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Grassmann and Weyl embeddings of orthogonal grassmannians

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Abstract Given a non-singular quadratic form q of maximal Witt index on $V := V(2n+1,\mathbb{F})$, let Δ be the building of type B_n formed by the subspaces of V totally singular for q and, for $1 \leq k \leq n$, let Δ_k be the k-grassmannian of Δ. Let ε_k be the embedding of Δ_k into $\mathrm{PG}(\bigwedge^k V)$ mapping every point $\langle v_1, v_2, \ldots, v_k\rangle$ of Δ_k to the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(\bigwedge^k V)$. It is known that if $\mathrm{char}(\mathbb{F}) \neq 2$ then $\dim(\varepsilon_k) = \binom{2n+1}{k}$. In this paper we give a new very easy proof of this fact. We also prove that if $\mathrm{char}(\mathbb{F}) = 2$ then $\dim(\varepsilon_k) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$. As a consequence, when $1 < k < n$ and $\mathrm{char}(\mathbb{F}) = 2$ the embedding ε_k is not universal. Finally, we prove that if $\mathbb{F}$ is a perfect field of characteristic $p > 2$ or a number field, $n > k$ and $k = 2$ or 3, then ε_k is universal.

Keywords Orthogonal grassmannians · Weyl modules · Veronesean embeddings · Orthogonal groups

1 Introduction

1.1 Definitions and notation

Let $V := V(2n+1,\mathbb{F})$ for a field $\mathbb{F}$ and let q be a non-singular quadratic form of V of Witt index n. Let Δ be the building of type B_n where the elements of type $k = 1, 2, \ldots, n$ are the k-dimensional subspaces of V totally singular for q.

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Let $\mathcal{G}_k$ and Δ_k be the k-grassmannians of $\mathrm{PG}(V)$ and Δ respectively. We recall that $\mathcal{G}_k$ is a point-line geometry where the points are the k-dimensional subspaces of V and the lines are the sets

$$l_{X,Y} := \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\}$$

for subspaces X and Y of V with $\dim(X) = k-1$, $\dim(Y) = k+1$ and $X \subset Y$. The grassmannian Δ_k is a subgeometry of $\mathcal{G}_k$. The points of Δ_k are the k-subspaces of V that are totally singular for q. When $k < n$ the lines of Δ_k are the lines $l_{X,Y}$ of $\mathcal{G}_k$ with Y totally singular. When $k = n$ the lines of Δ_n are the sets

$$l_X := \{Z \mid X \subset Z \subset X^{\perp},\ \dim(Z) = n,\ Z \text{ totally singular}\}$$

where X is a totally singular $(n-1)$-subspace of V and $X^{\perp}$ is the subspace orthogonal to X with respect to q. Note that the points of l_X form a conic in the projective plane $\mathrm{PG}(X^{\perp}/X)$. The geometry Δ_n is often called the dual of Δ_1. The latter is the polar space associated to the building Δ.

Let $W_k := \bigwedge^k V$. The *natural* projective embedding $e_k : \mathcal{G}_k \to \mathrm{PG}(W_k)$ of $\mathcal{G}_k$ maps every k-subspace $\langle v_1, v_2, \ldots, v_k\rangle$ of V to the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k\rangle$ of $\mathrm{PG}(W_k)$. Let $\varepsilon_k := e_k|_{\Delta_k}$ be the restriction of e_k to Δ_k. When $k < n$ the mapping ε_k is a projective embedding of Δ_k into the subspace $\langle \varepsilon_k(\Delta_k)\rangle$ of $\mathrm{PG}(W_k)$ spanned by $\varepsilon_k(\Delta_k)$. We call ε_k the *Grassmann embedding* of Δ_k.

If $k = n$ then ε_n maps lines of Δ_n onto non-singular conics of $\mathrm{PG}(W_n)$. So, ε_n is not a projective embedding. Indeed a projective embedding of a point-line geometry Γ into the projective space $\mathrm{PG}(W)$ of a vector space W is an injective mapping ε from the point-set of Γ to the set of points of $\mathrm{PG}(W)$ such that ε maps every line of Γ surjectively onto a line of $\mathrm{PG}(W)$ and $\varepsilon(\Gamma)$ spans $\mathrm{PG}(W)$ (see [19], for instance). The dimension of W is taken as the (vector) dimension $\dim(\varepsilon)$ of ε. Borrowing a word from [22], we say that an injective mapping ε from the point set of Γ to the set of points of $\mathrm{PG}(W)$ is a *veronesean embedding* if it maps every line of Γ onto a non-singular conic of $\mathrm{PG}(W)$ and $\varepsilon(\Gamma)$ spans $\mathrm{PG}(W)$. (Of course, the underlying division ring of W is assumed to be commutative.) We put $\dim(\varepsilon) = \dim(W)$, as for projective embeddings. With this terminology, ε_n is a veronesean embedding of Δ_n. We call it the *Grassmann veronesean embedding* of Δ_n, also the *Grassmann embedding* of Δ_n, for short. According to the previous conventions, $\dim(\varepsilon_n) := \dim(\langle \varepsilon_n(\Delta_n)\rangle)$.

We recall that Δ_n admits a projective embedding, namely the spin embedding. We shall denote it by the symbol $\varepsilon_{\mathrm{spin}}$. The embedding $\varepsilon_{\mathrm{spin}}$ is hosted by the so-called spin module, namely the Weyl module $V(\omega_n)$ (see below). Note that $\dim(V(\omega_n)) = 2^n$. Hence $\dim(\varepsilon_{\mathrm{spin}}) = 2^n$.

Henceforth $G := \mathrm{SO}(2n+1, \mathbb{F})$ is the stabilizer of the form q in $\mathrm{SL}(V) = \mathrm{SL}(2n+1, \mathbb{F})$. The group G also acts on W_k, according to the following rule:

$$g(v_1 \wedge \cdots \wedge v_k) = g(v_1) \wedge \cdots \wedge g(v_k) \quad \text{for } g \in G \text{ and } v_1, \ldots, v_k \in V.$$

Note that $\mathrm{SO}(2n+1, \mathbb{F}) = \mathrm{PSO}(2n+1, \mathbb{F})$, namely G is the adjoint Chevalley group of type B_n defined over $\mathbb{F}$. The universal Chevalley group of type B_n is the spin group

$\widetilde{G} = \mathrm{Spin}(2n+1, \mathbb{F})$. It acts faithfully on $V(\omega_n)$. If $\mathrm{char}(\mathbb{F}) = 2$ then $\widetilde{G} = G$. On the other hand, if $\mathrm{char}(\mathbb{F}) \neq 2$ then $\widetilde{G} = 2^{\cdot}G$, a non-split central extension of G by a group of order two.

Finally, we fix some notation for Weyl modules. Let $\omega_1, \omega_2, \dots, \omega_n$ be the fundamental dominant weights for the root system of type B_n, numbered in the usual way (see the picture at the beginning of this introduction). For a positive integral combination λ of $\omega_1, \dots, \omega_n$, we denote by $V(\lambda)$ the Weyl module over $\mathbb{F}$ with λ as the highest weight. The group $\widetilde{G}$ acts on $V(\lambda)$. If its action is unfaithful then $\widetilde{G}$ induces G on $V(\lambda)$, namely $V(\lambda)$ is a G-module. On the other hand, if $\widetilde{G}$ acts faithfully on $V(\lambda)$ and $\widetilde{G} \neq G$ then $V(\lambda)$ is a $\widetilde{G}$-module but not a G-module. For instance, if $\mathrm{char}(\mathbb{F}) \neq 2$ then $V(\omega_n)$ is a $\widetilde{G}$-module but not a G-module. On the other hand, $V(\omega_1), V(\omega_2), \dots, V(\omega_{n-1})$ and $V(2\omega_n)$ are G-modules.

Throughout this paper $\lambda_k := \omega_k$ for $k = 1, 2, \dots, n-1$ and $\lambda_n = 2\omega_n$. Note that $\dim(V(\lambda_k)) = \binom{2n+1}{k}$, as one can see by applying the Weyl dimension formula (see [18, 24.3], for instance).

1.2 Dimensions

The G-module $\langle \varepsilon_k(\Delta_k) \rangle$ is a homomorphic image of $V(\lambda_k)$ (see [12]; also Blok [3, Sect. 9]). We say that $V(\lambda_k)$ *hosts* ε_k (also that ε_k *lives* in $V(\lambda_k)$) if $\langle \varepsilon_k(\Delta_k) \rangle \cong V(\lambda_k)$ (isomorphism of G-modules). Equivalently, $\dim(\varepsilon_k) = \binom{2n+1}{k}$, namely $\varepsilon_k(\Delta_k)$ spans W_k.

When $\mathrm{char}(\mathbb{F}) \neq 2$ the G-module $V(\lambda_k)$ is irreducible, hence $\langle \varepsilon_k(\Delta_k) \rangle = V(\lambda_k)$ (see [12]; also Blok [3, Sect. 9]). We state this fact as a theorem.

Theorem 1.1 *Let* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then* $\dim(\varepsilon_k) = \binom{2n+1}{k}$. *In other words,* $V(\lambda_k)$ *hosts* ε_k.

In Sect. 2 we shall give a different and very easy proof of Theorem 1.1, relying only on elementary properties of quadratic forms in odd characteristic, without asking the irreducibility of $V(\lambda_k)$ for help.

By contrast, when $\mathrm{char}(\mathbb{F}) = 2$ the following holds:

Theorem 1.2 *Let* $\mathrm{char}(\mathbb{F}) = 2$. *Then* $\dim(\varepsilon_k) = \binom{2n+1}{k} - \binom{2n+1}{k-2}$, *where, if* $k = 1$, *we adopt the convention that* $\binom{2n+1}{-1} = 0$.

We shall prove this theorem in Sect. 3.3, obtaining it as a consequence of a more detailed statement valid when $\mathbb{F}$ is perfect (see below, Theorem 1.3). Before stating the latter result we must recall a few facts on symplectic grassmannians and their natural embeddings.

Put $V' := V(2n, \mathbb{F})$, let Δ^{sp} be the building of type C_n associated to the symplectic group $\mathrm{Sp}(2n, \mathbb{F})$ in its natural action on V' and, for $k = 1, 2, \dots, n$, let Δ_k^{sp} be the k-grassmannian of Δ^{sp}. Then Δ_k^{sp} is a subgeometry of the k-grassmannian $\mathcal{G}'_k$ of $\mathrm{PG}(V')$. Put $W'_k := \bigwedge^k V'$ and let $e'_k : \mathcal{G}'_k \to \mathrm{PG}(W'_k)$ be the natural embedding of $\mathcal{G}'_k$, sending every totally isotropic subspace $\langle v_1, \dots, v_k \rangle$ of V' to the point $\langle v_1 \wedge \cdots \wedge v_k \rangle$ of $\mathrm{PG}(W'_k)$. Let $\varepsilon_k^{sp} := e'_k|_{\Delta_k^{sp}}$ be the restriction of e'_k to Δ_k^{sp}. Then ε_k^{sp} is a projective embedding of Δ_k^{sp}, called the *natural* or *Grassmann* embedding of Δ_k^{sp}. It is well known that $\dim(\varepsilon_k^{sp}) = \binom{2n}{k} - \binom{2n}{k-2}$.

Let now $\mathrm{char}(\mathbb{F}) = 2$. If moreover $\mathbb{F}$ is perfect then $\Delta \cong \Delta^{sp}$. Indeed, denoted by N_0 the nucleus of the quadratic form q, namely the radical of the bilinear form associated to q, the projection from V to $V/N_0 \cong V'$ induces an isomorphism from Δ_1 to Δ_1^{sp}, which can be regarded as an isomorphism from Δ to Δ^{sp} and immediately induces an isomorphism from Δ_k to Δ_k^{sp} for every $k > 1$. Thus, both embeddings ε_k and ε_k^{sp} can be considered for Δ_k.

Let $k > 1$. Given an element X of Δ of type $k-1$, let $\mathrm{St}(X)$ be its upper residue, formed by the elements of Δ of type $k, k+1, \ldots, n$ that contain X. We call $\mathrm{St}(X)$ the *star* of X. Clearly, $\mathrm{St}(X)$ is (the building of) an orthogonal polar space of rank $n-k+1$ defined in $X^\perp/X$. Still assuming that $\mathrm{char}(\mathbb{F}) = 2$, let n_X be the nucleus of a quadratic form associated to the polar space $\mathrm{St}(X)$. Then $n_X = N_X/X$ where $N_X = \langle X, N_0\rangle$. Clearly, N_X is a point of $\mathcal{G}_k$ and, since n_X belongs to $X^\perp/X$, which is spanned by the 1-dimensional subspaces Y/X for Y ranging in the set of points of $\mathrm{St}(X)$, the point $e_k(N_X)$ of $\mathrm{PG}(W_k)$ belongs to $\langle \varepsilon_k(\Delta_k)\rangle$. Put $\mathcal{N}_k := \langle e_k(N_X)\rangle_{X\in\Delta_{k-1}}$. Clearly, $\mathcal{N}_k$ is stabilized by G.

In Sect. 3 we shall prove that the mapping $\iota_{k-1} : \Delta_{k-1} \to \mathrm{PG}(\mathcal{N}_k)$ sending every point X of Δ_{k-1} to $e_k(N_X)$ is a projective embedding and that $\mathcal{N}_k$ defines a quotient $\varepsilon_k/\mathcal{N}_k$ of ε_k. More precisely, when $k < n$ or $k = n$ but $\mathbb{F}$ is perfect then $\varepsilon_k/\mathcal{N}_k$ is a projective embedding in the usual sense, mapping lines of Δ_k onto lines of $\mathrm{PG}(\langle\varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k)$. On the other hand, let $k = n$ and let $\mathbb{F}$ be non-perfect. Then $\varepsilon_n/\mathcal{N}_n$ is a lax embedding as defined in [24], namely the image of a line of Δ_n under $\varepsilon_n/\mathcal{N}_n$ is properly contained in a line of $\mathrm{PG}(\langle\varepsilon_n(\Delta_n)\rangle/\mathcal{N}_n)$. In Sect. 3.3 we will prove the following:

Theorem 1.3 *Let $\mathbb{F}$ be a perfect field of characteristic 2 and let $k > 1$.*

(1) $\iota_{k-1} \cong \varepsilon_{k-1}^{sp}$. *Consequently,* $\dim(\mathcal{N}_k) = \binom{2n}{k-1} - \binom{2n}{k-3}$.
(2) $\varepsilon_k/\mathcal{N}_k \cong \varepsilon_k^{sp}$, *whence* $\dim(\langle\varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k) = \binom{2n}{k} - \binom{2n}{k-2}$.

Conjecture 1 *The equalities* $\dim(\mathcal{N}_k) = \binom{2n}{k-1} - \binom{2n}{k-3}$ *and* $\dim(\varepsilon_k/\mathcal{N}_k) = \binom{2n}{k} - \binom{2n}{k-2}$ *also hold if $\mathbb{F}$ is non-perfect.*

Both claims of Conjecture 1 hold true when $n \le 4$ and $\mathbb{F}$ is any field of characteristic 2, as one can check by crude computations.

Conjecture 2 *Let* $\mathrm{char}(\mathbb{F}) = 2$ *and $k > 2$. Then the kernel of the projection of $V(\lambda_k)$ onto $\langle\varepsilon_k(\Delta_k)\rangle$ is isomorphic to $V(\lambda_{k-2})$.*

1.3 Results and conjectures on universality

Following Kasikova and Shult [19], we say that a projective embedding of a point-line geometry Γ is *relatively universal* when it is not a proper quotient of any larger embedding of Γ. A projective embedding ε of Γ is *absolutely universal* if all embeddings of Γ defined over the same division ring as ε are quotients of ε. If all projective embeddings of Γ are defined over the same division ring (as is the case for Δ_k) then the absolutely universal embedding of Γ, if it exists, is uniquely determined up to isomorphisms. Clearly, every absolutely universal projective embedding is relatively universal. If Γ admits the absolutely universal embedding then the converse also holds true: all relatively universal embeddings of Γ are absolutely universal. In this case we may simply speak of *universal* embeddings, dropping the words 'absolutely' or 'relatively'. We can do so when dealing with Δ_k. Indeed Δ_k admits the absolutely universal projective embedding (Kasikova and Shult [19]).

As remarked earlier, the G-module $\langle\varepsilon_k(\Delta_k)\rangle$ is a quotient of $V(\lambda_k)$. Indeed an embedding $\widetilde{\varepsilon}_k$ of Δ_k can be created in $\mathrm{PG}(V(\lambda_k))$. More explicitly, if v_0 is a highest weight vector of $V(\lambda_k)$, then the G-orbit of $\langle v_0\rangle$ corresponds to the set of points of Δ_k and, if P_k is the minimal fundamental parabolic subgroup of G of type k and L_0 is the P_k-orbit of $\langle v_0\rangle$, then the G-orbit of L_0 corresponds to the set of lines of Δ_k. The embedding $\widetilde{\varepsilon}_k$ is projective when $k<n$ and veronesean when $k=n$. The projection of $V(\lambda_k)$ onto $\langle\varepsilon_k(\Delta_k)\rangle$ is a morphism from $\widetilde{\varepsilon}_k$ to the Grassmann embedding ε_k. Following Blok [3], when $k<n$ we call $\widetilde{\varepsilon}_k$ the *Weyl embedding* of Δ_k. We call $\widetilde{\varepsilon}_n$ the *Weyl veronesean embedding* of Δ_n.

It is well known that ε_1 is universal (Tits [23, Chap. 8]), no matter what $\mathrm{char}(\mathbb{F})$ is. Hence $\widetilde{\varepsilon}_1=\varepsilon_1$, for any field $\mathbb{F}$. By Theorem 1.1, if $\mathrm{char}(\mathbb{F})\neq 2$ then $\widetilde{\varepsilon}_k=\varepsilon_k$.

Let $\mathrm{char}(\mathbb{F})=2$ and $k>1$. Then $\dim(\varepsilon_k)<\dim(\widetilde{\varepsilon}_k)$ by Theorem 1.2. In this case ε_k is a proper quotient of $\widetilde{\varepsilon}_k$. We state this fact as a corollary, but keeping aside the case $k=n$ for the moment, since ε_n is not a projective embedding. We will turn back to ε_n in a few lines.

Corollary 1.4 *Let* $\mathrm{char}(\mathbb{F})=2$ *and* $1<k<n$. *Then* ε_k *is not universal.*

On the other hand, the following is quite plausible.

Conjecture 3 *The Weyl embedding* $\widetilde{\varepsilon}_k$ *is universal for any* $k=2,3,\dots,n-1$ *and any field* $\mathbb{F}$.

The following theorem, to be proved in Sect. 4, is one of the reasons that make us believe that the previous conjecture holds true.

Theorem 1.5 *Let* $\mathbb{F}$ *be a perfect field of positive characteristic or a number field.*

(1) *If* $n>2$ *then the Weyl embedding* $\widetilde{\varepsilon}_2$ *is universal.*
(2) *Let* $n>3$ *and* $\mathbb{F}\neq\mathbb{F}_2$. *Then the Weyl embedding* $\widetilde{\varepsilon}_3$ *is universal.*

The same conclusion as in (1) of Theorem 1.5 has been obtained by Cooperstein [14], but under the stronger assumption that $\mathbb{F}$ is a finite field of prime order. In fact Cooperstein [14] proves that when $|\mathbb{F}|$ is a prime integer, Δ_2 can be generated by $\binom{2n+1}{2}$ points. The universality of $\widetilde{\varepsilon}_2$ follows from this fact.

We now turn to the veronesean embeddings ε_n and $\widetilde{\varepsilon}_n$. Relative universality can be defined for veronesean embeddings just in the same way as for projective embeddings. Let ε be a veronesean embedding of a point-line geometry Γ. The linear hull of ε can be defined in the same way as for projective embeddings (see [21], for instance) and it is characterized as an initial object in the full sub-category of the category of veronesean embeddings of Γ formed by those embeddings e' for which $\mathrm{Hom}(e', e) \neq \emptyset$ (see [21] for details). We say that e is *relatively universal* if it is its own linear hull. Thus, it makes sense to ask whether ε_n or $\widetilde{\varepsilon}_n$ are relatively universal or not. By Theorem 1.2 we immediately obtain the following:

Corollary 1.6 *If* $\mathrm{char}(\mathbb{F}) = 2$ *then* ε_n *is not relatively universal.*

Actually, when $\mathrm{char}(\mathbb{F}) = 2$ the Weyl veronesean embedding $\widetilde{\varepsilon}_n$ is not relatively universal either (see [11]), but perhaps $\widetilde{\varepsilon}_n$ is relatively universal when $\mathrm{char}(\mathbb{F}) \neq 2$.

We warn that now we are not allowed to jump from relative universality to absolute universality as we can do when dealing with projective embeddings of Δ_k. Indeed we do not know if Δ_n admits an absolutely universal veronesean embedding when $\mathbb{F} \neq \mathbb{F}_2$. (If $\mathbb{F} = \mathbb{F}_2$ then Δ_n admits an absolutely universal veronesean embedding, obtained by taking the point-set of Δ_n as a basis of an $\mathbb{F}_2$-vector space.)

Another important difference exists between veronesean and projective embeddings: the dimension of a projective embedding of a point-line geometry Γ cannot be larger than the minimal number of points needed to generate Γ while the dimension of a veronesean embedding of Γ can be far larger than that number. For instance, if $\mathrm{char}(\mathbb{F}) \neq 2$ then Δ_n can be generated by 2^n points (Blok and Brouwer [4], Cooperstein and Shult [15]), whence every projective embedding of Δ_n is at most 2^n-dimensional. Actually $\dim(\varepsilon_{\mathrm{spin}}) = 2^n$. Therefore $\varepsilon_{\mathrm{spin}}$ is universal when $\mathrm{char}(\mathbb{F}) \neq 2$. By contrast, $\dim(\widetilde{\varepsilon}_n) = \binom{2n+1}{n} > 2^n$. In fact, the usual notion of generation is unfit for veronesean embeddings. We will say more on this point in Sect. 4.4.

As recalled above, $\varepsilon_{\mathrm{spin}}$ is universal when $\mathrm{char}(\mathbb{F}) \neq 2$. On the other hand, let $\mathbb{F}$ be a perfect field of characteristic 2. Then $\varepsilon_{\mathrm{spin}}$ is a quotient of ε_n^{sp} (Blok, Cardinali and De Bruyn [5]; see also Cardinali and Lunardon [10]). In this case $V(\lambda_n) = V(2\omega_n)$ admits a chain of submodules $V(\lambda_n) \supset A \supset B \supset C \supset 0$ where $\dim(A) = \binom{2n+1}{n} - 2^n$, $\dim(B) = \binom{2n+1}{n-1}$ and $\dim(C) = \binom{2n+1}{n-2}$, C is the kernel of the projection of $\widetilde{\varepsilon}_n$ onto ε_n, $V(\lambda_n)/B$ hosts ε_n^{sp} (by Lemma 1.3) and $V(\lambda_n)/A \cong V(\omega_n)$ hosts $\varepsilon_{\mathrm{spin}}$. Moreover, $B/C = \mathcal{N}_n$ hosts ε_{n-1}^{sp} by Lemma 1.3 and A/B hosts a projective embedding of Δ_{n-2}^{sp} (see [5], also [10]).

1.4 Non-universality of ε_k^{sp} when $\mathrm{char}(\mathbb{F}) = 2$ and $k < n$

It is known that Δ_k^{sp} admits the absolutely universal projective embedding, for every $k = 1, 2, \ldots, n$ (Kasikova and Shult [19]). When $\mathrm{char}(\mathbb{F}) \neq 2$, the absolutely universal projective embedding of Δ_k^{sp} is just ε_k^{sp} [1, 2, 6, 13]. On the other hand, it is well known that ε_1^{sp} is not universal when $\mathrm{char}(\mathbb{F}) = 2$ (Tits [23, Chap. 8]; see also De Bruyn and Pasini [17] for the non-perfect case). If $\mathbb{F}$ is a perfect field of characteristic 2 and $1 < k < n$ then $\varepsilon_k^{sp} \cong \varepsilon_k/\mathcal{N}_k$ by (1) of Lemma 1.3. Therefore:

Corollary 1.7 *Let* $\mathbb{F}$ *be a perfect field of characteristic* 2 *and let* $k < n$. *Then the embedding* ε_k^{sp} *is not universal.*

In this corollary, the restriction $k < n$ is essential. Indeed the isomorphism $\varepsilon_n^{sp} \cong \varepsilon_n/\mathcal{N}_n$ gives no information on the linear hull of ε_n^{sp}, since ε_n is not a projective embedding. In fact, if $|\mathbb{F}| > 2$ then ε_n^{sp} is universal (Cooperstein [13] in the finite case, De Bruyn and Pasini [16] for the infinite case). On the other hand, if $\mathbb{F} = \mathbb{F}_2$ then ε_n^{sp} is not universal (Li [20], Blokhuis and Brouwer [9]).

2 An elementary proof of Theorem 1.1

Throughout this section $\mathrm{char}(\mathbb{F}) \neq 2$. We also assume that $k > 1$, since when $k = 1$ there is nothing to prove. Indeed ε_1 is the natural embedding of the polar space Δ_1 into $W_1 = V$. Obviously, $\langle \varepsilon_1(\Delta_1)\rangle = W_1$.

From now on we will often take the liberty of using the symbol Δ_k to denote both the point-line geometry Δ_k and its point-set. However these little abuses will be harmless. The context will always help to avoid any confusion.

For $h = 0, 1, \dots, k$ let $\mathcal{G}_k^{(h)}$ be the set of k-subspaces X of V such that $\mathrm{cod}_X(X \cap X^\perp) \leq h$, where $\perp$ is the orthogonality relation defined by the bilinear form f_q associated to q. So,

$$\Delta_k = \mathcal{G}_k^{(0)} \subset \mathcal{G}_k^{(1)} \subset \cdots \subset \mathcal{G}_k^{(k-1)} \subset \mathcal{G}_k^{(k)} = \mathcal{G}_k.$$

Lemma 2.1 *For every $h = 1, \dots, k$, if $X \in \mathcal{G}_k^{(h)}$ then there exists a line l of $\mathcal{G}_k$ through X such that $|l \cap \mathcal{G}_k^{(h-1)}| \geq 2$.*

Proof Assume firstly that $h = k$ and let $X \in \mathcal{G}_k^{(k)} \setminus \mathcal{G}_k^{(k-1)}$, namely X is a k-subspace of V such that $X \cap X^\perp = 0$. Then q induces a non-singular quadratic form on X. Hence X contains at least one $(k-1)$-subspace Z such that q induces a non-singular form on Z. Consequently $Z \cap Z^\perp = 0$, because $\mathrm{char}(\mathbb{F}) \neq 2$. Therefore $V = Z \oplus Z^\perp$ and q induces a non-singular quadratic form q' on $Z^\perp$. Clearly, $\dim(Z^\perp) = (2n+1) - (k-1) = 2n+2-k$. By this fact and the well known Grassmann formula for dimensions of sums and intersections of subspaces one easily sees that every n-subspace of V meets $Z^\perp$ non-trivially. In particular, every maximal totally singular subspace of V has non-trivial intersection with $Z^\perp$. It follows that $Z^\perp$ contains at least one singular point of $\mathrm{PG}(V)$. On the other hand, q' is non-singular. It is also trace-valued, because $\mathrm{char}(\mathbb{F}) \neq 2$. Hence $Z^\perp$ is spanned by the singular points contained in it (compare Tits [23, Lemma 8.1.6]).

Clearly, $\dim(Z^\perp \cap X) = 1$. Let x be a non-zero vector in $Z^\perp \cap X$. Suppose that every singular point of $Z^\perp$ is orthogonal to x. Then $Z^\perp \subseteq X^\perp$ because $Z^\perp$ is spanned by its singular points and $X = \langle x, Z\rangle$. This forces $X \subseteq Z$, contrary to the choice of Z. It follows that $x \not\perp x_1$ for at least one singular point $\langle x_1\rangle$ of $\mathrm{PG}(Z^\perp)$. The non-degenerate projective line $\langle x, x_1\rangle$ of $\mathrm{PG}(Z^\perp)$ contains one more singular point $\langle x_2\rangle$ of $\mathrm{PG}(Z^\perp)$. Let $X_i := \langle Z, x_i\rangle$, $i = 1, 2$. Then $X_i \cap X_i^\perp = \langle x_i\rangle$. Therefore $X_i \in \mathcal{G}_k^{(k-1)}$. Moreover, X, X_1 and X_2 contain Z and are contained in the $(k+1)$-space $Y := \langle X, x_1\rangle = \langle X, x_2\rangle$. The line $l_{Z,Y}$ of $\mathcal{G}_k$ has the required properties: it contains X and two points of $\mathcal{G}_k^{(k-1)}$, namely X_1 and X_2.

Let now $h < k$. Put $R = X \cap X^{\perp}$, $X_R := X/R$ and $V_R := R^{\perp}/R$. Then $\dim(V_R) = 2n + 1 - 2(k - h)$ and q induces a non-singular quadratic form q_R on V_R, with maximal Witt index $n - k + h$. (We warn that the hypothesis that $\mathrm{char}(\mathbb{F}) \neq 2$ is implicitly used in this reduction.) We can now argue as in the previous case, replacing X with X_R, V with V_R and q with q_R. We leave the details for the reader. □

We recall that a set S of points of a point-line geometry Γ is a *subspace* of Γ if S contains every line l of Γ such that $|l \cap S| \geq 2$. Intersections of subspaces are still subspaces. So, given a set of points S of Γ we can consider the *span* $\langle S \rangle_{\Gamma}$ of S in Γ, namely the smallest subspace of Γ containing S, defined as the intersection of all subspaces containing S. We say that a set S of points of Γ *generates* Γ if $\langle S \rangle_{\Gamma} = \Gamma$.

Proposition 2.2 *The point-set of Δ_k generates $\mathcal{G}_k$.*

Proof By Lemma 2.1, every point of $\mathcal{G}_k^{(h)}$ belongs to at least one line meeting $\mathcal{G}_k^{(h-1)}$ in two distinct points. Hence $\mathcal{G}_k^{(h)} \subseteq \langle \mathcal{G}_k^{(h-1)} \rangle_{\mathcal{G}_k}$. So, $\mathcal{G}_k = \langle \mathcal{G}_k^{(0)} \rangle_{\mathcal{G}_k}$, namely Δ_k spans $\mathcal{G}_k$. □

By Proposition 2.2, $\langle \varepsilon_k(\Delta_k) \rangle = W_k$. Equivalently, $\dim(\varepsilon_k) = \dim(W_k) = \binom{2n+1}{k}$. This forces $V(\lambda_k) = W_k$, as claimed in Theorem 1.1.

3 A quotient of $\boldsymbol{\varepsilon_k}$ when $\mathbf{char}(\mathbb{F}) = 2$

Throughout this section, $\mathrm{char}(\mathbb{F}) = 2$ and $k > 1$. Up to rescaling the form q when $\mathbb{F}$ is non-perfect, we can assume to have chosen an ordered basis $B = (e_1, e_2, \ldots, e_{2n+1})$ of V with respect to which

$$q(x_1, \ldots, x_{2n+1}) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2.$$

We set $I := \{1, 2, \ldots, 2n + 1\}$ and $B_{\wedge} := (e_J)_{J \in \binom{I}{k}}$, where $\binom{I}{k}$ stands for the set of subsets of I of size k and $e_J = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}$ for every k-subset $J = \{j_1, \ldots, j_k\}$ of I, with the convention that $j_1 < j_2 < \cdots < j_k$.

The radical of the bilinear form associated to q is a 1-dimensional subspace N_0 of V. It is called the nucleus of the quadric $Q(2n, \mathbb{F})$ of $\mathrm{PG}(V)$ represented by the equation $q(x_1, \ldots, x_{2n+1}) = 0$, also the *nucleus* of q, for short. With B as above, N_0 is spanned by the vector $n_0 = (0, 0, \ldots, 0, 1)$.

As in Sect. 2, in the sequel we will freely use symbols as Δ_k and Δ_{k-1} to denote both point-line geometries and their point-sets. In order to avoid duplication of notation, we will also often use the same symbols for vector subspaces and the corresponding projective subspaces. Every time the context will make it clear if we are considering vector or projective spaces.

3.1 The subspace $\mathcal{N}_k$ and the embedding ι_{k-1}

Given a point X of Δ_{k-1} let $\mathrm{St}(X)$ be its star, defined as in the introduction of this paper. As noticed there, $\mathrm{St}(X)$ is isomorphic to an orthogonal polar space of rank $n-k+1$, naturally embedded in $X^\perp/X$. Let q_X be a quadratic form of $X^\perp/X$ associated to that polar space and let n_X be its nucleus. Then $n_X = N_X/X$ for a uniquely determined k-subspace N_X of V containing X and contained in $X^\perp$. On the other hand, $N_0 \subset X^\perp$ and $N_0 \cap X = 0$. Hence $\langle X, N_0\rangle$ is a k-subspace of $X^\perp$ containing X. Moreover, $\langle X, N_0\rangle \subseteq X^{\perp\perp}$. Therefore $n_X = \langle X, N_0\rangle/X$, namely

$$N_X = \langle X, N_0\rangle.$$

We warn that N_X is totally isotropic but it is not totally singular. Hence N_X is a point of $\mathcal{G}_k$ (actually it belongs to $\mathcal{G}_k^{(1)}$) but it is not a point of Δ_k. Put $\mathcal{N}_k := \langle e_k(N_X)\rangle_{X\in\Delta_{k-1}}$. We call $\mathcal{N}_k$ the *global nucleus* of ε_k.

Lemma 3.1 $\mathcal{N}_k \subseteq \langle\varepsilon_k(\Delta_k)\rangle$.

Proof The mapping sending every $Y \in \Delta_k \cap \mathrm{St}(X)$ to Y/X is isomorphic to the natural embedding of the polar space $\mathrm{St}(X)$. Hence the vector space $X^\perp/X$ is spanned by the 1-dimensional subspaces Y/X for $Y \in \Delta_k \cap \mathrm{St}(X)$. Consequently, $e_k(N_X) \in \langle\varepsilon_k(Y)\rangle_{Y\in\Delta_k\cap\mathrm{St}(X)} \subseteq \langle\varepsilon_k(\Delta_k)\rangle$. Therefore $\mathcal{N}_k \subseteq \langle\varepsilon_k(\Delta_k)\rangle$. □

For every $X \in \Delta_{k-1}$, put $\iota_{k-1}(X) := e_k(N_X)$.

Lemma 3.2 *The mapping ι_{k-1} is a projective embedding of Δ_{k-1} into* $\mathrm{PG}(\mathcal{N}_k)$.

Proof Let $\nu_{k-1} : \Delta_{k-1} \to \mathcal{G}_k$ be the mapping sending every point X of Δ_{k-1} to the point $N_X = \langle X, N_0\rangle$ of $\mathcal{G}_k$. Then $\iota_{k-1} = e_k \circ \nu_{k-1}$. It is easily seen that the mapping ν_{k-1} is an embedding of Δ_{k-1} into a subgeometry of $\mathcal{G}_k$, namely it is injective and it maps lines of Δ_{k-1} onto lines of $\mathcal{G}_k$. On the other hand e_k, being a projective embedding, is injective and maps lines of $\mathcal{G}_k$ onto lines of $\mathrm{PG}(W_k)$. Therefore ι_{k-1} is injective and maps lines of Δ_{k-1} onto lines of $\mathrm{PG}(W_k)$ (contained in $\mathrm{PG}(\mathcal{N}_k)$). □

We shall now give an explicit description of ι_{k-1}. In the sequel we regard a vector of V as the same thing as its sequence of coordinates with respect to the basis B. Coordinates in W_k are given with respect to the standard basis $B_\wedge$ of W_k, defined at the beginning of Sect. 3.

For $X \in \Delta_{k-1}$, let $\{x_1, \dots, x_{k-1}\}$ be a basis of the $(k-1)$-subspace X. Let $M_X = (x_1, \dots, x_{k-1})$ be the $[(k-1)\times(2n+1)]$-matrix with $x_1, \dots, x_{k-1}$ as the rows and let $M'_X = (x_1, \dots, x_{k-1}, n_0)$ be the $[k\times(2n+1)]$-matrix obtained by adding n_0 to M_X as a further row. Let $(X_J)_{J\in\binom{I}{k}}$ be the sequence of coordinates of a representative vector v_X of $\iota_{k-1}(X) = e_k(N_X)$. Since $N_X = \langle X, N_0\rangle = \langle x_1, \dots, x_{k-1}, n_0\rangle$, we can assume to have chosen v_X in such a way that X_J is the determinant of the $(k\times k)$-submatrix of M'_X formed by the columns indexed by the elements of J. Recall that $n_0 = (0, 0, \dots, 1)$. Hence $X_J = 0$ whenever $2n+1 \notin J$ while if $2n+1 \in J$ then X_J is the determinant of the $[(k-1)\times(k-1)]$-submatrix of M_X formed by the columns indexed by elements of $J \setminus \{2n+1\}$. So, regarding $N_X/N_0 = \langle X, N_0\rangle/N_0$ as a point of $\mathcal{G}_{k-1}$ and $(X_J)_{2n+1\in J\in\binom{I}{k}}$ as a vector of W_{k-1}, we have $(X_J)_{2n+1\in J\in\binom{I}{k}} = e_{k-1}(N_X/N_0)$. Suppose to have ordered the set $\binom{I}{k}$ in such a way that the k-subsets containing $2n+1$ come as last. Then we can rephrase the above as follows:

Lemma 3.3 *The last $\binom{2n}{k-1}$ coordinates of $\iota_{k-1}(X)$ are the same as the coordinates of $e_{k-1}(N_X/N_0)$. The remaining coordinates of $\iota_{k-1}(X)$ are null.*

Proposition 3.4 *Let $\mathbb{F}$ be perfect. Then $\iota_{k-1} \cong \varepsilon^{sp}_{k-1}$.*

Proof When $\mathbb{F}$ is perfect the mapping sending every totally singular subspace X of V to $\langle X, N_0\rangle/N_0$ is an isomorphism from Δ to Δ^{sp}. The isomorphism $\iota_{k-1} \cong \varepsilon^{sp}_{k-1}$ immediately follows from this remark and Lemma 3.3. □

3.2 The quotient $\varepsilon_k/\mathcal{N}_k$

By Lemma 3.1 we know that $\mathcal{N}_k \subseteq \langle \varepsilon_k(\Delta_k)\rangle$. In this subsection we shall prove that $\mathcal{N}_k$ satisfies both the following:

(Q1) $\mathrm{PG}(\mathcal{N}_k) \cap \varepsilon_k(\Delta_k) = \emptyset$;
(Q2) $\langle \varepsilon_k(X_1), \varepsilon_k(X_2)\rangle \cap \mathcal{N}_k = 0$ for any two distinct points X_1 and X_2 of Δ_k.

Properties (Q1) and (Q2) allow us to define the quotient $\varepsilon_k/\mathcal{N}_k$ as the composition of ε_k with the canonical projection of $\langle \varepsilon_k(\Delta_k)\rangle$ onto $\langle \varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k$. In view of (Q1), this composition is a mapping from the point-set of Δ_k to the set of points of the projective space $\mathrm{PG}(\langle \varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k)$ of $\langle \varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k$, sending $X \in \Delta_k$ to the point $\langle \varepsilon_k(X), \mathcal{N}_k\rangle/\mathcal{N}_k$. By (Q2), this mapping is injective.

When $k < n$ the mapping $\varepsilon_k/\mathcal{N}_k$ maps every line of Δ_k bijectively onto a line of $\mathrm{PG}(\langle \varepsilon_k(\Delta_k)\rangle/\mathcal{N}_k)$. Hence it is a projective embedding. As we shall see at the end of this subsection, when $\mathbb{F}$ is perfect $\varepsilon_n/\mathcal{N}_n$ is a projective embedding in the usual sense, mapping every line of Δ_n bijectively onto a line of $\mathrm{PG}(\langle \varepsilon_n(\Delta_n)\rangle/\mathcal{N}_n)$. If $\mathbb{F}$ is non-perfect then $\varepsilon_n/\mathcal{N}_n$ maps every line l of Δ_n into a line $\bar{l}$ of $\mathrm{PG}(\langle \varepsilon_n(\Delta_n)\rangle/\mathcal{N}_n)$, but not all points of $\bar{l}$ are images of points of l by $\varepsilon_n/\mathcal{N}_n$.

In view of the above, it is convenient to slightly revise our terminology. From now on we say that a projective embedding $\varepsilon : \Gamma \to \mathrm{PG}(W)$ as defined in Sect. 1.1 is a *full* projective embedding. On the other hand, following [24], if a mapping ε maps the lines of Γ injectively but possibly non-surjectively into lines of $\mathrm{PG}(W)$ then we say that ε is a *lax* projective embedding. So, we can rephrase as follows what we have said above: if $k < n$ then $\varepsilon_k/\mathcal{N}_k$ is full while $\varepsilon_n/\mathcal{N}_n$ is full when $\mathbb{F}$ is perfect and it is lax but not full when $\mathbb{F}$ is non-perfect.

Lemma 3.5 *Condition* (Q1) *holds.*

Proof By way of contradiction, let $\varepsilon_k(X) \in \mathrm{PG}(\mathcal{N}_k)$ for some $X \in \Delta_k$. The group $G = \mathrm{SO}(2n+1, \mathbb{F})$ acts transitively on Δ_k and stabilizes $\mathcal{N}_k$. Hence $\varepsilon_k(\Delta_k) \subseteq \mathrm{PG}(\mathcal{N}_k)$. This is a contradiction because every vector $(X_J)_{J\in\binom{I}{k}} \in \mathcal{N}_k$ has $X_J = 0$ whenever $2n+1 \notin J$, but only some of the vectors of $\varepsilon_k(\Delta_k)$ have this property. □

Lemma 3.6 *Let $k < n$. Then* (Q2) *holds.*

Proof By way of contradiction, let $\langle \varepsilon_k(X_1), \varepsilon_k(X_2)\rangle \cap \mathcal{N}_k \neq 0$ for two distinct points $X_1, X_2 \in \Delta_k$. As (Q1) holds, $\langle \varepsilon_k(X_1), \varepsilon_k(X_2)\rangle \cap \mathcal{N}_k$ does not contain any point of $\varepsilon_k(\Delta_k)$. Hence $\langle \varepsilon_k(X_1), \varepsilon_k(X_2)\rangle \cap \mathcal{N}_k$ is a point n_{X_1,X_2} of $\mathrm{PG}(W_k) \setminus \varepsilon_k(\Delta_k)$. Since ε_k is full, the points X_1 and X_2 cannot be collinear in Δ_k. Let $d = d(X_1, X_2)$ be the distance between X_1 and X_2 in the collinearity graph of Δ_k. We have $d > 1$, since X_1 and X_2 are non-collinear.

The group G acts transitively on the pairs of points of Δ_k at distance d and stabilizes $\mathcal{N}_k$. Hence $\langle \varepsilon_k(X), \varepsilon_k(Y)\rangle$ meets $\mathcal{N}_k$ in a point $n_{X,Y} \in \mathrm{PG}(W_k) \setminus \varepsilon_k(\Delta_k)$, for every pair of points $X, Y \in \Delta_k$ at distance d.

For any two collinear points Y_1, Y_2 of Δ_k we can pick a point X at distance d from both Y_1 and Y_2. Clearly, the point $\varepsilon_k(X)$ does not belong to the projective line $\langle \varepsilon_k(Y_1), \varepsilon_k(Y_2)\rangle$. Consequently, $n_{X,Y_1} \neq n_{X,Y_2}$ and the points $\varepsilon_k(X), \varepsilon_k(Y_1)$ and $\varepsilon_k(Y_2)$ span a projective plane which contains both of the lines $\langle \varepsilon_k(Y_1), \varepsilon_k(Y_2)\rangle$ and $\langle n_{X,Y_1}, n_{X,Y_2}\rangle$. These two lines, being coplanar, meet in a point, say z. On the one hand $z \in \varepsilon_k(\Delta_k)$, as $\langle \varepsilon_k(Y_1), \varepsilon_k(Y_2)\rangle \subset \varepsilon_k(\Delta_k)$ (recall that Y_1 and Y_2 are collinear in Δ_k). On the other hand, $z \in \mathrm{PG}(\mathcal{N}_k)$, since $\langle n_{X,Y_1}, n_{X,Y_2}\rangle \subseteq \mathcal{N}_k$. Hence $z \in \varepsilon_k(\Delta_k) \cap \mathrm{PG}(\mathcal{N}_k)$, contrary to (Q1). We have reached a final contradiction. □

We now turn to the case $k = n$. Recall that the lines of Δ_n are the stars of the elements of Δ of type $n-1$. For every $(n-1)$-element X of Δ, the image $\varepsilon_n(\mathrm{St}(X))$ of $\mathrm{St}(X)$ by ε_n is a conic C_X of $\mathrm{PG}(W_k)$, spanning a plane π_X of $\mathrm{PG}(W_n)$. Moreover, $\pi_X \cap \varepsilon_n(\Delta_n) = C_X$. The nucleus of C_X is the point $\nu_X := e_n(N_X)$. (Recall that $N_X = \langle X, N_0\rangle$.)

Lemma 3.7 *We have $\pi_X \cap \mathcal{N}_n = \nu_X$, for every $(n-1)$-element X of Δ.*

Proof Clearly, $\nu_X \in \pi_X \cap \mathcal{N}_n$. By way of contradiction, suppose that $\pi_X \cap \mathcal{N}_n$ is larger than ν_X. Since $C_X \cap \mathrm{PG}(\mathcal{N}_n) = \emptyset$ by (Q1), $\pi_X \cap \mathcal{N}_n$ is a projective line through ν_X. If $\mathbb{F}$ is perfect then the line $\pi_X \cap \mathcal{N}_n$ is tangent to C_X, namely it meets C_X in one point. This is impossible, since $C_X \cap \mathrm{PG}(\mathcal{N}_n) = \emptyset$.

Therefore $\mathbb{F}$ is non-perfect. Let $\widehat{\mathbb{F}}$ be the quadratic closure of $\mathbb{F}$. Put $\widehat{V} := \widehat{\mathbb{F}} \otimes_{\mathbb{F}} V$ (where vectors are linear combinations of the vectors of B with coefficients taken from $\widehat{\mathbb{F}}$) and $\widehat{W}_n := \bigwedge^n \widehat{V}$ $(= \widehat{\mathbb{F}} \otimes_{\mathbb{F}} W_n)$. The form q naturally extends to a non-singular quadratic form $\hat{q}$ of $\widehat{V}$, admitting the same expression as q with respect to B. Denoted by $\widehat{\Delta}$ the building of type B_n associated to $\hat{q}$, every element X of Δ is the intersection $X = V \cap \widehat{X}$ of V with a uniquely determined element $\widehat{X}$ of $\widehat{\Delta}$, of the same type as X (in fact $\widehat{X} = \widehat{\mathbb{F}} \otimes_{\mathbb{F}} X$). Accordingly, $\mathcal{G}_n$ and Δ_n can be regarded as $\mathbb{F}$-subgeometries of the n-grassmannians $\widehat{\mathcal{G}}_n$ and $\widehat{\Delta}_n$ of $\mathrm{PG}(\widehat{V})$ and $\widehat{\Delta}$ respectively and, if $\hat{e}_n$ and $\hat{\varepsilon}_n$ are the natural embeddings of $\widehat{\mathcal{G}}_n$ and the Grassmann embedding of $\widehat{\Delta}_n$, then e_n and ε_n are induced by $\hat{e}_n$ and $\hat{\varepsilon}_n$. Clearly, the global nucleus $\widehat{\mathcal{N}}_n$ of $\hat{\varepsilon}_n$ contains the $\widehat{\mathbb{F}}$-tensorization $\widehat{\mathbb{F}} \otimes_{\mathbb{F}} \mathcal{N}_n$ of the global nucleus $\mathcal{N}_n$ of ε_n.

Turning to the plane π_X and the conic C_X, we have $\pi_X = \mathrm{PG}(W_k) \cap \hat{\pi}_X$ for a uniquely determined plane $\hat{\pi}_X$ of $\mathrm{PG}(\widehat{W}_k)$ (in fact $\hat{\pi}_X = \widehat{\mathbb{F}} \otimes_{\mathbb{F}} \pi_X$) and $C_X = \pi_X \cap \widehat{C}_X$ for a uniquely determined conic $\widehat{C}_X$ of $\hat{\pi}_X$. The nucleus $\hat{\nu}_X$ of $\widehat{C}_X$ coincides with (the

1-subspace of $\widehat{W}_n$ spanned by) the nucleus ν_X of C_X. By assumption, $\pi_X \cap \mathcal{N}_n$ is a line of π_X through ν_X. Hence $\hat{\pi}_X \cap \widehat{N}_n$ contains a line of $\hat{\pi}_X$ through $\hat{\nu}_X$. However this is impossible, in view of the first paragraph of this proof. Indeed $\widehat{\mathbb{F}}$ is perfect. □

Lemma 3.8 *Let $k = n$. Then* (Q2) *holds.*

Proof By way of contradiction, let $\langle \varepsilon_n(X_1), \varepsilon_n(X_2)\rangle \cap \mathcal{N}_n \neq 0$ for two distinct points $X_1, X_2 \in \Delta_n$. By (Q1), $n_{X_1,X_2} := \langle \varepsilon_n(X_1), \varepsilon_n(X_2)\rangle \cap \mathcal{N}_n$ is a point of $\mathrm{PG}(W_n) \setminus \varepsilon_n(\Delta_n)$.

Let $d = d(X_1, X_2)$ be the distance between X_1 and X_2 in the collinearity graph of Δ_n. Suppose firstly that $d = 1$, namely X_1 and X_2 are collinear. Then $X_1 \cap X_2 \in \Delta_{n-1}$ and $\mathrm{St}(X_1 \cap X_2)$ is the line of Δ_n through X_1 and X_2. The image of $\mathrm{St}(X_1 \cap X_2)$ by ε_n is a conic $C_{X_1\cap X_2}$ spanning a plane $\pi_{X_1\cap X_2}$ of $\mathrm{PG}(W_n)$. The projective line $\langle \varepsilon_n(X_1), \varepsilon_n(X_2)\rangle$ is contained in $\pi_{X_1\cap X_2}$. Hence $n_{X_1,X_2} \in \pi_{X_1\cap X_2}$. However, $\pi_{X_1\cap X_2} \cap \mathcal{N}_n$ is the nucleus $\nu_{X_1\cap X_2}$ of the conic $C_{X_1\cap X_2}$, by Lemma 3.7. Hence $n_{X_1,X_2} = \nu_{X_1\cap X_2}$, namely $\langle \varepsilon_n(X_1), \varepsilon_n(X_2)\rangle$ is a line of $\pi_{X_1\cap X_2}$ through the nucleus of $C_{X_1\cap X_2}$. On the other hand, $\langle \varepsilon_n(X_1), \varepsilon_n(X_2)\rangle$ contains two distinct points of the conic $C_{X_1\cap X_2}$, namely $\varepsilon_n(X_1)$ and $\varepsilon_n(X_2)$. Thus, we have got a secant line of a conic passing through the nucleus of that conic. This is impossible. Therefore $d > 1$.

As in the proof of Lemma 3.6, the distance-transitivity of G on the collinearity graph of Δ_n implies that $\langle \varepsilon_n(X), \varepsilon_n(Y)\rangle \cap \mathcal{N}_n \neq 0$ for any two points $X, Y \in \Delta_n$ at mutual distance d. We now choose two collinear points Y_1, Y_2 of Δ_n and a point X at distance d from both of them. Then X has distance $d - 1$ from a unique point Y_0 of the line $\mathrm{St}(Y_1 \cap Y_2)$ of Δ_n through Y_1 and Y_2. The point $\varepsilon_n(X)$ does not belong to the plane $\pi_{Y_1\cap Y_2}$, since $\varepsilon_n(\mathrm{St}(Y_1 \cap Y_2)) = \pi_{Y_1\cap Y_2} \cap \varepsilon_n(\Delta_n)$, and $X \notin \mathrm{St}(Y_1 \cap Y_2)$. Hence $\varepsilon_n(X)$, $\varepsilon_n(Y_1)$ and $\varepsilon_n(Y_2)$ span a 3-dimensional subspace S of $\mathrm{PG}(W_n)$. We have $d(X, Y) = d$ for every $Y \in \mathrm{St}(Y_1 \cap Y_2) \setminus \{Y_0\}$. Hence $\langle \varepsilon_n(X), \varepsilon_n(Y)\rangle \cap \mathcal{N}_n \neq 0$ for every such Y. Let σ be the subspace of S spanned by the points $n_{X,Y}$ for $Y \in \mathrm{St}(Y_1 \cap Y_2) \setminus \{Y_0\}$. Clearly, $\sigma \subseteq \mathcal{N}_n$.

Suppose firstly that σ contains (or is) a plane. Then $\sigma \cap \pi_{Y_1\cap Y_2}$ has projective dimension at least 1. Accordingly, $\pi_{Y_1\cap Y_2} \cap \mathcal{N}_n$ contains at least a line. This contradicts Lemma 3.7. Hence σ must be a line. Suppose $|\mathbb{F}| > 2$. If $Y_3 \in \mathrm{St}(Y_1 \cap Y_2) \setminus \{Y_0, Y_1, Y_2\}$ ($\neq \emptyset$ because $|\mathbb{F}| > 2$) then $\varepsilon_n(Y_1)$, $\varepsilon_n(Y_2)$ and $\varepsilon_n(Y_3)$ are non-collinear in the projective plane $\pi_{Y_1\cap Y_2}$. Hence the points n_{X,Y_1}, n_{X,Y_2} and n_{X,Y_3} are non-collinear as well, contrary to the fact that σ is a line. We are forced to conclude that $\mathbb{F} = \mathbb{F}_2$. The line σ meets $\pi_{Y_1\cap Y_2}$ in a point. On the other hand, $\sigma \subseteq \mathcal{N}_n$. Hence $\sigma \cap \pi_{Y_1\cap Y_2}$ is the nucleus $\nu_{Y_1\cap Y_2}$ of the conic $C_{Y_1\cap Y_2}$, by Lemma 3.7. Let π be the plane spanned by n_{X,Y_1}, n_{X,Y_2} and $\varepsilon_n(X)$. Then $\pi \cap \pi_{Y_1\cap Y_2}$ is a line, say l. The line l belongs to $\pi_{Y_1\cap Y_2}$ and contains the nucleus $\nu_{Y_1\cap Y_2}$ of the conic $C_{Y_1\cap Y_2}$ as well as two points of it, namely $\varepsilon_n(Y_1)$ and $\varepsilon_n(Y_2)$. This is obviously impossible. We have reached a final contradiction. □

So, the mapping $\varepsilon_k/\mathcal{N}_k$ is well-defined and injective. As remarked at the beginning of this subsection, if $k < n$ then $\varepsilon_k/\mathcal{N}_k$ is a full projective embedding.

Let $k = n$. For every $(n-1)$-element X of Δ let $\bar{\lambda}_X$ be the set of lines of π_X through ν_X and λ_X the set of lines of π_X tangent to C_X. Clearly, $\lambda_X \subseteq \bar{\lambda}_X$. Moreover, by Lemma 3.7, the mapping θ_X sending every line $l \in \bar{\lambda}_X$ to $\langle l, \mathcal{N}_n\rangle$ is a bijection from $\bar{\lambda}_X$ to a line $\overline{L}_X$ of $\mathrm{PG}(\varepsilon_n(\Delta_n)/\mathcal{N}_n)$. The set $L_X := \theta_X(\lambda_X)$ is contained in $\overline{L}_X$. Moreover, if ζ_X is the bijection from $\mathrm{St}(X)$ to λ_X sending every $Y \in \mathrm{St}(X)$ to the line $\langle \varepsilon_n(Y), \nu_X\rangle$ of π_X, then the composite $\eta_X := \theta_X \circ \zeta_X$ is a bijection from $\mathrm{St}(X)$ to L_X. Clearly, η_X is the mapping induced by $\varepsilon_n/\mathcal{N}_n$ on the line $\mathrm{St}(X)$ of Δ_n.

If $\mathbb{F}$ is perfect then $\lambda_X = \bar{\lambda}_X$. In this case $L_X = \overline{L}_X$. Hence $\varepsilon_n/\mathcal{N}_n$ maps the line $\mathrm{St}(X)$ of Δ_n onto a line of $\mathrm{PG}(\varepsilon_n(\Delta_n)/\mathcal{N}_n)$. On the other hand, if $\mathbb{F}$ is non-perfect then λ_X is a proper subset of $\bar{\lambda}_X$. Accordingly, $L_X \subset \overline{L}_X$. In this case $\varepsilon_n/\mathcal{N}_n$ maps the line $\mathrm{St}(X)$ onto a proper subset of a line of $\mathrm{PG}(\varepsilon_n(\Delta_n)/\mathcal{N}_n)$. Summarizing:

Lemma 3.9 *Let $k = n$. If $\mathbb{F}$ is perfect then $\varepsilon_n/\mathcal{N}_n$ is a full projective embedding. If $\mathbb{F}$ is non-perfect then $\varepsilon_n/\mathcal{N}_n$ is a non-full lax embedding.*

Proposition 3.10 *Let $\mathbb{F}$ be perfect. Then $\varepsilon_k/\mathcal{N}_k \cong \varepsilon_k^{sp}$, for $k = 1, 2, \dots, n$.*

Proof We recall that, since $\mathbb{F}$ is assumed to be perfect, the mapping sending every element X of Δ to $\langle X, N_0\rangle/N_0$ is an isomorphism from Δ to a model of Δ^{sp} realized inside V/N_0. For $X \in \Delta_k$, let $(X_J)_{J\in\binom{I}{k}}$ be the family of coordinates of $\varepsilon_k(X)$ with respect to the basis $B_\wedge$ of W_k. If we take only those coordinates X_J with $2n+1 \notin J$ then we get a family of coordinates for the image of X by $\varepsilon_k/\mathcal{N}_{k-1}$. It is not difficult to see that these coordinates are just the same as those that we obtain if we apply ε_k^{sp} to $\langle X, N_0\rangle/N_0$. □

3.3 Proof of Theorems 1.2 and 1.3

Propositions 3.4 and 3.10 yield Theorem 1.3. Turning to the proof of Theorem 1.2, suppose firstly that $\mathbb{F}$ is perfect. We also assume $k > 1$, since the statement of Theorem 1.2 is trivial when $k = 1$. Under these hypotheses, Theorem 1.3 and the equality $[\binom{2n}{k-1} - \binom{2n}{k-3}] + [\binom{2n}{k} - \binom{2n}{k-2}] = \binom{2n+1}{k} - \binom{2n+1}{k-2}$ imply that

$$\dim(\varepsilon_k) = \binom{2n+1}{k} - \binom{2n+1}{k-2}. \tag{1}$$

Suppose now that $\mathbb{F}$ is non-perfect. Let $\widehat{\mathbb{F}}$ be a perfect extension of $\mathbb{F}$ (e.g. the quadratic closure of $\mathbb{F}$). Let $\widehat{V} = \widehat{\mathbb{F}} \otimes_{\mathbb{F}} V$ and define $\widehat{\Delta}$, $\widehat{\Delta}_k$ and $\hat{\varepsilon}_k$ accordingly (see the proof of Lemma 3.7). Then (1) holds for $\hat{\varepsilon}_k$. Dimensions cannot decrease when tensorizing with field extensions. Therefore:

$$\dim(\varepsilon_k) \le \binom{2n+1}{k} - \binom{2n+1}{k-2}. \tag{2}$$

On the other hand, $\mathbb{F}$ contains $\mathbb{F}_2$. Let Δ_k^0 be the subgeometry of Δ_k formed by the subspaces spanned by $\mathbb{F}_2$-linear combinations of the vectors of B and ε_k^0 the embedding induced by ε_k on Δ_k^0. All vectors of $\varepsilon_k^0(\Delta_k^0)$ are $\mathbb{F}_2$-linear combinations of vectors of $B_\wedge$. Thus (1) holds for ε_k^0, since $\mathbb{F}_2$ is perfect. It follows that $\langle\varepsilon_k(\Delta_k)\rangle$

contains an independent set of $\binom{2n+1}{k} - \binom{2n+1}{k-2}$ vectors. Consequently,

$$\dim(\varepsilon_k) \geq \binom{2n+1}{k} - \binom{2n+1}{k-2}. \tag{3}$$

Equation (1) follows from (2) and (3).

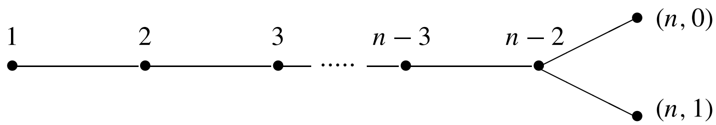
4 Proof of Theorem 1.5

In our proof of Theorem 1.5 we will go back and forth between B_n-buildings and D_n-buildings. So, we must firstly spend a few words on buildings of type D_n, their grassmannians and embeddings.

4.1 Grassmannians of D_n-buildings and their embeddings

Henceforth Δ^+ stands for the building of type D_n defined over $\mathbb{F}$. It can be constructed as follows. Given a non-singular quadratic form q^+ of Witt index n in $V' = V(2n, \mathbb{F})$, the non-trivial subspaces of V' totally singular for q^+, with their dimensions taken as types, form a non-thick building Δ' of Coxeter type C_n. The building Δ^+ is obtained from Δ' as follows: drop the elements of type $n-1$ and partition the set of n-elements in two families such that two n-elements X and Y are in the same family precisely if $\mathrm{codim}_X(X \cap Y)$ is even. Two elements X and Y not in the same family are declared to be incident precisely when $\dim(X \cap Y) = n-1$.

It is customary to choose the integers $n-1$ and n as types for these two families, but the following different convention better suits our needs in this section: we take the pairs $(n, 0)$ and $(n, 1)$ as types for them.



We allow $n = 3$. Recall that the diagram D_3 is the same as A_3, but with the usual types 1, 2, 3 replaced with $(3, 0)$, 1 and $(3, 1)$, respectively. If $n = 3$ then Δ^+ is isomorphic to $\mathrm{PG}(3, \mathbb{F})$ via the Klein correspondence, the elements of Δ^+ of type 1, $(3, 0)$ and $(3, 1)$ being respectively the lines, the points and the planes of $\mathrm{PG}(3, \mathbb{F})$ (or lines, planes and points, if we prefer so).

For $k < n$ the k-grassmannian of Δ' is defined just in the same way as the k-grassmannian Δ_k of Δ. We call it the *k-grassmannian* of Δ^+ and we denote it by the symbol Δ_k^+, although when $k = n-1$ this convention is not so consistent with the terminology commonly used in the literature. (If we followed the custom, we should rather call Δ_{n-1}^+ the $\{(n, 0), (n, 1)\}$-grassmannian of Δ^+.) The $(n, 0)$- and $(n, 1)$-grassmannian can also be defined, called *half spin geometries* in the literature, but we are not interested in them here.

The 1-grassmannian Δ_1^+ of Δ^+ is the polar space defined by q^+ on V'. Identify V' with a hyperplane of $V = (2n+1, \mathbb{F})$ suitably chosen so that Δ_1^+ is the polar space induced by Δ_1 on V'. Similarly, Δ_k^+ is the subgeometry induced by Δ_k on the set of k-subspaces of V'. Note that the points of Δ^+_{n-1} are the $\{(n,0),(n,1)\}$-flags of Δ^+ while the lines of Δ^+_{n-1} correspond to flags of Δ^+ type $\{n-2,(n,0)\}$ or $\{n-2,(n,1)\}$. In particular, if $n=3$ then Δ^+_2 is the so-called root-subgroup geometry of $\mathrm{SL}(4,\mathbb{F})$, with the point-plane flags of $\mathrm{PG}(3,\mathbb{F})$ as points and the line-plane and point-line flags of $\mathrm{PG}(3,\mathbb{F})$ as lines.

For $k=1,2,\dots,n-1$ we can define a projective embedding ε^+_k of Δ^+_k into a subspace $\langle\varepsilon_k(\Delta^+_k)\rangle$ of $W'_k=\bigwedge^k V'$ sending every point $\langle v_1,\dots,v_k\rangle$ of Δ^+_k to the point $\langle v_1\wedge\cdots\wedge v_k\rangle$ of $\mathrm{PG}(W'_k)$. We call ε^+_k the *Grassmann embedding* of Δ^+_k. Moreover, let $\mu_1,\dots,\mu_{n-2},\mu_{n,0}$ and $\mu_{n,1}$ be the fundamental dominant weights of the root system of type D_n, corresponding to the nodes $1,2,\dots,n-2,(n,0)$ and $(n,1)$ of the D_n-diagram in the obvious way. Put $\mu_{n-1}:=\mu_{n,0}+\mu_{n,1}$. Then for $k=1,2,\dots,n-1$ the Weyl module $V(\mu_k)$ hosts a projective embedding $\widetilde{\varepsilon}^+_k$ of Δ^+_k and ε^+_k is a quotient of $\widetilde{\varepsilon}^+_k$. We call $\widetilde{\varepsilon}^+_k$ the *Weyl embedding* of Δ^+_k.

By the Weyl dimension formula, $\dim(\widetilde{\varepsilon}^+_k)=\binom{2n}{k}$. Clearly, $\widetilde{\varepsilon}^+_1\cong\varepsilon^+_1$.

Proposition 4.1 *Let* $k>1$.

(1) *If* $\mathrm{char}(\mathbb{F})\neq 2$ *then* $\varepsilon^+_k\cong\widetilde{\varepsilon}^+_k$, *namely* $\dim(\varepsilon^+_k)=\binom{2n}{k}$.
(2) *Let* $\mathrm{char}(\mathbb{F})=2$. *Then* $\dim(\varepsilon^+_k)=\binom{2n}{k}-\binom{2n}{k-2}$. *In this case* ε^+_k *is a proper quotient of the Weyl embedding* $\widetilde{\varepsilon}^+_k$.

Proof Claim (1) can be proved by just the same argument used for Theorem 1.1. Indeed Lemma 2.1 still holds if we replace the $(2n+1)$-dimensional space V with the $2n$-dimensional space V', the quadratic form q with q^+ and $\mathcal{G}_k$ with the k-grassmannian $\mathcal{G}'_k$ of $\mathrm{PG}(V')$. The proof of Lemma 2.1 remains valid for this setting word for word. By that lemma, when $\mathrm{char}(\mathbb{F})\neq 2$ the point-set of Δ^+_k spans the k-grassmannian $\mathcal{G}'_k$ of $\mathrm{PG}(2n-1,\mathbb{F})$ (compare Proposition 2.2). Hence $\langle\varepsilon^+_k(\Delta^+_k)\rangle=W'_k$, namely $\dim(\varepsilon^+_k)=\binom{2n}{k}$.

Let $\mathrm{char}(\mathbb{F})=2$. Let f be the bilinearization of q^+ and Δ^{sp}_1 the polar space defined by f on V'. Then Δ^+_1 is a subgeometry (but not a subspace) of Δ^{sp}_1. Accordingly, Δ^+_k is a subgeometry of Δ^{sp}_k and the natural embedding ε^{sp}_k of Δ^{sp}_k (see Introduction) induces ε^+_k on Δ^+_k. We have $\dim(\varepsilon^{sp}_k)=\binom{2n}{k}-\binom{2n}{k-2}$. Hence $\dim(\varepsilon^+_k)\le\binom{2n}{k}-\binom{2n}{k-2}$. Moreover, claim (2) holds when $\mathbb{F}$ is perfect. Explicitly:

(2′) If $\mathbb{F}$ is perfect then $\dim(\varepsilon^+_k)=\binom{2n}{k}-\binom{2n}{k-2}$.

Indeed, let $\mathbb{F}$ be perfect. As Δ^+_k is a subgeometry of Δ^{sp}_k which in its turn is a subgeometry of $\mathcal{G}'_k$, in order to prove (2′) it suffices to prove that Δ^{sp}_k is contained in the subgeometry $\langle\Delta^+_k\rangle_{\mathcal{G}'_k}$ spanned by Δ^+_k. Let X be a k-dimensional subspace of V' totally isotropic for f. We must prove that $X\in\langle\Delta^+_k\rangle_{\mathcal{G}'_k}$. Since $\mathbb{F}$ is perfect $\Delta^{sp}_1\cong\Delta_1$. However Δ^+_1 is a geometric hyperplane of Δ_1. Hence Δ^+_1 is also a geometric hyperplane of Δ^{sp}_k too. Consequently either $X\in\Delta^+_k$ or $X_0:=\{x\in X\mid q^+(x)=0\}$ is a hyperplane of X and $X\subseteq X_0^\perp$. In the first case there is nothing to prove. Assume the latter. Then $X_0\in\Delta^+_{k-1}$ and $\dim(X_0^\perp/X_0)=2n-2k-2$. Given a flag F of Δ^+

of type $\{1, \ldots, k-1\}$ containing X_0, no matter which, $\mathrm{Res}_{\Delta^+}(F)$ is isomorphic to the building of type D_{n-k-1} associated to a non-degenerate quadratic form $q^+_{X_0}$ of $X_0^\perp/X_0$ of maximal Witt index $n-k-1$. The k-subspaces of V' containing X_0 and contained in $X_0^\perp$ form a subspace $\mathcal{G}'_k(X_0)$ of $\mathcal{G}'_k$ and the function mapping $Y \in \mathcal{G}'_k(X_0)$ onto Y/X_0 is an isomorphism π_{X_0} from the geometry induced by $\mathcal{G}'_k$ on $\mathcal{G}'_k(X_0)$ to $\mathrm{PG}(X_0^\perp/X_0)$. Clearly π_{X_0} maps the set of k-elements of Δ^+ containing X_0 onto the set of points of the quadric $Q^+_{X_0}$ associated to $q^+_{X_0}$. The point $\pi_{X_0}(X)$ of $\mathrm{PG}(X_0^\perp/X_0)$ does not belong to $Q^+_{X_0}$. On the other hand, every point of $\mathrm{PG}(X_0^\perp/X_0)$ belongs to a secant line of $Q^+_{X_0}$. Hence there is a line L of $\mathcal{G}'_k$ contained in $\mathcal{G}'_k(X_0)$ and containing X and two distinct points of Δ^+_k. Therefore $X \in \langle \Delta^+_k \rangle_{\mathcal{G}'_k}$, as we wished to prove. Claim (2′) is proved.

Having proved (2′) in the perfect case, claim (2) in the non-perfect case follows by descent to $\mathbb{F}_2$, as in the proof of Theorem 1.2 (see Sect. 3.3). □

The geometry Δ^+_k admits the absolutely universal embedding (Kasikova and Shult [19] for $k < n-1$ and Blok and Pasini [8] for $k = n-1$). Therefore, if $\widetilde{\varepsilon}^+_k$ is relatively universal then it is also absolutely universal. It follows from Tits [23, 8.4.3] that ε^+_1 ($= \widetilde{\varepsilon}^+_1$) is universal for any field $\mathbb{F}$. On the other hand, by Proposition 4.1, when $\mathrm{char}(\mathbb{F}) = 2$ and $k > 1$ the Grassmann embedding ε^+_k is not universal. In the sequel, as a by-product of our proof of Theorem 1.5, we shall show that, under the hypotheses assumed on $\mathbb{F}$ in Theorem 1.5, the Weyl embedding $\widetilde{\varepsilon}^+_k$ is universal for $k = 2$ or 3.

4.2 Back and forth between B_n and D_n

In this subsection k is 2 or 3, $n > k$ and

$$\eta_k : \Delta_k \to \mathrm{PG}(U_k), \qquad \eta^+_k : \Delta^+_k \to \mathrm{PG}\big(U'_k\big)$$

are given projective embeddings of Δ_k and Δ^+_k, for some $\mathbb{F}$-vector spaces U_k and U'_k. As in the previous sections, we will freely use the symbols Δ_k and Δ^+_k to denote the point-line geometries Δ_k and Δ^+_k as well as their point-sets. We will do the same with other symbols like $\Delta_{k,a}$, $\Delta_{k,H}$ etc. (see below). We denote spans in Δ_k, Δ^+_k, Δ_1 and Δ^+_1 by the symbols $\langle . \rangle_{\Delta_k}$, $\langle . \rangle_{\Delta^+_k}$, $\langle . \rangle_{\Delta_1}$ and $\langle . \rangle_{\Delta^+_1}$ respectively, keeping the symbol $\langle . \rangle$ for spans in U_k or U'_k.

Let H be a non-singular hyperplane of the polar space Δ_1 such that the polar space $\Delta_{1,H}$ induced by Δ_1 on H is isomorphic to Δ^+_1. Let $\Delta_{k,H}$ be the subgeometry of Δ_k induced on the set of totally singular k-subspaces of V contained in H. Then $\Delta_{k,H} \cong \Delta^+_k$. The embedding η_k induces on $\Delta_{k,H}$ a projective embedding $\eta_{k,H} : \Delta_{k,H} \to \mathrm{PG}(U_{k,H})$ where $U_{k,H} := \langle \eta_k(\Delta_{k,H}) \rangle$.

Let a be a point of Δ_1 exterior to H and $\Delta_{k,a}$ the subgeometry of Δ_k induced on the set of totally singular k-subspaces of V containing a. Then $\Delta_{k,a}$ is isomorphic to the $(k-1)$-grassmannian $\overline{\Delta}_{k-1}$ of a building $\overline{\Delta}$ of type B_{n-1} defined over $\mathbb{F}$. The embedding $\eta_{k,a} : \Delta_{k,a} \to \mathrm{PG}(U_{k,a})$ induced by η_k on $\Delta_{k,a}$, where $U_{k,a} := \langle \eta_k(\Delta_{k,a}) \rangle$, can be regarded as a projective embedding of $\overline{\Delta}_{k-1}$.

When $k = 2$ let l_0 be a line of Δ_1 not contained in $H \cup a^\perp$ and such that $a^\perp \cap l_0 \neq H \cap l_0$. Put $S_2 := \langle \{l_0\} \cup \Delta_{2,a} \cup \Delta_{2,H} \rangle_{\Delta_2}$.

When $k = 3$ (hence $n > 3$) the subgeometry $\Delta_{1,a,H}$ of Δ_1 induced on $a^\perp \cap H$ is isomorphic to the polar space associated with a non-singular quadratic form of $V(2n-1,\mathbb{F})$ of Witt index $n-1$. It is well known that the latter admits a generating set of $2n-1$ points. Hence the same holds for $\Delta_{1,a,H}$. Let $\{p_1,\dots,p_{2n-1}\}$ be a spanning set of $2n-1$ points of $\Delta_{1,a,H}$. For every $i = 1,\dots,2n-1$ let α_i be a plane of Δ_1 through p_i such that $\alpha_i \cap H \cap a^\perp = \{p_i\}$. Put $S_3 := \langle\{\alpha_i\}_{i=1}^{2n-1} \cup \Delta_{3,a} \cup \Delta_{3,H}\rangle_{\Delta_3}$.

Lemma 4.2 $S_2 = \Delta_2$.

Proof We firstly prove the following:

(1) All lines of Δ_1 coplanar with a in Δ_1 belong to S_2.

Let α be a plane of Δ_1 through a and l a line of α. If either $a \in l$ or $l \subset H$ there is nothing to prove. Let $a \notin l \not\subseteq H$ and let $p = l \cap H$. Then the lines of α through p are the points of a line L of Δ_2. The line L contains l, $\alpha \cap H$ and the line $\langle a, p\rangle_{\Delta_1}$ of Δ_1 through a and p. Clearly, $\alpha \cap H \in \Delta_{2,H}$ and $\langle a, p\rangle_{\Delta_1} \in \Delta_{2,a}$. Hence $l \in S_2$. Claim (1) is proved.

(2) All lines of Δ_1 coplanar with l_0 belong to S_2.

Let α be a plane of Δ_1 through l_0. Then α contains three lines of S_2, namely l_0, $\alpha \cap H$ and $\alpha \cap a^\perp$ (the latter belongs to S_2 by (1)). These three lines form a triangle, as $l_0 \cap a^\perp \notin H$ by assumption. It is now easy to see that all lines of α belong to S_2.

(3) If l is a line of Δ_1 meeting l_0 non-trivially, then l belongs to S_2.

Let $p := l \cap l_0$. Suppose firstly that $p \in H$. We can choose a plane α of Δ_1 through l such that $l_0^\perp \cap \alpha \neq \alpha \cap H$. We have $\alpha \cap H \in \Delta_{2,H}$ and $l_0^\perp \cap \alpha \in S_2$ by (2). Moreover, both $\alpha \cap H$ and $\alpha \cap l_0^\perp$ pass through $p \in l$. Therefore $l \in S_2$.

Let $p \notin H$. By (2), we can assume that l and l_0 are non-coplanar. We consider two cases. Suppose firstly that we can choose a plane α of Δ_1 through l_0 such that $a^\perp \cap \alpha \cap H \neq l^\perp \cap \alpha \cap H$. Let β be the plane of Δ_1 spanned by l and $\alpha \cap l^\perp$. Then $\alpha \cap l^\perp \in S_2$ by (2), $a^\perp \cap \beta \in S_2$ by (1) and $\beta \cap H \in \Delta_{2,H}$. Since $a^\perp \cap \alpha \cap H \neq l^\perp \cap \alpha \cap H$, these three lines form a triangle. Hence all lines of β belong to S_2. In particular, $l \in S_2$.

Assume now that $a^\perp \cap \alpha \cap H = l^\perp \cap \alpha \cap H$ for every plane α on l_0. Clearly, there is at least one plane β through l containing a line m through p, non-coplanar with l_0 and such that $a^\perp \cap \alpha \cap H \neq m^\perp \cap \alpha \cap H$. Then $m \in S_2$ by the previous paragraph. As $l_0^\perp \cap \beta \in S_2$ by (2), all lines of β through p belong to S_2.

(4) If l is a line of Δ_1 meeting $l_0^\perp$ non-trivially, then $l \in S_2$.

By (3) we can assume that $l \cap l_0 = \emptyset$. Suppose firstly that $l \subseteq l_0^\perp$. Pick a point $p \in l_0$, $p \not\perp a$. By (3), all lines of the plane $\alpha := \langle p, l\rangle$ through p are in S_2. On the other hand, the line $\alpha \cap a^\perp$ belongs to S_2 by (1). It does not pass through p, as $p \notin a^\perp$ by assumption. Therefore all lines of α belong to S_2. In particular, $l \in S_2$.

Suppose that $l \cap l_0^\perp$ is a point, say $p_1 := l \cap l_0^\perp$. We may also assume that $p_1 \notin l_0$, by (3). Then $p_2 := l_0 \cap l^\perp$ is a point. Let $\alpha = \langle p_2, l\rangle_{\Delta_1}$. All lines of the plane α

passing through p_2 belong to S_2, by (3). If $p_2 \notin H$ then $\alpha \cap H$ is a line of α in S_2 not through p_2. It follows that all lines of α belong to S_2. In particular, $l \in S_2$.

Let now $p_2 \in H$. The line $a^\perp \cap \alpha$ belongs to S_2 by (1). It does not pass through p_2, as $a^\perp \cap l_0 \notin H$ by assumption. As above, all lines of α belong to S_2. Whence $l \in S_2$.

We can now finish the proof of the lemma. Let l be any line of Δ_1. We may assume that $a \notin l \nsubseteq H$. In view of (1)–(4) we can also assume that $l \cap a^\perp$ is a point and $l \cap l_0^\perp = \emptyset$. Suppose that we can choose a plane α on l such that the point $p := l_0^\perp \cap \alpha$ does not belong to $H \cap a^\perp$. By (4), all lines of α through p belong to S_2. Moreover, $a^\perp \cap \alpha \in S_2$ by (1) and $\alpha \cap H \in \Delta_{2,H} \subseteq S_2$. Thus, α contains at least three lines of S_2 forming a triangle. Therefore all lines of α belong to S_2. In particular, $l \in S_2$.

Finally, suppose that $p = l_0^\perp \cap \alpha \in H \cap a^\perp$ for every plane α through l. Pick a point $p_1 \in l$ not in H and consider a line l_1 of α through p_1, different from either of the lines l and $l_2 := \langle p_1, p\rangle_{\Delta_1}$. Clearly, we can choose a plane α_1 on l_1 such that $l_0^\perp \cap \alpha_1 \notin H \cap a^\perp$. Hence $l_1 \in S_2$ by the previous paragraph. Moreover, $l_2 \in S_2$ by (4). Hence $l \in S_2$. □

Corollary 4.3 *Suppose that* $\dim(U_{2,H}) \le \binom{2n}{2}$. *Then* $\dim(U_2) \le \binom{2n+1}{2}$.

Proof This follows from Lemma 4.2, recalling that $\Delta_{2,a}$ is isomorphic to a polar space of type B_{n-1}, that every projective embedding of such a polar space has dimension equal to $2n-1$ or possibly $2n-2$ (the latter only when $\mathrm{char}(\mathbb{F}) = 2$) and noticing that $1 + (2n-1) + \binom{2n}{2} = \binom{2n+1}{2}$. □

Lemma 4.4 $S_3 = \Delta_3$.

Proof It will be useful to have fixed some terminology. In the sequel, a totally singular 4-subspace of V will be called a *space* of Δ_1, for short. We say that a point $p \in a^\perp \cap H$ is S_3-*full* if all planes of Δ_1 on p belong to S_3.

(1) All planes of Δ_1 contained in $a^\perp$ belong to S_3.

Let α be a plane of Δ_1 contained in $a^\perp$. If $a \in \alpha$ then $\alpha \in \Delta_{3,a} \subseteq S_3$. Suppose that $a \notin \alpha$ and let X be the space of Δ_1 spanned by a and α. Then all planes of X through a belong to S_3. Moreover, $X \cap H \in \Delta_{3,H} \subseteq S_3$. It follows that all planes of X belongs to S_3. In particular, $\alpha \in S_3$.

(2) Let $p \in a^\perp \cap H$ and let α_0 be a plane of Δ_1 on p such that $a^\perp \cap H \cap \alpha_0 = \{p\}$ and $\alpha_0 \in S_3$. Then p is S_3-full.

Let $\Delta_{3,p}$ be the subgeometry of Δ_3 induced on the set of planes of Δ_1 through p. Let $\overline{\Delta}_{3,p}$ be the induced subgeometry of $\Delta_{3,p}$ formed by those planes α such that $\alpha \cap a^\perp \cap H = \{p\}$. So, $\alpha_0 \in \overline{\Delta}_{3,p}$. It is not difficult to see that $\overline{\Delta}_{3,p}$ is connected.

Let now X be a space of Δ_1 through p containing α_0. Then X contains three planes of S_3 through p, namely α_0, $X \cap H$ and $X \cap a^\perp$ (which belongs to S_3 by (1)). The intersection of these three planes is the point p. Hence all planes of X through p belong to S_3. Therefore, every plane through p contained in a common space with α_0 belongs to S_3. Let now α be any plane in $\overline{\Delta}_{3,p}$ contained in a common space with α_0.

We can repeat the above argument with α_0 replaced by α, thus obtaining that all planes through p contained in a common space with α belong to S_3. In this way, exploiting the connectedness of $\overline{\Delta}_{3,p}$ we obtain the result that every plane through p contained in a common space with a plane $\alpha \in \overline{\Delta}_{3,p}$ belongs to S_3. However, every plane $\beta \in \Delta_{3,p}$ is contained in the same space as a plane $\alpha \in \overline{\Delta}_{3,p}$. Therefore $\Delta_{3,p} \subseteq S_3$, namely p is S_3-full.

(3) Let x and y be two points of $a^{\perp} \cap H$ collinear in $\Delta_{1,a,H}$ and z another point on the line l of $\Delta_{1,a,H}$ spanned by x and y. If both x and y are S_3-full then z is S_3-full.

It is easy to see that there exists at least one space X of Δ_1 containing l and such that $a^{\perp} \cap X = H \cap X = l$. Let X be such a space. As both x and y are S_3-full, all planes of Δ_1 contained in X and containing either x or y belong to S_3. It follows that all planes contained in X belong to S_3. In particular, all planes through z contained in X belong to S_3. On the other hand, at least one of these planes meets $a^{\perp}$ and H in distinct lines. Therefore z satisfies the hypotheses of (2). Hence z is S_3-full.

We can now finish the proof of the lemma. By (2), the points $p_1, \dots, p_{2n-1}$ considered in the definition of S_3 are S_3-full. Moreover they span $\Delta_{1,a,H}$. Hence all points of $a^{\perp} \cap H$ are S_3-full, by (3). On the other hand, every plane of Δ_1 meets $a^{\perp} \cap H$ non-trivially. Hence every plane of Δ_1 belongs to S_3. □

Corollary 4.5 *Suppose that* $\dim(U_{3,H}) \leq \binom{2n}{3}$ *and* $\dim(U_{3,a}) \leq \binom{2n-1}{2}$. *Then* $\dim(U_3) \leq \binom{2n+1}{3}$.

Proof Note that $(2n-1) + \binom{2n-1}{2} + \binom{2n}{3} = \binom{2n+1}{3}$. The conclusion follows from this equality and Lemma 4.4. □

We now turn to Δ_k^+ and its embedding $\eta_k^+ : \Delta_k^+ \to \mathrm{PG}(U'_k)$. Let H be a non-singular hyperplane of the polar space Δ_1^+. The polar space $\Delta_{1,H}^+$ induced by Δ_1^+ on H is isomorphic to the polar space $\overline{\Delta}_1$, for a building $\overline{\Delta}$ of type B_{n-1} defined over $\mathbb{F}$. Let $\Delta_{k,H}^+$ be the subgeometry of Δ_k^+ induced on H. Then $\Delta_{k,H}^+$ is isomorphic to the k-grassmannian $\overline{\Delta}_k$ of $\overline{\Delta}$. The embedding $\eta_k^+ : \Delta_k^+ \to \mathrm{PG}(U'_k)$ induces on $\Delta_{k,H}^+$ a projective embedding $\eta_{k,H}^+ : \Delta_{k,H}^+ \to \mathrm{PG}(U'_{k,H})$ where $U'_{k,H} := \langle \eta_k^+(\Delta_{k,H}^+) \rangle$.

Let a be a point of Δ_1^+ exterior to H and $\Delta_{k,a}^+$ the subgeometry of Δ_k^+ induced on the set of points $X \in \Delta_k^+$ such that $a \in X$. Then $\Delta_{k,a}^+$ is isomorphic to the $(k-1)$-grassmannian $\overline{\Delta}_{k-1}^+$ of a building $\overline{\Delta}^+$ of type D_{n-1} defined over $\mathbb{F}$. The embedding $\eta_{k,a}^+ : \Delta_{k,a}^+ \to \mathrm{PG}(U'_{k,a})$ induced by η_k^+ on $\Delta_{k,a}^+$, where $U'_{k,a} := \langle \eta_k^+(\Delta_{k,a}^+) \rangle$ can be regarded as a projective embedding of $\overline{\Delta}_{k-1}^+$.

When $k = 2$ let l_0 be a line of Δ_1^+ not contained in $H \cup a^{\perp}$ and such that $a^{\perp} \cap l_0 \neq H \cap l_0$. Put $S_2^+ := \langle \{l_0\} \cup \Delta_{2,a}^+ \cup \Delta_{2,H}^+ \rangle_{\Delta_2^+}$.

When $k = 3$ the subgeometry $\Delta_{1,a,H}^+$ of Δ_1^+ induced on $a^{\perp} \cap H$ is isomorphic to the polar space associated with a non-singular quadratic form of $V(2n-2, \mathbb{F})$ of Witt index $n-1$. It is well known that the latter can be spanned by $2n-2$ points. Hence the same holds for $\Delta_{1,a,H}^+$. Given a spanning set $\{p_1, \dots, p_{2n-2}\}$ of $\Delta_{1,a,H}^+$, for every

$i = 1, \dots, 2n-2$ we pick a plane α_i of Δ_1^+ through p_i such that $\alpha_i \cap H \cap a^\perp = \{p_i\}$. We put $S_3^+ := \langle \{\alpha_i\}_{i=1}^{2n-2} \cup \Delta_{3,a}^+ \cup \Delta_{3,H}^+ \rangle_{\Delta_3^+}$.

The next lemma can be proved by arguments very similar to those exploited in the proofs of Lemmas 4.2 and 4.3. We leave the details for the reader.

Lemma 4.6 *We have $S_2^+ = \Delta_2^+$ and $S_3^+ = \Delta_3^+$.*

Lemma 4.6 immediately implies the following:

Corollary 4.7

(1) *If* $\dim(U'_{2,H}) \le \binom{2n-1}{2}$ *then* $\dim(U'_2) \le \binom{2n}{2}$.
(2) *Let* $\dim(U'_{3,H}) \le \binom{2n-1}{3}$ *and* $\dim(U'_{3,a}) \le \binom{2n-2}{2}$. *Then* $\dim(U'_3) \le \binom{2n}{3}$.

4.3 Proof of Theorem 1.5. The case $k = 2$

Let $k = 2$. Assume firstly that $n = 3$. Then, as we have recalled before, the points of Δ_2^+ can be regarded as point-plane flags of $\mathrm{PG}(3, \mathbb{F})$. The Weyl embedding $\widetilde{\varepsilon}_2^+ : \Delta_2^+ \to \mathrm{PG}(W'_2)$ can be described as follows: W'_2 is the vector space of null-traced (4×4)-matrices and, for every non-zero vector v of $V(4, \mathbb{F})$ and every non-trivial linear functional f of $V(4, \mathbb{F})$ such that $f(v) = 0$, the flag $\{\langle v \rangle, \mathrm{Ker}(f)\}$ of $\mathrm{PG}(3, \mathbb{F})$ is mapped by $\widetilde{\varepsilon}_2^+$ onto the linear subspace of W'_2 spanned by the matrix $f \otimes v$. Note that $\dim(W'_2) = 4^2 - 1 = 15 = \binom{2 \cdot 3}{2}$. The next lemma is contained in the main result of Völklein [25] as a special case.

Lemma 4.8 *Let $\mathbb{F}$ be a perfect field of positive characteristic or a number field and let $n = 3$. Then the Weyl embedding $\widetilde{\varepsilon}_2^+$ is universal.*

The next theorem can be proved by induction on n, using Lemma 4.8 to start and Corollary 4.3 combined with part (1) of Corollary 4.7 to go on.

Theorem 4.9 *Let $\mathbb{F}$ be a perfect field of positive characteristic or a number field and let $n \ge 3$. Then every projective embedding of Δ_2 has dimension at most $\binom{2n+1}{2}$ and every projective embedding of Δ_2^+ has dimension at most $\binom{2n}{2}$.*

Since $\dim(\widetilde{\varepsilon}_2) = \binom{2n+1}{2}$ and $\dim(\widetilde{\varepsilon}_2^+) = \binom{2n}{2}$, Theorem 4.9 immediately implies the following corollary, which contains part (1) of Theorem 1.5.

Corollary 4.10 *Let $\mathbb{F}$ be a perfect field of positive characteristic or a number field and let $n \ge 3$. Then both $\widetilde{\varepsilon}_2$ and $\widetilde{\varepsilon}_2^+$ are universal.*

4.4 Quasi-veronesean embeddings of projective spaces

We can deal with the case $k = 3$ by induction just as we have done for $k = 2$ in the previous subsection, but in order to start the induction we need an analog of Lemma 4.8 for $n = 4$. We will obtain such a lemma in the next subsection. In the present subsection we prove a preliminary result to be exploited in the proof of that lemma. We firstly state a few definitions.

The following class of embeddings includes both projective and veronesean embeddings. Given a point-line geometry Γ and a vector space U defined over a commutative division ring, a *quasi-veronesean embedding* of Γ in $\mathrm{PG}(U)$ is an injective mapping ν from the point-set of Γ to the set of points of $\mathrm{PG}(U)$ such that $\nu(\Gamma)$ spans $\mathrm{PG}(U)$ and every line of Γ is mapped by ν onto either a non-singular conic or a line of $\mathrm{PG}(U)$. We set $\dim(\nu) := \dim(U)$, as usual.

Given a quasi-veronesean embedding $\nu : \Gamma \to \mathrm{PG}(U)$, the span $\langle \nu(l) \rangle$ in $\mathrm{PG}(U)$ of the image $\nu(l)$ of a line l of Γ is uniquely determined by any three of its points. This suggests to consider the following notions. A set of points X of Γ is a 3-*subspace* of Γ if every line meeting X in at least three points is contained in X. Intersections of 3-subspaces are 3-subspaces. So, for every set X of points, the intersection $\langle X \rangle^3_\Gamma$ of all 3-subspaces of Γ containing X is the smallest 3-subspace of Γ containing X. We call it the 3-*span* of X in Γ. Note that every subspace of Γ as defined in Sect. 2 is also a 3-subspace, but the converse is false in general. Hence $\langle X \rangle^3_\Gamma \subseteq \langle X \rangle_\Gamma$, possibly with strict inclusion. We say that X 3-*generates* Γ if $\langle X \rangle^3_\Gamma = \Gamma$.

Lemma 4.11 *Let $\mathbb{F} \neq \mathbb{F}_2$. Then, for every positive integer d, every quasi-veronesean embedding of* $\mathrm{PG}(d, \mathbb{F})$ *is at most $\binom{d+2}{2}$-dimensional.*

Proof It suffices to prove that $\mathrm{PG}(d, \mathbb{F})$ can be 3-generated by a set of $\binom{d+2}{2}$ points. Explicitly,

(1) $\mathrm{PG}(d, \mathbb{F})$ admits a 3-generating set of size $\binom{d+2}{2}$.

It is convenient to combine (1) with the following:

(2) If $X \subset \mathrm{PG}(d, \mathbb{F})$ is such that $|X| = \binom{d+1}{2}$ and $\langle X \rangle^3_{\mathrm{PG}(d,\mathbb{F})}$ is a hyperplane of $\mathrm{PG}(d, \mathbb{F})$, then we can find a set of points $Y \subset \mathrm{PG}(d, \mathbb{F})$ such that $|Y| = d + 1$, $Y \cap X = \emptyset$ and $X \cup Y$ 3-generates $\mathrm{PG}(d, \mathbb{F})$.

We shall prove the conjunction of (1) and (2) by induction on d. If $d = 1$ there is nothing to prove. Let $d > 1$. Assume that we have already proved that (2) holds for that d. Every hyperplane H of $\mathrm{PG}(d, \mathbb{F})$ admits a 3-generating set X of size $\binom{d+1}{2}$, by the inductive hypothesis on (1). By (2), we can enlarge X to a 3-generating set $X \cup Y$ of $\mathrm{PG}(d, \mathbb{F})$ of size $d + 1 + \binom{d+1}{2} = \binom{d+2}{2}$. So, (1) holds for d.

It remains to prove (2). Given X as in (2), let $H_1 = \langle X \rangle^3_{\mathrm{PG}(d,\mathbb{F})}$ and let H_2 be another hyperplane of $\mathrm{PG}(d, \mathbb{F})$, different from H_1. By the inductive hypothesis on (1), $H_1 \cap H_2$ admits a 3-generating set X' of size $\binom{d}{2}$. By the inductive hypothesis on (2), we can find $Y' \subset H_2$ such that $|Y'| = d$, $Y' \cap X' = \emptyset$ and $\langle X' \cup Y' \rangle^3_{\mathrm{PG}(d,\mathbb{F})} = H_2$. Thus, $\langle X \cup Y' \rangle^3_{\mathrm{PG}(3,\mathbb{F})} \supseteq H_1 \cup H_2$. Pick now a point $p \in \mathrm{PG}(d, \mathbb{F}) \setminus (H_1 \cup H_2)$ and let H_3 be the unique hyperplane of $\mathrm{PG}(d, \mathbb{F})$ containing $\{p\} \cup H_1 \cap H_2$. Every point x of $\mathrm{PG}(d, \mathbb{F})$ exterior to $H_1 \cup H_2 \cup H_3$ belongs to a line through p meeting $H_1 \cup H_2$ in two distinct points. Hence $\langle X \cup Y' \cup \{p\} \rangle^3_{\mathrm{PG}(d,\mathbb{F})}$ contains all points of $\mathrm{PG}(d, \mathbb{F})$ except possibly those of H_3 different from p and exterior to $H_1 \cap H_2$. As $\mathbb{F} \neq \mathbb{F}_2$, this set of points is enough to 3-generate $\mathrm{PG}(d,\mathbb{F})$. So, $Y := Y' \cup \{p\}$ has the properties required in (2). □

Remarks

1. The hypothesis $\mathbb{F} \neq \mathbb{F}_2$ cannot be dropped from Lemma 4.11. Indeed let Γ be a point-line geometry where every line has just 3 points and let P be the point-set of Γ. Then every subset of P is a 3-subspace of Γ and Γ admits a (universal) veronesean embedding $\nu : \Gamma \to \mathrm{PG}(\mathbb{F}_2^P)$ where $\mathbb{F}_2^P$ is the $\mathbb{F}_2$-vector space of all functions $f : P \to \mathbb{F}_2$ and ν sends $p \in P$ to the characteristic function of $\{p\}$. Clearly, $\dim(\nu) = |P|$.
2. Note that $\binom{d+2}{2}$ is indeed the dimension of the usual veronesean embedding of $\mathrm{PG}(d,\mathbb{F})$, sending a vector $(x_i)_{i=0}^d \in V(d+1,\mathbb{F})$ to the vector $(x_i x_j)_{i\le j} \in V(\binom{d+2}{2},\mathbb{F})$. So, by Lemma 4.11, that embedding is relatively universal when $\mathbb{F} \neq \mathbb{F}_2$.

4.5 An analog of Lemma 4.8 for $n = 4$

In this subsection $\mathbb{F}$ is either a perfect field of positive characteristic different from $\mathbb{F}_2$ or a number field, $n = 4$ and $\eta_3^+ : \Delta_3^+ \to \mathrm{PG}(U_3')$ is a projective embedding of Δ_3^+.

For a point p of Δ_1^+, we denote by $\Delta_{3,p}^+$ the subgeometry of Δ_3^+ induced on the set of planes of Δ_1^+ containing p and $\eta_{3,p}^+$ is the restriction of η_3^+ to $\Delta_{3,p}^+$. Clearly, $\Delta_{3,p}^+$ is isomorphic to the 2-grassmannian $\overline{\Delta}_2^+$ of a building $\overline{\Delta}^+$ of type D_3 defined over $\mathbb{F}$ and $\eta_{3,p}^+$ can be regarded as a projective embedding of $\overline{\Delta}_2^+$ into $U_{3,p}' := \langle \eta_3^+(\Delta_{3,p}^+)\rangle$. By Corollary 4.10, $\eta_{3,p}^+$ is a quotient of the Weyl embedding of $\overline{\Delta}_2^+$. Hence $\dim(U_{3,p}') \le 15$.

Given two non-collinear points a and b of Δ_1^+, let $\Delta_{3,a,b}^+$ be the set of planes of Δ_1^+ contained in $a^\perp \cap b^\perp$ and $U_{3,a,b}' := \langle \eta_3^+(\Delta_{3,a,b}^+)\rangle$.

Lemma 4.12 $\dim(U_{3,a,b}') \le 20$.

Proof The polar space Δ_1^+ induces on $a^\perp \cap b^\perp$ the line grassmannian of a projective geometry $\Pi_{a,b} \cong \mathrm{PG}(3,\mathbb{F})$. So, the set $\Delta_{3,a,b}^+$ is partitioned in two sets P_0 and P_1, corresponding to the points and the planes of $\Pi_{a,b}$.

Let $p \in a^\perp \cap b^\perp$. The residue $\mathrm{res}(p)$ of p in Δ^+ is a D_3-building, $\Delta_{3,p}^+$ is the 2-grassmannian of the building $\mathrm{res}(p)$ and the subgeometry $\Delta_{2,p}^+$ induced by Δ_2^+ on the set of lines of Δ_1^+ through p is the polar space associated to $\mathrm{res}(p)$. The lines l_a and l_b of Δ_1^+ joining p with a or b respectively, are points of the polar space $\Delta_{2,p}^+$ and $S_p := \Delta_{3,p}^+ \cap \Delta_{3,a,b}^+$ is the set of lines of $\Delta_{2,p}^+$ contained in $l_a^\perp \cap l_b^\perp$. The set S_p is partitioned in two families $P_{p,0} = S_p \cap P_0$ and $P_{p,1} = S_p \cap P_1$. In the polar space $\Delta_{2,p}^+$, the sets $P_{p,0}$ and $P_{p,1}$ form the two families of lines of a grid. On the other hand, p is a line of the projective geometry $\Pi_{a,b}$. Accordingly, $P_{p,0}$ is the set of points of the line p of $\Pi_{a,b}$ and $P_{p,1}$ is the set of planes of $\Pi_{a,b}$ through p.

The polar space $\Delta^+_{2,p}$ can also be regarded as the line-grassmannian of a projective geometry $\Pi_p \cong \mathrm{PG}(3,\mathbb{F})$. The lines l_a and l_b appear as two skew lines in Π_p while the elements of $P_{p,0}$ and $P_{p,1}$ are point-plane flags, formed by a point in l_a and a plane through l_b or a point of l_b and a plane on l_a. We can assume that the elements of $P_{p,0}$ are flags $\{x, X\}$ with x a point of l_a and X a plane through l_b while those of $P_{p,1}$ are flags $\{x, X\}$ with $x \in l_b$ and $X \supset l_a$.

The embedding $\eta^+_{3,p} : \Delta^+_{3,p} \to \mathrm{PG}(U'_{3,p})$ is a quotient of the Weyl embedding $\widetilde{\varepsilon}^+_{2,p} : \Delta^+_{3,p} \to \mathrm{PG}(\widetilde{U})$, where $\widetilde{U}$ is the vector space of null-traced 4×4-matrices with entries in $\mathbb{F}$. We have given an explicit description of this embedding in Sect. 4.3. Comparing that description with the above characterization of $P_{p,0}$ and $P_{p,1}$ as point-plane flags of Π_p, one can see that for $i = 0, 1$ the embedding $\widetilde{\varepsilon}^+_{2,p}$ maps $P_{p,i}$ onto a non-singular conic $C_{p,i}$ of $\mathrm{PG}(\widetilde{U})$. Explicitly, if $\{u_1, u_2, u_3, u_4\}$ is a basis of $V(4,\mathbb{F})$ such that l_a and l_b correspond to the lines $\langle u_1, u_2\rangle$ and $\langle u_3, u_4\rangle$ of $\mathrm{PG}(3,\mathbb{F}) \cong \Pi_p$, then $C_{p,0} = \{\langle M_0(s,t)\rangle\}_{s,t\in\mathbb{F}}$ and $C_{p,1} = \{\langle M_1(s,t)\rangle\}_{t,s\in\mathbb{F}}$ where

$$M_0(t,s) = \begin{bmatrix} -ts & s^2 & 0 & 0 \\ -t^2 & ts & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_1(t,s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -ts & s^2 \\ 0 & 0 & -t^2 & ts \end{bmatrix}.$$

Let φ be the projection of $\widetilde{U}$ over $U'_{3,p}$. Since $\eta^+_{3,p}$ is injective, $\langle C_{p,i}\rangle \cap \ker(\varphi)$ is either the null subspace or the nucleus of the conic $C_{p,i}$, the latter case possibly occurring only if $\mathrm{char}(\mathbb{F}) = 2$. In the first case φ maps $C_{p,i}$ onto a conic of $\mathrm{PG}(U'_{3,p})$. In the second case, since $\mathbb{F}$ is perfect by assumption, every line of $\langle C_{p,i}\rangle$ through the nucleus of $C_{p,i}$ meets $C_{p,i}$ in exactly one point. Hence φ maps $C_{p,i}$ onto a line of $\mathrm{PG}(U'_{3,p})$. Thus, $\eta^+_{3,p}$ maps $P_{p,i}$ onto either a conic or a line.

Let now $\eta^+_{3,a,b}$ be the restriction of η^+_3 to $\Delta^+_{3,a,b}$. Clearly, $\eta^+_{3,a,b}$ and $\eta^+_{3,p}$ induce the same mapping on $\Delta^+_{3,a,b} \cap \Delta^+_{3,p}$. As remarked above, every point $p \in a^\perp \cap b^\perp$ is a line of $\Pi_{a,b}$ and $P_{p,0}$ is the set of points of that line. Moreover P_0 is the set of points of $\Pi_{a,b}$. As $\eta^+_{3,p}$ maps $P_{p,0}$ onto either a conic or a line, $\eta^+_{3,a,b}$ is a quasi-veronesean embedding of $\Pi_{a,b}$. By Lemma 4.11, $\dim(\langle \eta^+_{3,a,b}(P_0)\rangle) \le 10$. By a dual argument, $\dim(\langle \eta^+_{3,a,b}(P_1)\rangle) \le 10$. Hence $\dim(\langle \eta^+_{3,a,b}(P_0 \cup P_1)\rangle) \le 20$. □

The polar space $\Delta^+_{1,a,b}$ induced by Δ^+_1 on $a^\perp \cap b^\perp$ can be generated by six points, say $p_1, p_1, \ldots, p_6$. For every $i = 1, 2, \ldots, 6$, let α_i be a plane of Δ^+_1 on p_i such that $a^\perp \cap b^\perp \cap \alpha_i = \{p_i\}$. Put $S := \langle \{\alpha_i\}_{i=1}^6 \cup \Delta^+_{3,a} \cup \Delta^+_{3,b} \cup \Delta^+_{3,a,b}\rangle_{\Delta^+_3}$.

Lemma 4.13 $S = \Delta^+_3$.

Proof Throughout the proof of this lemma the words 'point', 'line', 'plane' and 'space' refer to a point, a line, a plane or a 3-space, respectively, of the polar space Δ^+_1. We say that a point $p \in a^\perp \cap b^\perp$ is *S-full* if all planes on p belong to S. We chop our proof in a series of steps.

(1) Every plane contained in a common space with either a or b belongs to S.

Let X be a space on a. Then $b^\perp \cap X$ is a plane. It belongs to $\Delta^+_{3,a,b}$. Hence it belongs to S. On the other hand, $a \notin b^\perp \cap X$ since $a \not\perp b$ by assumption. Moreover, all planes of X through a belong to $\Delta^+_{3,a}$, whence they belong to S. It follows that all planes of X are in S. Claim (1) follows.

(2) Given a point $p \in a^\perp \cap b^\perp$, if there is a plane α_0 on p such that $\alpha_0 \in S$ and $\alpha_0 \cap a^\perp \cap b^\perp = \{p\}$, then p is S-full.

Let X be a space on α_0. By (1), both planes $a^\perp \cap X$ and $b^\perp \cap X$ belong to S. These two planes meet α_0 in distinct lines passing through p. Therefore, and since $\alpha_0 \in S$, all planes of X through p belong to S. Let H be the set of planes through p that meet $a^\perp \cap b^\perp$ in at least a line. The complement $\Delta^+_{3,p} \setminus H$ of H in $\Delta^+_{3,p}$ is a connected subgeometry of $\Delta^+_{3,p}$. It contains α_0, which belongs to S. Hence, by the above, $\Delta^+_{3,p} \setminus H \subseteq S$. Moreover, still by the above, every plane through p contained in a common subspace with a plane of $\Delta^+_{3,p} \setminus H$ belongs to S. On the other hand, a plane through p is not contained in a common space with any of the planes of $\Delta^+_{3,p} \setminus H$ only if it belongs to $\Delta^+_{3,a,b}$. If so, it belongs to S. Therefore S contains all planes through p.

(3) Let x, y, z be three points of a line $l \subset a^\perp \cap b^\perp$ and suppose that both x and y are S-full. Then z is S-full too.

There exists at least one space X containing l and such that $a^\perp \cap b^\perp \cap X = l$. As x and y are S-full, all planes of X through either x or y belong to S. Hence all planes of X belong to S. On the other hand $a^\perp \cap b^\perp \cap X = l$. Therefore there exists at least one plane α of X containing z and such that $\alpha \cap a^\perp \cap b^\perp = \{z\}$. This plane belongs to S, as all planes of X belong to S. So, z satisfies the hypotheses of (2). By (2), z is S-full.

We can now finish the proof of the lemma. For every $i = 1, 2, \ldots, 6$, we have chosen the plane α_i on p_i in such a way that the hypotheses of (2) hold for p_i and α_i. Hence p_i is S-full. On the other hand, $p_1, \ldots, p_6$ span $\Delta^+_{1,a,b}$, by assumption. Therefore every point of $a^\perp \cap b^\perp$ is S-full, by (3). Finally, every plane meets $a^\perp \cap b^\perp$ in at least a point. Hence every plane belongs to S. □

Theorem 4.14 *Let $\mathbb{F}$ be either a perfect field of positive characteristic, different from $\mathbb{F}_2$, or a number field. Let $n = 4$. Then the Weyl embedding $\widetilde{\varepsilon}^+_3$ is universal.*

Proof By Lemma 4.13, for every projective embedding η^+_3 of Δ^+_3 we have

$$\dim(\eta^+_3) \le 6 + \dim(\eta^+_{3,a}) + \dim(\eta^+_{3,b}) + \dim(U'_{3,a,b}).$$

On the other hand, $\dim(\eta^+_{3,a})$ and $\dim(\eta^+_{3,b})$ are less or equal to 15 by Lemma 4.8 and $\dim(U'_{3,a,b}) \le 20$ by Lemma 4.12. Hence $\dim(\eta^+_3) \le 56$. However $\dim(\widetilde{\varepsilon}^+_3) = 56$ and Δ^+_3 admits the absolutely universal embedding [8]. Hence $\widetilde{\varepsilon}^+_3$ is universal. □

4.6 Proof of Theorem 1.5. The case $k = 3$

By an inductive argument on n, using Theorem 4.14 to start and Corollary 4.5 combined with part (2) of Corollary 4.7 to go on, we obtain the following:

Theorem 4.15 *Let $\mathbb{F}$ be a perfect field of positive characteristic, different from $\mathbb{F}_2$ or a number field and let $n > 3$. Then every projective embedding of Δ_3 has dimension at most $\binom{2n+1}{3}$ and every projective embedding of Δ_3^+ has dimension at most $\binom{2n}{3}$.*

Since $\dim(\widetilde{\varepsilon}_3) = \binom{2n+1}{3}$ and $\widetilde{\varepsilon}_3^+ = \binom{2n}{3}$, Theorem 4.15 immediately implies the following corollary, which contains part (2) of Theorem 1.5.

Corollary 4.16 *Let $\mathbb{F}$ be a perfect field of positive characteristic, different from $\mathbb{F}_2$, or a number field and let $n > 3$. Then both $\widetilde{\varepsilon}_3$ and $\widetilde{\varepsilon}_3^+$ are universal.*

4.7 Remarks

1. The assumptions $n > 2$ when $k = 2$ and $n > 3$ when $k = 3$ cannot be removed from Theorem 1.5. Indeed when $n = k = 2$ or 3 the Weyl embedding $\widetilde{\varepsilon}_k$ is veronesean. Regretfully, we do not know so much on veronesean embeddings. We guess that $\widetilde{\varepsilon}_n$ is relatively universal when $\mathrm{char}(\mathbb{F}) \neq 2$, but so far we have not found a way to prove this conjecture, even in the case of $n = 2$. On the other hand, if $\mathbb{F}$ is a perfect field of characteristic 2 then $\widetilde{\varepsilon}_n$ is not universal, for any n (see [11]).
2. One might believe that the ideas exploited in Sect. 4.3 can be re-used to obtain results similar to Corollaries 4.3, 4.5 and 4.7 for any $k < n$, but things are not so easy as they look at first glance. For instance, let $k = 4$. Instead of choosing a generating set $p_1, p_2, \dots, p_{2n-1}$ for the polar space induced by Δ_1 on $a^\perp \cap H$ and suitable planes $\alpha_1, \alpha_2, \dots, \alpha_{2n-1}$ on $p_1, p_2, \dots, p_{2n-1}$ as we have done for Δ_3, we could consider a generating set $\{l_1, \dots, l_m\}$ of the 2-grassmannian $\overline{\Delta}_2$ of that polar space and a suitable 4-space on each of $l_1, \dots, l_m$. However, for this move to be effective we need $m = \binom{2n-1}{2}$. Thus, we should know that $\overline{\Delta}_2$ admits a generating set of size $\binom{2n-1}{2}$. However we do not know if this is true in general. It is true when $\mathbb{F}$ is a finite prime field (Cooperstein [14]), but perhaps it is false for other fields (compare Blok and Pasini [7]). Anyway, Theorem 1.5 is of no help here. That theorem only tells us that every projective embedding of $\overline{\Delta}_2$ is at most $\binom{2n-1}{2}$-dimensional. It says nothing on generating sets.

   We face similar difficulties if, in the attempt to generalize Theorem 4.14 to Δ_{n-1}^+ with $n > 4$, we try to rephrase the proof of Lemma 4.13. Besides this, in order to generalize Theorem 4.14 we must preliminarily prove an analog of Lemma 4.11 for quasi-veronesean embeddings of half spin geometries, obtaining an upper bound for the dimension of such an embedding, but this does not look so easy to do. Perhaps, it is equivalent to determine an upper bound for the dimension of a quasi-veronesean embedding of Δ_n.

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