

Combinatorial Hopf algebra of superclass functions of type D

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Abstract We provide a Hopf algebra structure on the space of superclass functions on the unipotent upper triangular group of type D over a finite field based on a supercharacter theory constructed by André and Neto in J. Algebra 305, 394–429 (2006) and 322 (2009). Also, we make further comments with respect to types B and C . Type A was explored by M. Aguiar et al. in Adv. Math. 229 (2012); thus this paper is a contribution to understand combinatorially the supercharacter theory of the other classical Lie types.

Keywords Supercharacters · Set partitions of type D · Hopf algebras · Unipotent upper triangular groups

1 Introduction

The problem of simultaneously reducing to canonical form two linear operators on a finite-dimensional space is a “wild” problem in representation theory. This problem contains all classification matrix problems given by quivers (see [9]). In this sense, the classical representation theory for the type A group $U_n(q)$ of unipotent $n \times n$ upper triangular matrices over a finite field is known to be wild. This makes, in some sense, hopeless any attempt to study the representation theory of the group $U_n(q)$. In his Ph.D. thesis C. André started to develop a theory that approximates the representation theory of $U_n(q)$. Roughly speaking, by using certain linear combinations of irreducible characters and lumping together conjugacy classes under certain conditions, the resulting theory behaves very nicely (see [5, 18, 19]). This gave rise to the concept of “supercharacter theory”. Later on, P. Diaconis and I.M. Isaacs extended this concept to arbitrary algebra groups (see [14]). Supercharacter theory of

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the group $U_n(q)$ has a rich combinatorics which connects this beautiful theory with classical combinatorial objects. As a matter of fact, in [3] a Hopf algebra structure is provided on the graded vector space \mathbf{SC} of superclass characteristic functions over $U_n(q)$, for $n \geq 0$. Moreover, when $q = 2$ this Hopf algebra is a realization of a well-known combinatorial Hopf algebra, namely, the Hopf algebra of symmetric functions in noncommuting variables (see [11, 17]).

The reader familiar with the classical representation theory of the symmetric group S_n will notice how this resembles the relationship between symmetric functions and the character theory of S_n . Also, supercharacters of $U_n(2)$ are indexed by set partitions of the set $[n] = \{1, 2, \dots, n\}$ and by *labeled* set partitions for general q .

In this paper, we study combinatorially the supercharacter theory corresponding to the other classical Lie types B , C and D , making emphasis on the latter. This study is based on the supercharacter theory constructed by André and Neto in [6, 7]. These groups fail to be algebra groups unlike type A . However, we can regard them as subgroups of the convenient group of type A and restrict the supercharacter theory of type A to the respective subgroup.

The paper is organized as follows. In Sect. 2 we provide the reader with the basic definitions concerning combinatorial Hopf algebras and supercharacters. In Sect. 3, we give a combinatorial interpretation for the supercharacter theory of the group $U_{2n}^D(q)$ of even orthogonal unipotent upper triangular matrices with coefficients in the field \mathbb{F}_q of characteristic ≥ 3 . This combinatorial interpretation uses *labeled* D_{2n} -partitions of the set $[\pm n] := \{1, \dots, n, -n, \dots, -1\}$. More specifically, we use these partitions to index orbit representatives for superclasses and supercharacters of the group $U_{2n}^D(q)$. Then, we define the analog of \mathbf{SC} for type D as follows:

$$\begin{aligned} \mathbf{SC}^D &= \bigoplus_{n \geq 0} \mathbf{SC}_{2n}^D \\ &= \bigoplus_{n \geq 0} \text{span}_{\mathbb{C}} \{ \kappa_{\lambda} : \lambda \in D_{2n}(q) \} \end{aligned}$$

where κ_{λ} denotes the superclass characteristic function indexed by the labeled D_{2n} -partition λ . Using a change of basis, we prove that the space \mathbf{SC}^D is endowed with a Hopf algebra structure. This Hopf algebra structure is in analogy to the one given for type A in [3]. However, the product structure on \mathbf{SC}^D is not raised directly from representation theory. Yet, Proposition 3.15 suggests that the algebra structure holds a similar connection with representation theory, as in the type A case. This suggested connection awaits exploration. The coalgebra structure is raised directly from representation theory by using restriction.

In the final section, we discuss briefly the supercharacter theory for types B and C . Also we make some remarks concerning forthcoming work on the Hopf monoid structure that \mathbf{SC}^D carries, following the results in [4].

2 Preliminaries

We start by defining supercharacter theory for a finite group G . This definition, which can be stated in different ways, is due to Diaconis and Isaacs [14].

Definition 2.1 A supercharacter theory for G consists of:

- A partition \mathcal{K} of G
- A set \mathcal{X} of characters of G

such that the following holds:

1. $|\mathcal{K}| = |\mathcal{X}|$
2. every irreducible character of G is a constituent of a unique $\chi \in \mathcal{X}$
3. the characters in \mathcal{X} are constant on members of \mathcal{K} .

The elements in \mathcal{X} are scalar multiples of linear combinations of the form $\sum_{\psi \in X} \psi(1)\psi$ where X is a subset of irreducible characters of G , by [14, Lemma 2.1].

Remark Definition 2.1 is equivalent to say that $\text{span}_{\mathbb{C}}\{\sum_{g \in K} g : K \in \mathcal{K}\}$ is a subalgebra of $Z(\mathbb{C}G)$ with unit 1. Given such a partition \mathcal{K} , there exists a unique \mathcal{X} , up to isomorphism, with the desired properties.

Examples 2.2

- Every group is endowed with the supercharacter theory where the set of superclasses \mathcal{K} consists of the usual conjugacy classes and the set of supercharacters \mathcal{X} is formed by the irreducible characters of G .
- Similarly, the trivial supercharacter theory of G is such that $\mathcal{K} = \{\{1\}, G - \{1\}\}$ and $\mathcal{X} = \{1, \rho_G - 1\}$, where ρ_G is the regular representation.
- A less trivial example is given by the cyclic group of order 2^n , where $n \geq 2$. It is not hard to see that lumping together the elements of G by their order, gives a set of superclasses K , whose corresponding supercharacters are formed by adding together all the d -primitive roots of unity for each $d|n$.

This paper explores the particular supercharacter theory constructed by André and Neto in [7] of the classical group $U_{2n}^D(q)$ of $2n \times 2n$ unipotent upper triangular matrices of type D . Here, we refer to this construction as *the* supercharacter theory of type D , since it is the one we are interested in. We regard $U_{2n}^D(q)$ as a subgroup of the group $U_{2n}(q)$ of $2n \times 2n$ unipotent upper triangular matrices, which is an *algebra group* as defined below (see [15]).

Definition 2.3 Let J be a finite dimensional associative nilpotent \mathbb{F} -algebra and let G denote the set consisting of formal objects of the form $1 + a$ where $a \in J$. Then $G = 1 + J$ is a group, where the multiplication is given by $(1 + a)(1 + b) = 1 + a + b + ab$. The group G is the *algebra group* based on J .

As an example, denote by u_n the algebra of nilpotent upper triangular matrices associated to the group $U_n(q)$. Then we see that $U_n(q) = I + u_n$, and thus $U_n(q)$ is an algebra group.

The supercharacter theory for the group $U_n(q)$ has a very nice combinatorial interpretation. Its superclasses are indexed by labeled set partitions of type A as well as its supercharacters (see [3]). In analogy with type A , in the next section we describe

the supercharacter theory for the group $U_{2n}^D(q)$ using *labeled D_{2n} -partitions*, though as mentioned in the introduction, $U_{2n}^D(q)$ is not an algebra group. Before that, we give a quick intro to combinatorial Hopf algebras. For a further reading on this topic, see [2].

2.1 Combinatorial Hopf algebras

Let \mathcal{A} be a vector space over a field \mathbb{K} . We say that \mathcal{A} is an associative algebra with unit 1 if \mathcal{A} has a linear map $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ such that $m \circ (m \otimes \text{Id}) = m \circ (\text{Id} \otimes m)$ where Id is the identity map in \mathcal{A} . The unit can also be associated with a linear map $u : \mathbb{K} \rightarrow \mathcal{A}$ such that $t \mapsto t \cdot 1$. The maps m and u must be compatible in the sense that

$$m \circ (\text{Id} \otimes u) = m \circ (u \otimes \text{Id}) = \text{Id}$$

On the other hand, a *coalgebra* is a vector space D over \mathbb{K} with a coproduct $\Delta : D \otimes D \rightarrow D$ and a counit $\epsilon : D \rightarrow \mathbb{K}$ which are \mathbb{K} -linear maps. The coproduct must be coassociative in the sense that $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$ and must be compatible with ϵ :

$$(\epsilon \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \epsilon) \circ \Delta = \text{Id}$$

If an algebra (\mathcal{A}, m, u) has also a coalgebra structure given by Δ, ϵ , we say that \mathcal{A} is a *bialgebra* if Δ, ϵ are algebra homomorphisms.

Definition 2.4 A *Hopf algebra* \mathcal{A} is a bialgebra together with a linear map $S : \mathcal{A} \rightarrow \mathcal{A}$ called antipode. The map S satisfies

$$\sum_k S(a_k)b_k = \epsilon(a) \cdot 1 = \sum_k a_k S(b_k) \quad \text{where } \Delta(a) = \sum_k a_k \otimes b_k$$

We say that a bialgebra \mathcal{A} is *graded* if there exists a direct sum decomposition

$$\mathcal{A} = \bigoplus_{k \geq 0} A_k$$

such that $A_i \otimes A_j \subseteq A_{i+j}$, $u(\mathbb{K}) \subseteq A_0$, $\Delta(A_i) \subseteq \bigoplus_{j=0}^i A_j \otimes A_{i-j}$ and $\epsilon(A_n) = 0$ for $n \geq 1$. Finally, we say that \mathcal{A} is *connected* if $A_0 \cong \mathbb{K}$.

There are few different notations of a combinatorial Hopf algebra in the literature, but in this paper we say that \mathcal{A} is a *combinatorial Hopf algebra* if \mathcal{A} is a graded and connected bialgebra with antipode and such that \mathcal{A} has a distinguished basis with positive structure constants [12], i.e., a distinguished basis that multiplies/comultiplies positively.

3 Supercharacter theory of type D

The supercharacter theory of type D this paper considers is due to André and Neto [6, 7]. Here, we give a combinatorial interpretation of their algebro-geometric

construction. From now on, \mathbb{F}_q will denote a field of characteristic $p \geq 3$ and order $q = p^r$ for some integer $r \geq 1$. Also, denote by \mathbb{F}_q^* the multiplicative group of non-zero elements of the field \mathbb{F}_q . The group $U_{2n}^D(q)$ corresponds to even orthogonal unipotent upper triangular matrices with coefficients in \mathbb{F}_q and can be described as the following set (see [13]):

$$U_{2n}^D(q) = \left\{ \begin{pmatrix} P & PQ \\ 0 & JP^{-t}J \end{pmatrix} : P \in U_n(q), Q \in M_n(q), JQ^tJ = -Q \right\},$$

where $M_n(q)$ is the set of $n \times n$ matrices over \mathbb{F}_q and J is the $n \times n$ matrix with ones in the antidiagonal and zeros elsewhere.

We will drop the subindex $2n$ from our notation when the size of matrices is clear from context. In order to describe the supercharacters of $U^D(q)$, we make use of an associated nilpotent algebra $\mathfrak{u}^D(q)$. The algebra $\mathfrak{u}^D(q)$ is given by

$$\mathfrak{u}^D(q) = \left\{ \begin{pmatrix} R & Q \\ 0 & -JR^tJ \end{pmatrix} : R \in U_n(q) - I_n, Q \in M_n(q), JQ^tJ = -Q \right\}$$

with $M_n(q)$ and J as before. We will make use of the total order

$$1 < \dots < n < -n < \dots < -1$$

to index the columns and rows of matrices in $U^D(q)$ and in $\mathfrak{u}^D(q)$, from left to right and top to bottom.

A vector space basis for $\mathfrak{u}^D(q)$ over \mathbb{F}_q is given by the matrices $\{y_\alpha\}_\alpha$ where α runs over the set of positive roots Φ^+ of type D , given by

$$\Phi^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\}$$

and y_α denotes the matrix

$$y_\alpha = \begin{cases} e_{i,j} - e_{-j,-i} & \text{if } \alpha = e_i - e_j \\ e_{i,-j} - e_{j,-i} & \text{if } \alpha = e_i + e_j \end{cases}$$

where $e_{i,j} \in \mathfrak{u}^D(q)$ has 1 in position i, j and zeros elsewhere. Now define the support of y_α by

$$\text{supp}(y_\alpha) = \begin{cases} (i, j), (-j, -i) & \text{if } \alpha = e_i - e_j \\ (i, -j), (j, -i) & \text{if } \alpha = e_i + e_j \end{cases}$$

Notice that this definition can be extended linearly to the whole of $\mathfrak{u}^D(q)$.

Denote by $[\pm n]$ the set $\{\pm 1, \pm 2, \dots, \pm n\}$. Combinatorially, linear combinations of the matrices y_α with at most one non-zero entry in every row and column can be seen as *labeled D_{2n} -partitions* or simply *$D_{2n}(q)$ -partitions*, which consists of triples (i, j, a) where $i, j \in [\pm n]$ and $a \in \mathbb{F}_q^*$. Any triple of this form is called a *labeled arc* and will be represented as $i \overset{a}{\frown} j$. Thus, we have the following definition.

Definition 3.1 A $D_{2n}(q)$ -partition λ of $[\pm n]$ is a set of labeled arcs in $[\pm n]$ such that for $j \neq -i$:

- (a) If $i \overset{a}{\frown} j \in \lambda$ then $-j \overset{-a}{\frown} -i \in \lambda$
- (b) If $i \overset{a}{\frown} j \in \lambda$ and $i < k < j$ then $i \overset{b}{\frown} k \notin \lambda, k \overset{b}{\frown} j \notin \lambda$.

We write $\lambda \in D_{2n}(q)$ to indicate that λ is a $D_{2n}(q)$ -partition.

The number of $D_{2n}(q)$ -partitions is given in [16] where the notion of labeled D partitions was previously defined, as well as their analog in type B .

For $\lambda \in D_{2n}(q)$, we define the corresponding matrix $y_\lambda \in u^D(q)$ by

$$y_\lambda = \sum_{i \overset{a}{\frown} j \in \lambda} a e_{i,j} \quad (3.1)$$

The map $\lambda \mapsto y_\lambda$ defines a bijection between set partitions in $D_{2n}(q)$ and matrices in $u^D(q)$ with at most one non-zero element in every row and column.

Every $\lambda \in D_{2n}(q)$ can be written uniquely as $\lambda = \lambda^+ \cup \lambda^-$, where:

- $\lambda^+ \cap \lambda^- = \emptyset$
- $i \overset{a}{\frown} j \in \lambda^+$ if and only if $-j \overset{-a}{\frown} -i \in \lambda^-$ where $i > 0$ and $i < |j|$

In view of this, λ is completely determined by λ^+ (or λ^-). Thus, every arc $i \overset{a}{\frown} j \in \lambda^+$ can be represented by the triple $\{(i, j, a)\}$. In this case, the triple $\{(-j, -i, -a)\} \in \lambda^-$.

3.1 Superclasses and supercharacters

In this section we describe combinatorially the superclasses and supercharacters of $U_{2n}^D(q)$ using $D_{2n}(q)$ -partitions and keeping in mind that $U_{2n}^D(q)$ is a subgroup of $U_{2n}(q)$. Using algebraic varieties, André and Neto proved that supercharacters and superclasses of the group $U_{2n}^D(q)$ are indexed by matrices in $u^D(q)$ with at most one non-zero element in every row and column (see [7]). Thus, they can be indexed using $D_{2n}(q)$ -partitions as well.

The group $U_{2n}(q)$ acts on its nilpotent algebra $u_{2n}(q)$ by left and right multiplication. It can be shown that when adding the identity matrix I_{2n} to each one of these orbits we get the superclasses of $U_{2n}(q)$ (see [14]).

Let $\lambda \in D_{2n}(q)$ and let y_λ as in (3.1). Since $u^D(q) \subset u_{2n}(q)$ we can consider the orbit

$$V_\lambda = U_{2n}(q)y_\lambda U_{2n}(q) \in u_{2n}(q)$$

Notice that V_λ is not necessarily in $u^D(q)$. However, since $V_\lambda + I_{2n}$ is a superclass in $U_{2n}(q)$ and $U_{2n}^D(q)$ is a subgroup of $U_{2n}(q)$, then we have the following definition.

Definition 3.2 Let V_λ as above. The superclass in $U_{2n}^D(q)$ associated to λ is denoted by K_λ and is defined as $K_\lambda = U_{2n}^D(q) \cap (V_\lambda + I_{2n})$.

As mentioned in the introduction $U_{2n}^D(q)$ is not an algebra group, i.e., $U_{2n}^D(q) \neq I_{2n} + u_{2n}^D(q)$. Yet there is a bijective correspondence between $U_{2n}^D(q)$ and $u^D(q)$. This bijection is provided by the following lemma of André and Neto.

Lemma 3.3 ([7, Lemma 2.3]) *Let λ be a D_{2n} -partition and let I denote the identity matrix of the corresponding size. Put x and y as*

$$x = \left(\begin{array}{c|c} P & PQ \\ \hline 0 & JP^{-t}J \end{array} \right) \in U^D(q) \quad \text{and} \quad y = \left(\begin{array}{c|c} P-I & Q \\ \hline 0 & -J(P-I)^tJ \end{array} \right) \in u^D(q)$$

Then $x \in K_\lambda$ if and only if $y \in V_\lambda$.

To illustrate this lemma, let us consider the following example:

Example 3.4 Let $n = 5$ and let λ be the $D_{2n}(q)$ -partition given by

$$\lambda = \begin{array}{ccccccccc} & a & b & c & -c & -b & -a & & \\ & \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright & & \\ 1 & 2 & 3 & 4 & -4 & -3 & -2 & -1 & \end{array}$$

A natural representative for the orbit V_λ is given by the corresponding y_λ . In this example we have

$$y_\lambda = \left(\begin{array}{cccc|cccc} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & b & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & c & 0 & 0 & 0 \\ & & & 0 & 0 & -c & 0 & 0 \\ \hline & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & -b & 0 \\ & & & & & & 0 & -a \\ & & & & & & & 0 \end{array} \right) \quad \text{and}$$

$$x_\lambda = \left(\begin{array}{cccc|cccc} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & b & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & c & bc & 0 & 0 \\ & & & 1 & 0 & -c & 0 & 0 \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & -b & 0 \\ & & & & & & 1 & -a \\ & & & & & & & 1 \end{array} \right)$$

where x_λ is the matrix in K_λ given by Lemma 3.3.

Henceforth we denote by θ a fixed nontrivial homomorphism from the additive group of \mathbb{F}_q to \mathbb{C}^* . The following theorem of André and Neto defines the supercharacters of type D .

Theorem 3.5 ([8, Theorem 5.3]) *Let λ be a $D_{2n}(q)$ -partition and let x_μ be the superclass associated to the $D_{2n}(q)$ -partition μ . Then the complex valued function χ^λ*

which is constant on superclasses of $U_{2n}^D(q)$ is given by

$$\chi^\lambda(x_\mu) = \begin{cases} \frac{\chi^\lambda(1)}{q^{|\{k \curvearrowright l \in \mu^+ | i \prec k \prec l \prec j, i \curvearrowright j \in \lambda^+\}|}} \prod_{i \curvearrowright j \in \lambda^+, i \curvearrowright j \in \mu^+} \theta(ab) & \text{if } i \curvearrowright k \in \lambda \text{ and } i \prec l \prec k, \text{ then } i \curvearrowright l, l \curvearrowright k \notin \mu, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

The set of functions $\{\chi^\lambda : \lambda \in D_{2n}(q)\}$ forms a supercharacter theory for the group $U_{2n}^D(q)$.

A few remarks are worth mentioning about some of the combinatorial properties of supercharacters. For an algebraic proof see [7]:

- $\chi^\lambda(1) = \begin{cases} q^{j-i-1} & \text{if } j \leq n \\ q^{2n-i-j} & \text{otherwise} \end{cases}$ when $\lambda^+ = \{(i, j, a)\}$ is a single arc.
- $\chi^\lambda = \prod_{\lambda_{ij} \in \lambda} \chi^{\lambda_{ij}}$ where $\lambda_{ij} = \{i \curvearrowright j, -j \curvearrowright -i\} \in \lambda$. Thus, $\chi^\lambda(1) = 1$ if and only if $j = i + 1$ for every λ_{ij} .

3.2 Product and coproduct

Let \mathbf{SC}_{2n}^D be the vector space of superclass functions over the group $U_{2n}^D(q)$. This is the space of functions $\alpha : U_{2n}^D(q) \rightarrow \mathbb{C}$ that are constant on superclasses. Now that we know how superclasses and supercharacters look like as matrices and as partitions, we will define a product and a coproduct on the graded vector space $\mathbf{SC}^D = \bigoplus_{n \geq 0} \mathbf{SC}_{2n}^D$. By convention, $\mathbf{SC}_0^D = \mathbb{C}$. As proved in [8, Theorem 4.1], the supercharacters of $U_{2n}^D(q)$ form a basis for \mathbf{SC}_{2n}^D .

Given $\lambda \in D_{2n}(q)$ define $\kappa_\lambda \in \mathbf{SC}_{2n}^D$ as the function with the formula

$$\kappa_\lambda(x_\mu) = \begin{cases} 1 & \text{if } x_\mu \text{ is in the superclass of } x_\lambda \\ 0 & \text{otherwise} \end{cases}$$

for $\mu \in D_{2n}(q)$. These *superclass characteristic functions* of course form another basis of \mathbf{SC}^D .

First, we will endow the vector space \mathbf{SC}^D with a coalgebra structure and thus we want to define a coproduct. As before we denote by $[\pm n]$ the set $\{\pm 1, \pm 2, \dots, \pm n\}$ so that if $A \subseteq [n]$ then $[\pm A] = A \cup -A$ where $-A = \{-i : i \in A\}$. We start by giving the following definition.

Definition 3.6 Let $J = (J_1 | \dots | J_r)$ be a sequence of subsets of $[\pm n]$. We call J a *D-set partition* of $[\pm n]$ if for all $i \neq j$ such that $1 \leq i, j \leq r$ we have $J_i = -J_j$, $J_i \cap J_j = \emptyset$ and $\bigcup_i J_i = [\pm n]$.

Remark that for a *D-set partition* $J = (J_1 | \dots | J_r)$, each part J_i is completely determined by the set $J_i \cap [n]$. In particular, if $J = (J_1 | J_2)$ and $J_1 \cap [n] = A$ then

$J_2 \cap [n] = A^c$, where $A^c = [n] \setminus A$. For this reason, we will write such a D -set partition as $J = (A|A^c)$. As an example, with $n = 4$ we see that the D -set partition $J = (1\bar{3}\bar{1}\bar{3}|2\bar{4}\bar{2}\bar{4})$ can be written as $J = (A|A^c)$ where $A = \{1, 3\}$.

Let $J = (A|A^c)$ be an ordered D -set partition of $[\pm n]$ and let

$$S_J(q) = \{\lambda \in D_{2n}(q) : i \overset{a}{\curvearrowright} j \text{ in } \lambda \text{ implies } i, j \text{ are in the same part of } J\}$$

and define the *standardization map* by the bijection

$$\text{st}_J : S_J(q) \rightarrow S_{[\pm|A|]}(q) \times S_{[\pm|A^c|]}(q) \quad (3.3)$$

that relabels the indices of partitions in $S_J(q)$ according to the unique order-preserving map

$$\text{st}_A : \pm A \rightarrow [\pm|A|] \quad (3.4)$$

where A is a part of J .

As an example, let $J = \{134|25\}$ and let λ be given by

$$\lambda = \begin{array}{cccccccccccc} & a & & b & & c & -c & & -b & & -a & \\ \bullet & \curvearrowright & \bullet & & \bullet & \curvearrowright & \bullet & & \bullet & \curvearrowright & \bullet & \\ 1 & 2 & 3 & 4 & 5 & -5 & -4 & -3 & -2 & -1 \end{array} \quad (3.5)$$

then

$$\text{st}_J(\lambda) = \begin{array}{ccccccc} a & & c & -c & & -a & \\ \bullet & \curvearrowright & \bullet & & \bullet & \curvearrowright & \bullet \\ 1 & 2 & 3 & -3 & -2 & -1 \end{array} \times \begin{array}{ccccc} b & & & -b & \\ \bullet & & \bullet & & \bullet \\ 1 & 2 & -2 & -1 \end{array} \in S_{[\pm 3]}(q) \times S_{[\pm 2]}(q)$$

Definition 3.7 Let $J = (A|A^c)$ be an ordered D -set partition of $[\pm n]$. Define $U_J^D \subseteq U^D$, where $U^D = U_{2n}^D(q)$, as

$$U_J^D = \{x \in U^D : x_{ij} \neq 0 \text{ implies } i, j \text{ are in the same part of } J\}$$

The map in (3.3) can be extended to produce an isomorphism $\text{st}_J : U_J^D \rightarrow U_{2|A|}^D(q) \times U_{2|A^c|}^D(q)$ by reordering the rows and columns as in (3.4).

The restriction map on $\mathbf{SC}_{2n}^D(q)$ is given by

$$\text{Res}_{\text{st}_J(U_J^D)}^{U^D} : \mathbf{SC}_{2n}^D(q) \rightarrow \mathbf{SC}_{2|A|}^D(q) \otimes \mathbf{SC}_{2|A^c|}^D(q) \quad (3.6)$$

where $\text{Res}_{\text{st}_J(U_J^D)}^{U^D}(\chi)(u) = \chi(\text{st}_J^{-1}(u))$ for $u \in U_{2|A|}^D(q) \times U_{2|A^c|}^D(q)$.

We must show that this map is well-defined, i.e., takes superclass functions to superclass functions. Since supercharacters of type D are restrictions of supercharacters of type A (see [7, Proposition 2.2]), Eq. (3.6) can be written as

$$\text{Res}_{\text{st}_J(U_J^D)}^{U^D}(\zeta_{U^D})(u) = \zeta_{U^D}(\text{st}_J^{-1}(u))$$

for some ζ supercharacter of type A such that its restriction ζ_{U^D} to U^D is precisely χ . Also, in [14, Theorem 6.4], Diaconis and Isaacs prove that superclass functions of type A restrict to superclass functions. Putting these facts together we conclude that the restriction map sends superclass functions of type D to superclass functions.

Now, define the coproduct on supercharacters as

$$\Delta(\chi) := \sum_J \text{Res}_{\text{st}_J(U_J^D)}^{U^D}(\chi)$$

summing over all ordered D -set partitions $J = (A|A^c)$ of $[\pm n]$.

Given a subset A of $[n]$ and $\lambda \in D_{2n}(q)$ let $\lambda|_A$ denote the restriction of λ to the set $[\pm A]$. This is the ground set of $\lambda|_A$ is $[\pm A]$ and $i \overset{a}{\curvearrowright} j \in \lambda|_A$ if $i \overset{a}{\curvearrowright} j \in \lambda$. Now, denoting by $\mathcal{A}(\lambda)$ the set of arcs of λ we see that $\mathcal{A}(\lambda|_A) \sqcup \mathcal{A}(\lambda|_{A^c}) \subseteq \mathcal{A}(\lambda)$. When the equality $\mathcal{A}(\lambda|_A) \cup \mathcal{A}(\lambda|_{A^c}) = \mathcal{A}(\lambda)$ holds we write $\lambda = \lambda|_A \sqcup \lambda|_{A^c}$.

The following proposition tells us how to compute the coproduct in the superclass characteristic functions.

Proposition 3.8 *Let λ be a $D_{2n}(q)$ -partition. Then*

$$\Delta(\kappa_\lambda) = \sum_{\lambda = \lambda|_A \sqcup \lambda|_{A^c}} \kappa_{\text{st}_A(\lambda|_A)} \otimes \kappa_{\text{st}_{A^c}(\lambda|_{A^c})}$$

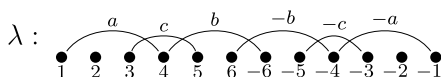
summing over all $A \subseteq [n]$ such that $\lambda = \lambda|_A \sqcup \lambda|_{A^c}$.

Proof For an ordered D -set composition $J = (A|A^c)$ we have $\text{st}_A(\lambda|_A) \in D_{2|A|}(q)$ and $\text{st}_{A^c}(\lambda|_{A^c}) \in D_{2|A^c|}(q)$. Given $\mu \in D_{2|A|}(q)$ and $\nu \in D_{2|A^c|}(q)$ denote by $x_{\mu \times \nu}$ the natural orbit superclass representative indexed by $\mu \times \lambda$ in the group $U_{2|A|}^D(q) \times U_{2|A^c|}^D(q)$. Then we have

$$\text{Res}_{\text{st}_J(U_J^D)}^{U^D}(\kappa_\lambda)(x_{\mu \times \nu}) = \begin{cases} 1 & \text{if } \lambda|_A = \text{st}_J^{-1}(\mu) \text{ and } \lambda|_{A^c} = \text{st}_J^{-1}(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

This means that $\text{Res}_{\text{st}_J(U_J^D)}^{U^D}(\kappa_\lambda)(x_{\mu \times \nu}) \neq 0$ when $\text{st}_J(\lambda) = \mu \times \nu$. This concludes the proof. \square

Example 3.9 Let $\lambda \in D_{12}(q)$ given by



Then,

$$\begin{aligned} \Delta(\kappa_\lambda) = & \kappa_\lambda \otimes \kappa_\emptyset + \kappa_{\begin{smallmatrix} a & b-b-a \\ 1 & 2 & 3-3-2-1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} c & -c \\ 1 & 2 & 3-3-2-1 \end{smallmatrix}} + \kappa_{\begin{smallmatrix} c & -c \\ 1 & 2 & 3-3-2-1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} a & b-b-a \\ 1 & 2 & 3-3-2-1 \end{smallmatrix}} \\ & + \kappa_{\begin{smallmatrix} a & b-b-a \\ 1 & 2 & 3 & 4-4-3-2-1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} c & -c \\ 1 & 2-2-1 \end{smallmatrix}} + \kappa_{\begin{smallmatrix} c & -c \\ 1 & 2-2-1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} a & b-b-a \\ 1 & 2 & 3 & 4-4-3-2-1 \end{smallmatrix}} \\ & + \kappa_{\begin{smallmatrix} 1 & -1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} a & c & b & -b & -c & -a \\ 1 & 2 & 3 & 4 & 5-5-4-3-2-1 \end{smallmatrix}} + \kappa_{\begin{smallmatrix} a & c & b & -b & -c & -a \\ 1 & 2 & 3 & 4 & 5-5-4-3-2-1 \end{smallmatrix}} \otimes \kappa_{\begin{smallmatrix} 1 & -1 \end{smallmatrix}} + \kappa_\emptyset \otimes \kappa_\lambda \end{aligned}$$

It is not hard to see that the coproduct is coassociative. Also, notice that some of the beauty of this coalgebra structure is that it is directly connected to representation theory, as is the case in type A . We now define a multiplication in the space \mathbf{SC}^D as follows.

Definition 3.10 For λ, μ labelled D -partitions of $[\pm k], [\pm(n-k)]$, respectively, define

$$\kappa_\lambda \cdot \kappa_\mu := \sum_{v \in D_{2n}(q)} \kappa_v$$

summing over all $D_{2n}(q)$ -partitions v such that $v|_{[\pm k]} = \lambda$ and $v|_{[\pm(k+1, \dots, n)]} = \mu \uparrow^k$ where, for $1 \leq i, j \leq n-k$

$$\mu \uparrow^k = \{(k+i) \frown (k+j) : i \frown j \in \mu\} \cup \{(k+i) \frown (-k-j) : i \frown -j \in \mu\}$$

Example 3.11 Denote $-i$ by \bar{i} , then

$$\begin{aligned} \kappa_{\begin{array}{c} a \quad \bar{a} \\ \bullet \quad \bullet \\ 1 \quad 2 \quad 2 \quad 1 \end{array}} \cdot \kappa_{\begin{array}{c} b \quad \bar{b} \\ \bullet \quad \bullet \\ 1 \quad 2 \quad 2 \quad 1 \end{array}} &= \kappa_{\begin{array}{c} a \quad b \quad \bar{b} \quad \bar{a} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 3 \quad 2 \quad 1 \end{array}} + \sum_{c \in \mathbb{R}_q^*} \kappa_{\begin{array}{c} a \quad c \quad b \quad \bar{b} \quad \bar{c} \quad \bar{a} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 3 \quad 2 \quad 1 \end{array}} \\ &+ \sum_{c \in \mathbb{R}_q^*} \kappa_{\begin{array}{c} a \quad c \quad b \quad \bar{b} \quad \bar{c} \quad \bar{a} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 3 \quad 2 \quad 1 \end{array}} \end{aligned}$$

Since we want to induce an algebra structure on \mathbf{SC}^D , we need to prove that Definition 3.10 is indeed a product in the sense that it should be associative. This will be shown once we introduce the P -basis. In order to motivate somehow this definition of the product, besides being a “natural” way of doing it, in type A the product structure is raised from the *inflation* map on superclass functions of that type. When trying to obtain the product from representation theory in type D , the analogous inflation map in this case failed in the sense that superclass functions are not mapped to superclass functions anymore. For this reason, instead of deducing Definition 3.10 from a representation-theory point of view, the product was directly defined in this way. Nevertheless, in Proposition 3.15, we will see that the connection with representation theory remains strong. As a final remark, before proving the main result of this paper, this product differs from the one defined for \mathbf{SC} in [3]. The difference is that here we do not concatenate λ and μ . Instead, we put μ in between λ^+ and λ^- . This resembles the product defined in [1, Sect. 3.5] for the Hopf monoid \mathbf{Pal} of palindromic set compositions. In Sect. 4.2 we point out that this supercharacter theory of type D , in particular, carries a Hopf monoid structure. Yet this Hopf monoid is different from the Hopf monoid \mathbf{Pal} , since their coproduct structures are different. We will give a brief description of the Hopf monoid \mathbf{SC}^D in Sect. 4.2.

Next, we define a different basis for \mathbf{SC} , in order to make computations easier.

Definition 3.12 Let λ, μ be $D_{2n}(q)$ -partitions. We say that $\lambda \leq \mu$ if $\mathcal{A}(\lambda) \subseteq \mathcal{A}(\mu)$ where $\mathcal{A}(\lambda)$ denotes the set of arcs in λ .

Given λ , we denote by P_λ the superclass function defined as

$$P_\lambda := \sum_{\mu \geq \lambda} \kappa_\mu$$

From here, we see that $\{P_\lambda\}_{\lambda \in D_{2n}(q)}$ forms a basis for \mathbf{SC}^D as $n \geq 0$. This basis is called *P-basis*.

Proposition 3.13 *The P-basis multiplies and comultiplies as follows:*

(a) *For μ, ν labelled D-partitions of $[\pm k], [\pm(n-k)]$, respectively, we have*

$$P_\mu \cdot P_\nu = P_{\mu \sqcup \nu \uparrow^k} \quad (3.7)$$

(b) *For $\lambda \in D_{2n}(q)$ we have*

$$\Delta(P_\lambda) = \sum P_{\text{st}_A(\mu)} \otimes P_{\text{st}_{A^c}(\nu)} \quad (3.8)$$

summing over all subsets $A \subseteq [n]$ such that $\lambda = \mu \sqcup \nu$ where $\lambda|_A = \mu$ and $\lambda|_{A^c} = \nu$.

Proof (a) The left hand side of (3.7) gives us

$$P_\mu \cdot P_\nu = \left(\sum_{\sigma \geq \mu} \kappa_\sigma \right) \cdot \left(\sum_{\delta \geq \nu} \kappa_\delta \right) = \sum_{\sigma \geq \mu} \sum_{\delta \geq \nu} \kappa_\sigma \cdot \kappa_\delta$$

Notice that the minimal element in this last equality corresponds to $\kappa_{\mu \sqcup \nu \uparrow^k}$, where \sqcup stands for disjoint union, and every other term in each $\kappa_\sigma \cdot \kappa_\delta$ is associated to a $D_{2n}(q)$ -partition τ such that $\tau > \mu \sqcup \nu \uparrow^k$. On the other hand

$$P_{\mu \sqcup \nu \uparrow^k} = \sum_{\tau \geq \mu \sqcup \nu \uparrow^k} \kappa_\tau$$

This concludes part (a). To prove part (b) notice that from the left hand side of (3.8) we have

$$\Delta(P_\lambda) = \Delta\left(\sum_{\delta \geq \lambda} \kappa_\delta\right) = \sum_{\delta \geq \lambda} \sum_{\substack{A \subseteq [n] \\ \delta = \tau \sqcup \sigma}} \kappa_{\text{st}_A(\tau)} \otimes \kappa_{\text{st}_{A^c}(\sigma)}$$

On the other hand, the right hand side of (3.8) gives us

$$\sum_{\lambda = \mu \sqcup \nu} P_{\text{st}_A(\mu)} \otimes P_{\text{st}_{A^c}(\nu)} = \sum_{\lambda = \mu \sqcup \nu} \sum_{\tau \geq \mu} \sum_{\sigma \geq \nu} \kappa_{\text{st}_A(\tau)} \otimes \kappa_{\text{st}_{A^c}(\sigma)}$$

Now, every $\delta \geq \lambda$ such that $\delta = \tau \sqcup \sigma$ is such that $\lambda = (\tau \cap \lambda) \sqcup (\sigma \cap \lambda)$. This last decomposition of λ can be written as $\lambda = \mu \sqcup \nu$ and we see that $\tau \geq \mu$, $\sigma \geq \nu$. Similarly, If $\lambda = \mu \sqcup \nu$ then $\delta = \tau \sqcup \sigma$ is such that $\delta \geq \lambda$, for $\tau \geq \mu$, $\sigma \geq \nu$.

The proposition follows. \square

Notice that by the simplicity of the multiplication in the P -basis, we see that the Definition 3.10 gives an associative operation. Indeed, for λ, ν, μ labelled D -partitions of $[\pm k], [\pm l], [\pm m]$, respectively, we have

$$\begin{aligned}(P_\lambda \cdot P_\mu) \cdot P_\nu &= P_{\lambda \sqcup \mu \uparrow^k} \cdot P_\nu \\ &= P_{\lambda \sqcup \mu \uparrow^k \sqcup \nu \uparrow^{k+l}} \\ &= P_{\lambda \sqcup (\mu \sqcup \nu \uparrow^l) \uparrow^k} \\ &= P_\lambda \cdot P_{\mu \sqcup \nu \uparrow^l} = P_\lambda \cdot (P_\mu \cdot P_\nu)\end{aligned}$$

Also, it follows that the space \mathbf{SC} is free. The cofreeness is also guaranteed, following arguments analog to the ones exposed in [11], but since this is not too relevant for our main results we skip the details.

We have that \mathbf{SC}^D is graded, connected, has a unit $\kappa_\emptyset \in \mathbf{SC}_0^D$ and a counit $\epsilon : \mathbf{SC} \rightarrow \mathbb{C}$ obtained by taking the coefficient of κ_\emptyset . In order to get a bialgebra structure, as stated in the preliminaries, most of the compatibilities coming from the requirement on the maps m, u, Δ, ϵ are straightforward to check. The compatibility between the product and the coproduct is less obvious and is what will allow us to conclude the main result of this paper. Namely, we want to prove that $\Delta(P_\mu \cdot P_\nu) = \Delta(P_\mu) \cdot \Delta(P_\nu)$. Now we are ready to prove the main theorem.

Theorem 3.14 *The product and coproduct given in Proposition 3.13, provides the space \mathbf{SC}^D with a Hopf algebra structure.*

Proof We prove only the compatibility relation between the product and the coproduct as explained in the previous paragraph. Let $\lambda \in D_{2k}(q)$, $\mu \in D_{2(n-k)}(q)$, then

$$\begin{aligned}\Delta(P_\lambda) \cdot \Delta(P_\mu) &= \left(\sum_{\substack{\lambda = \tau_1 \sqcup \sigma_1 \\ B \subseteq [k]}} P_{\text{st}_B(\tau_1)} \otimes P_{\text{st}_{B^c}(\sigma_1)} \right) \left(\sum_{\substack{\mu = \tau_2 \sqcup \sigma_2 \\ C \subseteq [n-k]}} P_{\text{st}_C(\tau_2)} \otimes P_{\text{st}_{C^c}(\sigma_2)} \right) \\ &= \sum_{\substack{\lambda = \tau_1 \sqcup \sigma_1 \\ B \subseteq [k]}} \sum_{\substack{\mu = \tau_2 \sqcup \sigma_2 \\ C \subseteq [n-k]}} P_{\text{st}_B(\tau_1)} P_{\text{st}_C(\tau_2)} \otimes P_{\text{st}_{B^c}(\sigma_1)} P_{\text{st}_{C^c}(\sigma_2)} \\ &= \sum_{\substack{\lambda = \tau_1 \sqcup \sigma_1 \\ B \subseteq [k]}} \sum_{\substack{\mu = \tau_2 \sqcup \sigma_2 \\ C \subseteq [n-k]}} P_{\text{st}_B(\tau_1) \sqcup \text{st}_C(\tau_2 \uparrow^{|B|})} \otimes P_{\text{st}_{B^c}(\sigma_1) \sqcup \text{st}_{C^c}(\sigma_2 \uparrow^{|C|})} \quad (3.9)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Delta(P_\lambda \cdot P_\mu) &= \Delta(P_\gamma) \quad \text{where } \gamma = \lambda \sqcup \mu \uparrow^k \\ &= \sum_{\substack{\gamma = \tau \sqcup \sigma \\ A \subseteq [n]}} P_{\text{st}_A(\tau)} \otimes P_{\text{st}_{A^c}(\sigma)} \quad \text{with } \gamma|_A = \tau, \gamma|_{A^c} = \sigma \quad (3.10)\end{aligned}$$

Now, since $\gamma = \lambda \sqcup \mu \uparrow^k$, then we can decompose τ and σ in the form

$$\tau = \tau_1 \sqcup \tau_2, \quad \sigma = \sigma_1 \sqcup \sigma_2$$

such that τ_1, σ_1 only intersect λ and similarly, τ_2, σ_2 only intersect $\mu \uparrow^k$. This decomposition induces a decomposition on the set $A = B \sqcup C$, so that the last equality in (3.10) becomes

$$\begin{aligned} \Delta(P_\lambda \cdot P_\mu) &= \sum_{\substack{\gamma=(\tau_1 \sqcup \tau_2) \sqcup (\sigma_1 \sqcup \sigma_2) \\ B \sqcup C \subseteq [n]}} P_{\text{st}_{B \sqcup C}(\tau_1 \sqcup \tau_2)} \otimes P_{\text{st}_{B^c \sqcup C^c}(\sigma_1 \sqcup \sigma_2)} \\ &= \sum_{\substack{\gamma=(\tau_1 \sqcup \tau_2) \sqcup (\sigma_1 \sqcup \sigma_2) \\ B \sqcup C \subseteq [n]}} P_{\text{st}_B(\tau_1) \sqcup \text{st}_C(\tau_2 \uparrow^{|B|})} \otimes P_{\text{st}_{B^c}(\sigma_1) \sqcup \text{st}_{C^c}(\sigma_2 \uparrow^{|B^c|})} \\ &= \sum_{\substack{\lambda=\tau_1 \sqcup \sigma_1 \\ B \subseteq [k]}} \sum_{\substack{\mu=\tau_2 \sqcup \sigma_2 \\ C \subseteq [n-k]}} P_{\text{st}_B(\tau_1) \sqcup \text{st}_C(\tau_2 \uparrow^{|B|})} \otimes P_{\text{st}_{B^c}(\sigma_1) \sqcup \text{st}_{C^c}(\sigma_2 \uparrow^{|C|})} \end{aligned}$$

Putting together this last equality with Eq. (3.9), we can conclude that the desired compatibility holds. \square

This allows us to conclude that the space \mathbf{SC}^D is indeed a combinatorial Hopf algebra as defined in the preliminaries.

We want to point out that different definitions of a combinatorial Hopf algebra can be given depending on the purposes. An alternative definition is as follows. A Hopf algebra \mathcal{A} is a combinatorial Hopf algebra if it is graded, connected and has a distinguished character $\zeta : \mathcal{A} \rightarrow \mathbb{K}$. This singled out character is given by the trivial character in the case when the Hopf structure on \mathcal{A} arises from representation theory (see [2]).

As mentioned in the introduction, the product structure on the space \mathbf{SC}^D has a very interesting behaviour in the supercharacter basis.

Proposition 3.15 *Let $\lambda \in D_{2n}(q)$ and $\mu \in D_{2m}(q)$. Then,*

$$\chi^\lambda \cdot \chi^\mu = \chi^{\lambda \sqcup \mu \uparrow^n}$$

Proof Let us consider the following expansions in the κ basis. Let $\#\text{nest}_\mu^\lambda = |\{k \prec l \in \mu^+ | i \prec k \prec l \prec j, i \prec j \in \lambda^+\}|$ then by Eq. (3.2) we have

$$\begin{aligned} \chi^\lambda &= \chi^\lambda(1) \left[\sum_{\alpha} \frac{1}{q^{\#\text{nest}_{\alpha}^{\lambda}}} \prod_{i \prec j \in \lambda, i \prec j \in \alpha} \theta(ab) \kappa_{\alpha} \right] \\ \chi^\mu &= \chi^\mu(1) \left[\sum_{\beta} \frac{1}{q^{\#\text{nest}_{\beta}^{\mu}}} \prod_{i \prec j \in \mu, i \prec j \in \beta} \theta(cd) \kappa_{\beta} \right] \\ \chi^{\lambda \sqcup \mu \uparrow^n} &= \chi^{\lambda \sqcup \mu \uparrow^n}(1) \left[\sum_{\gamma} \frac{1}{q^{\#\text{nest}_{\gamma}^{\lambda \sqcup \mu \uparrow^n}}} \prod_{i \prec j \in \lambda \sqcup \mu \uparrow^n, i \prec j \in \gamma} \theta(e f) \kappa_{\gamma} \right] \end{aligned}$$

where the sum is over every α, β, γ such that $\chi^\lambda(x_\alpha), \chi^\mu(x_\beta), \chi^{\lambda \sqcup \mu \uparrow n}(x_\gamma)$ are non-zero, respectively. Here, x_v is the superclass representative corresponding to the partition v , as before.

By using the previous expansions we get

$$\begin{aligned} \chi^\lambda \cdot \chi^\mu &= \chi^\lambda(1) \chi^\mu(1) \left[\sum_{\alpha, \beta} \frac{1}{q^{\#\text{nest}_\alpha^\lambda + \#\text{nest}_\beta^\mu}} \left(\prod_{\substack{i^a j \in \lambda, i^b j \in \alpha \\ i^c j \in \mu, i^d j \in \beta}} \theta(ab) \theta(cd) \right) \kappa_\alpha \cdot \kappa_\beta \right] \\ &= \chi^{\lambda \sqcup \mu \uparrow n}(1) \left[\sum_{\gamma} \frac{1}{q^{\#\text{nest}_\gamma^{\lambda \sqcup \mu \uparrow n}}} \prod_{\substack{i^a j \in \lambda, i^b j \in \gamma|[\pm n] \\ i^c j \in \mu, i^d j \in \gamma|[\pm m] \uparrow n}} \theta(ab) \theta(cd) \kappa_\gamma \right] \\ &= \chi^{\lambda \sqcup \mu \uparrow n}(1) \left[\sum_{\gamma} \frac{1}{q^{\#\text{nest}_\gamma^{\lambda \sqcup \mu \uparrow n}}} \prod_{i^e j \in \lambda \sqcup \mu \uparrow n, i^f j \in \gamma} \theta(e f) \kappa_\gamma \right] \\ &= \chi^{\lambda \sqcup \mu \uparrow n}. \end{aligned} \quad \square$$

Although the analog of the inflation map of type A does not work in this case, in the sense that a similar projection map does not hold the wanted properties, this proposition indicates that certain “inflation” underlines the product structure in the supercharacter basis. This result gives a stronger connection of this combinatorial Hopf algebra with representation theory. Now the question that remains to be explored is, what representation theoretic functor is playing the role of inflation in this type? The author would be happy to hear any answer in this direction.

We finish this paper by giving a brief outline concerning the types B and C .

4 Final comments

4.1 Type B and type C

Following the construction in [6], supercharacters and superclasses for types B and C are also indexed by labelled partitions of the corresponding type. The unipotent upper triangular matrices of type B is the group of $m \times m$ orthogonal matrices where $m = 2n + 1$ for some $n \in \mathbb{Z}_{\geq 0}$. We define $B_m(q)$ -partitions as labelled set partitions on the set $\{1, \dots, n, 0, -n, \dots, -1\}$ satisfying the same properties as $D_{2n}(q)$ -partitions with the additional property that we allow at most one arc of the form $i \xrightarrow{a} 0$ together with $0 \xrightarrow{-a} -i$.

Unfortunately, any attempt from the author to construct a product on \mathbf{SC}^B fails since dealing with odd-size matrices make impossible an embedding from $\mathbf{SC}_{2k+1}^B \times \mathbf{SC}_{2l+1}^B$ to $\mathbf{SC}_{2(k+l)+1}^B$, although a different grading and suitable changes could make it possible. On the other hand, we have a structure of \mathbf{SC}^D -module on \mathbf{SC}^B , since it is clear that $\mathbf{SC}_{2k}^D \times \mathbf{SC}_{2l+1}^B$ embeds into $\mathbf{SC}_{2(k+l)+1}^B \in \mathbf{SC}^B$.

For unipotent upper triangular matrices of type C the situation is better. This type corresponds to the group of $2n \times 2n$ symplectic matrices and the combinatorial description for its supercharacter theory resembles the one for type D . In this case $C_{2n}(q)$ -partitions are defined as in Definition 3.1 but we also allow arcs $i \overset{a}{\curvearrowright} -i$. Similar arguments can be used in this case, producing a similar definition for product and coproduct over the graded vector space \mathbf{SC}^C endowing it with a Hopf algebra structure.

4.2 Forthcoming work

We remind the reader that this paper has considered only the case when $\text{char}(\mathbb{F}_q) \geq 3$. The case $p = 2$ requires a different description of the elements of the group $U_{2n}^D(q)$. We want to understand this case as well, since this might allow us to have an unlabelled version of what we have done here.

On the other hand, a coarser version of the supercharacter theory of type D as exposed here could have some connection with the case $q = 2$. Namely, by lumping together conjugacy classes in $U_{2n}^D(q)$ through the action $(T_{2n}U_{2n}(q)AU_{2n}(q)T_{2n}^{-1} + I_{2n}) \cap U_{2n}^D(q)$, where T_{2n} is the subgroup of diagonal matrices of $GL_{2n}(q)$, $A \in \mathfrak{u}^D(q)$, gives a coarser superclass theory, whose supercharacter values are integers. Hence, the unlabelled version of the Hopf algebra constructed here would realize the version given by this super-theory. This is inspired by the work done in [10].

Finally, we want to point out that types C and D not only have a Hopf algebra structure, a Hopf monoid structure can be provided too. For a basic background on Hopf monoid in species the reader can consult [4].

Briefly, let the species \mathbf{SC}^D be such that for a finite set K

$$\mathbf{SC}^D[K] = \bigoplus_{\phi \in L[K]} \mathbf{SC}^{(\phi, D)}[K]$$

where $L[K]$ is the set of linear orders on K and $\mathbf{SC}^{(\phi, D)}[K]$ being the set of $D_{2|K|}(q)$ -partitions that respect the order given by ϕ . In other words, let the set $K \sqcup \bar{K}$ be ordered by $\phi \cdot \overleftarrow{\phi}$ where \cdot denotes concatenation and \bar{K} is a second copy of K with $\overleftarrow{\phi}$ being the order of K reversed. Now, after drawing the arcs of λ on top of $K \sqcup \bar{K}$ and putting $\bar{\phi} = \phi \cdot \overleftarrow{\phi}$ we ask that if $i \overset{a}{\curvearrowright} j \in \lambda$ then $i \leq_{\bar{\phi}} j$; also λ must satisfy the analog of condition (b) in Definition 3.1, replacing $<$ by $\leq_{\bar{\phi}}$. Then we can check that for nonintersecting finite sets I, J the following maps

$$\begin{aligned} m_{I,J} : \mathbf{SC}^{(\phi, D)}[I] \otimes \mathbf{SC}^{(\tau, D)}[J] &\rightarrow \mathbf{SC}^{(\phi \cdot \tau, D)}[I \sqcup J] \\ \Delta_{I,J} : \mathbf{SC}^{(\phi, D)}[I \sqcup J] &\rightarrow \mathbf{SC}^{(\phi|_I, D)}[I] \otimes \mathbf{SC}^{(\phi|_J, D)}[J] \end{aligned}$$

defined in analogy with the structure presented here, satisfy all the axioms required to make of the species \mathbf{SC}^D a Hopf monoid. All of this is done following [4] and it is part of a future paper by the author.

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