The set $\mathrm{HS}^{\{1,\dots,n\}}_k$ is then a basis of the vector space $C^n_{k-1}$.

We define the following species:

**Definition 2.14** The species $\mathcal{H}_k$ of large $k$-chains of hypertrees is defined by

$$I \mapsto \mathrm{HL}^I_k.$$

The species $\mathcal{HS}_k$ of strict $k$-chains of hypertrees is the species defined by

$$I \mapsto \mathrm{HS}^I_k.$$

**Definition 2.15** Let $M_{k,s}$ be the set of words on $\{0,1\}$ of length $k$, containing $s$ letters "1". The species $\mathcal{M}_{k,s}$ is defined by

$$\begin{cases} \emptyset \mapsto M_{k,s}, \\ V \neq \emptyset \mapsto \emptyset. \end{cases}$$

Let us describe the link between these species:

**Proposition 2.16** *The species $\mathcal{H}_k$ and $\mathcal{HS}_i$ are related by*

$$\mathcal{H}_k \cong \sum_{i\geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$

*Proof* Let $(a_1, \dots, a_k)$ be a large $k$-chain. It can be factorized into an ordered pair formed by a strict $s$-chain $(a_{i_1}, \dots, a_{i_s})$, obtained by deleting repetitions and minimum $\hat{0}$, if it is possible, and an element $u_1 \dots u_k$ of $M_{k,s}$ such that:

- $u_1 = 0$ if $a_1 = \hat{0}$, 1 otherwise;
- $u_j = 0$ if $a_j = a_{j-1}$, 1 otherwise.

From a strict $i$-chain and a word $u_1 \dots u_k$ of $M_{k,i}$, a large $k$-chain can be reconstructed.

This establishes the desired species isomorphism. □

**Corollary 2.17** *Consider the action by permutation of $\mathfrak{S}_n$ on $\{1, \dots, n\}$. The characters $\chi_k$ and $\chi_i^s$ of the induced action on the vector spaces $\mathcal{H}_k(\{1, \dots, n\})$ and $\mathcal{HS}_i(\{1, \dots, n\})$ satisfy*

$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s. \tag{2.4}$$

*Proof* The isomorphism of Proposition 2.16 is a species isomorphism, so it preserves the symmetric group action.

This gives

$$\mathcal{H}_k\bigl(\{1, \dots, n\}\bigr) \cong \sum_{i\geq 0} \mathcal{HS}_i\bigl(\{1, \dots, n\}\bigr) \times \mathcal{M}_{k,i}(\emptyset).$$

Moreover, the action of $\mathfrak{S}_n$ on $\mathcal{M}_{k,i}(\emptyset)$ is trivial, so that we obtain

$$\chi_k = \sum_{i\geq 0} \chi_i^s \times \#M_{k,i}.$$

The cardinality of $M_{k,i}$ is $\binom{k}{i}$. As the maximal length of a strict chain in $\mathrm{HT_n}$ is $n-2$, the sum is finite. □

As the expression of $\sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s$ is polynomial in $k$, of degree bounded by $n$, it enables us to extend $\chi_k$ to integers. Equation (2.3) shows that the character $\chi_k$ evaluated at $k = -1$ is the negation of the character given by the action of $\mathfrak{S}_n$ induced on poset homology.

**Proposition 2.18** *Let us write $P_n(X)$ for the polynomial whose value at $k$ gives the number of large $k$-chains in the poset $\widehat{HT_n}$. The negation of the character given by the action of $\mathfrak{S}_n$ induced on the homology of poset $\widehat{HT_n}$ is given by $P_n(-1)$.*

## 3 Relations between species and auxiliary species

In this section, we define new species and establish connections between them. The reader may consult Appendix A for definitions of some usual species used in this part.

### 3.1 Rooted and pointed hypertrees

Let $k$ be a natural number.

We define the following pointed hypertrees:

**Definition 3.1** A *rooted hypertree* is a hypertree $H$ together with a vertex $s$ of $H$. The hypertree $H$ is said to be *rooted at* $s$ and $s$ is called the root of $H$.

An example is presented in Fig. 4.

Let us recall that the minimum of a chain is the hypertree with the smallest number of edges on the chain.

The species associated with rooted hypertrees is denoted by $\mathcal{H}^p$. The one associated with large $k$-chains of hypertrees, whose minimum is a rooted hypertree, is denoted by $\mathcal{H}_k^p$. This vertex is then distinguished in the other hypertrees of the chain, so that all hypertrees in the chain can be considered as rooted at this vertex. In the following, the species $\mathcal{H}_k^p$ will be called "species of large rooted $k$-chains".

**Definition 3.2** An *edge-pointed hypertree* is a hypertree $H$ together with an edge $a$ of $H$. The hypertree $H$ is said to be *pointed at* $a$.

An example is presented in Fig. 5.

The species associated with edge-pointed hypertrees is denoted by $\mathcal{H}^a$. The one associated with large $k$-chains of hypertrees whose minimum is an edge-pointed hypertree is denoted by $\mathcal{H}_k^a$.

**Definition 3.3** An *edge-pointed rooted hypertree* is a hypertree $H$ on at least two vertices, together with an edge $a$ of $H$ and a vertex $v$ of $a$. The hypertree $H$ is said to be *pointed at a and rooted at s*.

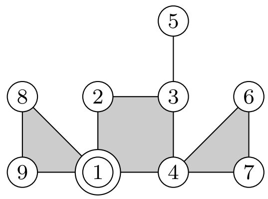



**Fig. 4** A hypertree on nine vertices, rooted at 1

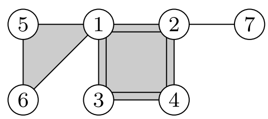



**Fig. 5** A hypertree on seven vertices, pointed at the edge $\{1, 2, 3, 4\}$

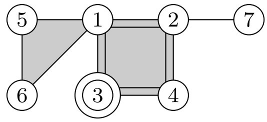



**Fig. 6** A hypertree on seven vertices, pointed at edge $\{1, 2, 3, 4\}$ and rooted at 3

An example is presented in Fig. 6.

The species associated with edge-pointed rooted hypertrees is denoted by $\mathcal{H}^{pa}$. The one associated with large $k$-chains of hypertrees whose minimum is an edge-pointed rooted hypertree is denoted by $\mathcal{H}_k^{pa}$.

### 3.2 Dissymmetry principle

The reader may consult book [2, Chap. 2.3] for a deeper explanation on the dissymmetry principle. In a general way, a *dissymmetry principle* is the use of a natural center to obtain the expression of a non-pointed species in terms of pointed species. An example of this principle is the use of the center of a tree to express unrooted trees in terms of rooted trees. The expression of the hypertrees species in terms of pointed and rooted hypertrees species is the following:

**Proposition 3.4** *The species of hypertrees and of rooted hypertrees are related by*

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}. \tag{3.1}$$

For the proof, we need the following notions which use Proposition 2.5:

**Definition 3.5** The *eccentricity* of a vertex or an edge is the maximal number of vertices and edges on the minimal walk from it to another vertex. The *center* of a hypertree (edge-pointed or not, rooted or not) is the vertex or the edge with minimal eccentricity.

**Proposition 3.6** *The center is unique.*

*Proof* We prove this proposition *ad absurdum*.

Let us consider a hypertree $H$ such that there are two different vertices or edges $a$ and $b$ of same eccentricity $e$ which are centers of $H$. The number of vertices or edges on a walk from an edge to a vertex is even. The number of vertices or edges on a walk from a vertex to a vertex is odd. Therefore, either $a$ and $b$ are vertices, or they are edges, according to the parity of $e$. As they are different, there is a nontrivial minimal walk of odd length from $a$ to $b$ with at least one element $c$ on it different from $a$ and $b$.

We consider a walk $(b, \dots, e_n, v_n = f)$ from $b$ to a vertex $f$ such that $c$ is not in the walk. If $c$ is not in the unique minimal walk $(a, \dots, e'_p, v'_p = f)$ from $a$ to $f$, then the concatenation $(b, \dots, e_n, f, e'_p, \dots, a)$ is a walk from $b$ to $a$ and $c$ is not in it. The edges of type $e_i$ (respectively $e'_j$) are all different. If this walk is not minimal, there is a minimal $i$ such that $e_i$ and $e'_j$ are equals for an integer $j$. Then the walk

$(b, \dots, e_i, v'_j, \dots, a)$ is minimal and $c$ is not on it. It means that there are two different minimal walks from $b$ to $a$, which is not possible.

Therefore, for every vertex $f$, $c$ is either in the walk from $b$ to $f$ or in the walk from $a$ to $f$. The eccentricity of $c$ is then strictly less than $e$, which is in contradiction with the minimality of $e$. □

We can now prove Proposition 3.4:

*Proof* The following maps are bijections, inverse one of the other:

$$\phi : \mathcal{H} + \mathcal{H}^{pa} \to \mathcal{H}^a + \mathcal{H}^p,$$

$$\psi : \mathcal{H}^a + \mathcal{H}^p \to \mathcal{H} + \mathcal{H}^{pa}.$$

If $T$ belongs to $\mathcal{H}$, $\phi(T)$ is the hypertree obtained by pointing the center of $T$. We thus obtain a rooted hypertree if the center is a vertex and an edge-pointed hypertree otherwise (case A).

If $T$ belongs to $\mathcal{H}^{pa}$, $\phi(T)$ is the hypertree obtained from $T$ by:

- forgetting the root of $T$ if it is its center, obtaining an edge-pointed hypertree (case B),
- forgetting the pointed edge of $T$ if it is its center, obtaining a rooted hypertree (case C),
- forgetting the pointed edge or root which is the nearest from the center of the hypertree (case D).

If $T$ belongs to $\mathcal{H}^a$, $\psi(T)$ is the hypertree obtained from $T$ by:

- forgetting the pointed edge of $T$ if it is its center (reverse of case A),
- rooting the center of $T$ if it belongs to the pointed edge of $T$ (reverse of case B),
- rooting the nearest vertex of the pointed edge from the center of $T$ (reverse of case D).

Otherwise, $T$ belongs to $\mathcal{H}^p$, $\psi(T)$ is the hypertree obtained from $T$ by:

- forgetting the root of $T$ if it is its center (reverse of case A),
- pointing the center if it is an edge containing the root of $T$ (reverse of case C),
- pointing the nearest edge containing the root from the center of $T$ (reverse of case D). □

Let $k$ be a natural number.

The following proposition links large $k$-chains of hypertrees, rooted hypertrees, edge-pointed hypertrees and edge-pointed rooted hypertrees.

**Proposition 3.7** (Dissymmetry principle for hypertrees chains) *The following relation holds*:

$$\mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^p + \mathcal{H}_k^a. \tag{3.2}$$

*Proof* We apply the dissymmetry principle to the minimum of the chain. □

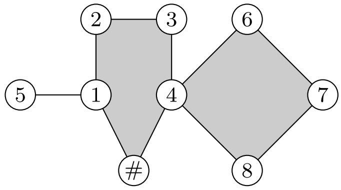



**Fig. 7** A hollow hypertree on eight vertices

## 3.3 Relations between species

### 3.3.1 Relations for $\mathcal{H}_k^p$

To determine a functional equation for $\mathcal{H}_k^p$, we introduce another type of hypertree.

**Definition 3.8** A *hollow hypertree* on $n$ vertices ($n \geq 2$) is a hypertree on the set $\{\#, 1, \dots, n\}$, such that the vertex labeled by #, called the gap, belongs to one and only one edge.

An example is presented in Fig. 7.

**Definition 3.9** A *hollow hypertrees $k$-chain* is a chain of length $k$ in the poset of hypertrees on $\{\#, 1, \dots, n-1\}$, whose minimum is a hollow hypertree. The species of hollow hypertrees $k$-chains is denoted by $\mathcal{H}_k^c$. The species of hollow hypertrees $k$-chains whose minimum has only one edge, is denoted by $\mathcal{H}_k^{cm}$. Remark that the other hypertrees of the chain are not necessarily hollow hypertrees because the vertex labeled by # is in one and only one edge in the minimum of the chain but can be in two or more edges then.

These species are linked by the following proposition:

**Proposition 3.10** *The species $\mathcal{H}_k^p$, $\mathcal{H}_k^c$ and $\mathcal{H}_k^{cm}$ satisfy*

$$\mathcal{H}_k^p = X + X \times \mathrm{Comm} \circ \mathcal{H}_k^c, \tag{3.3}$$

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \tag{3.4}$$

$$\mathcal{H}_k^{cm} = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c. \tag{3.5}$$

*Proof*

1. A $k$-chain of rooted hypertrees on one vertex is just the same as one vertex repeated $k$ times. Thus, it is the same object as a singleton, so the associated species is the species $X$.

   We now consider $k$-chains of rooted hypertrees on at least two vertices. Each such chain can be separated into a singleton and a set of hollow hypertrees $k$-chains. The singleton is the root of the minimum hypertree. The set of hollow hypertrees $k$-chains is obtained by:

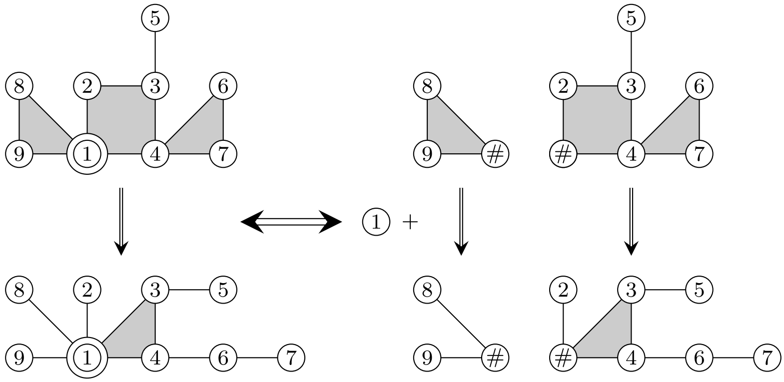



**Fig. 8** Decomposition of a rooted hypertrees chain

- deleting the root in every hypertree,
- putting a gap # where the root was,
- separating in the minimum the edges containing gaps, so that we obtain a set of hollow hypertrees.

The third point induces a decomposition of the chain into a set of hollow hypertrees chains. Indeed, it gives a partition of the set of edges such that every vertex different from the root appears exactly one time, and this partition is preserved during the chain. This gives the result (3.3).

*Example 3.11* A rooted hypertrees chain decomposed into a singleton and a set of hollow hypertrees $k$-chains is represented in Fig. 8. Here are drawn only the minima (at the top) and the maxima (at the bottom) of the chains.

2. Let $S$ be a hollow hypertrees $k$-chain as defined in Definition 3.9.

   The hollow edge in the minimum, i.e. the edge containing the gap, gives at each stage $l$ of the chain a set of distinguished edges $D_e^l$. Considering only these distinguished edges, we obtain a hollow hypertrees $k$-chain $D$ whose minimum has only one edge.

   Deleting the hollow edge $D_e^1$ in the minimum of $S$ gives a hypertree forest, i.e. a list of hypertrees $(h_1, \dots, h_f)$. Each hypertree $h_i$ has a distinguished vertex $s_i$ which was in the hollow edge. Let us say that $h_i$ is rooted at $s_i$. The evolution of edges of the hypertree $h_i$ in $S$ induces a chain $S_{h_i}$. The rootedness of $h_i$ induces a rootedness of $S_{h_i}$.

   Note that the hypertrees forest $(h_1^l, \dots, h_f^l)$ obtained at stage $l$ of the chain by deleting $D_e^l$ is the same as the hypertrees forest obtained by taking the hypertrees at stage $l$ in chains $S_{h_1}, \dots, S_{h_f}$.

   Thus the chain $S$ is the chain $D$, where at stage $l$, on vertex $i$, we have grafted hypertree $h_i^l$. The grafting consists of replacing vertex $i$ by the root of $h_i^l$ in the hypertree.

   The chain $S$ can also be seen as chain $D$, where the rooted hypertrees chain $S_{h_i}$ has been inserted in vertex $i$. This gives the result (3.4).

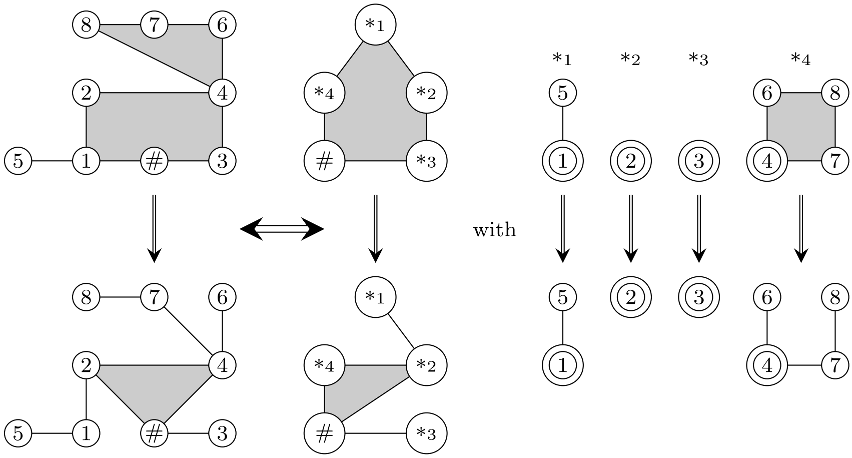



**Fig. 9** Decomposition of a hollow hypertrees chain

*Example 3.12* A hollow hypertrees chain is equivalent to a hollow hypertrees chain whose minimum has only one edge, and whose vertices are rooted hypertrees chains. An example is presented in Fig. 9.

3. A hollow hypertrees $k$-chain, whose minimum has only one edge can be seen as a $(k-1)$-chain $C_{k-1}$ with a vertex labeled by #. Separating the edges containing the label # in the minimum of $C_{k-1}$ is the same as separating this chain in a non-empty set of hollow hypertrees $(k-1)$-chains. This gives the result (3.5).

□

As the species $\mathcal{H}_k^p$ can be factored by the species $X$, the map $\frac{\mathcal{H}_k^p - X}{X}$ is a species. We obtain the following corollary:

**Corollary 3.13** *The species $\mathcal{H}_k^p$ satisfies*

$$\mathcal{H}_k^p = X + X \times \text{Comm} \circ \left( \frac{\mathcal{H}_{k-1}^p - X}{X} \circ \mathcal{H}_k^p \right). \tag{3.6}$$

### *3.3.2 Relations for $\mathcal{H}_k^a$*

We have the following relation:

**Proposition 3.14** *The species $\mathcal{H}_k^a$ satisfies*

$$\mathcal{H}_k^a = (\mathcal{H}_{k-1} - X) \circ \mathcal{H}_k^p. \tag{3.7}$$

*Proof* Let $S$ be an edge-pointed hypertrees $k$-chain. The pointed edge in the minimum of $S$ gives at each stage $l$ of the chain a set of distinguished edges $D_e^l$, obtained from the fission of the pointed edge. Considering only these distinguished edges, we

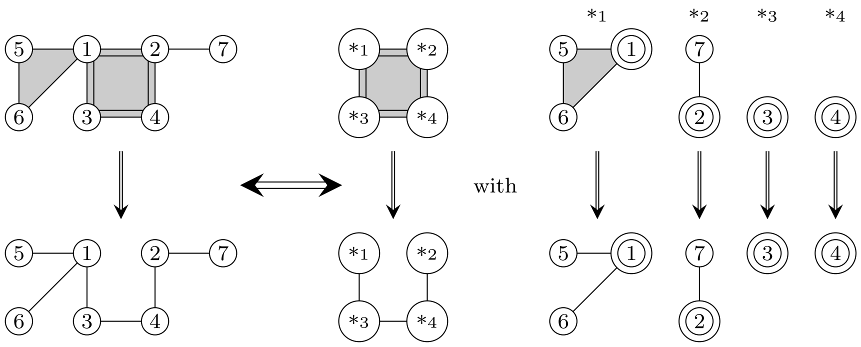



**Fig. 10** Decomposition of an edge-pointed hypertrees chain

obtain a hypertrees $k$-chain $D$ whose minimum has only one edge. That chain can be seen as a $(k-1)$-chain of hypertrees on at least two vertices.

Deleting the pointed edge $D_e^1$ in the minimum of $S$ gives a hypertrees forest, i.e. a list of hypertrees $(h_1, \dots, h_f)$. Each hypertree $h_i$ has a distinguished vertex $s_i$ which was in the pointed edge. Let us say that $h_i$ is rooted at $s_i$. The evolution of edges of hypertree $h_i$ at $S$ induces a chain $S_{h_i}$. The rootedness of $h_i$ induces a rootedness of $S_{h_i}$.

Note that the hypertrees forest $(h_1^l, \dots, h_f^l)$ obtained at stage $l$ of the chain by deleting $D_e^l$ is the same as the hypertrees forest obtained by taking hypertrees at stage $l$ in chains $S_{h_1}, \dots, S_{h_f}$

Thus the chain $S$ is the chain $D$, where at stage $l$, on vertex $i$, we have grafted the hypertree $h_i^l$. The grafting consists of replacing vertex $i$ by the root of $h_i^l$ in the hypertree.

The chain $S$ can also be seen as the chain $D$, where the rooted hypertrees chain $S_{h_i}$ has been inserted in vertex $i$. This gives the result, as in the proof of Proposition 3.10. □

*Example 3.15* An edge-pointed hypertrees chain can be separated into a hypertrees chain whose vertices are rooted hypertrees chains. An example is presented in Fig. 10.

### *3.3.3 Relations for $\mathcal{H}_k^{pa}$*

We have the following proposition:

**Proposition 3.16** *The species $\mathcal{H}_k^{pa}$ satisfies the functional equation*:

$$\mathcal{H}_k^{pa} = \left(\mathcal{H}_{k-1}^{p} - X\right) \circ \mathcal{H}_k^{p}. \tag{3.8}$$

*Proof* Forgetting the rootedness gives the decomposition of Proposition 3.14.

Rooting edge-pointed hypertrees chain is the same as pointing out a vertex in the hypertrees $k-1$-chain. This gives the result. □

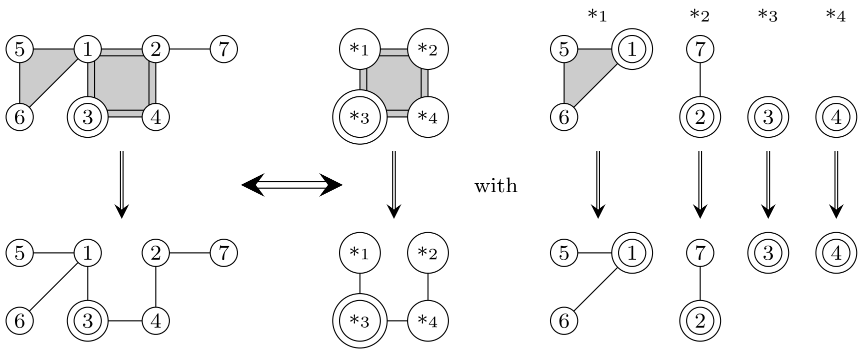



**Fig. 11** Decomposition of an edge-pointed rooted hypertrees chain

*Example 3.17* An edge-pointed rooted hypertrees chain can be seen as a rooted hypertrees chain, whose vertices are labeled by rooted hypertrees chains. An example is presented in Fig. 11.

#### 3.3.4 Relations for $\mathcal{H}_k$

Rootedness gives the following proposition:

**Proposition 3.18** *The species $\mathcal{H}_k$ satisfies*

$$X \times \mathcal{H}'_k = \mathcal{H}^p_k, \tag{3.9}$$

*where $'$ is the differentiation of species.*

### 3.4 Back to strict and large chains

The rootedness of a chain does not change the polynomial nature of the character, shown in Sect. 2.4. Consequently, generating series and cycle index series of $\mathcal{H}^p$ are polynomial in $k$.

Moreover, as the substitution of formal power series with polynomial coefficients is a formal power series with polynomial coefficients, generating series and cycle index series associated with $\mathcal{H}^a$, $\mathcal{H}^{pa}$, $\mathcal{H}^c$ and $\mathcal{H}^{cm}$ are polynomials in $k$.

Consequently, for all considered species, we can take the value of the cycle index series at $-1$ and this will give the character of the symmetric group on the homology associated with pointed hypertrees poset.

## 4 Dimension of the poset homology

Generating series associated with species $\mathcal{H}_k$, $\mathcal{H}^p_k$, $\mathcal{H}^a_k$, $\mathcal{H}^{pa}_k$, $\mathcal{H}^c_k$ and $\mathcal{H}^{cm}_k$ are denoted by $\mathcal{C}_k$, $\mathcal{C}^p_k$, $\mathcal{C}^a_k$, $\mathcal{C}^{pa}_k$, $\mathcal{C}^c_k$ and $\mathcal{C}^{cm}_k$. We compute them here.

## 4.1 Connections between generating series

The equalities between species of Sect. 3 give equalities in terms of generating series:

**Proposition 4.1** *The series $\mathcal{C}_k^p$ satisfies*

$$\mathcal{C}_k^p = x \times \exp\left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1\right). \tag{4.1}$$

*The series $\mathcal{C}_k^a$ satisfies*

$$\mathcal{C}_k^a = (\mathcal{C}_{k-1} - x)\left(\mathcal{C}_k^p\right). \tag{4.2}$$

*The series $\mathcal{C}_k^{pa}$ satisfies*

$$\mathcal{C}_k^{pa} = \left(\mathcal{C}_{k-1}^p - x\right)\left(\mathcal{C}_k^p\right). \tag{4.3}$$

*The series $\mathcal{C}_k$ satisfies*

$$x \times \mathcal{C}_k' = \mathcal{C}_k^p. \tag{4.4}$$

*Moreover, according to the dissymmetry principle of Proposition 3.7, these series also satisfy*

$$\mathcal{C}_k + \mathcal{C}_k^{pa} = \mathcal{C}_k^p + \mathcal{C}_k^a. \tag{4.5}$$

## 4.2 Values of the series for $k = 0$ and $k = -1$

### *4.2.1 Computation of $\mathcal{C}_0$ and $\mathcal{C}_0^p$*

There is only one hypertrees 0-chain: the empty chain. This gives

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1. \tag{4.6}$$

Relation (4.4) gives

$$\mathcal{C}_0^p = x e^x. \tag{4.7}$$

### *4.2.2 Computation of $\mathcal{C}_{-1}$*

Using Proposition 2.18, it is sufficient to study the value at $-1$ of the polynomial whose value at $k$ gives the number of large $k$-chains to obtain the dimension on the homology group. Therefore we study the value at $-1$ of the exponential generating series whose coefficients are these polynomials. The series $\mathcal{C}_{-1}$ is given by the following theorem. This result was first proved by McCammond and Meier in [10]. We give here another proof:

**Theorem 4.2** ([10, Theorem 5.1]) *The dimension of the only nontrivial homology group of the poset of hypertrees on n vertices is $(n-1)^{n-2}$.*

*Proof* According to (4.2) and (4.3), applied at $k=0$, the dissymmetry principle of Corollary 3.7 is

$$\begin{aligned}\mathcal{C}_0-\mathcal{C}_0^p&=\mathcal{C}_0^a-\mathcal{C}_0^{pa}\\&=(\mathcal{C}_{-1}-\mathcal{C}_{-1}^p)\circ\mathcal{C}_0^p.\end{aligned}$$

With (4.4), (4.6) and (4.7), this equality is equivalent to

$$\left(\mathcal{C}_{-1}-x\mathcal{C}'_{-1}\right)\circ xe^x=e^x-xe^x-1. \tag{4.8}$$

We define a new series:

**Definition 4.3** Let $\Sigma W$ be the series given by

$$\Sigma W(x)=\sum_{n\geq 1}(-1)^{n-1}n^{n-1}\frac{x^n}{n!}.$$

This series is the suspension of the generating series $W$ of rooted hypertrees species, associated with the PreLie operad. It satisfies the following equation, obtained from the decomposition of rooted trees (see [2, page 2] for instance):

$$\Sigma W(x)e^{\Sigma W(x)}=x.$$

We compute its differential:

$$(\Sigma W)'(x)=\frac{1}{x+e^{\Sigma W}}.$$

Composing (4.8) by $\Sigma W$, we get:

$$\mathcal{C}_{-1}-x\mathcal{C}'_{-1}=e^{\Sigma W}-x-1.$$

To conclude, we need the following lemma:

**Lemma 4.4** *Computing the term $e^{\Sigma W}-x-1$ gives*

$$e^{\Sigma W}-x-1=\sum_{n\geq 2}(-1)^{n-1}(n-1)^{n-1}\frac{x^n}{n!}.$$

*Proof of Lemma 4.4* Both parts of the equation vanish at 0.

On the one hand, differentiation gives

$$\left(e^{\Sigma W}-x-1\right)'=\Sigma W'e^{\Sigma W}-1=\frac{e^{\Sigma W}-x-e^{\Sigma W}}{x+e^{\Sigma W}}=-x\Sigma W'.$$

On the other hand, we get

$$\left(\sum_{n\geq 2}(-1)^{n-1}(n-1)^{n-1}\frac{x^n}{n!}\right)'=\sum_{n\geq 2}(-1)^{n-1}(n-1)^{n-1}\frac{x^{n-1}}{(n-1)!}.$$

It gives

$$\left(\sum_{n\geq 2}(-1)^{n-1}(n-1)^{n-1}\frac{x^n}{n!}\right)' = \sum_{n\geq 1}(-1)^n n^n \frac{x^n}{n!} = -x\Sigma W'.$$

The derivatives of these formal series are the same and they both vanish at 0, so they are equal. □

We conclude thanks to Lemma 4.4, by considering $\mathcal{C}_{-1} = \sum_{n\geq 1} a_n \frac{x^n}{n!}$. Thus coefficients $a_n$ satisfy, for all integers $n > 0$:

$$a_n - na_n = -(n-1)a_n = (-1)^{n-1}(n-1)^{n-1}.$$ □

**Corollary 4.5** *The derivative of series $\mathcal{C}_{-1}$ is given by*

$$(\mathcal{C}_{-1} - x)' = \Sigma W.$$

*Proof* We differentiate the expression of $\mathcal{C}_{-1}$ obtained in the previous theorem:

$$(\mathcal{C}_{-1} - x)' = \sum_{n\geq 2}(-1)^n (n-1)^{n-2}\frac{x^{n-1}}{(n-1)!}.$$

This gives the result. □

*4.2.3 Back to $\mathcal{C}_0^a$ and $\mathcal{C}_0^{pa}$*

The series $\mathcal{C}_0^a$ and $\mathcal{C}_0^{pa}$ are given by the following proposition.

**Proposition 4.6**

1. *The series $\mathcal{C}_0^a$ satisfies*

$$\mathcal{C}_0^a = \sum_{n\geq 2}(n-1)^2\frac{x^n}{n!}. \tag{4.9}$$

2. *The series $\mathcal{C}_0^{pa}$ satisfies*

$$\mathcal{C}_0^{pa} = \sum_{n\geq 2} n(n-1)\frac{x^n}{n!}. \tag{4.10}$$

*Proof* According to (4.2) and (4.7), $\mathcal{C}_0^a$ satisfies

$$\mathcal{C}_0^a(x) = (\mathcal{C}_{-1} - x)\circ \mathcal{C}_0^p(x) = (\mathcal{C}_{-1} - x)\circ xe^x.$$

Differentiating this equality and using Corollary 4.5, we get:

$$\left(\mathcal{C}_0^a\right)'(x) = \Sigma W \circ xe^x \times (x+1)e^x = x(x+1)e^x.$$

So it gives

$$\left(\mathcal{C}_0^a\right)'(x) = \sum_{n\geq 1} n^2 \frac{x^n}{n!}.$$

As $\mathcal{C}_0^a(0) = 0$, we obtain the first result.

According to (4.3) and (4.7), $\mathcal{C}_0^{pa}$ satisfies

$$\mathcal{C}_0^{pa}(x) = \left(\mathcal{C}_{-1}^p - x\right) \circ \mathcal{C}_0^p(x) = \left(x\left(\mathcal{C}_{-1}' - 1\right)\right) \circ xe^x.$$

With Corollary 4.5, we get:

$$\mathcal{C}_0^{pa}(x) = (x\Sigma W) \circ xe^x = x^2 e^x.$$

This gives the second result. □

## 5 Action of the symmetric group on the poset homology

The reader may consult Appendix B for basic definitions on cycle index series and Appendix A for definitions of usual species used in this section and the following.

### 5.1 Description of the action

Let us consider a hypertree poset on $n$ vertices, as described previously. The symmetric group $\mathfrak{S}_n$ acts on the set of vertices by permutation. This action preserves the number of edges and the poset order, so it induces an action on the homology associated with poset $\widehat{\mathrm{HT}_\mathrm{n}}$. We will determine in this section the character of this action on poset homology.

In the sequel, $\mathbf{C}_k$, $\mathbf{C}_k^p$, $\mathbf{C}_k^a$ and $\mathbf{C}_k^{pa}$ will stand for cycle index series associated with species $\mathcal{H}_k$, $\mathcal{H}_k^p$, $\mathcal{H}_k^a$ and $\mathcal{H}_k^{pa}$.

### 5.2 Connection between cycle index series

Relations between species of Sect. 3 give the following proposition:

**Proposition 5.1** *The series $\mathbf{C}_k$, $\mathbf{C}_k^p$, $\mathbf{C}_k^a$ and $\mathbf{C}_k^{pa}$ satisfy the following relations*:

$$\mathbf{C}_k + \mathbf{C}_k^{pa} = \mathbf{C}_k^p + \mathbf{C}_k^a, \tag{5.1}$$

$$\mathbf{C}_k^p = p_1 + p_1 \times \mathbf{C}_{\mathrm{Comm}} \circ \left(\frac{\mathbf{C}_{k-1}^p \circ \mathbf{C}_k^p - \mathbf{C}_k^p}{\mathbf{C}_k^p}\right), \tag{5.2}$$

$$\mathbf{C}_k^a + \mathbf{C}_k^p = \mathbf{C}_{k-1} \circ \mathbf{C}_k^p, \tag{5.3}$$

$$\mathbf{C}_k^{pa} + \mathbf{C}_k^p = \mathbf{C}_{k-1}^p \circ \mathbf{C}_k^p, \tag{5.4}$$

*and*

$$p_1 \frac{\partial \mathbf{C}_k}{\partial p_1} = \mathbf{C}_k^p. \tag{5.5}$$

These relations hold for $k$ in $\mathbb{Z}$. Indeed the coefficients of the $p_\lambda$ are polynomials in $k$, so we can extend the previous relations holding for $k$ in $\mathbb{N}$.

## 5.3 Computation of the symmetric group character

### *5.3.1 Computation of* $\mathbf{C}_{-1}$

Using Proposition 2.18, it is sufficient to study the value at $-1$ of the polynomial whose value at $k$ gives the character of the action of symmetric group on large $k$-chains to obtain the character on the homology group. Therefore we study the value at $-1$ of the cycle index series whose coefficients are these polynomials.

The PreLie operad is anti-cyclic as proven in the article of F. Chapoton [4]. It means that the usual action of the symmetric group $\mathfrak{S}_n$ on the module PreLie$(n)$, whose basis is the set of rooted trees on $n$ vertices, can be extended into an action of the symmetric group $\mathfrak{S}_{n+1}$. We write $M$ for the cycle index series associated with this anti-cyclic structure.

The reader may consult the article [5, Part 5.4] for more information on this series.

We will prove the following theorem, which describes the action of the symmetric group on the homology of the hypertree poset in terms of cycle index series associated with the Comm and PreLie operads:

**Theorem 5.2** *The cycle index series* $\mathbf{C}_{-1}$, *which gives the character of the action of the symmetric group on the homology of the hypertree poset, is related to the cycle index series* $M$ *associated with the anti-cyclic structure of the* PreLie *operad by*

$$\mathbf{C}_{-1} = p_1 - \Sigma M = \mathbf{C}_{\mathrm{Comm}} \circ \Sigma \mathbf{C}_{\mathrm{PreLie}} + p_1(\Sigma \mathbf{C}_{\mathrm{PreLie}} + 1). \tag{5.6}$$

*The cycle index series* $\mathbf{C}_{-1}^p$ *is given by*

$$\mathbf{C}_{-1}^p = p_1(\Sigma \mathbf{C}_{\mathrm{PreLie}} + 1). \tag{5.7}$$

*Proof* We first compute $\mathbf{C}_0$ and $\mathbf{C}_0^p$. There is only one 0-chain: the empty chain. It is fixed by every permutation. A quick computation gives

$$\mathbf{C}_0 = \mathbf{C}_{\mathrm{Comm}}.$$

We derive from (5.5):

$$\mathbf{C}_0^p = p_1 \frac{\partial \mathbf{C}_{\mathrm{Comm}}}{\partial p_1} = \mathbf{C}_{\mathrm{Perm}} = p_1(1 + \mathbf{C}_{\mathrm{Comm}}). \tag{5.8}$$

Equation (5.2) gives

$$\mathbf{C}_0^p = p_1 + p_1 \times \mathbf{C}_{\mathrm{Comm}} \circ \left(\frac{\mathbf{C}_{-1}^p \circ \mathbf{C}_0^p - \mathbf{C}_0^p}{\mathbf{C}_0^p}\right),$$

so the substitution of $\mathbf{C}_0^p$ by its expression gives

$$p_1 + p_1 \times \mathbf{C}_{\mathrm{Comm}} = p_1 + p_1 \times \mathbf{C}_{\mathrm{Comm}} \circ \left(\frac{\mathbf{C}_{-1}^p \circ \mathbf{C}_{\mathrm{Perm}} - \mathbf{C}_{\mathrm{Perm}}}{\mathbf{C}_{\mathrm{Perm}}}\right).$$

Recall that $\Sigma \mathbf{C}_{\mathrm{PreLie}} \circ \mathbf{C}_{\mathrm{Perm}} = \mathbf{C}_{\mathrm{Perm}} \circ \Sigma \mathbf{C}_{\mathrm{PreLie}} = p_1$, according to [5] .[1]
We obtain

$$\Sigma \mathbf{C}_{\mathrm{PreLie}} = \frac{\mathbf{C}_{-1}^p - p_1}{p_1},$$

hence the result:

$$\mathbf{C}_{-1}^p = p_1(\Sigma \mathbf{C}_{\mathrm{PreLie}} + 1). \tag{5.9}$$

The dissymmetry equation (5.1), combined with relations (5.3) and (5.4) in $k = 0$ gives

$$\mathbf{C}_{\mathrm{Comm}} + \mathbf{C}_{-1}^p \circ \mathbf{C}_{\mathrm{Perm}} - \mathbf{C}_{\mathrm{Perm}} = \mathbf{C}_{\mathrm{Perm}} + \mathbf{C}_{-1} \circ \mathbf{C}_{\mathrm{Perm}} - \mathbf{C}_{\mathrm{Perm}}.$$

Composing by $\Sigma \mathbf{C}_{\mathrm{PreLie}}$ and replacing $\mathbf{C}_{-1}^p$ by its expression in (5.7), we obtain

$$\mathbf{C}_{-1} = \mathbf{C}_{\mathrm{Comm}} \circ \Sigma \mathbf{C}_{\mathrm{PreLie}} + p_1(\Sigma \mathbf{C}_{\mathrm{PreLie}} + 1) - p_1.$$

As $(p_1(\mathbf{C}_{\mathrm{Comm}} + 1)) \circ \Sigma \mathbf{C}_{\mathrm{PreLie}} = p_1$,
we thus obtain

$$\mathbf{C}_{\mathrm{Comm}} \circ \Sigma \mathbf{C}_{\mathrm{PreLie}} = \frac{p_1 - \Sigma \mathbf{C}_{\mathrm{PreLie}}}{\Sigma \mathbf{C}_{\mathrm{PreLie}}}.$$

Therefore, the cycle index series $\mathbf{C}_{-1}$ satisfies

$$\mathbf{C}_{-1} = -1 + \frac{p_1}{\Sigma \mathbf{C}_{\mathrm{PreLie}}} + p_1 \times \Sigma \mathbf{C}_{\mathrm{PreLie}}. \tag{5.10}$$

According to [4, Eq. (50)], composing by the suspension, we get

$$\Sigma M - 1 = -p_1\left(-1 + \Sigma \mathbf{C}_{\mathrm{PreLie}} + \frac{1}{\Sigma \mathbf{C}_{\mathrm{PreLie}}}\right).$$

The result is obtained by using the following equality, obtained from (5.10):

$$(p_1 - \mathbf{C}_{-1}) - 1 = p_1 - \frac{p_1}{\Sigma \mathbf{C}_{\mathrm{PreLie}}} - p_1 \times \Sigma \mathbf{C}_{\mathrm{PreLie}}.$$ □

[1] This is a consequence of Koszul duality for operads.

### *5.3.2 Back to $\mathbf{C}_0^a$ and $\mathbf{C}_0^{pa}$*

In this part, we refine the results obtained at Proposition 4.6.

**Theorem 5.3** *Cycle index series associated with species of large* 0*-chains, whose minimum is an edge-pointed hypertree and species of large* 0*-chains, whose minimum is an edge-pointed rooted hypertree, satisfy*

$$\mathbf{C}_0^a = \mathbf{C}_{\mathrm{Comm}} + (p_1 - 1) \times \mathbf{C}_{\mathrm{Perm}}, \tag{5.11}$$

*and*

$$\mathbf{C}_0^{pa} = p_1 \mathbf{C}_{\mathrm{Perm}}. \tag{5.12}$$

*For a cycle index series* $\mathbf{C}$, *we write* $(\mathbf{C})_n$ *for the part of* $\mathbf{C}$ *corresponding to a representation of the symmetric group* $\mathfrak{S}_n$.

*Therefore, for all* $n \geq 2$, *writing* $S^{(n-1,1)}$ *for the irreducible representation of the symmetric group* $\mathfrak{S}_n$ *associated with the partition* $(n-1, 1)$ *of* $n$, *we obtain*

1. $(\mathbf{C}_0^a)_n$ *is the character of the representation* $S^{(n-1,1)} \otimes S^{(n-1,1)}$;
2. $(\mathbf{C}_0^{pa})_n$ *is the character of the representation* $S^{(n-1,1)} \otimes S^{(n-1,1)} \oplus S^{(n-1,1)}$.

*Proof* The equalities come from Relations (5.3) and (5.4), replacing $\mathbf{C}_0^p$ by its expression in (5.8), $\mathbf{C}_{-1}^p$ by its expression in (5.7) and $\mathbf{C}_{-1}$ by its expression in Theorem 5.2. We obtain

$$\left(\mathbf{C}_0^a\right)_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} + \sum_{\lambda \vdash n-2} p_1^2 \frac{p_\lambda}{z_\lambda} - \sum_{\lambda \vdash n-1} p_1 \frac{p_\lambda}{z_\lambda}. \tag{5.13}$$

Denote now by $f_\lambda$ the number of fixed points in a permutation of type $\lambda$.

The coefficient in front of $\frac{p_\lambda}{z_\lambda}$ in $(\mathbf{C}_0^a)_n$ is

$$1 - f_\lambda + f_\lambda(f_\lambda - 1) = (f_\lambda - 1)^2.$$

In the same way, we obtain

$$\left(\mathbf{C}_0^{pa}\right)_n = \sum_{\lambda \vdash n-2} p_1^2 \frac{p_\lambda}{z_\lambda}. \tag{5.14}$$

The coefficient in front of $\frac{p_\lambda}{z_\lambda}$ in $(\mathbf{C}_0^{pa})_n$ is

$$(f_\lambda - 1)^2 + f_\lambda - 1 = f_\lambda(f_\lambda - 1).$$

We conclude thanks to the following lemma:

**Lemma 5.4** *The character of the irreducible representation* $S^{(n-1,1)}$ *on the conjugacy class* $C_\sigma$ *is equal to* $p-1$, *where* $p$ *is the number of fixed points of every element in* $C_\sigma$.

Indeed, according to the previous lemma the character of the representation $S^{(n-1,1)} \otimes S^{(n-1,1)}$ on the conjugacy class $C_\sigma$ is equal to $(p-1)^2$, where $p$ is the number of fixed points of every element in $C_\sigma$. This gives the first relation.

The second one is obtained by computing the character of the representation $S^{(n-1,1)} \otimes S^{(n-1,1)} \oplus S^{(n-1,1)}$, equal to $f_\sigma(f_\sigma - 1)$ on a conjugacy class whose elements have $f_\sigma$ fixed points. □

*Proof of the lemma* The natural representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$ is the direct sum of the trivial representation and the representation $S^{(n-1,1)}$.

The character of this representation on a conjugacy class $C_\sigma$ is equal to the number of fixed points of every element of $C_\sigma$.

The character of the trivial representation is equal to 1. The result is obtained by difference. □

# 6 Action of the symmetric group on Whitney homology

## 6.1 Definition and properties of Whitney homology

The reader may consult the article [13] for definitions and properties of Whitney homology.

**Definition 6.1** The *Whitney homology* of a poset $\mathcal{P}$ with minimum $\hat{0}$ is the collection of spaces:

$$\mathrm{WH_i}(P) = \bigoplus_{x \in P} \tilde{H}_{i-2}\big([\hat{0}, x]\big), \quad i \geq 2. \tag{6.1}$$

**Theorem 6.2** [13] *If a poset $P$ is Cohen–Macaulay, its Whitney homology satisfies*

$$\mathrm{WH_i}(P) = \bigoplus_{x \in P_{i-1}} \tilde{H}_{i-2}\big([m, x]\big) \tag{6.2}$$

*where $P_{i-1} = \{x \in P | r(x) = i-1\}$ and $r(x)$ is the* rank *of $x$.*

As $\widehat{\mathrm{HT_n}}$ is Cohen–Macaulay, according to Theorem 2.12, Theorem 6.2 can be applied.

To compute the Whitney homology of $\widehat{\mathrm{HT_n}}$, we define a weight on large $k$-chains:

**Definition 6.3** The *weight* of a hypertrees chain $S$, denoted by $w(S)$, is

$$w(S) = \#\mathrm{edge}\big(\max(S)\big) - 1$$

where $\#\mathrm{edge}(\max(S))$ is the number of edges of the maximum in $S$.

Note that in $\widehat{\mathrm{HT_n}}$, the weight of a chain is equal to the rank of its maximum.

For $\mathcal{E}$ a species with cycle index series $\mathbf{C}$, we will denote by $\mathcal{E}_t$ the associated weighted species with cycle index series $\mathbf{C}_t$.

Thus, the species $\mathcal{H}_{k,t}$ is the species which associates to a set $A$ the set of all pairs of large hypertrees $k$-chain with the weight of its maximum. Therefore, we have

$$\mathbf{C}_{k,t} = \sum_{n\geq 1}\sum_{i\geq 0}\chi\left(\mathrm{HL}^{\mathrm{n}}_{\mathrm{k,i}}\right)t^i\frac{x^n}{n!},$$

where $\chi(\mathrm{HL}^{\mathrm{n}}_{\mathrm{k,i}})$ is the character given by the action of symmetric group $\mathfrak{S}_n$ on the space of large $k$-chains whose maximum have rank $i$.

We argue as we did in Sect. 2.4: our aim is to find polynomial relations in $k$ between large $k$-chains, and then evaluate them at $k=-1$. Therefore, we will obtain

$$\mathbf{C}_{-1,t} = -\sum_{n\geq 1}\sum_{i\geq 0}\mathrm{WH_i}(\widehat{\mathrm{HT_n}})t^i\frac{x^n}{n!}. \tag{6.3}$$

## 6.2 Connections between cycle index series

Relations between species of Sect. 3 give the following relations when we take the weight into account:

**Proposition 6.4** *Series $\mathbf{C}_{k,t}$, $\mathbf{C}^p_{k,t}$, $\mathbf{C}^a_{k,t}$ and $\mathbf{C}^{pa}_{k,t}$ satisfy the following relations*:

$$\mathbf{C}_{k,t} + \mathbf{C}^{pa}_{k,t} = \mathbf{C}^p_{k,t} + \mathbf{C}^a_{k,t}, \tag{6.4}$$

$$\mathbf{C}^p_k = \frac{p_1}{t}\times(1+\mathbf{C}_{\mathrm{Comm}})\circ\left(\frac{t\mathbf{C}^p_{k-1,t}-p_1}{p_1}\circ t\mathbf{C}^p_{k,t}\right), \tag{6.5}$$

$$\mathbf{C}^a_{k,t} = \left(\mathbf{C}_{k-1,t}-\frac{p_1}{t}\right)\circ\left(t\mathbf{C}^p_{k,t}\right), \tag{6.6}$$

$$\mathbf{C}^{pa}_{k,t} = \left(\mathbf{C}^p_{k-1,t}-\frac{p_1}{t}\right)\circ\left(t\mathbf{C}^p_{k,t}\right), \tag{6.7}$$

$$p_1\frac{\partial\mathbf{C}_{k,t}}{\partial p_1} = \mathbf{C}^p_{k,t}. \tag{6.8}$$

## 6.3 New pointed chains

We need two new kinds of pointed chain. Therefore, we will denote:

- by $\mathcal{H}^A_{k,t}$, the species associated with large weighted hypertrees $k$-chains, whose maximum is an edge-pointed hypertree, and by $\mathbf{C}^A_{k,t}$ the associated cycle index series.
- by $\mathcal{H}^{pA}_{k,t}$, the species associated with large weighted hypertrees $k$-chains whose maximum is an edge-pointed rooted hypertree and $\mathbf{C}^{pA}_{k,t}$ the associated cycle index series.

Note that, by definition, the species $\mathcal{H}^a_{1,t}$ coincides with the species $\mathcal{H}^A_{1,t}$ and that the species $\mathcal{H}^{pa}_{1,t}$ coincides with species $\mathcal{H}^{pA}_{1,t}$.

The previous species are related with the other pointed hypertrees species by the following theorem:

**Theorem 6.5** *The species* $\mathcal{H}^{A}_{k,t}$ *and* $\mathcal{H}^{pA}_{k,t}$ *satisfy*

$$\mathcal{H}^{A}_{k,t} = \mathcal{H}^{A}_{k-1,t} \circ \left(t\mathcal{H}^{p}_{k,t}\right), \tag{6.9}$$

$$\mathcal{H}^{pA}_{k,t} = \mathcal{H}^{pA}_{k-1,t} \circ \left(t\mathcal{H}^{p}_{k,t}\right), \tag{6.10}$$

$$\mathcal{H}_{k,t} + \mathcal{H}^{pA}_{k,t} = \mathcal{H}^{p}_{k,t} + \mathcal{H}^{A}_{k,t}. \tag{6.11}$$

*Proof* Pointing an edge in the maximum is the same as pointing an edge in the minimum and pointing an edge in the set of distinguished edges thus obtained in the maximum of the chain. Using the proof of Proposition 3.14 and the previous statement gives the first relation.

If we distinguish a vertex (root) in the chain, we obtain the second relation.

The third relation is obtained by the same reasoning as in Sect. 3.2 on the dissymmetry principle. □

This implies the following relations:

**Corollary 6.6** *Series* $\mathbf{C}^{A}_{k,t}$ *and* $\mathbf{C}^{pA}_{k,t}$ *satisfy*

$$\mathbf{C}^{A}_{k,t} = \mathbf{C}^{A}_{k-1,t} \circ \left(t\mathbf{C}^{p}_{k,t}\right), \tag{6.12}$$

$$\mathbf{C}^{pA}_{k,t} = \mathbf{C}^{pA}_{k-1,t} \circ \left(t\mathbf{C}^{p}_{k,t}\right). \tag{6.13}$$

### 6.4 The HAL series

We recall here the definitions of HAL series introduced in the article of F. Chapoton [5].

**Definition 6.7** The series HAL, $\mathrm{HAL}^{\mathrm{p}}$, $\mathrm{HAL}^{\mathrm{pA}}$ and $\mathrm{HAL}^{\mathrm{A}}$ are the series defined by the following functional equations:

$$\mathrm{HAL}^{\mathrm{pA}} = p_1\left(\frac{p_1}{1+tp_1} \circ \mathbf{C}_{\mathrm{Comm}} \circ \left(p_1 + (-t)\,\mathrm{HAL}^{\mathrm{pA}}\right)\right), \tag{6.14}$$

$$\mathrm{HAL}^{\mathrm{p}} = p_1\left(\Sigma_t \mathbf{C}_{\mathrm{Lie}} \circ \mathbf{C}_{\mathrm{Comm}} \circ \left(p_1 + (-t)\,\mathrm{HAL}^{\mathrm{pA}}\right)\right), \tag{6.15}$$

$$\mathrm{HAL}^{\mathrm{A}} = (\mathbf{C}_{\mathrm{Comm}} - p_1) \circ \left(p_1 + (-t)\,\mathrm{HAL}^{\mathrm{pA}}\right), \tag{6.16}$$

$$\mathrm{HAL} = \mathrm{HAL}^{\mathrm{p}} + \mathrm{HAL}^{\mathrm{A}} - \mathrm{HAL}^{\mathrm{pA}}. \tag{6.17}$$

We introduce the series $\Sigma W_t$, defined by

$$(t\mathbf{C}_{\mathrm{Perm}} - tp_1 + p_1) \circ \Sigma W_t = \Sigma W_t \circ (t\mathbf{C}_{\mathrm{Perm}} - tp_1 + p_1) = p_1.$$

**Proposition 6.8** *Series $\Sigma W_t$ satisfies*

$$\mathbf{C}_{\mathrm{Comm}} \circ \Sigma W_t = \frac{p_1 - \Sigma W_t}{t \Sigma W_t}. \tag{6.18}$$

*Proof* By definition, we have

$$(\mathbf{C}_{\mathrm{Perm}} - p_1) \circ \Sigma W_t = \frac{p_1 - \Sigma W_t}{t}.$$

However, $\mathbf{C}_{\mathrm{Perm}}$ satisfies $\mathbf{C}_{\mathrm{Perm}} = p_1(1 + \mathbf{C}_{\mathrm{Comm}})$, hence the result. □

The following theorem gives explicit expressions for HAL series in terms of $\Sigma W_t$.

**Theorem 6.9** *The series* $\mathrm{HAL}$, $\mathrm{HAL}^{\mathrm{p}}$, $\mathrm{HAL}^{\mathrm{pA}}$ *and* $\mathrm{HAL}^{\mathrm{A}}$ *satisfy*

$$\mathrm{HAL}^{\mathrm{pA}} = \frac{p_1 - \Sigma W_t}{t}, \tag{6.19}$$

$$\mathrm{HAL}^{\mathrm{A}} = (\mathbf{C}_{\mathrm{Comm}} - p_1) \circ \Sigma W_t, \tag{6.20}$$

$$\mathrm{HAL}^{\mathrm{p}} = \frac{p_1}{t}\left(\Sigma \mathbf{C}_{\mathrm{Lie}} \circ \frac{p_1 - \Sigma W_t}{\Sigma W_t}\right). \tag{6.21}$$

*where $\Sigma \mathbf{C}_{\mathrm{Lie}}$ is the series satisfying $\Sigma \mathbf{C}_{\mathrm{Lie}} \circ \mathbf{C}_{\mathrm{Comm}} = \mathbf{C}_{\mathrm{Comm}} \circ \Sigma \mathbf{C}_{\mathrm{Lie}} = p_1$.*

*Proof*

1. Applying (6.18), a computation gives

$$p_1\left(\frac{\mathbf{C}_{\mathrm{Comm}} \circ \Sigma W_t}{1 + t\mathbf{C}_{\mathrm{Comm}} \circ \Sigma W_t}\right) = p_1 \frac{p_1 - \Sigma W_t}{t \Sigma W_t + t p_1 - t \Sigma W_t},$$

   hence the relation:

$$\frac{p_1 - \Sigma W_t}{t} = p_1\left(\frac{p_1}{1 + t p_1} \circ \mathbf{C}_{\mathrm{Comm}} \circ \left(p_1 + (-t)\frac{p_1 - \Sigma W_t}{t}\right)\right).$$

   The series $\mathrm{HAL}^{\mathrm{pA}}$ and $\frac{p_1 - \Sigma W_t}{t}$ satisfy the same functional equation. Moreover if we know the first $n$ terms of a solution of this equation, the equation gives the $n+1$th one: there is a unique solution of this equation, such that the coefficient of $x^0$ vanishes. Therefore, $\mathrm{HAL}^{\mathrm{pA}}$ and $\frac{p_1 - \Sigma W_t}{t}$ are equal.
2. The second equality results from the first one and (6.16) because the series $\Sigma W_t$ satisfies

$$p_1 + (-t)\,\mathrm{HAL}^{\mathrm{pA}} = \Sigma W_t.$$

3. According to the first relation of the proposition, the series $\mathrm{HAL}^{\mathrm{p}}$ satisfies

$$\mathrm{HAL}^{\mathrm{p}} = p_1(\Sigma_t \mathbf{C}_{\mathrm{Lie}} \circ \mathbf{C}_{\mathrm{Comm}} \circ \Sigma W_t).$$

The equality $\Sigma_t \mathbf{C}_{\mathrm{Lie}} = \frac{1}{t} \Sigma \mathbf{C}_{\mathrm{Lie}} \circ (t p_1)$ implies

$$\mathrm{HAL}^{\mathrm{p}} = \frac{p_1}{t} (\Sigma \mathbf{C}_{\mathrm{Lie}} \circ t \mathbf{C}_{\mathrm{Comm}} \circ \Sigma W_t).$$

Applying (6.18), we get the result. □

### 6.5 Character computation

#### *6.5.1 Computation of series for $k = 0$*

We can compute the following series:

**Proposition 6.10**

1. *The series $\mathbf{C}_{0,t}$ can be expressed as*:

$$\mathbf{C}_{0,t} = \mathbf{C}_{\mathrm{Comm}} - p_1 + \frac{p_1}{t}. \tag{6.22}$$

2. *The series $\mathbf{C}^{p}_{0,t}$ can be expressed as*:

$$\mathbf{C}^{p}_{0,t} = \mathbf{C}_{\mathrm{Perm}} - p_1 + \frac{p_1}{t} = p_1 \mathbf{C}_{\mathrm{Comm}} + \frac{p_1}{t}. \tag{6.23}$$

   *The series $t\mathbf{C}^{p}_{0,t}$ is then the inverse of series $\Sigma W_t$ for substitution.*
3. *The series $\mathbf{C}^{A}_{0,t}$ can be expressed as*:

$$\mathbf{C}^{A}_{0,t} = \mathbf{C}_{\mathrm{Comm}} - p_1. \tag{6.24}$$

4. *The series $\mathbf{C}^{pA}_{0,t}$ can be expressed as*:

$$\mathbf{C}^{pA}_{0,t} = \mathbf{C}_{\mathrm{Perm}} - p_1 = p_1 \mathbf{C}_{\mathrm{Comm}}. \tag{6.25}$$

*Proof*

1. The only hypertrees chain fixed by the action of an element $\sigma$ of the symmetric group $\mathfrak{S}_n$ is the empty chain. Nevertheless, the weight of the empty chain is 1, except for $n = 1$, where it is equal to $\frac{1}{t}$. Therefore the series $\mathbf{C}_{0,t}$ only differs from $\mathbf{C}_{\mathrm{Comm}}$ for $n = 1$, hence the result.
2. As $p_1 \frac{\partial \mathbf{C}_{\mathrm{Comm}}}{\partial p_1} = \mathbf{C}_{\mathrm{Perm}}$, the result comes from Relation (6.8) with $k = 0$.
3. By definition, $\mathbf{C}^{A}_{1,t} = \mathbf{C}^{a}_{1,t}$, with Relations (6.12) and (6.6), the series $\mathbf{C}^{A}_{0,t}$ satisfies

$$\mathbf{C}^{A}_{0,t} = \mathbf{C}_{0,t} - \frac{p_1}{t} = \mathbf{C}_{\mathrm{Comm}} - p_1.$$

4. By definition, $\mathbf{C}^{pA}_{1,t} = \mathbf{C}^{pa}_{1,t}$, with Relations (6.13) and (6.7), the series $\mathbf{C}^{pA}_{0,t}$ satisfies

$$\mathbf{C}^{pA}_{0,t} = \mathbf{C}^{p}_{0,t} - \frac{p_1}{t} = \mathbf{C}_{\mathrm{Perm}} - p_1 = p_1 \mathbf{C}_{\mathrm{Comm}}.$$

□

### 6.5.2 *Computation of the series for $k=-1$*

The following theorem refines the computation of the characteristic polynomial in [5], proves the conjecture of [5, Conjecture 5.3] and links the action of the symmetric group on Whitney homology of the hypertree poset with the action of the symmetric group on a set of hypertrees decorated by the Lie operad.

**Theorem 6.11**

1. *The series* $\mathbf{C}^{pA}_{-1,t}$ *satisfies*

$$\mathbf{C}^{pA}_{-1,t} = \frac{p_1 - \Sigma W_t}{t} = \mathrm{HAL}^{\mathrm{pA}}. \tag{6.26}$$

2. *The series* $\mathbf{C}^{A}_{-1,t}$ *satisfies*

$$\mathbf{C}^{A}_{-1,t} = (\mathbf{C}_{\mathrm{Comm}} - p_1) \circ \Sigma W_t = \mathrm{HAL}^{\mathrm{A}}. \tag{6.27}$$

3. *The series* $\mathbf{C}^{p}_{-1,t}$ *satisfies*

$$\mathbf{C}^{p}_{-1,t} = \frac{p_1}{t}\left(1 + \Sigma \mathbf{C}_{\mathrm{Lie}} \circ \frac{p_1 - \Sigma W_t}{\Sigma W_t}\right) = \mathrm{HAL}^{\mathrm{p}} + \frac{p_1}{t}. \tag{6.28}$$

4. *The series* $\mathbf{C}_{-1,t}$ *satisfies*

$$\mathbf{C}_{-1,t} = \mathrm{HAL} + \frac{p_1}{t}. \tag{6.29}$$

*Proof* The right part of equalities is given by Theorem 6.9.

1. Relation (6.13) with $k=0$ gives, together with (6.23) and (6.25):

$$\mathbf{C}^{pA}_{-1,t} = (p_1 \mathbf{C}_{\mathrm{Comm}}) \circ \Sigma W_t.$$

   We then conclude thanks to (6.18).
2. Relation (6.13) with $k=0$ gives, together with (6.23) and (6.24):

$$\mathbf{C}^{A}_{-1,t} = (\mathbf{C}_{\mathrm{Comm}} - p_1) \circ \Sigma W_t,$$

   hence the result.
3. As $\Sigma W_t$ is the inverse of $t\mathbf{C}^{p}_{0,t}$, Relation (6.5) with $k=0$ gives

$$p_1 = \Sigma W_t \left(1 + \mathbf{C}_{\mathrm{Comm}} \circ \frac{t\mathbf{C}^{p}_{-1,t} - p_1}{p_1}\right)$$

   as $\Sigma \mathbf{C}_{\mathrm{Lie}} \circ \mathbf{C}_{\mathrm{Comm}} = p_1$ according to [5], we obtain

$$\Sigma \mathbf{C}_{\mathrm{Lie}} \circ \frac{p_1 - \Sigma W_t}{\Sigma W_t} = \frac{t\mathbf{C}^{p}_{-1,t} - p_1}{p_1}.$$

   We thus obtain the result.

4. This relation comes from previous relations associated with the dissymmetry principle. □

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## Appendix A: Reminder on species

We give in this part only a brief reminder on species. The reader will find more on this subject in the book [2].

**Definition 7.1** A *species* $F$ is a functor from the category of finite sets and bijections to the category of finite sets. To a finite set $I$, the species $F$ associates a finite set $F(I)$ independent of the nature of $I$.

*Example 7.2*

- The map which associates to a finite set $I$ the set of total orders on $I$ is a species, called the linear order species and denoted by $L$.
- The map which associates to a finite set $I$ the set $\{I\}$ is a species, called the set species and denoted by $E$.
- The map defined for all finite set $I$ by

$$I \mapsto \begin{cases} \{I\}, & \text{if } \#I = 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

  is a species, called the singleton species and denoted by $X$.
- The map defined for all finite set $I$ by

$$I \mapsto \begin{cases} \{I\}, & \text{if } \#I \geq 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

  is a species denoted by *Comm*, and called the species associated with the Comm operad.
- The map which associates to a finite set $I$ the set $I$ is a species, called the pointed set species and denoted by *Perm*. It is associated with the Perm operad.
- The map which associates to a finite set $I$ the set of labeled rooted trees with labels in $I$ is a species denoted by *PreLie*, associated with the PreLie operad.

To each species $F$, we can associate the following generating series:

$$C_F(x) = \sum_{n \geq 0} \#F\bigl(\{1, \dots, n\}\bigr) \frac{x^n}{n!}.$$

*Example 7.3* The generating series of species defined previously are:

- $C_L(x) = \frac{1}{1-x}$,
- $C_E(x) = \exp(x)$,

- $C_X(x) = x$,
- $C_{\mathrm{Comm}}(x) = \exp(x) - 1$.

The following operations can be defined on species:

**Definition 7.4** Let $F$ and $G$ be two species. We define the following operations on species:

- $F'(I) = F(I \sqcup \{\bullet\})$, (differentiation)
- $(F + G)(I) = F(I) \sqcup G(I)$, (addition)
- $(F \times G)(I) = F(I) \times G(I)$, (product)
- $(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, where $\mathcal{P}(I)$ runs on the set of partitions of $I$ (substitution).

We have the following property:

**Proposition 7.5** *Let $F$ and $G$ be two species. Their generating series satisfy*

- $C_{F'} = C'_F$,
- $C_{F+G} = C_F + C_G$,
- $C_{F \times G} = C_F \times C_G$,
- $C_{F \circ G} = C_F \circ C_G$.

## Appendix B: Reminder on cycle index series

Let $F$ be a species. We can associate a formal power series to it: its cycle index series. The reader can consult the book [2] for a reference on this subject. This formal power series is a symmetric function defined as follow:

**Definition 8.1** The *cycle index series* of a species $F$ is the formal power series in an infinite number of variables $\mathbf{p} = (p_1, p_2, p_3, \dots)$ defined by

$$\mathbf{C}_F(\mathbf{p}) = \sum_{n \geq 0} \frac{1}{n!} \biggl( \sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \dots \biggr),$$

where $F^\sigma$ stands for the set of $F$-structures fixed under the action of $\sigma$ and where $\sigma_i$ is the number of cycles of length $i$ in the decomposition of $\sigma$ into disjoint cycles.

We can define the following operations on cycle index series.

**Definition 8.2** The operations $+$ and $\times$ on cycle index series are the same as on formal series.

For $f = f(\mathbf{p})$ and $g = g(\mathbf{p})$, plethystic substitution $f \circ g$ is defined by

$$f \circ g(\mathbf{p}) = f\bigl(g(p_1, p_2, p_3, \dots), g(p_2, p_4, p_6, \dots), \dots, g(p_k, p_{2k}, p_{3k}, \dots), \dots\bigr).$$

It is left-linear.

This operations satisfy

**Proposition 8.3** *Let $F$ and $G$ be two species. Their cycle index series satisfy*

$$\mathbf{C}_{F+G} = \mathbf{C}_F + \mathbf{C}_G, \qquad \mathbf{C}_{F\times G} = \mathbf{C}_F \times \mathbf{C}_G,$$
$$\mathbf{C}_{F\circ G} = \mathbf{C}_F \circ \mathbf{C}_G, \qquad \mathbf{C}_{F'} = \frac{\partial \mathbf{C}_F}{\partial p_1}.$$

Moreover, we define the following operation:

**Definition 8.4** The *suspension* $\Sigma_t$ of a cycle index series $f(p_1, p_2, p_3, \dots)$ is defined by

$$\Sigma_t f = -\frac{1}{t} f\left(-tp_1, -t^2 p_2, -t^3 p_3, \dots\right).$$

By convention, we will write $\Sigma$ for the suspension in $t = 1$.

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