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Remarks on the paper "Skew Pieri rules for Hall–Littlewood functions" by Konvalinka and Lauve

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Abstract In a recent paper Konvalinka and Lauve proved several skew Pieri rules for Hall–Littlewood polynomials. In this note we show that q-analogues of these rules are encoded in a q-binomial theorem for Macdonald polynomials due to Lascoux and the author.

Keywords Pieri rules · Hall–Littlewood polynomials · Macdonald polynomials

1 The Konvalinka–Lauve formulas and their q-analogues

We refer the reader to [15] for definitions concerning Hall–Littlewood and Macdonald polynomials.

Let $P_{\lambda/\mu} = P_{\lambda/\mu}(X;t)$ and $Q_{\lambda/\mu} = Q_{\lambda/\mu}(X;t)$ be the skew Hall–Littlewood polynomials, $e_r = P_{(1^r)}$ the rth elementary symmetric function, h_r the rth complete symmetric function and $q_r = Q_{(r)}$. Then the ordinary Pieri formulas for Hall–Littlewood polynomials are given by [15]

$$P_\mu e_r = \sum_\lambda \mathrm{vs}_{\lambda/\mu}(t) P_\lambda, \tag{1.1a}$$

$$P_\mu q_r = \sum_\lambda \mathrm{hs}_{\lambda/\mu}(t) P_\lambda, \tag{1.1b}$$

where the sums on the right are over partitions λ such that $|\lambda| = |\mu| + r$. The Pieri coefficient $\mathrm{vs}_{\lambda/\mu}(t)$ is given by [15, p. 215, (3.2)]

$$\mathrm{vs}_{\lambda/\mu}(t) = \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t, \tag{1.2}$$

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so that $\mathrm{vs}_{\lambda/\mu}(t)$ is zero unless $\mu \subseteq \lambda$ with $\lambda - \mu$ a vertical strip. Similarly, $\mathrm{hs}_{\lambda/\mu}(t)$ vanishes unless $\mu \subseteq \lambda$ with $\lambda - \mu$ a horizontal strip, in which case [15, p. 218, (3.10)]

$$\mathrm{hs}_{\lambda/\mu}(t) = \prod_{\substack{\lambda'_i = \mu'_i + 1 \\ \lambda'_{i+1} = \mu'_{i+1}}} \bigl(1 - t^{\lambda'_i - \lambda'_{i+1}}\bigr). \tag{1.3}$$

To express the skew Pieri formulas, Konvalinka and Lauve [9] (see also [8]) introduced a third Pieri coefficient

$$\mathrm{sk}_{\lambda/\mu}(t) := t^{n(\lambda/\mu)} \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_t, \tag{1.4}$$

where $n(\lambda/\mu) := \sum_{i\geq 1} \binom{\lambda'_i - \mu'_i}{2}$. Note that $\mathrm{sk}_{\lambda/\mu}(t) = 0$ if $\mu \not\subseteq \lambda$.

It seems Konvalinka and Lauve have been unaware that the above function has appeared in the literature before. Indeed, in exactly the above form and denoted as $g^{\lambda}_{\mu}(t)$, it was used by Kirillov to prove the Pieri rule [7, Lemma 4.1]

$$P_\mu h_r = \sum_\lambda \mathrm{sk}_{\lambda/\mu}(t) P_\lambda. \tag{1.5}$$

Moreover, $\mathrm{sk}_{\lambda/\mu}(t)$ arose in [20, Eq. (4.3)] as a formula for the modified Hall–Littlewood polynomial $Q'_{\lambda/\mu}(1) = Q_{\lambda/\mu}(1, t, t^2, \dots)$—a result first stated in [12, Theorem 3.1], albeit in the not-so-easily-recognisable form

$$Q'_{\lambda/\mu}(1) = \begin{cases} t^{n(\lambda/\mu)} \displaystyle\prod_{i=1}^{l(\mu)} \frac{1 - t^{\lambda'_{\mu_i} - i + 1}}{(t;t)_{\mu'_i - \mu'_{i+1}}} & \text{for } \mu \subseteq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In a more general form pertaining to Macdonald polynomials it also appeared in [18, p. 173, Remark 2] and [19, Proposition 3.2], see (1.8) below. Prior to the above-mentioned papers $\mathrm{sk}_{\lambda/\mu}(t)$ appeared in the theory of abelian p-groups:

$$\mathrm{sk}_{\lambda/\mu}(t) = t^{n(\lambda) - n(\mu)} \alpha_\lambda\bigl(\mu; t^{-1}\bigr),$$

where $\alpha_\lambda(\mu; p)$ is the number of subgroups of type μ in a finite abelian p-group of type λ, [2–4, 21].

Theorem 1.1 (Konvalinka–Lauve [9, Theorems 2–4]) *For partitions $\nu \subseteq \mu$,*

$$P_{\mu/\nu} e_r = \sum_{\lambda,\eta} (-1)^{|\nu - \eta|} \mathrm{vs}_{\lambda/\mu}(t)\, \mathrm{sk}_{\nu/\eta}(t) P_{\lambda/\eta}, \tag{1.6a}$$

$$P_{\mu/\nu} h_r = \sum_{\lambda,\eta} (-1)^{|\nu - \eta|} \mathrm{sk}_{\lambda/\mu}(t)\, \mathrm{vs}_{\nu/\eta}(t) P_{\lambda/\eta}, \tag{1.6b}$$

$$P_{\mu/\nu} q_r = \sum_{\lambda,\eta,\omega} (-1)^{|\nu - \omega|} t^{|\omega - \eta|} \mathrm{hs}_{\lambda/\mu}(t)\, \mathrm{vs}_{\nu/\omega}(t)\, \mathrm{sk}_{\omega/\eta}(t) P_{\lambda/\eta}, \tag{1.6c}$$

where each of the multiple sums is subject to the restriction $|\lambda| + |\eta| = |\mu| + |\nu| + r$.

For $\nu = 0$ the first and third skew Pieri formulas reduce to (1.1a) and (1.1b), respectively, whereas the second formula simplifies to (1.5) (see also [9, Theorem 1]). Theorem 1.1 for $t = 0$ gives the skew Pieri rules for Schur functions due to Assaf and McNamara [1] who, more generally, conjectured a skew Littlewood–Richardson rule. The identities (1.6a) and (1.6b) were first conjectured by Konvalinka in [8]. The subsequent proof of the theorem by Konvalinka and Lauve combines Hopf algebraic techniques in the spirit of the proof of the Assaf–McNamara conjecture [10] with intricate manipulations involving t-binomial coefficients.

The aim of this note is to point out that all of the skew Pieri formulas (1.6a)–(1.6c) are implied by a generalized q-binomial theorem for Macdonald polynomials and, consequently, have simple q-analogues.

From here on let $P_{\lambda/\mu} = P_{\lambda/\mu}(X; q, t)$ and $Q_{\lambda/\mu} = Q_{\lambda/\mu}(X; q, t)$ denote skew Macdonald polynomials. Let f be an arbitrary symmetric function. Adopting plethystic or λ-ring notation, see, e.g., [5, 11], we define $f((a-b)/(1-t))$ in terms of the power sums with positive index r as

$$p_r\left(\frac{a-b}{1-t}\right) = \frac{a^r - b^r}{1-t^r}.$$

In other words, $p_r((a-b)/(1-t)) = a^r \epsilon_{b/a,t}(p_r)$ with $\epsilon_{u,r}$ Macdonald's evaluation homomorphism [15, p. 338, (6.16)]. Equivalently, in terms of complete symmetric functions,

$$h_r\left(\frac{a-b}{1-t}\right) = \left[z^r\right] \frac{(bz;t)_\infty}{(az;t)_\infty}.$$

We now define the following five Pieri coefficients for Macdonald polynomials:

$$\mathrm{vs}_{\lambda/\mu}(q,t) := \psi'_{\lambda/\mu}(q,t) = (-1)^{|\lambda-\mu|} Q_{\lambda/\mu}\left(\frac{q-1}{1-t}\right), \tag{1.7a}$$

$$\mathrm{hs}_{\lambda/\mu}(q;t) := \varphi_{\lambda/\mu}(q,t) = Q_{\lambda/\mu}(1), \tag{1.7b}$$

$$\mathrm{sk}_{\lambda/\mu}(q,t) := Q_{\lambda/\mu}\left(\frac{1-q}{1-t}\right), \tag{1.7c}$$

$$\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t) := Q_{\lambda/\mu}\left(\frac{1-q/t}{1-t}\right), \tag{1.7d}$$

$$\mathrm{ks}_{\lambda/\mu}(q,t) := Q_{\lambda/\mu}(-1), \tag{1.7e}$$

where $\psi'_{\lambda/\mu}(q,t)$ and $\varphi_{\lambda/\mu}(q,t)$ is notation used by Macdonald, and where the -1 in $Q_{\lambda/\mu}(-1)$ is a plethystic -1, i.e., applied to the power sum p_r of positive index r it gives the number -1. The Pieri coefficients $\mathrm{vs}_{\lambda/\mu}(q,t)$ and $\mathrm{hs}_{\lambda/\mu}(q,t)$ have nice factorized forms generalising (1.2) and (1.3), see [16, pp. 336–342]. So does

$\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t)$ [18, p. 173, Remark 2], [19, Proposition 3.2]:

$$\widehat{\mathrm{sk}}_{\lambda/\mu}(q,t)=\begin{cases} t^{n(\lambda)-n(\mu)}\prod_{i,j=1}^{l(\lambda)}\frac{(qt^{j-i-1};q)_{\lambda_i-\mu_j}(qt^{j-i};q)_{\mu_i-\mu_j}}{(qt^{j-i-1};q)_{\mu_i-\mu_j}(qt^{j-i};q)_{\lambda_i-\mu_j}} & \text{for } \mu\subseteq\lambda,\\ 0 & \text{otherwise,}\end{cases} \tag{1.8}$$

where $(a;q)_k:=(a;q)_\infty/(aq^k;q)_\infty$ for all $k\in\mathbb{Z}$. We leave it to the reader to verify that the above right-hand side for $q=0$ reduces to the right-hand side of (1.4). The remaining two Pieri coefficients do not factor into binomials. For example

$$\mathrm{sk}_{(2,1)/(1,0)}(q,t)=\frac{1-q-q^2+t+qt-q^2t}{1-q^2t},$$

$$\mathrm{ks}_{(2,1)/(1,0)}(q,t)=\frac{(1-t)(1+q-t+qt-t^2-qt^2)}{(1-q)(1-q^2t)}.$$

Of course, $\mathrm{sk}_{\lambda/\mu}(0,t)=\mathrm{sk}_{\lambda/\mu}(t)$ so it does factorize in the classical limit. This is, however, not the case for $\mathrm{ks}_{\lambda/\mu}(0,t)$, and

$$\mathrm{ks}_{(2,1)/(1,0)}(0,t)=(1-t)\big(1-t-t^2\big).$$

Let $g_r=g_r(X;q,t)=Q_{(r)}(X;q,t)$, so that $g_r(X;0,t)=q_r(X;t)$. Then the following q-analogue of Theorem 1.1 holds.

Theorem 1.2 *For partitions $\nu\subseteq\mu$,*

$$P_{\mu/\nu}e_r=\sum_{\lambda,\eta}(-1)^{|\nu-\eta|}\,\mathrm{vs}_{\lambda/\mu}(q,t)\,\mathrm{sk}_{\nu/\eta}(q,t)P_{\lambda/\eta}, \tag{1.9a}$$

$$P_{\mu/\nu}h_r=\sum_{\lambda,\eta}(-1)^{|\nu-\eta|}\,\mathrm{sk}_{\lambda/\mu}(q,t)\,\mathrm{vs}_{\nu/\eta}(q,t)P_{\lambda/\eta}, \tag{1.9b}$$

$$P_{\mu/\nu}g_r=\sum_{\lambda,\eta}\mathrm{hs}_{\lambda/\mu}(q,t)\,\mathrm{ks}_{\nu/\eta}(q,t)P_{\lambda/\eta} \tag{1.9c}$$

$$=\sum_{\lambda,\eta,\omega}(-1)^{|\nu-\omega|}t^{|\omega-\eta|}\,\mathrm{hs}_{\lambda/\mu}(q,t)\,\mathrm{vs}_{\nu/\omega}(q,t)\,\widehat{\mathrm{sk}}_{\omega/\eta}(q,t)P_{\lambda/\eta}, \tag{1.9d}$$

where each of the multiple sums is subject to the restriction $|\lambda|+|\eta|=|\mu|+|\nu|+r$.

2 The q-binomial theorem for Macdonald polynomials

In [14, Eq. (2.11)] Lascoux and the author proved the following q-binomial theorem for Macdonald polynomials:

$$\sum_{\lambda}Q_{\lambda/\nu}\left(\frac{a-b}{1-t}\right)P_{\lambda/\mu}(X)=\left(\prod_{x\in X}\frac{(bx;q)_\infty}{(ax;q)_\infty}\right)\sum_{\lambda}Q_{\mu/\lambda}\left(\frac{a-b}{1-t}\right)P_{\nu/\lambda}(X). \tag{2.1}$$

For $\mu=\nu=0$ and $(a,b)\mapsto(1,a)$ this is the well-known Kaneko–Macdonald q-binomial theorem [6, 16]

$$\sum_{\lambda} t^{n(\lambda)}\frac{(a)_\lambda}{c'_\lambda}P_\lambda(X)=\prod_{x\in X}\frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{2.2}$$

where we have used [15, p. 338, (6.17)]

$$Q_\lambda\left(\frac{1-a}{1-t}\right)=t^{n(\lambda)}\frac{(a)_\lambda}{c'_\lambda}.$$

Here $(a)_\lambda=(a;q,t)_\lambda:=\prod_{i\geq 1}(at^{1-i};q)_{\lambda_i}$ and $c'_\lambda=c'_\lambda(q,t)$ is the generalized hook polynomial $c'_\lambda=\prod_{s\in\lambda}(1-q^{a(s)+1}t^{l(s)})$ with $a(s)$ and $l(s)$ the arm-length and leg-length of the square $s\in\lambda$.

To show that (2.1) encodes the skew Pieri formulas (1.9a)–(1.9d) we first consider the $\mu=0$ case

$$\sum_{\lambda} Q_{\lambda/\nu}\left(\frac{a-b}{1-t}\right)P_\lambda(X)=P_\nu(X)\prod_{x\in X}\frac{(bx;q)_\infty}{(ax;q)_\infty}. \tag{2.3}$$

If we multiply this by $Q_{\nu/\mu}((b-a)/(1-t))$ and sum over ν using (2.3) with $(\lambda,\nu,a,b)\mapsto(\nu,\mu,b,a)$ we obtain

$$\sum_{\lambda,\nu} Q_{\lambda/\nu}\left(\frac{a-b}{1-t}\right)Q_{\nu/\mu}\left(\frac{b-a}{1-t}\right)P_\lambda(X)=P_\mu(X).$$

This implies the orthogonality relation (implicit in [17] and given in its more general nonsymmetric form in [13, Eq. (6.5)])

$$\sum_{\nu} Q_{\lambda/\nu}\left(\frac{a-b}{1-t}\right)Q_{\nu/\mu}\left(\frac{b-a}{1-t}\right)=\delta_{\lambda\mu}. \tag{2.4}$$

Thanks to (2.4), identity (2.1) is equivalent to

$$\sum_{\lambda,\eta} Q_{\nu/\eta}\left(\frac{a-b}{1-t}\right)Q_{\lambda/\mu}\left(\frac{b-a}{1-t}\right)P_{\lambda/\eta}(X)=P_{\mu/\nu}(X)\prod_{x\in X}\frac{(ax;q)_\infty}{(bx;q)_\infty}.$$

There are now three special cases to consider. First, if $b=aq$ then

$$P_{\mu/\nu}(X)\prod_{x\in X}(1-ax)=\sum_{\lambda,\eta} a^{|\lambda-\mu|+|\nu-\eta|}Q_{\lambda/\mu}\left(\frac{q-1}{1-t}\right)Q_{\nu/\eta}\left(\frac{1-q}{1-t}\right)P_{\lambda/\eta}(X).$$

Equating coefficients of $(-a)^r$ and using definition (1.7a) and (1.7c) yields (1.9a). Next, if $a=bq$

$$P_{\mu/\nu}(X)\prod_{x\in X}\frac{1}{1-bx}=\sum_{\lambda,\eta} b^{|\lambda-\mu|+|\nu-\eta|}Q_{\lambda/\mu}\left(\frac{1-q}{1-t}\right)Q_{\nu/\eta}\left(\frac{q-1}{1-t}\right)P_{\lambda/\eta}(X).$$

Equating coefficients of b^r and again using (1.7a) and (1.7c) yields (1.9b). Finally, if $a = bt$

$$P_{\mu/\nu}(X)\prod_{x\in X}\frac{(btx;q)_\infty}{(bx;q)_\infty}=\sum_{\lambda,\eta}b^{|\lambda-\mu|+|\nu-\eta|}Q_{\lambda/\mu}(1)Q_{\nu/\eta}(-1)P_{\lambda/\eta}(X).$$

Equating coefficients of b^r and using (1.7b) and (1.7e) gives (1.9c). To show that (1.9c) and (1.9d) are equivalent, we recall Rains' q-Pfaff–Saalschütz summation for Macdonald polynomials [17, Corollary 4.9]:

$$\sum_\nu\frac{(a)_\nu}{(c)_\nu}Q_{\lambda/\nu}\left(\frac{a-b}{1-t}\right)Q_{\nu/\mu}\left(\frac{b-c}{1-t}\right)=\frac{(a)_\mu(b)_\lambda}{(b)_\mu(c)_\lambda}Q_{\lambda/\mu}\left(\frac{a-c}{1-t}\right),\tag{2.5}$$

which for $c = a$ is (2.4). Setting $b = a/q$ and $c = a/t$ and using (1.7a), (1.7d) and (1.7e) yields

$$\mathrm{ks}_{\lambda/\mu}(q,t)=(t/q)^{|\lambda-\mu|}\frac{(a/q)_\mu(a/t)_\lambda}{(a)_\mu(a/q)_\lambda}\sum_\nu(-1)^{|\lambda-\nu|}\frac{(a)_\nu}{(a/t)_\nu}\mathrm{vs}_{\lambda/\nu}(q,t)\widehat{\mathrm{sk}}_{\nu/\mu}(q,t).$$

Taking the $a \to \infty$ limit this further simplifies to

$$\mathrm{ks}_{\lambda/\mu}(q,t)=\sum_\nu(-1)^{|\lambda-\nu|}t^{|\nu-\mu|}\,\mathrm{vs}_{\lambda/\nu}(q,t)\,\widehat{\mathrm{sk}}_{\nu/\mu}(q,t),$$

which proves the equality between (1.9c) and (1.9d).

To conclude let us mention that all other identities of [9] admit simple q-analogues. For example, if we take (2.5) and specialize $b = a/q$ and $c = at$ then

$$\sum_\mu\frac{(a)_\mu}{(at)_\mu}(-1)^{|\lambda-\mu|}\,\mathrm{vs}_{\lambda/\mu}(q,t)Q_{\mu/\nu}\left(\frac{1-qt}{1-t}\right)=\frac{(a)_\nu(a/q)_\lambda}{(a/q)_\nu(at)_\lambda}q^{|\lambda-\nu|}\,\mathrm{hs}_{\lambda/\nu}(q,t).$$

Letting $a \to \infty$ this reduces to

$$\sum_\mu(-t)^{|\lambda-\mu|}\,\mathrm{vs}_{\lambda/\mu}(q,t)Q_{\mu/\nu}\left(\frac{1-qt}{1-t}\right)=\mathrm{hs}_{\lambda/\nu}(q,t).$$

For $q = 0$ this is [9, Lemma 5]

$$\sum_\mu(-t)^{|\lambda-\mu|}\,\mathrm{vs}_{\lambda/\mu}(t)\,\mathrm{sk}_{\mu/\nu}(t)=\mathrm{hs}_{\lambda/\nu}(t).$$

Similarly, according to [13, Eq. (6.23)]

$$\sum_\nu t^{n(\nu)}\frac{(a)_\nu}{c'_\nu}f^\lambda_{\mu\nu}(q,t)=Q_{\lambda/\mu}\left(\frac{1-a}{1-t}\right).\tag{2.6}$$

For $a = q = 0$ this is [7, Corollary 4.2], [9, Corollary 6]

$$\sum_\nu t^{n(\nu)}f^\lambda_{\mu\nu}(t)=\mathrm{sk}_{\lambda/\mu}(t).$$

Finally, to obtain a q-analogue of [9, Theorem 7] we have to work a little harder. First note that

$$\begin{aligned} P_\nu(X)e_m(X)\sum_{r=0}^{\infty}h_r(X) &= \sum_{\mu}\mathrm{sk}_{\mu/\nu}(q,t)P_\mu(X)e_m(X) \\ &= \sum_{\mu}\sum_{\substack{\lambda\\ |\lambda-\mu|=m}}\mathrm{vs}_{\lambda/\mu}(q,t)\,\mathrm{sk}_{\mu/\nu}(q,t)P_\lambda(X). \end{aligned} \tag{2.7}$$

To compute this in a different way, observe that if we set $a=q$ in (2.2) then

$$\sum_{\lambda}t^{n(\lambda)}\frac{(q)_\lambda}{c'_\lambda}P_\lambda(X)=\prod_{x\in X}\frac{1}{1-x}=\sum_{r=0}^{\infty}h_r(X).$$

Using this as well as $e_m=P_{(1^m)}$ we get

$$P_\nu(X)e_m(X)\sum_{r=0}^{\infty}h_r(X)=\sum_{\eta}t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}P_\nu(X)P_\eta(X)P_{(1^m)}(X).$$

By a double use of $P_\mu P_\nu=f^{\lambda}_{\mu\nu}P_\lambda$ this leads to

$$\begin{aligned} P_\nu(X)e_m(X)\sum_{r=0}^{\infty}h_r(X) &= \sum_{\eta}t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\,P_\nu(X)P_\eta(X)P_{(1^m)}(X) \\ &= \sum_{\mu,\eta}t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\,f^{\mu}_{\eta,(1^m)}(q,t)P_\mu(X)P_\nu(X) \\ &= \sum_{\lambda,\mu,\eta}t^{n(\eta)}\frac{(q)_\eta}{c'_\eta}\,f^{\mu}_{\eta,(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t)P_\lambda(X) \\ &= \sum_{\lambda,\mu}\mathrm{sk}_{\mu/(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t)P_\lambda(X), \end{aligned} \tag{2.8}$$

where the final equality follows from the $a=q$ case of (2.6). Equating coefficients of $P_\lambda(X)$ in (2.7) and (2.8) yields

$$\sum_{\substack{\mu\\ |\lambda-\mu|=m}}\mathrm{vs}_{\lambda/\mu}(q,t)\,\mathrm{sk}_{\mu/\nu}(q,t)=\sum_{\mu}\mathrm{sk}_{\mu/(1^m)}(q,t)f^{\lambda}_{\mu\nu}(q,t).$$

By (1.4),

$$\mathrm{sk}_{\lambda/(1^m)}(0,t)=\mathrm{sk}_{\lambda/(1^m)}(t)=t^{n(\lambda/(1^m))}\begin{bmatrix}\lambda'_1\\ m\end{bmatrix}_t=t^{n(\lambda)-\binom{m}{2}}\begin{bmatrix}\lambda'_1\\ m\end{bmatrix}_{t^{-1}},$$

so that for $q=0$ we obtain [9, Theorem 7]

$$\sum_{\substack{\mu \\ |\lambda-\mu|=m}} \mathrm{vs}_{\lambda/\mu}(t)\,\mathrm{sk}_{\mu/\nu}(t) = \sum_{\mu} t^{n(\lambda)-\binom{m}{2}} f^{\lambda}_{\mu\nu}(t) \begin{bmatrix} \lambda'_1 \\ m \end{bmatrix}_{t^{-1}}.$$

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